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# CONTACT PROBLEM FOR A WEDGE-SHAPED ELASTIC SOLID IN ANTIPLANE SHEAR STRESS DISTRIBUTION

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**Abstract.** The three-part mixed boundary value contact problem for a wedge-shaped region has been solved with the aid of Mellin transforms. Closed form expression for shear stress has been obtained. Finally, numerical results for shear stress and the resultant contact pressure have been obtained and interpreted.

**1. Introduction.** In this paper we consider the problem of distribution of shear stress in an infinite wedge of an isotropic elastic solid under torsion. The shear stress field is produced by the indentation of the plane face of the wedge by the rigid punch. The solution of the mixed boundary value problem is reduced to the solution of triple integral equations involving Mellin transforms. Closed form solution of triple integral equations is obtained. After solving the triple integral equations, the closed form expression for shear stress is obtained. The numerical results for shear stress and the resultant contact pressure under the punch are tabulated. The two-part mixed boundary value problems of distribution of shear stress in an infinite wedge of homogeneous elastic isotropic solid under the usual assumption of plane strain were considered by Srivastav and Narain [1]. In [1] they considered the crack and contact problems in a wedge-shaped region and reduced the solution of problems into Fredholm integral equations of the second kind, which were solved numerically. The method presented here to solve the three-part mixed boundary value contact problem in a closed form and the expressions obtained for the physical quantities in closed form are new in the literature.

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**2. Formulation of the problem.** We consider the shear stress distribution in a wedge-shaped region when the plane faces $\theta = \pm\alpha$ are constrained so that the displacement vanishes and the region $\theta = 0$, $a < r < b$ is indented symmetrically by a rigid punch; therefore, in this region, the displacement is described. Due to symmetry of the problem about $\theta = 0$, we consider the region $0 < \theta < \alpha$, $0 < r < \infty$.

For the anti-plane stress shear problem, the only non-zero displacement components along the $z$ direction should be governed by the following partial differential equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial[u_z(r,\theta)]}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 u_z(r,\theta)}{\partial\theta^2} = 0. \tag{1}$$

The corresponding shear stress components are:

$$\sigma_{rz}(r,\theta) = \mu\frac{\partial u_z(r,\theta)}{\partial r}, \tag{2}$$

$$\sigma_{\theta z}(r,\theta) = \mu\frac{\partial u_z(r,\theta)}{r\partial\theta}, \tag{3}$$

where $\mu$ is the shear modulus.

The boundary conditions are:

$$(u_z)_{\theta=0} = f(r), \quad a < r < b, \tag{4}$$

$$(\sigma_{\theta z})_{\theta=0} = 0, \qquad 0 < r < a, \quad b < r < \infty, \tag{5}$$

$$(u_z)_{\theta=\alpha} = 0, \qquad 0 < r < \infty. \tag{6}$$

The solution of the differential equation (1) may be written in the following form:

$$u_z(r,\theta) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} A(s)r^{-s}\frac{\sin[(\alpha-\theta)s]}{\cos\alpha s}ds, \tag{7}$$

where $A(s)$ is an unknown function.

The above solution (7) satisfies the boundary condition (6) identically.

With the help of equations (3) and (7), the boundary conditions (4) and (5) can be reduced to solution of the following triple integral equations:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sA(s)r^{-s}ds = 0, \quad 0 < r < a, \tag{8}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tan(\alpha s)A(s)r^{-s}ds = f(r), \quad a < r < b, \tag{9}$$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sA(s)r^{-s}ds = 0, \quad r > b. \tag{10}$$

**3. Solution of triple integral equations and physical quantities.** For solving the triple integral equations (8), (9), and (10), we adopt a method with affinities to the elementary solution given by Sneddon [2] and assume the function $A(s)$ as:

$$A(s) = h^{-1}\beta\left(\frac{1}{2},\frac{s}{h}\right)\int_0^b t^{s-\frac{h}{2}}g(t)dt, \quad h = \pi/\alpha, \tag{11}$$

where $g(t)$ is an unknown function to be determined and $\beta(\ )$ is the Beta function.

Writing

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sA(s)r^{-s}ds = -\frac{r}{2\pi i}\frac{d}{dr}\int_{c-i\infty}^{c+i\infty} A(s)r^{-s}ds, \tag{12}$$

substituting the expression for $A(s)$ from equation (11) into equation (12), interchanging the order of integrations, and using the following result given in ([3], p. 311), for $c > 0$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^{-1}\beta\left(\frac{1}{2},\frac{s}{h}\right) r^{-s}t^{s}ds = \begin{cases} t^{h/2}(t^h - r^h)^{-\frac{1}{2}}, & 0 < r < t, \\ 0, & r > t, \end{cases} \tag{13}$$

we find that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sr^{-s}A(s)ds = \begin{cases} -r\frac{d}{dr}\int_r^b \frac{g(t)dt}{(t^h-r^h)^{\frac{1}{2}}}, & 0 < r < b, \\ 0, & r > b. \end{cases} \tag{14}$$

Now it is obvious that the form of $A(s)$ given by equation (11) satisfies equation (10). If we substitute the value of $A(s)$ from equation (11) into equations (8) and (9), interchange the order of integrations, and evaluate the infinite integrals, we obtain the following integral equations:

$$\frac{d}{dr}\int_r^b \frac{g(t)dt}{(t^h - r^h)^{\frac{1}{2}}} = 0, \quad 0 < r < a, \tag{15}$$

$$\int_0^r \frac{g(t)dt}{(r^h - t^h)^{\frac{1}{2}}} = f(r), \quad a < r < b, \tag{16}$$

where we have used the following integral given in ([3], p. 311) for $c < \frac{h}{2}$:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} h^{-1}\beta\left(\frac{1}{2},\frac{1}{2}-\frac{s}{h}\right) t^{s}r^{-s}ds = \begin{cases} 0, & 0 < r < t, \\ (r^h - t^h)^{-\frac{1}{2}}t^{\frac{h}{2}}, & r > t. \end{cases} \tag{17}$$

To solve the integral equations (15) and (16), we differentiate equation (16) with respect to $r$ to obtain

$$\frac{d}{dr}\int_0^r \frac{g(t)dt}{(r^h - t^h)^{\frac{1}{2}}} = f'(r), \quad a < r < b, \tag{18}$$

where prime denotes the derivative with respect to $r$. Let us assume that

$$\frac{d}{dr}\int_0^r \frac{g(t)dt}{(r^h - t^h)^{\frac{1}{2}}} = R(r), \quad 0 < r < a. \tag{19}$$

By making use of the solution of Abel's type integral equations from ([4], pp. 40–42), we find from equations (18) and (19) that

$$g(t) = \frac{h}{\pi}t^{h-1}\left[\int_0^a \frac{R(x)dx}{(t^h - x^h)^{\frac{1}{2}}} + \int_a^t \frac{f'(x)dx}{(t^h - x^h)^{\frac{1}{2}}}\right], \quad a < t < b. \tag{20}$$

Again, making use of the solution of Abel's type integral equation, we find from equation (19) that:

$$g(t) = \frac{h}{\pi}t^{h-1}\int_0^t \frac{R(x)dx}{(t^h - x^h)^{\frac{1}{2}}}, \quad 0 < t < a. \tag{21}$$

Equation (16) can be written in the following form:

$$\int_0^a \frac{g(t)dt}{(r^h - t^h)^{\frac{1}{2}}} + \int_a^r \frac{g(t)dt}{(r^h - t^h)^{\frac{1}{2}}} = f(r), \quad a < r < b. \tag{22}$$

Substituting the values of $g(t)$ from equations (20) and (21) into equation (22), we find that

$$\begin{aligned}&\frac{h}{\pi}\int_0^a \frac{t^{h-1}dt}{(r^h-t^h)^{\frac{1}{2}}}\int_0^t \frac{R(u)du}{(t^h-u^h)^{\frac{1}{2}}}\\&+\frac{h}{\pi}\int_a^r \frac{t^{h-1}dt}{(r^h-t^h)^{\frac{1}{2}}}\left\{\int_0^a \frac{R(u)du}{(t^h-u^h)^{\frac{1}{2}}}+\int_a^t \frac{f'(u)du}{(t^h-u^h)^{\frac{1}{2}}}\right\}\\&=f(r),\quad a<r<b.\end{aligned}\tag{23}$$

Changing the order of integrations in the integrals in equation (23), we find that

$$\begin{aligned}&\frac{h}{\pi}\int_0^a R(u)du\int_u^a \frac{t^{h-1}dt}{[(r^h-t^h)(t^h-u^h)]^{\frac{1}{2}}}\\&+\frac{h}{\pi}\int_0^a R(u)du\int_a^r \frac{t^{h-1}dt}{[(r^h-t^h)(t^h-u^h)]^{\frac{1}{2}}}\\&+\frac{h}{\pi}\int_a^r f'(u)du\int_u^r \frac{t^{h-1}dt}{[(r^h-t^h)(t^h-u^h)]^{\frac{1}{2}}}=f(r),\quad a<r<b.\end{aligned}\tag{24}$$

Equation (24) can be written in the form

$$\begin{aligned}&\frac{h}{\pi}\int_0^a R(u)du\int_u^r \frac{t^{h-1}dt}{[(r^h-t^h)(t^h-u^h)]^{\frac{1}{2}}}\\&\frac{h}{\pi}\int_a^r f'(u)du\int_u^r \frac{t^{h-1}dt}{[(r^h-t^h)(t^h-u^h)]^{\frac{1}{2}}}=f(r),\quad a<r<b.\end{aligned}\tag{25}$$

Making use of the integral

$$\int_u^r \frac{ht^{h-1}dt}{[(r^h-t^h)(t^h-u^h)]^{\frac{1}{2}}}=\pi,\tag{26}$$

we can find from equation (25) that

$$\int_0^a R(u)du=f(a).\tag{27}$$

Equation (15) can be written in the following form:

$$\frac{d}{dr}\int_r^a \frac{g(t)dt}{(t^h-r^h)^{\frac{1}{2}}}+\frac{d}{dr}\int_a^b \frac{g(t)dt}{(t^h-r^h)^{\frac{1}{2}}}=0,\quad 0<r<a.\tag{28}$$

Substituting the value of $g(t)$ from equations (20) and (21) into equation (28), we find that

$$\begin{aligned}&\frac{h}{\pi}\left\{\frac{d}{dr}\int_r^a \frac{t^{h-1}dt}{(t^h-r^h)^{\frac{1}{2}}}\int_0^t \frac{R(x)dx}{(t^h-x^h)^{\frac{1}{2}}}\right\}\\&+\frac{h}{\pi}\frac{d}{dr}\left[\int_a^b \frac{t^{h-1}dt}{(t^h-r^h)^{\frac{1}{2}}}\int_0^a \frac{R(x)dx}{(t^h-x^h)^{\frac{1}{2}}}+\int_a^b \frac{dt}{(t^h-r^h)^{\frac{1}{2}}}\int_a^t \frac{f'(x)dx}{(t^h-x^h)^{\frac{1}{2}}}\right]\\&=0,\quad 0<r<a.\end{aligned}\tag{29}$$

Making use of the following results:

$$\int_0^a dx \int_{\max(x,r)}^b dt = \int_r^a dt \int_0^t dx + \int_a^b dt \int_0^a dx, \tag{30}$$

$$\int_a^b dt \int_a^t dx = \int_a^b dx \int_x^b dt, \tag{31}$$

we find from equation (29) that

$$\frac{d}{dr}\left[\int_0^a R(x)dx\frac{h}{\pi}\int_{\max(x,r)}^b \frac{t^{h-1}dt}{[(t^h-r^h)(t^h-x^h)]^{\frac{1}{2}}}\right.$$
$$\left.+\int_a^b f'(x)dx\frac{h}{\pi}\int_x^b \frac{t^{h-1}dt}{[(t^h-r^h)(t^h-x^h)]^{\frac{1}{2}}}\right] = 0, \quad 0<r<a. \tag{32}$$

Evaluating the integrals of equation (32), we find that

$$\frac{d}{dr}\left[\int_0^a R(x)\log\left|\frac{\sqrt{b^h-x^h}+\sqrt{b^h-r^h}}{\sqrt{b^h-x^h}-\sqrt{b^h-r^h}}\right|dx\right.$$
$$\left.+\int_a^b f'(x)\log\left|\frac{\sqrt{b^h-x^h}+\sqrt{b^h-r^h}}{\sqrt{b^h-x^h}-\sqrt{b^h-r^h}}\right|dx\right] = 0, \quad 0<r<a, \tag{33}$$

and interchanging the order of differentiation and integration, we find that

$$\int_0^a \frac{(\sqrt{b^h-x^h})R(x)dx}{x^h-r^h} = -\int_a^b \frac{(\sqrt{b^h-x^h})f'(x)dx}{x^h-r^h}$$
$$= M(r)(\text{say}), \quad 0<r<a. \tag{34}$$

Making use of a suitable Tricomi theorem [5], we obtain the solution of the integral equation (34) in the following form:

$$R(x) = \frac{x^{h-1}}{\sqrt{(b^h-x^h)(a^h-x^h)}}$$
$$\times\left[\frac{-h^2}{\pi^2}\int_0^a \frac{(a^h-r^h)^{\frac{1}{2}}r^{\frac{h}{2}-1}M(r)dr}{r^h-x^h}+\frac{C}{x^{\frac{h}{2}}}\right], \; 0<x<a, \tag{35}$$

where $C$ is an arbitrary constant. Substituting for $R(x)$ from (35) in (27), we find the value of $C$ in the following form:

$$C = \frac{hb^{h/2}}{2K\left(\frac{a^{a/2}}{b^{h/2}}\right)}\left[f(a)+\frac{h^2}{\pi^2}\int_0^a \frac{x^{h-1}dx}{\sqrt{(b^h-x^h)(a^h-x^h)}}\right.$$
$$\left.\times\int_0^a \frac{(a^h-r^h)^{\frac{1}{2}}r^{h/2-1}M(r)dr}{(r^h-x^h)}\right], \tag{36}$$

where $K\left(\frac{a^{a/2}}{b^{h/2}}\right)$ denotes the complete elliptic integral. If we assume that $f(x)=p_0$, where $p_0$ is a constant, we find that

$$C = \frac{hb^{h/2}p_0}{2K\left(\frac{a^{a/2}}{b^{h/2}}\right)} \tag{37}$$

and

$$R(x) = \frac{x^{h/2-1}C}{[(b^h - x^h)(a^h - x^h)]^{\frac{1}{2}}}, \quad 0 < x < a. \tag{38}$$

If $f(x)$ is constant, by making use of equations (14) and (20), we find that

$$\begin{aligned} &\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sA(s)r^{-s}ds \\ &= -\frac{r}{\pi}\frac{d}{dr}\int_0^a R(x)\log\left|\frac{\sqrt{b^h - x^h} + \sqrt{b^h - r^h}}{\sqrt{b^h - x^h} - \sqrt{b^h - r^h}}\right| dx, \quad a < x < b. \end{aligned} \tag{39}$$

Interchanging the order of differentiation and integration, we find that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sA(s)r^{-s}ds = -\frac{hr^h}{\pi\sqrt{b^h - r^h}}\int_0^a \frac{(\sqrt{b^h - x^h})R(x)dx}{r^h - x^h}. \tag{40}$$

Substituting the value of $R(x)$ from equation (38) into equation (40) and making use of the following integral

$$\int_0^a \frac{x^{\frac{h}{2}-1}dx}{(r^h - x^h)\sqrt{a^h - x^h}} = \frac{\pi}{hr^{h/2}\sqrt{r^h - a^h}}, \quad a < r < b, \tag{41}$$

and equation (37), we find that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} sA(s)r^{-s}ds = \frac{hr^{h/2}p_0 b^{h/2}}{2K\left(\frac{a^{a/2}}{b^{h/2}}\right)\sqrt{(r^h - a^h)(b^h - r^h)}}, \quad a < r < b. \tag{42}$$

Making use of equations (3), (7), and (42), we find that

$$\begin{aligned} \sigma_{\theta z}(r,0) &= -\frac{\mu}{2\pi i r}\int_{c-i\infty}^{c+i\infty} sA(s)r^{-s}ds \\ &= \frac{-hr^{h/2-1}p_0\mu b^{h/2}}{2K\left(\frac{a^{a/2}}{b^{h/2}}\right)\sqrt{(r^h - a^h)(b^h - r^h)}}, \quad a < r < b. \end{aligned} \tag{43}$$

The resultant contact pressure under the rigid punch is:

$$\begin{aligned} P &= -\int_a^b \sigma_{\theta z}(r,0)dr \\ &= \frac{hp_0 b^{h/2}\mu}{2K\left(\frac{a^{a/2}}{b^{h/2}}\right)}\int_a^b \frac{r^{h/2-1}dr}{\sqrt{(r^h - a^h)(b^h - r^h)}}. \end{aligned} \tag{44}$$

Evaluating the above integral, we find that

$$P = \frac{p_0\mu F[\frac{\pi}{2}, q]}{K\left(\frac{a^{a/2}}{b^{h/2}}\right)}, \tag{45}$$

where

$$q = \frac{\sqrt{b^h - a^h}}{b^{h/2}} \tag{46}$$

and $F$ is the elliptic integral of the first kind defined in the book ([6], p. 904).

The numerical results for shear stress and contact pressure are presented in the form of tables.

TABLE 1. Shear stress $\left(\frac{\sigma_{\theta z}}{\mu p_0}\right)$ along $\theta = 0,\ a < r < b$.

| $b = 1,\ r = 0.9$ | | | | |
|---|---|---|---|---|
| $a$ \ $h$ | 2 | 3 | 4 | 6 |
| 0.2 | -1.48279 | -1.84082 | -2.17295 | -2.79022 |
| 0.4 | -1.56158 | -1.88925 | -2.20052 | -2.79804 |
| 0.6 | -1.75806 | -2.05955 | -2.34175 | -2.88669 |
| 0.8 | -2.50976 | -2.83274 | -3.11844 | -3.63789 |

TABLE 2. Contact pressure $\left(\frac{P}{\mu p_0}\right)$ along $\theta = 0,\ a < r < b$.

| $b = 1$ | | | | |
|---|---|---|---|---|
| $a$ \ $h$ | 2 | 3 | 4 | 6 |
| 0.2 | 0.989872 | 1.25525 | 1.51646 | 2.03391 |
| 0.4 | 0.957802 | 1.16264 | 1.36015 | 1.74809 |
| 0.6 | 0.897211 | 1.04895 | 1.18949 | 1.45641 |
| 0.8 | 0.787247 | 0.88742 | 0.97519 | 1.13200 |

Numerical results indicate that the absolute value of the shear stress under the punch increases as the wedge angle $\alpha$ decreases (i.e., $h$ increases) and also the shear stress increases as the punch length Decreases. We also notice that the value of the contact pressure $P$ under the punch increases as the wedge angle decreases and it decreases as the punch length decreases.

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