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ON FREQUENCIES OF STRINGS AND DEFORMATIONS OF BEAMS

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Abstract. The paper deals with the ratio of the first two frequencies of the vibrating string and a monotonicity property of deformations of beams under symmetrization method applied to its loads.

1. Introduction. This paper deals with the ratio of the first two frequencies of the Sturm-Liouville equation. This topic and related topics have been the object of considerable attention recently (see [2], [4]–[7], [9], [11]–[16] and many others). The paper also deals with a priori estimates for, and monotonicity property of, deformation of beams.

Let $\rho(x) \geq 0$, $x \in [0, a]$ be the density of a string of unit tension, which is fixed at the end points. Then the natural frequencies of the string are determined by the eigenvalues of the differential system

$$u''(x) + \lambda\rho(x)\, u(x) = 0, \qquad u(\pm 1) = 0. \tag{1.1}$$

In §2 we are concerned with the ratio of the first two eigenvalues of (1.1) for a certain class of densities.

Let $u_n(x)$ be the nth eigenfunction of (1.1) corresponding to $\lambda_n(\rho(x))$, normalized so that

$$\int_0^a \rho(x)\, u_n^2(x)\, dx = 1. \tag{1.2}$$

We assume that $u_1(x) > 0$ on $(0, a)$ and $u_2(x) < 0$ on (x_0, a) where $u_2(x_0) = 0$, $0 < x_0 < a$. We first note that if $\rho > 0$, there are points x_-, x_+ with

$$0 \leq x_- < x_0 < x_+ < a \tag{1.3}$$

such that

$$\begin{aligned} u_2^2(x) &> u_1^2(x) \quad \text{on } (0, x_-) \cup (x_+, a) \\ u_2^2(x) &< u_1^2(x) \quad \text{on } (x_-, x_+). \end{aligned} \tag{1.4}$$

Since $u_1(x)$ and $u_2(x)$ are normalized, the set $(0, x_-) \cup (x_+, a)$ is not empty (see [12, pp. 1806-1807]).

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Next, suppose $\rho(\bullet, t)$ is a one-parameter family of piecewise continuous densities such that $\frac{\partial \rho}{\partial t}(x,t)$ exists. Let $\lambda_n(t)$ be the nth eigenvalue of (1.1) with $\rho = \rho(x,t)$ and let $u_n(x,t)$ be the corresponding normalized eigenfunction.

The formula

$$\frac{d}{dt}\left[\frac{\lambda_n(t)}{\lambda_m(t)}\right] = \frac{\lambda_n(t)}{\lambda_m(t)} \int_0^a \frac{\partial \rho(x,t)}{\partial t}\left[u_m^2(x,t) - u_n^2(x,t)\right] dx \tag{1.5}$$

where

$$\int_0^a \rho(x,t)\, u_n^2(x,t)\, dx = 1 \tag{1.6}$$

is satisfied, was derived by Keller in [12], and is used in §2.

The second part of the paper, §3, deals with the equation

$$w^4(x) = f(x), \qquad f \geq 0, \qquad x \in (-1,1). \tag{1.7}$$

This equation is the governing differential equation for the deformation $w(x)$ of a cantilever beam with a load $f \geq 0$. The boundary conditions considered in connection with the beam are:

Clamped at $x = -1$ and free at $x = 1$:

$$w(-1) = w^{'}(-1) = w^{''}(1) = w^{'''}(1) = 0. \tag{1.8}$$

Clamped at $x = \pm 1$:

$$w(\pm 1) = w^{'}(\pm 1) = 0. \tag{1.9}$$

We establish the monotonicity of the deformation of the beams under a symmetrization method applied to the load f.

A single well function is a piecewise continuous function $[-1,1]$, not increasing on $[-1,l]$ and not decreasing on $[l,1]$, $l \in [-1,1]$.

A single barrier function is a piecewise continuous function on $[-1,1]$, not decreasing on $[-1,l]$ and not increasing on $[l,1]$, $l \in [-1,1]$.

A function $f(x)$ defined on $[-1,1]$ is called left balanced if for every x, $x \in [0,1]$, $f(-x) \geq f(x)$; see [1] and [3].

For a single well function we denote the inverse function of $f(x)$ by $x_1(y)$ for $x \in [-1,l]$ and by $x_2(y)$ for $x \in [l,1]$.

We define a class of functions $f(x,a)$ by a procedure called continuous symmetrization.

For $0 \leq a \leq \frac{1}{2}$ and $x \in [-1, 2al]$ we denote the inverse function of $f(x,a)$ by $x_1(y,a)$, and for $x \in [2al, 1]$ we denote the inverse function by $x_2(y,a)$ [5, p. 200]:

$$x_1(y,a) = \left(a + \frac{1}{2}\right) x_1(y) + \left(a - \frac{1}{2}\right) x_2(y).$$

$$x_2(y,a) = \left(a + \frac{1}{2}\right) x_2(y) + \left(a - \frac{1}{2}\right) x_1(y).$$

If $f(x)$ is a single well and also a left balanced function, we can extend the class of the functions $f(x,a)$ for $-1 \leq a \leq 1$ as follows; see [3].

For $\frac{1}{2} \leq a \leq 1$ and $x \in [-1, l(2-2a) + (2a-1)]$ we denote the inverse function of $f(x,a)$ by $x_1(y,a)$ and for $x \in [l(2-2a) + 2a - 1, 1]$ we denote the inverse function

by $x_2(y,a)$:

$$x_1(y,a) = x_1(y) + (2a-1)(1-x_2(y)),$$
$$x_2(y,a) = x_2(y) + (2a-1)(1-x_2(y)).$$

For $-1 \le a \le 0$ and $-1 \le x \le 1$ we denote

$$x_1(y,a) = -x_2(y,-a),$$
$$x_2(y,a) = -x_1(y,-a),$$

If $f(-1) > f(1)$ we enlarge the interval of definition of $x_2(y)$ by adding the interval $f(1) \le y \le f(-1)$ on which we define $x_1(y) = 1$. Obviously, $f\left(x, \frac{1}{2}\right) = f(x)$, $f(x,0) = f_{\#}(x)$ is the symmetrical increasing rearrangement, $f(x,1)$ is the decreasing rearrangement, and $f(x,-1)$ is the increasing rearrangement of $f(x)$.

The functions $f(x,a)$ are equimeasurable; i.e., for each y, $m(x, f(x,a) \ge y, -1 \le x \le 1) = m(x : f(x \ge y, -1 \le x \le 1))$ (see [10, Chap. X] and [16, Chap. VI]).

For single barrier functions the definition of $f(x,a)$ $0 \le a \le \frac{1}{2}$ is the same as for a single well function. In the case where $f(x)$ is a single barrier and also right balanced it can be extended in the same way as for single well left balanced functions.

Remark. If $f(x)$ is a symmetric single well or single barrier, then $f(x,a) = f(x)$, $-\frac{1}{2} \le a \le \frac{1}{2}$.

If $f(x)$ is monotone, then $f(x,a) = f(x)$, $\frac{1}{2} \le a \le 1$ and $f(x) = f(-x)$, $-1 \le a \le -\frac{1}{2}$.

2. The ratio of frequencies of a vibrating string. In Theorem 1 and Theorem 2, we use the same methods as in [12].

Theorem 1. Let $\rho(x)$ and $\varepsilon(x)$ be positive symmetric densities on $[0,a]$ and let

$$\rho(x,t) = \rho(x)\,t + \varepsilon(x)(1-t), \qquad 0 \le t \le 1. \tag{2.1}$$

If

$$\left(\frac{\rho(x)}{\varepsilon(x)}\right)' < 0, \qquad 0 < x \le \frac{a}{2}, \tag{2.2}$$

then

$$\frac{d}{dt}\left(\frac{\lambda_2(t)}{\lambda_1(t)}\right) < 0, \tag{2.3}$$

and especially

$$\frac{\lambda_2(\rho)}{\lambda_1(\rho)} = \frac{\lambda_2(1)}{\lambda_1(1)} \le \frac{\lambda_2(0)}{\lambda_1(0)} = \frac{\lambda_2(\varepsilon)}{\lambda_1(\varepsilon)}. \tag{2.4}$$

If

$$\left(\frac{\rho(x)}{\varepsilon(x)}\right)' > 0, \qquad 0 < x < \frac{a}{2}, \tag{2.5}$$

then

$$\frac{d}{dt}\left(\frac{\lambda_2(t)}{\lambda_1(t)}\right) > 0 \tag{2.6}$$

and

$$\frac{\lambda_2(\rho)}{\lambda_1(\rho)} = \frac{\lambda_2(1)}{\lambda_1(1)} \ge \frac{\lambda_2(0)}{\lambda_1(0)} = \frac{\lambda_2(\varepsilon)}{\lambda_1(\varepsilon)}. \tag{2.7}$$

Corollary 1. (See also [12].) By choosing $\varepsilon(x) = const$, $0 \le x \le a$, we get that if $\rho(x)$ is a symmetric single well on $[0,a]$, then

$$\frac{\lambda_2(\rho)}{\lambda_1(\rho)} \le 4. \tag{2.8}$$

If $\rho(x)$ is a symmetric single barrier density on $[0,a]$, then

$$\frac{\lambda_2(\rho)}{\lambda_1(\rho)} \ge 4. \tag{2.9}$$

Equalities in both cases hold only if $\rho = const$.

Proof. We show first that under the condition that $\left(\frac{\rho(x)}{\epsilon(x)}\right)' < 0$ on $\left[0, \frac{a}{2}\right]$ we get that

$$\int_0^{\frac{a}{2}} \epsilon(x)\left(u_1^2(x,t) - u_2^2(x,t)\right)dx > 0. \tag{2.10}$$

As $\rho(x)$ and $\epsilon(x)$ are symmetric on $[0,a]$ we get from (1.6) that

$$\int_0^{\frac{a}{2}} \rho(x,t)\left(u_1^2(x,t) - u_2^2(x,t)\right)dt = 0.$$

Because of the symmetry of $\rho(x,t)$ on $[0,a]$ we get from (1.3) and (1.4) that

$$u_1^2(x) < u_2^2(x), \quad 0 < x < x_-,$$

and

$$u_2^2(x) < u_1^2(x) \quad \text{on } x_- < x < \frac{a}{2}.$$

Hence

$$\int_x^{\frac{a}{2}} \rho(\xi,t)\left(u_1^2(\xi,t) - u_2^2(\xi,t)\right)d\xi > 0.$$

As $\varepsilon(x)$ and $\rho(x)$ are positive and $\left(\frac{\rho(x)}{\varepsilon(x)}\right)' < 0$ on $\left(0, \frac{a}{2}\right)$ and therefore also $\left(\frac{\varepsilon(x)}{\rho(x,t)}\right)' > 0$, we get that

$$\int_0^{\frac{a}{2}} \varepsilon(x)\left(u_1^2(x,t) - u_2^2(x,t)\right)dx \tag{2.11}$$
$$= \int_0^{\frac{a}{2}} \frac{\varepsilon(x)}{\rho(x,t)}\left(\rho(x,t)\left(u_1^2(x,t) - u_2^2(x,t)\right)dt\right) \ge 0.$$

The last inequality is obtained by simple integration by part. Hence (2.10) is proved.

From (1.6) and (2.1) and the symmetry of $\rho(x)$ and $\varepsilon(x)$ on $[0,a]$ we get from (2.10) that

$$t\int_0^{\frac{a}{2}} (\rho(x) - \varepsilon(x))\left(u_2^2(x,t) - u_1^2(x,t)\right)dx \tag{2.12}$$
$$= \int_0^{\frac{a}{2}} \varepsilon(x)\left(u_1^2(x,t) - u_2^2(x,t)\right)dx > 0$$

and as $\frac{\partial\rho(x,t)}{\partial t} = \rho(x) - \varepsilon(x)$ we get from (2.12) and (1.5) that (2.3) holds. (2.4) is an immediate result of (2.3) and so is (2.8).

The proof that under condition (2.5) the inequalities (2.6), (2.7), and (2.9) follow, is similar. □

For the next theorem we need the following lemma that appeared in [12].

LEMMA 1. (See [12].) If g is three times differentiable and u satisfies (1.1) where ρ is differentiable, then

$$\begin{aligned} & g(a)\left(u'(a)\right)^2 - g(0)\left(u'(0)\right)^2 \\ = & \int_0^a \left[2\lambda g'(x)\rho(x) + \lambda g(x)\rho'(x) + \frac{1}{2}g''(x)\right] u^2(x)\,dx. \end{aligned} \tag{2.13}$$

THEOREM 2. Consider the one parameter family of densities $\rho(x,t) = tx^m + bx^{m-1}$, $t > 0$, where $b \geq 0$ and $m \geq 1$ are constants.

Let $\lambda_n(t)$ be the nth eigenvalue of (1.1) with $\rho = \rho(x,t)$. Then the ratio $\frac{\lambda_2(t)}{\lambda_1(t)}$ is a strictly increasing function of t.

Proof. By (1.5) we get

$$\frac{d}{dt}\left(\frac{\lambda_2(t)}{\lambda_1(t)}\right) = \frac{\lambda_2(t)}{\lambda_1(t)} \int_0^a x^m \left[u_1^2(x,t) - u_2^2(x,t)\right] dx. \tag{2.14}$$

Therefore it is sufficient to show that

$$\langle x(t)\rangle = \int_0^a x^m \left[u_1^2(x,t) - u_2^2(x,t)\right] dx > 0 \quad \text{for all } t > 0. \tag{2.15}$$

Taking $g(x) = x$ and $g(x) = x^2$ in (2.13) and $\rho(x,t) = tx^m + bx^{m-1}$, $t \geq 0$, where $b \geq 0$ and $m \geq 1$ are given constants, the same procedure as in [12, Lemma 4.3] leads to (2.15) and therefore from (2.14) we get that $\frac{\lambda_2(t)}{\lambda_1(t)}$ is strictly increasing with t. □

From the equality

$$\frac{\lambda_2\left(x^m + tx^{m-1}\right)}{\lambda_1\left(x^m + tx^{m-1}\right)} = \frac{\lambda_2\left(\frac{x^m}{t} + x^{m-1}\right)}{\lambda_1\left(\frac{x^m}{t} + x^{m-1}\right)}$$

we get that $\frac{\lambda_2\left(x^m+tx^{m-1}\right)}{\lambda_1\left(x^m+tx^{m-1}\right)}$ is strictly decreasing with $t > 0$ for $m \geq 1$.

Hence an immediate result is:

THEOREM 3. Let $k = [m] \geq 1$ be an integer and let $t_i \geq 0$, $b_i \geq 0$, $i = 1, ..., k$. Then

$$\frac{\lambda_2\left(t_1x^m + b_1x^{m-1}\right)}{\lambda_1\left(t_1x^m + b_1x^{m-1}\right)} \geq \frac{\lambda_2\left(t_2x^{m-1} + b_2x^{m-2}\right)}{\lambda_1\left(t_2x^{m-1} + b_2x^{m-2}\right)} \geq ... \geq \frac{\lambda_2\left(t_kx^{m+1-k} + b_kx^{m-k}\right)}{\lambda_1\left(t_kx^{m+1-k} + b_kx^{m-k}\right)}.$$

If m is a positive integer, then

$$\frac{\lambda_2\left(t_1x^m + b_1x^{m-1}\right)}{\lambda_1\left(t_1x^m + b_1x^{m-1}\right)} \geq ... \geq \frac{\lambda_2\left(t_mx + b_m\right)}{\lambda_1\left(t_mx + b_m\right)} \geq \frac{\lambda_2\left(b_m\right)}{\lambda_1\left(b_m\right)} \geq 4.$$

3. Monotonicity of deformation and of the average deformation. Ferone and Kawhl proved the following theorems in [8]. We will use some of their methods to prove our theorems for single well and single barrier functions.

THEOREM A ([8, Theorem 2]). Compare the solution u of

$$u^{(4)} = f \quad \text{in } (-1,1), \qquad u(-1) = u'(-1) = 0 = u''(1) = u'''(1)$$

with the solution v of

$$v^{(4)} = f_* \quad \text{in } (-1,1), \qquad v(-1) = v'(-1) = 0 = v''(1) = v'''(1),$$

where f_* denotes the monotone increasing rearrangement of f. Then

$$|u(x)| \le v(x) \quad \text{for} \quad x \in (-1,1)\,.$$

Moreover, if $f \ge 0$ in $(-1,1)$, then $z \le |u(x)| = u(x)$. Here z solves

$$z^{(4)} = f^* \quad \text{in } (-1,1)\,, \qquad z(-1) = z^{'}(-1) = 0 = z^{''}(1) = z^{'''}(1)$$

and f^* denotes the monotone decreasing rearrangement of f.

Theorem B ([8, Theorem 3]). Compare the solution u of

$$u^{(4)} = f \ge 0 \quad \text{in } (-1,1)\,, \qquad u(\pm 1) = u^{'}(\pm 1) = 0,$$

with the solution v of

$$v^{(4)} = f^{\#} \ge 0 \quad \text{in } (-1,1)\,, \qquad v(\pm 1) = v^{'}(\pm 1) = 0,$$

where $f^{\#}$ denotes the symmetrical decreasing rearrangements of f.

a) If this is the case, then $\|u\|_1 \le \|v\|_1$ and $\|u\|_\infty \le \frac{1}{24}\|f\|_1$. The last estimate becomes sharp as f approaches $\delta_0(x)$.

b) If $f = \delta_b(x)$ for $b \in (0,1)$, then the solution is given by

$$u_b(x) = \frac{1}{12}\left[|x-b|^3 - (1-bx)\left(1-x^3\right) + \frac{3}{2}(1-bx)\left(1-x^3\right)\left(1-b^2\right)\right]$$

and $\|u_b\|_\infty = u_b\left(\frac{b}{b+2}\right)$ is decreasing in b. Its maximum is attained for $b = 0$.

c) If $f = 0.5$ is uniform, then $u(x) = \frac{1}{48}\left(1-|x|^2\right)^2$.

We also use the following results from [1, Theorem 1] and [1, Remarks 2,3]. There the proofs deal with a simple well, continuous function f but the results hold also for single well piecewise continuous functions.

In Theorem C we state that part of [1, Theorem 1, Remark 2, and Remark 3] that we use here.

Theorem C. Let $f(x) \ge 0$ be single well continuous in $[-1,1]$; then:

a) For $-\frac{1}{2} \le a \le \frac{1}{2}$, $f(x,a)$ is single well too, and if $f(x)$ is left balanced, so is $f(x,a)$, for $0 \le a \le 1$, and $f(x,a)$ is single well for $-1 \le a \le 1$.

b) If $f(x)$ is left balanced, then $f(x,a)$ is not decreasing in a when $f(x,a)$ is not increasing in x, and $f(x,a)$ is not increasing with a when $f(x,a)$ is not decreasing in x, $0 \le a \le 1$.

c) Define

$$I(a) = \int_{-s}^{s} f(x,a)dx \qquad 0 \le s \le 1;$$

then $I(\alpha)$ is not decreasing in a, $0 \le a \le 1$.

Corollary 2. From Theorem C it is easily verified that when $f(x)$ is a left balanced single well function, then $f(x,a_1)$ and $f(x,a_2)$, $-1 \le a \le 1$, cross each other exactly once, and if $-1 \le a \le b \le 1$, then $\int_s^1 f(x,b)\,dx \le \int_s^1 f(x,a)\,dx$, $-1 \le s \le 1$.

Combining the methods used in the proofs of Theorems A and B [8, Theorem 2, Theorem 3] and in the proof of Theorem C, we get the following two theorems.

THEOREM 4. Let $f(x) \geq 0$ be a single well function on $[-1,1]$. Let $u(x,a)$ be the deformation of the clamped beam with a load $f(x,a)$ determined by

$$\frac{\partial^{(4)} u(x,a)}{\partial x^4} = f(x,a) \qquad u(\pm 1,a) = u'(\pm 1,a) = 0.$$

Then the average deformation $\|u(x,a)\|_1$ is monotone increasing in a, $0 \leq a \leq \frac{1}{2}$.

If $f(x)$ is also left balanced, then the average deformation is monotone increasing in a, $0 \leq a \leq 1$.

In particular when $f(x)$ is single well and left balanced on $[-1,1]$, then

$$\|u(x,0)\|_1 \leq \|u(x,a)\|_1 \leq \|u(x,1)\|_1, \quad 0 \leq a \leq 1, \tag{3.1}$$

which means that the minimum average deformation is attained when the load is $f_{\#}$; the symmetrical increasing rearrangement of $f(x)$ and the maximum deformation is attained when the load is a monotone rearrangement of $f(x)$.

If $f(x)$ is a single barrier function, then the average deformation $\|u(x,a)\|_1$ is monotone decreasing in a, $0 \leq a \leq \frac{1}{2}$, which means that the maximum average deformation is attained when the load is $f^{\#}$, the symmetrical decreasing rearrangement of $f(x)$.

If $f(x)$ is a single barrier as well as a right balanced function, then $\|u(x,a)\|_1$ is monotone decreasing in a, $0 \leq a \leq 1$, which means that the average deformation is minimal when the load is the monotone rearrangement of $f(x)$.

Proof. In [8, Theorem 3] it was established that given a load $f(x)$ with the deformation $u(x)$ we get in the case of the clamped beam that

$$\|u\|_1 = \frac{1}{24} \int_{-1}^{1} \left(1-x^2\right)^2 f(x)dx.$$

Hence, for our class of loads $f(x,a)$ we get that

$$\|u(x,a)\|_1 = \frac{1}{24} \int_{-1}^{1} \left(1-x^2\right)^2 f(x,a)dx. \tag{3.2}$$

$\left(1-x^2\right)^2$ is symmetrical decreasing, and according to Theorem C,

$$\int_{-s}^{s} f(x,a)dx, \qquad 0 \leq s \leq 1$$

for a single well function $f(x)$ is not decreasing in a, $0 \leq a \leq \frac{1}{2}$; therefore it is obvious that $\int_{-1}^{1} \left(1-x^2\right)^2 f(x,a)dx$ is also not decreasing in a, $0 \leq a \leq \frac{1}{2}$ and therefore, we get from (3.3) that the average deformation is not decreasing for $0 \leq a \leq \frac{1}{2}$.

If $f(x)$ is also left balanced, then $f(x,a)$, $\frac{1}{2} \leq a \leq 1$ is defined and the result holds for $0 \leq a \leq 1$; hence (3.2) holds.

This completes the proof of the theorem for single well functions. The proof of the theorem for single barrier functions is similar. □

THEOREM 5. Let $f(x) \geq 0$ be a single well left balanced function on $[-1,1]$. Let $u(x,a)$ be the deformation of a cantilever beam which is clamped at one end and free at the

other end with a load $f(x,a)$ determined by

$$\frac{\partial u\,(x,a)}{\partial x} = f(x,a), \qquad 0 = u(-1,a) = \frac{\partial u(-1,a)}{\partial x} = \frac{\partial^2(1,a)}{\partial x^2} = \frac{\partial^3 u(1,a)}{\partial x^3}.$$

Then $u(x,a)$ is decreasing with a, $-1 \le a \le 1$. In particular, Theorem 5 says that this beam undergoes minimum deformation if the load is decreasing on $x \in [-1,1]$ and maximum deformation if the load is increasing.

$$u\,(x,-1) \ge u\,(x,a) \ge u\,(x,1)$$

holds where $u\,(x,-1)$ is the deformation due to f_*, the increasing rearrangement of f, and $u\,(x,1)$ is the deformation due to the load f^*, the decreasing rearrangement of f.

The same holds if $f(x)$ is a single barrier function.

Proof. As in the proof of [8, Theorem 2], it is obvious that under these boundary conditions and for nonnegative f that

$$u\,(x,a) = \int_0^x \int_0^w \int_\zeta^1 \left[\int_\eta^1 f\,(\xi,a)\,d\xi\right] d\eta d\zeta dw.$$

This together with Corollary 2 implies Theorem 5. □

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