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$\mathbf{L}^1$ STABILITY OF SEMIGROUPS WITH RESPECT TO THEIR GENERATORS

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Abstract. This note is concerned with the $\mathbf{L}^1$–theory for the system $\partial_t u = \operatorname{div}_x A(u) + B \cdot \Delta u + C(u)$ in several space dimensions. First, an existence result is proved for data in $\mathbf{L}^1 \cap \mathbf{L}^\infty \cap \mathbf{BV}$. Then, the $\mathbf{L}^1$–Lipschitz dependence of the solutions with respect to the natural norms of A, B and C is achieved. As a corollary, the vanishing viscosity limit for conservation laws in 1D recently obtained in a work by Bianchini and Bressan is slightly extended.

1. Introduction. We are concerned with the $\mathbf{L^1}$–theory for the equation

$$\partial_t u = \operatorname{div}_x A(u) + B \cdot \Delta u + C(u), \tag{1.1}$$

where $u\colon [0, +\infty[\times \mathbb{R}^n \mapsto \mathbb{R}^m$ and with initial data chosen in $\mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m) \cap \mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^m) \cap \mathbf{BV}(\mathbb{R}^n; \mathbb{R}^m)$. First, we provide suitable estimates and, as a byproduct, an existence result. Then, the $\mathbf{L^1}$–Lipschitz dependence of solutions with respect to A, B and C is achieved. A key role is played by our choice of the $\mathbf{L^1}$ norm to measure the distance between solutions. A motivation for this choice is the widely studied hyperbolic limit $B \to 0$. In the case $C = 0$, B multiple of identity and $n = 1$, this limit was fully computed in the recent paper [2] and an estimate on the convergence rate was provided in [5].

Here, as a byproduct of our main result, we slightly extend the limit in [2, Theorem 1] to nondiagonal viscosity matrices.

Different from the cited papers above, our stability result does not require any *smallness* assumption: the initial data and the solution need merely to have bounded $\mathbf{L}^\infty$ and $\mathbf{L^1}$ norms as well as bounded total variation.

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Moreover, the stability with respect to the initial datum proved below yields the estimate (2.7) of the type

$$\|u(t) - w(t)\|_{\mathbf{L^1}} \leq (1 + \delta(t, B)) \cdot \|u_o - w_o\|_{\mathbf{L^1}} \quad \text{with} \quad \lim_{t \to 0} \delta(t, B) = 0\,,$$

so that this estimate is optimal for small times, B being assumed only positive definite and symmetric. On the other hand, δ blows up to $+\infty$ as $B \to 0$. The estimate above may thus be seen as a completion to [2, (1.16)], where it is proved that if $B = \varepsilon\,\mathrm{Id}$, then

$$\|u(t) - w(t)\|_{\mathbf{L^1}} \leq L \cdot \|u_o - w_o\|_{\mathbf{L^1}}$$

with $L > 1$ independent from ε.

We prove below that the semigroup generated by (1.1) depends on A, B and C through the $\mathbf{L}^\infty$ norm of $\nabla_u A$, through the norm of B and through the Lipschitz distance (2.12) on C. A classical issue related to (1.1) is the minimal regularity of the initial data that allows us to prove this stability with respect to A, B and C. More precisely, we seek an estimate of the form

$$\|u_1(t) - u_2(t)\|_{\mathbf{L^1}} \leq f(t) \cdot \|(A_1, B_1, C_1) - (A_2, B_2, C_2)\|\,, \tag{1.2}$$

f being a continuous function that vanishes at $t = 0$. Above, u_1 (respectively u_2) is the solution to (1.1) corresponding to A_1, B_1, C_1 (respectively A_2, B_2, C_2), and $\|\cdot\|$ is a suitable norm. Here, we choose initial data in $\mathbf{L^1}(\mathbb{R}^n; \mathbb{R}^m) \cap \mathbf{L}^\infty(\mathbb{R}^n; \mathbb{R}^m) \cap \mathbf{BV}(\mathbb{R}^n; \mathbb{R}^m)$. The slightly weaker choice of data being Radon measures makes the estimate (1.2) impossible. Indeed, consider the scalar 1D case $m = n = 1$, $A_1 = A_2 = 0$, $C_1 = C_2 = 0$ with Dirac's delta as initial data. Then, by the scaling properties of the heat kernel, $\|u_1(t) - u_2(t)\|_{\mathbf{L^1}}$ is a nonzero constant independent of t. A similar phenomenon is shown in [3, Theorem 2.6], in the case $n = m = 1$, $B_1 = B_2 = 0$, and $C_1 = C_2 = 0$, and A_1, A_2 strictly convex.

In the case $B_1 = B_2$ and $C_1 = C_2$, the present estimate extends that obtained in [3, Corollary 2.5] for systems of conservation laws in one space dimension; see also [2, Corollary 16.1]. If $A_1 = A_2$ and $B_1 = B_2$, we get classical o.d.e.-like estimates. When $A_1 = A_2 = 0$ and $C_1 = C_2 = 0$, then Theorem 2.3 reduces to the Lipschitz dependence in $\mathbf{L^1}$ of the linear parabolic semigroup from the matrices B_1 and B_2.

The difficulties in the limit $B \to 0$ that were recently overcome in [2] stem from the role played in the present paper by the minimal eigenvalue β_1 of B. Indeed, most of the estimates below blow up as β_1 vanishes. On the other hand, in [2], B is required to be multiple of the identity, while here, B need not be even diagonal.

Section 2 is devoted to the statement of the results, while the technical details are deferred to Section 3.

2. Main result. Throughout this paper, on equation (1.1), we make the following assumptions:

(A) $A \in \mathbf{W^{2,\infty}}(\mathbb{R}^m; \mathbb{R}^{m \times n})$, i.e. there exist constants α_1, α_2 such that, uniformly in $u \in \mathbb{R}^m$, $\|\nabla_u A(u)\| \leq \alpha_1$ and $\left\|M^T\, \nabla_u^2 A(u)\, M\right\| \leq \alpha_2 \|M\|^2$ for all $M \in \mathbb{R}^{m \times n}$.

(B) $B \in \mathbb{R}^{m \times m}$ is symmetric and positive definite, i.e. there exist constants β_1, β_2 such that $\beta_1 \cdot \|v\|^2 \leq v^T B v \leq \beta_2 \cdot \|v\|^2$ for all $v \in \mathbb{R}^m$.

(C) $C \in \mathbf{W^{1,\infty}}(\mathbb{R}^m;\mathbb{R}^m)$ satisfies $C(0) = 0$, and there exists a suitable constant γ such that $\|\nabla_u C(u)\| \leq \gamma$, uniformly in $u \in \mathbb{R}^m$.

(In the present paper, $\| \ \|$ denotes the Euclidean norm in $\mathbb{R}^n$, or $\mathbb{R}^m$, or $\mathbb{R}^{m\times m}$, i.e. the Frobenius norm for matrices.) The above conditions are required to hold uniformly in $u \in \mathbb{R}^n$ only for notational simplicity. Indeed, below we provide $\mathbf{L}^\infty$ estimates on the solution.

As a first step we need a slight generalization of a standard result on Lipschitz semigroups in metric spaces; see [4, Theorem 2.9].

Proposition 2.1. Let $\mathcal{D}$ be a metric space with distance d. Let $S\colon \mathcal{D} \times [0,+\infty[\mapsto \mathcal{D}$ and $w\colon [0,T] \mapsto \mathcal{D}$ be such that

1. S is a semigroup. Moreover there exist a positive constant $\mathcal{L}$ and a continuous map $\varphi\colon [0,+\infty[\mapsto [0,+\infty[$ such that for all $u_1, u_2 \in \mathcal{D}$ and $t_1, t_2 \in [0,+\infty[$,
$$d\left(S_{t_1}u_1, S_{t_2}u_2\right) \leq \mathcal{L} \cdot d(u_1,u_2) + |\varphi(t_1) - \varphi(t_2)|\,;$$
2. w is continuous. Moreover, for all $\delta \in]0,T[$ there exists a positive constant $\mathcal{L}_\delta$ such that for all $t \in [\delta, T]$
$$d\left(w(t_1), w(t_2)\right) \leq \mathcal{L}_\delta \cdot |t_1 - t_2|\,.$$

Then, for all $t \in [0,T]$,
$$d\left(S_t w(0), w(t)\right) \leq \mathcal{L} \cdot \int_0^t \liminf_{h\to 0} \frac{1}{h} d\left(w(\tau+h), S_h w(\tau)\right)\, d\tau\,. \tag{2.3}$$

The proof is deferred to Section 3.

We introduce for later use the space
$$\mathbf{X} = \mathbf{L^1}(\mathbb{R}^n,\mathbb{R}^m) \cap \mathbf{L}^\infty(\mathbb{R}^n,\mathbb{R}^m) \cap \mathbf{BV}(\mathbb{R}^n,\mathbb{R}^m)$$
which we equip with the norm
$$\|u\|_{\mathbf{X}} = \|u\|_{\mathbf{L^1}} + \|u\|_{\mathbf{L}^\infty} + \mathrm{TV}(u),$$
where by $\mathrm{TV}(u)$ we understand
$$\mathrm{TV}(u) = \sup\left\{\int_{\mathbb{R}^n} u \,\mathrm{div}_x\, v\, dx\colon v \in \mathbf{C^\infty_c} \text{ and } \|v\|_{\mathbf{L}^\infty} \leq 1\right\}.$$
In the case $n = 1$, a slightly different definition of TV is often used, leading to a larger space $\mathbf{BV}$. In the present context, this wider generality is useless.

We define below a nonlinear semigroup $S\colon [0,+\infty[\times \mathbf{X} \mapsto \mathbf{X}$ whose trajectories $t \mapsto u(t) = S_t u_o$ are solutions to (1.1) with initial datum $u_o \in \mathbf{X}$. Moreover, we prove that
$$u \in \mathbf{C^0}\left([0,+\infty[\,;\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right) \cap \mathbf{L^\infty_{loc}}\left([0,+\infty[\,;\mathbf{X}\right).$$
Note that solutions to (1.1) can be understood with two different meanings: *mild* and *weak* solutions.

Fix $p \in [1,+\infty[$. By a *mild solution* we mean a function $u = u(t,x)$ in
$$\mathbf{C^0}\left([0,+\infty[\,;\mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m)\right)$$

that satisfies

$$\begin{aligned} u(t) &= u_o * \Phi_B(t) + \int_0^t A\left(u(\tau)\right) * \nabla_x \Phi_B(t-\tau)\, d\tau \\ &\quad + \int_0^t C\left(u(\tau)\right) * \Phi_B(t-\tau)\, d\tau\,, \end{aligned} \tag{2.4}$$

Φ_B being the fundamental solution of the parabolic system $\partial_t u = B\Delta u$. See [8, Chapter 9] or [7, § 2.3] as general references on fundamental solutions. We mention here for later use the following elementary estimates.

LEMMA 2.2. Let B satisfy **(B)**. Then, the fundamental solution Φ_B of $\partial_t u = B \cdot \Delta u$ on $\mathbb{R}^n$ satisfies

$$\begin{aligned} \|\Phi_B(t)\|_{\mathbf{L^1}} &= 1\,, \\ \|\nabla_x \Phi_B(t)\|_{\mathbf{L^1}} &\leq \frac{K(\beta_1)}{\sqrt{t}}, \end{aligned} \tag{2.5}$$

where

$$K(\beta_1) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \frac{\sqrt{m}}{\sqrt{\beta_1}} \leq \sqrt{\frac{nm}{2}} \cdot \frac{1}{\sqrt{\beta_1}}\,.$$

The proof is deferred to Section 3.

A map $u = u(t,x)$ in $\mathbf{C^0}\left([0,T]; \mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m)\right)$ is a *weak solution* to (1.1) with initial datum u_o at time $t=0$ if for any $\psi \in \mathbf{C^2_c}(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^m)$ it holds that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} u\, \partial_t \psi\, dx\, dt + \int_{\mathbb{R}^n} u_o\, \psi(0)\, dx \\ &= \int_0^T \int_{\mathbb{R}^n} \left(A(u)\, \nabla_x \psi - Bu\, \Delta\psi - C(u)\, \psi\right)\, dx\, dt\,. \end{aligned} \tag{2.6}$$

For functions in $\mathbf{C^0}\left([0,+\infty[\,; \mathbf{L^2}(\mathbb{R}^n;\mathbb{R}^m)\right)$, the two concepts of solution coincide, as it follows from Lemma 3.1 and Lemma 3.2. Note that, if $p \in [1,2]$, then

$$\mathbf{C^0}\big([0,T]; \mathbf{L^2}(\mathbb{R}^n;\mathbb{R}^m)\big) \supset \mathbf{C^0}([0,T]; \mathbf{L^p}(\mathbb{R}^n;\mathbb{R}^m)) \cap \mathbf{L^\infty_{loc}}([0,T]; \mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^m))\,.$$

Remark that the space on the left-hand side above reflects the parabolic nature of (1.1), while the space on the right-hand side, when $p=1$, reminds us of its hyperbolic nature. Therefore, below, we do not specify if the solution of (1.1) has to be understood in its mild or weak sense.

In the following, we explicitly mention the dependence of the various quantities on β_1, omitting the other parameters $\alpha_1, \alpha_2, \beta_2$ and γ.

THEOREM 2.3. Let A, B and C satisfy **(A)**, **(B)** and **(C)**. Then, equation (1.1) generates a semigroup $S\colon\ [0,+\infty[\, \times \mathbf{X} \mapsto \mathbf{X}$ with the following properties:

1. for all $u_o \in \mathbf{X}$, the orbit $t \mapsto S_t u_o$ yields a solution to (1.1).
2. $S \in \mathbf{C^0}\left([0,+\infty[\,; \mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right) \cap \mathbf{L^\infty_{loc}}\left([0,+\infty[\,; \mathbf{X}\right)$. More precisely, there exist $L_1\colon\ [0,+\infty[\, \mapsto [0,+\infty[$ and $\varphi\colon\ [0,+\infty[\, \times [0,+\infty[\, \mapsto [0,+\infty[$ positive, absolutely

continuous and increasing in each argument, such that for all $s, t \in [0, +\infty[$ with $s < t$ and for all $u_o, w_o \in \mathbf{X}$,

$$\begin{aligned} \|S_t u_o - S_s w_o\|_{\mathbf{L^1}} &\leq L_1(s;\beta_1) \cdot \|u_o - w_o\|_{\mathbf{L^1}} \\ &\quad + (\|u_o\|_{\mathbf{L^1}} + \mathrm{TV}(u_o)) \\ &\quad \cdot |\varphi(t, \|u_o\|_{\mathbf{L}^\infty}; \beta_1) - \varphi(s, \|u_o\|_{\mathbf{L}^\infty}; \beta_1)|, \end{aligned} \tag{2.7}$$

$$\|S_t u_o\|_{\mathbf{L^1}} \leq L_1(t;\beta_1) \cdot \|u_o\|_{\mathbf{L^1}}, \tag{2.8}$$

$$\|S_t u_o\|_{\mathbf{L}^\infty} \leq L_1(t;\beta_1) \cdot \|u_o\|_{\mathbf{L}^\infty}, \tag{2.9}$$

$$\mathrm{TV}(S_t u_o) \leq L_1(t;\beta_1) \cdot \mathrm{TV}(u_o). \tag{2.10}$$

Explicit expressions for L_1 and φ are in (3.36) and (3.41).

3. For $t > 0$, and $u_o \in \mathbf{X}$, $S_t u_o$ is in $\mathbf{W^{2,1}}(\mathbb{R}^n; \mathbb{R}^m)$, and there exists a $L_2\colon [0, +\infty[\times [0, +\infty[\mapsto [0, +\infty[$ positive, increasing and absolutely continuous in each variable, such that for a.e. t

$$\|S_t u_o\|_{\mathbf{W^{2,1}}} \leq \left(\frac{K(\beta_1)}{\sqrt{t}} + L_2(t, \|u_o\|_{\mathbf{L}^\infty}; \beta_1) \right) \cdot \mathrm{TV}(u_o). \tag{2.11}$$

An explicit expression for L_2 is in (3.39).

Remark that in the hyperbolic limit $\beta_1 \to 0$, by (2.5), we have $K(\beta_1) \to +\infty$ and also $L_1(t;\beta_1) \to +\infty$, $L_2 \to +\infty$, $\varphi(t, \|u_o\|_{\mathbf{L}^\infty}; \beta_1) \to +\infty$ for any t.

The next step consists of showing that the semigroup generated by (1.1) fits in the abstract framework of Proposition 2.1. To this aim, we need to introduce the following distance for Lipschitz functions that vanish at 0:

$$d_{\mathbf{Lip}}(C_1, C_2) = \sup_{u \in \mathbb{R}^m} \frac{\|C_2(u) - C_1(u)\|}{\|u\|}. \tag{2.12}$$

As a result we obtain:

THEOREM 2.4. Let A_1, A_2, B_1, B_2 and C_1, C_2 satisfy **(A)**, **(B)** and **(C)**. Call S^i the semigroup generated by $\partial_t u = \mathrm{div}_x A_i(u) + B_i \Delta u + C_i(u)$. Then, there exist positive $\mathcal{L}_1\colon [0, +\infty[\mapsto [0, +\infty[$ and $\mathcal{L}_2\colon [0, +\infty[\times [0, +\infty[\mapsto [0, +\infty[$ absolutely continuous and increasing in each argument, such that for all $u_o \in X$ and for all $t \in [0, +\infty[$,

$$\begin{aligned} \left\|S_t^1 u - S_t^2 u\right\|_{\mathbf{L^1}} &\leq \mathcal{L}_1(t;\beta_1) \cdot \|\nabla_u A_1 - \nabla_u A_2\|_{\mathbf{L}^\infty} \cdot \mathrm{TV}(u_o) \\ &\quad + 2 \cdot K(\beta_1) \cdot \sqrt{t} \cdot \|B_1 - B_2\| \cdot \mathrm{TV}(u_o) \\ &\quad + \mathcal{L}_2(t, \|u_o\|_{\mathbf{L}^\infty}; \beta_1) \cdot \|B_1 - B_2\| \cdot \mathrm{TV}(u_o) \\ &\quad + \mathcal{L}_1(t;\beta_1) \cdot d_{\mathbf{Lip}}(C_1, C_2) \cdot \|u_o\|_{\mathbf{L^1}}. \end{aligned} \tag{2.13}$$

A classical problem related to equations of the form (1.1) is the behavior of the solutions in the limit $B \to 0$. For completeness, we quote here the following consequence of [2, Theorem 1] and [5, Theorem 1].

THEOREM 2.5. Let $n = 1$, A satisfy **(A)** and be strictly hyperbolic. There exist constants L, L' and a domain $\mathcal{D} \subset \mathbf{L^1}(\mathbb{R}; \mathbb{R}^m)$ such that for all sufficiently small $\varepsilon > 0$ the equation

$$\partial_t u_\varepsilon = \partial_x A(u_\varepsilon) + \varepsilon \cdot \Delta u_\varepsilon \tag{2.14}$$

generates a semigroup $S^\varepsilon\colon [0,+\infty[\,\times \mathbf{X} \mapsto \mathbf{X}$ with the properties:

1. for all $u_o \in \mathbf{X}$, the orbit $t \mapsto S^\varepsilon u_o$ yields a solution to (2.14) with initial data u_o;
2. S^ε is Lipschitz uniformly in ε, i.e.

$$\begin{aligned} \left\|S^\varepsilon_t u_o - S^\varepsilon_s w_o\right\|_{\mathbf{L^1}} \quad \leq \quad & L \cdot \left\|u_o - w_o\right\|_{\mathbf{L^1}} \\ & + L \cdot \left|\left(t+\sqrt{\varepsilon t}\right) - \left(s+\sqrt{\varepsilon s}\right)\right|; \end{aligned} \tag{2.15}$$

3. as $\varepsilon \to 0$, the semigroup S^ε converges to the Standard Riemann Semigroup $S\colon [0,+\infty[\,\times \mathcal{D} \mapsto \mathcal{D}$ with $\mathcal{D} \supseteq \{w \in \mathbf{X}\colon \mathrm{TV}(w) < \delta\}$, for a suitable positive δ, and with convergence rate

$$\left\|S^\varepsilon_t w_o - S_t w_o\right\|_{\mathbf{L^1}} \leq L' \cdot (1+t) \cdot \sqrt{\varepsilon} \cdot |\ln \varepsilon| \cdot \mathrm{TV}(w_o)\,. \tag{2.16}$$

For the definition and properties of Standard Riemann Semigroups, see [4].

The techniques developed in the present work allow a slight extension of the latter result to nondiagonal matrices.

Corollary 2.6. Let $n = 1$, A satisfy **(A)** and be strictly hyperbolic. For all small positive ε, let $B_\varepsilon \in \mathbb{R}^{m\times m}$ be symmetric, positive definite and call $S^{B_\varepsilon}\colon [0,+\infty[\times \mathbf{X} \mapsto \mathbf{X}$ the semigroup generated by

$$\partial_t u_\varepsilon = \partial_x A(u_\varepsilon) + B_\varepsilon \cdot \Delta u_\varepsilon$$

as constructed in Theorem 2.3. Let S be the SRS as in Theorem 2.5. If there exists a positive δ such that

$$\|B_\varepsilon - \varepsilon\,\mathrm{Id}\| \leq \exp\left(-\exp\frac{1}{\varepsilon^{\frac{1}{2}+\delta}}\right)$$

for all sufficiently small ε, then, for all $u_o \in \mathcal{D}$,

$$\lim_{\varepsilon\to 0}\left\|S^{B_\varepsilon}_t u_o - S_t u_o\right\|_{\mathbf{L^1}} = 0 \tag{2.17}$$

uniformly in any bounded time interval.

An estimate on the convergence rate in (2.17) can be deduced through (2.16).

3. Technical proofs.

Proof of Lemma 2.2. The first equality follows from the well-known conservation of the $\mathbf{L^1}$ norm. By **(B)**, there exists an orthonormal matrix R such that $B' = R^T \cdot B \cdot R$ is diagonal. Then,

$$\left\|\nabla_x \Phi_B(t)\right\|_{\mathbf{L^1}} = \left\|\nabla_x \Phi_{B'}(t)\right\|_{\mathbf{L^1}} \leq \sqrt{m} \cdot \left\|\nabla_x \Phi_{\beta_1}(t)\right\|_{\mathbf{L^1}}\,,$$

where by Φ_{β_1} we denote the fundamental solution to the scalar equation $\partial_t u = \beta_1 \cdot \Delta u$. Now compute

$$\begin{aligned}
\Phi_{\beta_1}(t,x) &= \frac{1}{(4\pi\beta_1 t)^{n/2}} e^{-\beta_1 \|x\|^2/(4t)}\,, \\
\|\nabla_x \Phi_{\beta_1}(t)\|_{\mathbf{L^1}} &= \frac{1}{2^{n+1}\pi^{n/2}\beta_1^{-1+n/2} t^{1+n/2}} \int_{\mathbb{R}^n} \|x\|\, e^{-\beta_1\|x\|^2/(4t)}\, dx \\
&= \frac{1}{2^n \beta_1^{-1+n/2} t^{1+n/2}} \cdot \frac{1}{\Gamma(n/2)} \cdot \int_0^{+\infty} r^n e^{-\beta_1 r^2/(4t)}\, dr \\
&= \frac{\Gamma\left((n+1)/2\right)}{\Gamma(n/2)} \cdot \frac{1}{\sqrt{\beta_1}} \cdot \frac{1}{\sqrt{t}}\,.
\end{aligned}$$

Now note that the sequence $a_n = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)} \cdot \sqrt{\frac{2}{n}}$ is strictly increasing. Indeed, let $f(x) = \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x)} \cdot \sqrt{x}$ and observe that

$$f'(x) \geq \left(\frac{\Gamma'\left(x+\frac{1}{2}\right)}{\Gamma\left(x+\frac{1}{2}\right)} - \frac{\Gamma'(x)}{\Gamma(x)} \right) \sqrt{x} \geq 0\,.$$

Moreover, by Stirling approximation, $\lim_{n\to+\infty} a_n = 1$, completing the proof. □

Proof of Proposition 2.1. Fix a positive δ and let $t \in [\delta, T]$. Then, by Theorem 2.9 in [4] and the assumptions on S, w,

$$\begin{aligned}
d\left(S_t w(0), w(t)\right) &\leq d\left(S_{t-\delta} S_\delta w(0), S_{t-\delta} w(\delta)\right) + d\left(S_{t-\delta} w(\delta), w(t)\right) \\
&\leq \mathcal{L} \cdot \left(d\left(S_\delta w(0), w(0)\right) + d\left(w(\delta), w(0)\right)\right) \\
&\quad + \mathcal{L} \cdot \int_\delta^t \liminf_{h\to 0+} \frac{1}{h} d\left(w(\tau+h), S_h w(\tau)\right)\, d\tau \\
&\leq \mathcal{L} \cdot \varphi(\delta) + \mathcal{L} \cdot d\left(w(\delta), w(0)\right) \\
&\quad + \mathcal{L} \cdot \int_\delta^t \liminf_{h\to 0+} \frac{1}{h} d\left(w(\tau+h), S_h w(\tau)\right)\, d\tau\,.
\end{aligned}$$

Now letting $\delta \to 0$, the Monotone Convergence Theorem yields the desired result. □

Aiming at the proof of equivalence between mild and weak solutions of (1.1), for a fixed $u = u(t,x)$ we introduce the function

$$f(t,x) = \operatorname{div}_x A\left(u(t,x)\right) + C\left(u(t,x)\right)$$

and consider the linear problem

$$\partial_t u = B \cdot \Delta u + f\,. \tag{3.18}$$

As a first step, we need a slight generalization of a result in [7, Paragraph 2.3.1].

Lemma 3.1. Fix $p \in [1, +\infty[$. Choose $f \in \mathbf{C^0}\left([0,+\infty[\,; \mathbf{W^{-1,p}}(\mathbb{R}^n;\mathbb{R}^m)\right)$ and $g \in \mathbf{L^p}(\mathbb{R}^n, \mathbb{R}^m)$. Then, any mild solution to (3.18) with initial datum $u(0) = g$ is also a weak solution.

For the definitions of mild and weak solutions to the linear Cauchy problem for (3.18), see [7].

Proof. Call $\check{\varphi}^\varepsilon$, resp. $\hat{\varphi}^\varepsilon$, a $\mathbf{C^\infty_c}$ mollifier having $\mathbf{L^1}$-norm equal to 1 and with support in the ball centered at 0 with radius ε in $\mathbb{R}^n$, resp. in $\mathbb{R}^{n+1}$. Define $g^\varepsilon = g * \check{\varphi}^\varepsilon$, $f^\varepsilon = f * \hat{\varphi}^\varepsilon$ and

$$u^\varepsilon(t) = g^\varepsilon * \Phi_B(t) + \int_0^t f^\varepsilon(\tau) * \Phi_B(t-\tau)\, d\tau\,. \tag{3.19}$$

Then, by [7, Paragraph 2.3.1], u^ε is a weak solution of

$$\begin{cases} \partial_t u^\varepsilon - B \cdot \Delta u^\varepsilon = f, \\ u^\varepsilon(0) = g^\varepsilon \end{cases}$$

so that for all $\psi \in \mathbf{C^\infty_c}\left([0,+\infty[\, \times \mathbb{R}^n; \mathbb{R}^m\right)$,

$$\int_0^{+\infty} \int_{\mathbb{R}^n} \left(u^\varepsilon \partial_t \psi + u^\varepsilon \Delta \psi + f^\varepsilon \psi\right)\, dx\, dt = \int_{\mathbb{R}^n} g^\varepsilon\, \psi(0)\, dx\,.$$

As $\varepsilon \to 0$, we have that $g^\varepsilon \to g$ in $\mathbf{L^p}(\mathbb{R}^n; \mathbb{R}^m)$ and $f^\varepsilon \to f$ strongly in

$$\mathbf{C^0}\left([0,+\infty[\,; \mathbf{W^{-1,p}}(\mathbb{R}^n; \mathbb{R}^m)\right).$$

Moreover, since $\|\Phi_B(\tau)\|_{\mathbf{W^{1,1}}} \le 1 + \frac{K(\beta_1)}{\sqrt{t}}$ in (3.19) and $f(\tau) \in \mathbf{W^{-1,p}}(\mathbb{R}^n; \mathbb{R}^m)$ for all $\tau \ge 0$, it follows that $u^\varepsilon \to u$ uniformly on compact sets in $\mathbf{C^0}\left([0,+\infty[\,; \mathbf{L^p}(\mathbb{R}^n; \mathbb{R}^m)\right)$; see also [6, Proposition IV.21]. □

The other implication, namely from weak to mild solution, is obtained through the uniqueness of weak solutions which, in turn, follows from the stability result below.

LEMMA 3.2. Let A, B and C satisfy **(A)**, **(B)** and **(C)**. Fix a positive T and call u, w two weak solutions of (1.1) in $\mathbf{C^0}\left([0,T]; \mathbf{L^2}(\mathbb{R}^n; \mathbb{R}^m)\right)$. Then, for any $t, s \in [0,T]$ with $t \ge s$,

$$\|u(t) - w(t)\|_{\mathbf{L^2}} \le \|u(s) - w(s)\|_{\mathbf{L^2}}\, e^{\frac{1}{2}\left(\frac{\alpha_1{}^2}{2\beta_1} + \gamma^2 + 1\right)(t-s)}. \tag{3.20}$$

Usual uniqueness proofs essentially rely on using $u - w$ as a test function in the definitions of weak solutions for u and for w. Here, due to the low regularity

$$\mathbf{C^0}\left([0,T]; \mathbf{L^2}(\mathbb{R}^n; \mathbb{R}^m)\right),$$

more care in the choice of the test function is necessary. Indeed, we extend the technique in [10, Chapter 2, (5.44)] to improve the regularity in both time and space variables.

We recall for later use the following estimate. If $f \in \mathbf{L^1}(\mathbb{R}^n, \mathbb{R}^m)$ and $g \in \mathbf{L^p}(\mathbb{R}^n, \mathbb{R}^m)$, then

$$\|f * g\|_{\mathbf{L^p}} \le \|f\|_{\mathbf{L^1}} \cdot \|g\|_{\mathbf{L^p}} \tag{3.21}$$

for all $p \in [1, \infty]$; see [6, Theorem IV.15].

Proof of Lemma 3.2. By (2.6) to u and w, for any $\psi \in \mathbf{C_c^2}(]0,+\infty[\times\mathbb{R}^n;\mathbb{R}^m)$,

$$\int_0^{+\infty}\int_{\mathbb{R}^n}(u-w)\partial_t\psi dx\,d\tau \tag{3.22}$$

$$+\quad \int_0^{+\infty}\int_{\mathbb{R}^n}B(u-w)\Delta\psi\,dx\,d\tau \tag{3.23}$$

$$-\quad \int_0^{+\infty}\int_{\mathbb{R}^n}\left(A(u)-A(w)\right)\nabla_x\psi\,dx\,d\tau \tag{3.24}$$

$$+\quad \int_0^{+\infty}\int_{\mathbb{R}^n}\left(C(u)+C(w)\right)\psi\,dx\,d\tau \quad = \quad 0\,. \tag{3.25}$$

Let $\rho\in\mathbf{C_c^\infty}(\mathbb{R};[0,+\infty[)$ satisfy $\|\rho\|_{\mathbf{L^1}}=1$, $\operatorname{spt}\rho\subseteq[-1,1]$ and for all $\xi\in\mathbb{R}$, $\rho(\xi)=\rho(-\xi)$. Fix s,t in $]0,T[$. For any $\varepsilon\in]0,\min\{s,T-t\}[$ define $\varphi^\varepsilon\in\mathbf{C_c^\infty}(\mathbb{R}^{n+1};\mathbb{R})$ by

$$\varphi^\varepsilon(t,x)=\frac{1}{\varepsilon}\rho(t/\varepsilon)\cdot\prod_{i=1}^n\frac{1}{\varepsilon}\rho(x_i/\varepsilon)\,. \tag{3.26}$$

The above properties of ρ ensure that for any $f,g\in\mathbf{L^2}(\mathbb{R}^{n+1};\mathbb{R}^m)$,

$$\int_{\mathbb{R}^{n+1}}(\varphi^\varepsilon*g)f\,dt\,dx=\int_{\mathbb{R}^{n+1}}(\varphi^\varepsilon*f)g\,dt\,dx; \tag{3.27}$$

see [6, Proposition IV.16]. Moreover, for any $f\in\mathbf{L^2}(\mathbb{R}^{n+1};\mathbb{R}^m)$ and any differential operator D,

$$D(f*\varphi^\varepsilon)=\varphi^\varepsilon*Df \tag{3.28}$$

pointwise; see [9, § 4.2, (4.2.5)]. Finally, choose the test function

$$\psi=\varphi^\varepsilon*\left(\chi_{]s,t[}(u-w)*\varphi^\varepsilon\right),$$

where $\chi_{]s,t[}$ is the characteristic function of the set $]s,t[\times\mathbb{R}^n$. Consider the terms in (3.22)–(3.25) separately. Apply (3.27) and (3.28) to obtain

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n}(u-w)\partial_t\psi dx\,d\tau\\
=\;&\int_0^{+\infty}\int_{\mathbb{R}^n}(u-w)\,\partial_t\left(\varphi^\varepsilon*\left(\chi_{]s,t[}(u-w)*\varphi^\varepsilon\right)\right)dx\,d\tau\\
=\;&\int_0^{+\infty}\int_{\mathbb{R}^n}(u-w)\,\varphi^\varepsilon*\partial_t\left(\chi_{]s,t[}(u-w)*\varphi^\varepsilon\right)dx\,d\tau\\
=\;&\int_0^{+\infty}\int_{\mathbb{R}^n}(u-w)*\varphi^\varepsilon\,\partial_t\left(\chi_{]s,t[}(u-w)*\varphi^\varepsilon\right)dx\,d\tau\\
=\;&\int_s^t\int_{\mathbb{R}^n}\partial_t\left((u-w)*\varphi^\varepsilon\right)\chi_{]s,t[}(u-w)*\varphi^\varepsilon\,dx\,d\tau\\
=\;&\frac{1}{2}\int_s^t\int_{\mathbb{R}^n}\partial_t\left((u-w)*\varphi^\varepsilon\right)^2\,dx\,d\tau\\
=\;&\frac{1}{2}\left(\|(u(t)-w(t))*\varphi^\varepsilon\|^2_{\mathbf{L^2}}-\|(u(s)-w(s))*\varphi^\varepsilon\|^2_{\mathbf{L^2}}\right).
\end{aligned} \tag{3.29}$$

Concerning (3.23), similarly

$$
\begin{aligned}
& \int_0^{+\infty}\int_{\mathbb{R}^n} B\cdot(u-w)\Delta\psi\,dx\,d\tau \\
= & \int_s^t\int_{\mathbb{R}^n} B\cdot\left(\nabla_x\left((u-w)*\varphi^\varepsilon\right)\right)\nabla_x\left(\nabla_x\left((u-w)*\varphi^\varepsilon\right)\right)\,dx\,d\tau \\
\geq & \ \beta_1\int_s^t \left\|\nabla_x\left((u-w)*\varphi^\varepsilon\right)\right\|^2_{\mathbf{L^2}}\,d\tau\,. \qquad (3.30)
\end{aligned}
$$

The treatment of (3.24) is similar, also with the aid of Young inequality and (3.21).

$$
\begin{aligned}
& \int_0^{+\infty}\int_{\mathbb{R}^n} \left(A(u)-A(w)\right)\nabla_x\psi\,dx\,d\tau \\
= & \int_s^t\int_{\mathbb{R}^n} \left((A(u)-A(w))*\varphi^\varepsilon\right)\,\left(\nabla_x\left((u-w)*\varphi^\varepsilon\right)\right)\,dx\,d\tau \\
\leq & \int_s^t \frac{1}{4\beta_1}\|(A(u)-A(w))*\varphi^\varepsilon\|^2_{\mathbf{L^2}}\,d\tau \\
& \quad + \int_s^t \beta_1\|\nabla_x\left((u-w)*\varphi^\varepsilon\right)\|^2_{\mathbf{L^2}}\,d\tau \\
\leq & \int_s^t \left(\frac{1}{4\beta_1}\|A(u)-A(w)\|^2_{\mathbf{L^2}} + \beta_1\|\nabla_x((u-w)*\varphi^\varepsilon)\|^2_{\mathbf{L^2}}\right)\,d\tau \\
= & \int_s^t \left(\frac{{\alpha_1}^2}{4\beta_1}\|u-w\|^2_{\mathbf{L^2}} + \beta_1\|\nabla_x\left((u-w)*\varphi^\varepsilon\right)\|^2_{\mathbf{L^2}}\right)\,d\tau\,. \qquad (3.31)
\end{aligned}
$$

Finally, concerning (3.25),

$$
\begin{aligned}
& \int_0^{+\infty}\int_{\mathbb{R}^n} \left(C(u)+C(w)\right)\psi\,dx\,d\tau \\
= & \int_s^t\int_{\mathbb{R}^n} \left((C(u)-C(w))*\varphi^\varepsilon\right)\,\left((u-w)*\varphi^\varepsilon\right)\,dx\,d\tau \\
= & \frac{1}{2}\int_s^t \left(\|C(u)-C(w)\|^2_{\mathbf{L^2}} + \|(u-w)*\varphi^\varepsilon\|^2_{\mathbf{L^2}}\right)\,d\tau\,. \\
= & \frac{1}{2}(1+\gamma^2)\int_s^t \|u-w\|^2_{\mathbf{L^2}}\,d\tau\,. \qquad (3.32)
\end{aligned}
$$

Now, collect (3.29), (3.30), (3.31) and (3.32) to obtain

$$
\begin{aligned}
& \frac{1}{2}\left(\|(u(t)-w(t))*\varphi^\varepsilon\|^2_{\mathbf{L^2}} - \|(u(s)-w(s))*\varphi^\varepsilon\|^2_{\mathbf{L^2}}\right) \\
& +\beta_1\int_s^t \|\nabla_x\left((u-w)*\varphi^\varepsilon\right)\|^2_{\mathbf{L^2}}\,d\tau \\
\leq & \int_s^t \left(\frac{{\alpha_1}^2}{4\beta_1}\|u-w\|^2_{\mathbf{L^2}} + \beta_1\|\nabla_x\left((u-w)*\varphi^\varepsilon\right)\|^2_{\mathbf{L^2}}\right)\,d\tau \\
& +\frac{1}{2}(1+\gamma^2)\int_s^t \|u-w\|^2_{\mathbf{L^2}}\,d\tau
\end{aligned}
$$

so that

$$\begin{aligned}\|(u(t)-w(t))*\varphi^\varepsilon\|^2_{\mathbf{L^2}} &\leq \|(u(s)-w(s))*\varphi^\varepsilon\|^2_{\mathbf{L^2}} \\ &\quad + \int_s^t \left(\frac{{\alpha_1}^2}{2\beta_1}+\gamma^2+1\right)\|u-w\|^2_{\mathbf{L^2}}\,d\tau\,.\end{aligned}$$

Since both terms in the first line above are in $\mathbf{C^0}\left([0,T];\mathbf{L^2}(\mathbb{R}^n;\mathbb{R}^m)\right)$, as $\varepsilon\to 0$ they converge strongly in $\mathbf{L^2}$. Hence

$$\begin{aligned}\|(u(t)-w(t))\|^2_{\mathbf{L^2}} &\leq \|(u(s)-w(s))\|^2_{\mathbf{L^2}} \\ &\quad + \int_s^t \left(\frac{{\alpha_1}^2}{2\beta_1}+\gamma^2+1\right)\|u-w\|^2_{\mathbf{L^2}}\,d\tau\,.\end{aligned}$$

Now apply the Gronwall Lemma and obtain (3.20) in the open interval $]0,T[$. By continuity, the proof is completed. □

The next lemma is a modification of [1, Chapter 2, Theorem 3.3.1].

LEMMA 3.3. Fix $a\in[0,1[$. Then, there exists a positive $c=c(a)$ with the following property: if $\delta\colon [0,+\infty[\,\mapsto[0,+\infty[$ is such that $t\mapsto t^a\,\delta(t)$ belongs to $\mathbf{L^\infty_{loc}}\left([0,+\infty[\,;\mathbb{R}\right)$ and for positive H_1,H_2,H_3

$$\delta(t)\leq H_1t^{-a}+\int_0^t\left(H_2+\frac{H_3}{\sqrt{t-\tau}}\right)\delta(\tau)\,d\tau$$

for a.e. $t\in[0,+\infty[$, then

$$\delta(t)\leq H_1t^{-a}\left(1+(H_2+H_3)c\sqrt{t}e^{(H_2+H_3)\pi t}\right)\,. \tag{3.33}$$

If $H_2=0$, the proof is in [1]. The present case is a straightforward generalization and, hence, its proof is omitted. Define

$$\mathbf{Y}=\mathbf{W^{1,\infty}}(\mathbb{R}^n,\mathbb{R}^m)\cap\mathbf{W^{2,1}}(\mathbb{R}^n,\mathbb{R}^m)$$

which we equip with the norm

$$\|u\|_{\mathbf{Y}}=\|u\|_{\mathbf{W^{1,\infty}}}+\|u\|_{\mathbf{W^{2,1}}}\,.$$

LEMMA 3.4. Let A, B and C satisfy **(A)**, **(B)** and **(C)**. Then, there exists a strictly positive time $\hat{t}=\hat{t}(\|u_o\|_{\mathbf{Y}})$ such that for every initial datum $u_o\in\mathbf{Y}$, system (1.1) admits a solution $u\in\mathbf{C^0}\big([0,\hat{t}];\mathbf{Y}\big)$.

Proof. Recursively define the sequence

$$\begin{aligned}u_1(t) &= u_o*\Phi_B(t),\\ u_{k+1}(t) &= u_o*\Phi_B(t)+\int_0^t A\left(u_k(\tau)\right)*\nabla_x\Phi_B(t-\tau)\,d\tau\\ &\quad+\int_0^t C\left(u_k(\tau)\right)*\Phi_B(t-\tau)\,d\tau \qquad \text{for } k\geq 1\,.\end{aligned}$$

Note that $u_k \in \mathbf{Y}$ for all k and moreover, since

$$\begin{aligned} u_{k+1}(t) - u_k(t) &= \int_0^t \left(A(u_k(\tau) - A(u_{k-1}))\right) * \nabla_x \Phi_B(t-\tau)\, d\tau \\ &\quad + \int_0^t \left(C(u_k(\tau) - C(u_{k-1}))\right) * \Phi_B(t-\tau)\, d\tau\,, \end{aligned}$$

there exists a $\hat{t} = \hat{t}(\|u_o\|_{\mathbf{Y}})$ sufficiently small such that the standard Banach Fixed Point argument can be applied in $\mathbf{C^0}\big([0,\hat{t}];\mathbf{Y}\big)$. Therefore, the sequence $u_k(t)$ converges in $\mathbf{Y}$ uniformly on $[0,\hat{t}]$ to a u in $\mathbf{C^0}\big([0,\hat{t}];\mathbf{Y}\big)$. Finally, the Dominated Convergence Theorem allows us to show that $u(t)$ satisfies (2.4). □

LEMMA 3.5. Fix p in $[1,+\infty]$. Let A, B and C satisfy **(A)**, **(B)** and **(C)**. Let u, w be two solutions to (1.1) in $\mathbf{C^0}\left([0,+\infty[\,;\mathbf{Y}\right)$. Then, there exists a positive, absolutely continuous and increasing function $L_1\colon\ [0,+\infty[\ \mapsto [0,+\infty[$ such that

$$\|u(t) - w(t)\|_{\mathbf{L^p}} \leq L_1(t;\beta_1) \cdot \|u_o - w_o\|_{\mathbf{L^p}}, \tag{3.34}$$

$$\|\nabla_x u(t)\|_{\mathbf{L^p}} \leq L_1(t;\beta_1) \cdot \|\nabla_x u_o\|_{\mathbf{L^p}}, \tag{3.35}$$

where

$$L_1(t;\beta_1) = 1 + (\gamma + K(\beta_1))\, c\sqrt{t} e^{(\gamma+\alpha_1 K(\beta_1))\pi t} \tag{3.36}$$

with K as in (2.5) and c as in (3.33).

Proof. By (2.4),

$$\begin{aligned} & \|u(t) - w(t)\|_{\mathbf{L^p}} \\ \leq\ & \|\left((u_o - w_o) * \Phi_B\right)(t)\|_{\mathbf{L^p}} \\ & + \int_0^t \|\left(A\left(u(\tau)\right) - A\left(w(\tau)\right)\right) * \nabla_x \Phi_B(t-\tau)\|_{\mathbf{L^p}}\, d\tau \\ & + \int_0^t \|\left(C\left(u(\tau)\right) - C\left(w(\tau)\right)\right) * \Phi_B(t-\tau)\|_{\mathbf{L^p}}\, d\tau \\ \leq\ & \|u_o - w_o\|_{\mathbf{L^p}} \cdot \|\Phi_B(t)\|_{\mathbf{L^1}} \\ & + \|\nabla_u A\|_{\mathbf{L^\infty}} \int_0^t \|u(\tau) - w(\tau)\|_{\mathbf{L^p}} \|\nabla_x \Phi_B(t-\tau)\|_{\mathbf{L^1}}\, d\tau \\ & + \|\nabla_u C\|_{\mathbf{L^\infty}} \int_0^t \|u(\tau) - w(\tau)\|_{\mathbf{L^p}} \|\Phi_B(t-\tau)\|_{\mathbf{L^1}}\, d\tau \\ \leq\ & \|u_o - w_o\|_{\mathbf{L^p}} \\ & + \alpha_1 \int_0^t \frac{K(\beta_1)}{\sqrt{t-\tau}} \|u(\tau) - w(\tau)\|_{\mathbf{L^p}}\, d\tau \\ & + \gamma \int_0^t \|u(\tau) - w(\tau)\|_{\mathbf{L^p}}\, d\tau \\ \leq\ & \|u_o - w_o\|_{\mathbf{L^p}} + \int_0^t \left(\gamma + \frac{\alpha_1 K(\beta_1)}{\sqrt{t-\tau}}\right) \|u(\tau) - w(\tau)\|_{\mathbf{L^p}}\, d\tau\,. \end{aligned}$$

Apply Lemma 3.3 with $a = 0$, $\delta(t) = \|u(t) - w(t)\|_{\mathbf{L^p}}$, $H_1 = \|u_o - w_o\|_{\mathbf{L^p}}$, $H_2 = \gamma$ and $H_3 = \alpha_1 K(\beta_1)$ to obtain

$$\|u(t) - w(t)\|_{\mathbf{L^p}} \leq \left(1 + (\gamma + K(\beta_1))\, c\sqrt{t} e^{(\gamma+\alpha_1 K(\beta_1))\pi t}\right) \cdot \|u_o - w_o\|_{\mathbf{L^p}}.$$

To prove (3.35), consider the cases $p \in [1, +\infty[$ and $p = \infty$ separately. Then, compute the difference quotients:

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(\frac{\|u(t,x) - u(t,x+he_i)\|}{h}\right)^p dx \\
\leq\ & \left(1 + (\gamma + K(\beta_1))\, c\sqrt{t} e^{(\gamma+\alpha_1 K(\beta_1))\pi t}\right)^p \int_{\mathbb{R}^n} \left(\frac{\|u_o(x) - u_o(x+he_i)\|}{h}\right)^p dx \\
&\cdot \sup_{\mathbb{R}^n} \frac{\|u(t,x) - u(t,x+he_i)\|}{h} \\
\leq\ & \left(1 + (\gamma + K(\beta_1))\, c\sqrt{t} e^{(\gamma+\alpha_1 K(\beta_1))\pi t}\right) \cdot \sup_{\mathbb{R}^n} \frac{\|u_o(x) - u_o(x+he_i)\|}{h}.
\end{aligned}$$

Above, e_i is the i-th vector in the standard basis of $\mathbb{R}^n$. Passing to the norms in the limit $h \to 0^+$, one gets (3.35) for $p \in [1, +\infty]$. □

Below we use the Gagliardo-Nirenberg inequality [11, Formula (2.2)], which ensures that for a suitable constant K_{G-N} we have

$$\|\nabla_x u(t)\|^2_{\mathbf{L^2}} \leq K_{G-N} \cdot \left\|\nabla^2_x u(t)\right\|_{\mathbf{L^1}} \cdot \|u(t)\|_{\mathbf{L}^\infty}. \tag{3.37}$$

Lemma 3.6. Let A, B and C satisfy **(A)**, **(B)** and **(C)**. Then, there exists a positive function $L_2\colon [0, +\infty[\times [0, +\infty[\mapsto [0, +\infty[$ absolutely continuous and increasing in each argument, such that for any function $u \in \mathbf{C^0}([0, +\infty[, \mathbf{Y})$ solving (1.1),

$$\|u(t)\|_{\mathbf{W^{2,1}}} \leq \left(\frac{K(\beta_1)}{\sqrt{t}} + L_2\left(t, \|u_o\|_{\mathbf{L}^\infty}; \beta_1\right)\right) \cdot \|\nabla_x u_o\|_{\mathbf{L^1}}, \tag{3.38}$$

where

$$\begin{aligned}
L_2\left(t, \|u_o\|_{\mathbf{L}^\infty}; \beta_1\right) &= cK^2(\beta_1)(\alpha_1 + \gamma) \\
&+ cK^2(\beta_1)\, K_{G-N}\, L_1(t;\beta_1) \|u_o\|_{\mathbf{L}^\infty} \\
&\quad \cdot e^{\pi K(\beta_1)(\alpha_1+\gamma+K_{G-N} L_1(t;\beta_1)\|u_o\|_{\mathbf{L}^\infty})t},
\end{aligned} \tag{3.39}$$

with $L_1(t;\beta_1)$ as in (3.36), $K(\beta_1)$ as in (2.5), and K_{G-N} as in (3.37).

Proof. Let u be the solution of (2.4). Then

$$\begin{aligned}
\left\|\nabla_x^2 u(t)\right\|_{\mathbf{L^1}} &\leq \left\|\nabla_x^2\left((u_o * \Phi_B)(t)\right)\right\|_{\mathbf{L^1}} \\
&\quad + \int_0^t \left\|\nabla_x^2 A\left(u(\tau)\right) * \nabla_x \Phi_B(t-\tau)\right\|_{\mathbf{L^1}} d\tau \\
&\quad + \int_0^t \left\|\nabla_x^2\left(C\left(u(\tau)\right) * \Phi_B(t-\tau)\right)\right\|_{\mathbf{L^1}} d\tau \\
&\leq \left\|\nabla_x u_o\right\|_{\mathbf{L^1}} \cdot \left\|\nabla_x \Phi_B(t)\right\|_{\mathbf{L^1}} \\
&\quad + \alpha_2 \int_0^t \left\|\nabla_x u(\tau)\right\|_{\mathbf{L^2}}^2 \cdot \left\|\nabla_x \Phi_B(t-\tau)\right\|_{\mathbf{L^1}} d\tau \\
&\quad + \alpha_1 \int_0^t \left\|\nabla_x^2 u(\tau)\right\|_{\mathbf{L^1}} \cdot \left\|\nabla_x \Phi_B(t-\tau)\right\|_{\mathbf{L^1}} d\tau \\
&\quad + \gamma \int_0^t \left\|\nabla_x u(\tau)\right\|_{\mathbf{L^1}} \cdot \left\|\nabla_x \Phi_B(t-\tau)\right\|_{\mathbf{L^1}} d\tau.
\end{aligned}$$

Then, by (2.5), (3.37) and (3.34),

$$\begin{aligned}
\left\|\nabla_x^2 u(t)\right\|_{\mathbf{L^1}} &\leq \frac{K(\beta_1)}{\sqrt{t}} \left\|\nabla_x u_o\right\|_{\mathbf{L^1}} \\
&\quad + K_{G-N} \cdot \sup_{\tau\in[0,t]} \left\|u(\tau)\right\|_{\mathbf{L^\infty}} \cdot \int_0^t \frac{K(\beta_1)\alpha_2}{\sqrt{t-\tau}} \left\|\nabla_x^2 u(\tau)\right\|_{\mathbf{L^1}} d\tau \\
&\quad + \int_0^t \frac{K(\beta_1)\alpha_1}{\sqrt{t-\tau}} \left\|\nabla_x^2 u(\tau)\right\|_{\mathbf{L^1}} d\tau \\
&\quad + \int_0^t \frac{K(\beta_1)\gamma}{\sqrt{t-\tau}} \left\|\nabla_x u(\tau)\right\|_{\mathbf{L^1}} d\tau
\end{aligned}$$

so that

$$\begin{aligned}
&\left\|u(t)\right\|_{\mathbf{W^{2,1}}} \\
\leq\ & \frac{K(\beta_1)}{\sqrt{t}} \left\|\nabla_x u_o\right\|_{\mathbf{L^1}} \\
&+ K(\beta_1)\left(\alpha_1 + \gamma + K_{G-N} \sup_{\tau\in[0,t]} \left\|u(t)\right\|_{\mathbf{L^\infty}}\right) \int_0^t \frac{\left\|u(\tau)\right\|_{\mathbf{W^{2,1}}}}{\sqrt{t-\tau}} d\tau \\
\leq\ & \frac{K(\beta_1)}{\sqrt{t}} \left\|\nabla_x u_o\right\|_{\mathbf{L^1}} \\
&+ K(\beta_1)\left(\alpha_1 + \gamma + K_{G-N}\, L_1(t;\beta_1)\left\|u_o\right\|_{\mathbf{L^\infty}}\right) \int_0^t \frac{\left\|u(\tau)\right\|_{\mathbf{W^{2,1}}}}{\sqrt{t-\tau}} d\tau.
\end{aligned}$$

Now apply Lemma 3.3 with

$$\begin{aligned}
&a = \frac{1}{2} \qquad \delta(t) = \left\|u(t)\right\|_{\mathbf{W^{2,1}}} \qquad H_1 = K(\beta_1)\left\|\nabla_x u_o\right\|_{\mathbf{L^1}} \qquad H_2 = 0, \\
&H_3 = K(\beta_1)\cdot\left(\alpha_1 + \gamma + K_{G-N}\, L_1(t;\beta_1)\left\|u_o\right\|_{\mathbf{L^\infty}}\right)
\end{aligned}$$

to obtain (3.38). □

LEMMA 3.7. Let A, B and C satisfy **(A)**, **(B)** and **(C)**. Then, there exists a positive function $\varphi\colon [0,+\infty[\times[0,+\infty[\mapsto[0,+\infty[$ absolutely continuous and increasing in each argument, such that for all $s,t\in[0,+\infty[$ with $s<t$ and for all functions $u,w\in\mathbf{C^0}([0,T],\mathbf{Y})$ solving (1.1) with initial data u_o,w_o, we have

$$\begin{aligned} & \|u(t)-w(s)\|_{\mathbf{L^1}} \\ \leq\ & L_1(s;\beta_1)\cdot\|u_o-w_o\|_{\mathbf{L^1}} \\ & +\|u_o\|_{\mathbf{W^{1,1}}}\cdot\left|\varphi\left(t,\|u_o\|_{\mathbf{L}^\infty};\beta_1\right)-\varphi\left(s,\|u_o\|_{\mathbf{L}^\infty};\beta_1\right)\right|, \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} & \varphi\left(t,\|u_o\|_{\mathbf{L}^\infty};\beta_1\right) \\ =\ & 2\beta_2\cdot K(\beta_1)\cdot\sqrt{t} \\ & +\int_0^t\left((\alpha_1+\gamma)L_1(\tau;\beta_1)+\beta_2L_2\left(\tau,\|u_o\|_{\mathbf{L}^\infty};\beta_1\right)\right)d\tau \end{aligned} \tag{3.41}$$

and K is as in (2.5), L_1 is as in (3.36) and L_2 as in (3.39).

Proof. Note first that

$$\|u(t)-w(s)\|_{\mathbf{L^1}}\leq\|u(t)-u(s)\|_{\mathbf{L^1}}+\|u(s)-w(s)\|_{\mathbf{L^1}}\,.$$

The latter term above follows from (3.34) with $p=1$. The estimate on the former term follows from (1.1), using (3.35), (3.38) and (3.34):

$$\begin{aligned} & \|u(t)-u(s)\|_{\mathbf{L^1}} \\ \leq\ & \int_s^t\|\mathrm{div}_x\,A\left(u(\tau)\right)+B\Delta u+C\left(u(\tau)\right)\|_{\mathbf{L^1}}\,d\tau \\ \leq\ & \int_s^t\left(\alpha_1\|\nabla_xu(\tau)\|_{\mathbf{L^1}}+\beta_2\left\|\nabla_x^2u(\tau)\right\|_{\mathbf{L^1}}+\gamma\|u(\tau)\|_{\mathbf{L^1}}\right)d\tau \\ \leq\ & \int_s^t\left(\alpha_1\|\nabla_xu(\tau)\|_{\mathbf{L^1}}+\beta_2\|u(\tau)\|_{\mathbf{W^{2,1}}}+\gamma\|u(\tau)\|_{\mathbf{L^1}}\right)d\tau \\ \leq\ & \int_s^t\alpha_1L_1(\tau;\beta_1)\,d\tau\,\|\nabla_xu_o\|_{\mathbf{L^1}} \\ & +\int_s^t\beta_2\left(\frac{K(\beta_1)}{\sqrt{\tau}}+L_2\left(\tau,\|u_o\|_{\mathbf{L}^\infty};\beta_1\right)\right)d\tau\,\|\nabla_xu_o\|_{\mathbf{L^1}} \\ & +\int_s^t\gamma L_1(\tau;\beta_1)\,d\tau\,\|u_o\|_{\mathbf{L^1}} \\ \leq & \int_s^t\left((\alpha_1+\gamma)L_1(\tau;\beta_1)+\beta_2\left(\frac{K(\beta_1)}{\sqrt{\tau}}+L_2\left(\tau,\|u_o\|_{\mathbf{L}^\infty};\beta_1\right)\right)\right)d\tau\|u_o\|_{\mathbf{W^{1,1}}}, \end{aligned}$$

which completes the proof. □

Proof of Theorem (2.3). First consider the case $u_o\in\mathbf{Y}$. Then, Lemma 3.4 ensures that the solution to (1.1) exists on a time interval whose length depends only on the norm of the initial data. Therefore, the estimates provided by Lemma 3.5 and Lemma 3.6 allow us to obtain a solution in $\mathbf{C^0}([0,+\infty[;\mathbf{Y})$ that satisfies the bounds (2.8), (2.9), (2.10) and (2.11).

Regularize an initial datum $u_o \in \mathbf{X}$ with a mollifier $\varphi \in \mathbf{C^\infty_c}(\mathbb{R}^n;\mathbb{R})$ similar to the one defined in (3.26), to obtain a sequence $u_o^k \in \mathbf{Y}$ such that

$$\left\|u_o^k\right\|_{\mathbf{X}} \leq \|u_o\|_{\mathbf{X}} \text{ and as } k \to +\infty \qquad \begin{array}{ll} u_o^k \to u_o & \text{in } \mathbf{L^1}(\mathbb{R}^n,\mathbb{R}^m)\,, \\ u_o^k \overset{*}{\rightharpoonup} u_o & \text{in } \mathbf{L^\infty}(\mathbb{R}^n,\mathbb{R}^m)\,, \\ u_o^k \overset{*}{\rightharpoonup} u_o & \text{in } \mathbf{BV}(\mathbb{R}^n,\mathbb{R}^m)\,. \end{array} \tag{3.42}$$

Let $u^k(t)$ be the corresponding solution to (1.1). Lemma 3.5 implies that the sequence u^k is in $\mathbf{C^0}\left([0,+\infty[\,;\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right)$ and is Lipschitz in u_o^k uniformly in k on any compact time interval. Therefore, as $k \to +\infty$, the sequence u^k converges uniformly to a map u in $\mathbf{C^0}\left([0,+\infty[\,;\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right)$. By (3.35) and (3.42), this strong convergence allows us to obtain, in the limit, (2.7) from (3.40).

Moreover, (3.34) and (3.35) ensure that for any positive T,

$$\begin{array}{rcl} u^k \overset{*}{\rightharpoonup} u & \text{in} & \mathbf{L^\infty_w}\left([0,T];\mathbf{L^\infty}(\mathbb{R}^n,\mathbb{R}^m)\right), \\ \nabla_x u^k \overset{*}{\rightharpoonup} \nabla_x u & \text{in} & \mathbf{L^\infty_w}\left([0,T];\mathcal{M}(\mathbb{R}^n;\mathbb{R}^{m\times n})\right) \end{array} \tag{3.43}$$

and, hence, they yield (2.8), (2.9) and (2.10) in the limit. As a consequence, we also have that for all t, $u(t) \in \mathbf{L^\infty_{loc}}\left([0,+\infty[\,;\mathbf{X}\right)$. Here, $\mathcal{M}(\mathbb{R}^p;\mathbb{R}^q)$ is the space of $\mathbb{R}^q$-valued Radon measure on $\mathbb{R}^p$.

Now note that (3.38) implies that, for all positive t, as $k \to +\infty$ one has $\sqrt{t}\cdot\nabla_x^2 u^k(t) \overset{*}{\rightharpoonup} \sqrt{t}\cdot\nabla_x^2 u(t)$ in $\mathbf{L^\infty_w}\left([0,T];\mathcal{M}(\mathbb{R}^n;\mathbb{R}^{m\times n^2})\right)$. Define the sequence w^k by

$$\begin{aligned} \nabla_x^2 w^k(t) &= \nabla_x^2\left((u_o * \Phi_B)(t)\right) \\ &\quad + \int_0^{t-1/k} \nabla_x^2 A\left(u(\tau)\right) * \nabla_x \Phi_B(t-\tau)\,d\tau \\ &\quad + \int_0^{t-1/k} \nabla_x^2\left(C\left(u(\tau)\right) * \Phi_B(t-\tau)\right)\,d\tau \end{aligned}$$

and note that $w^k(t) \in \mathbf{C^\infty}(\mathbb{R}^n;\mathbb{R}^m)$ for all $t > 0$. Moreover, $\nabla_x^2 w^k$ is a Cauchy sequence in $\mathbf{L^1}$ and as $k \to +\infty$ converges to $\nabla_x^2 u$, by (2.4). Hence, for all $t > 0$, $u(t)$ is in $\mathbf{W^{2,1}}(\mathbb{R}^n;\mathbb{R}^m)$. Using (3.42), it is then justified to pass to the limit in (3.38) to obtain (2.11).

Now define, for $u_o \in \mathbf{X}$, $S_t u_o = u(t)$. By (2.8), (2.9) and (2.10), S is well defined on $[0,+\infty[\,\times \mathbf{X} \mapsto \mathbf{X}$. By (3.43), $S \in \mathbf{L^\infty_{loc}}\left([0,+\infty[\,;\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^m)\right)$. The regularity $S \in \mathbf{C^0}\left([0,+\infty[\,;\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right)$ follows from (2.7).

We conclude this proof with the semigroup property which follows from the autonomous nature of (1.1) together with the uniqueness proved in Lemma 3.2. □

Proof of Theorem 2.4. By Theorem 2.3, we can apply Proposition 2.1 with $\mathcal{D} = \mathbf{X}$, $d(u,w) = \|w-u\|_{\mathbf{L^1}}$, $w(t) = S^2 u_o$ and $S = S^1$. Compute the integrand in (2.3) as follows:

$$\begin{aligned} \frac{1}{h} d\left(S^2_{t+h}u_o, S^1_h S^2_t u_o\right) &= \frac{1}{h}\left\|S^2_h S^2_t u_o - S^1_h S^2_t u_o\right\|_{\mathbf{L^1}} \\ &= \left\|\frac{S^2_h S^2_t u_o - S^2_t u_o}{h} - \frac{S^1_h S^2_t u_o - S^2_t u_o}{h}\right\|_{\mathbf{L^1}}. \end{aligned}$$

Denote $w = S_t^2 u_o$ and use the fact that for $t > 0$, $S_t^i u_o \in \mathbf{Y}$ and hence $\partial_t S_t^i u_o \in \mathbf{C^0}\left([0,+\infty[\,;\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^m)\right)$:

$$\begin{aligned}
& \liminf_{h\to 0} \frac{1}{h} d\left(S_{t+h}^2 u_o, S_h^1 S_t^2 u_o\right) && (3.44)\\
= \; & \lim_{h\to 0} \left\| \frac{S_h^2 w - w}{h} - \frac{S_h^1 w - w}{h} \right\|_{\mathbf{L^1}} \\
= \; & \left\| \left(\partial_t S^2\right)_{|t=0} w - \left(\partial_t S^1\right)_{|t=0} w \right\|_{\mathbf{L^1}} \\
= \; & \left\| \mathrm{div}_x \left(A_1(w) - A_2(w)\right)\right\|_{\mathbf{L^1}} \\
& + \|B_1 \cdot \Delta w - B_2 \cdot \Delta w\|_{\mathbf{L^1}} + \|C_1(w) - C_2(w)\|_{\mathbf{L^1}} \\
\leq \; & \|\nabla_u A_1 - \nabla_u A_2\|_{\mathbf{L}^\infty} \cdot \mathrm{TV}(w) \\
& + \|B_1 - B_2\| \cdot \|w\|_{\mathbf{W^{2,1}}} + d_{\mathbf{Lip}}(C_1, C_2) \cdot \|w\|_{\mathbf{L^1}}, && (3.45)
\end{aligned}$$

where we used (2.12). Insert the above result in (2.3), and use (2.10), (2.11) and (2.8) to obtain

$$\begin{aligned}
& \left\|S_t^1 u_o - S_t^2 u_o\right\|_{\mathbf{L^1}} \\
\leq \; & L_1(t;\beta_1) \cdot \|\nabla_u A_1 - \nabla_u A_2\|_{\mathbf{L}^\infty} \cdot \int_0^t \mathrm{TV}(S_t^2 u_o)\, d\tau \\
& + L_1(t;\beta_1) \cdot \|B_1 - B_2\| \cdot \int_0^t \left\|S_t^2 u_o\right\|_{\mathbf{W^{2,1}}} \\
& + L_1(t;\beta_1) \cdot d_{\mathbf{Lip}}(C_1, C_2) \cdot \int_0^t \left\|S_t^2 u_o\right\|_{\mathbf{L^1}} \\
\leq \; & \left(L_1(t;\beta_1) \int_0^t L_1(\tau;\beta_1)\, d\tau\right) \cdot \|\nabla_u A_1 - \nabla_u A_2\|_{\mathbf{L}^\infty} \cdot \mathrm{TV}(u_o) \\
& + L_1(t;\beta_1) \cdot \|B_1 - B_2\| \left(2K(\beta_1)\sqrt{t} + \int_0^t L_2\left(\tau, \|u_o\|_{\mathbf{L}^\infty}; \beta_1\right)\right) \mathrm{TV}(u_o) \\
& + \left(L_1(t;\beta_1) \int_0^t L_1(\tau;\beta_1)\, d\tau\right) \cdot d_{\mathbf{Lip}}(C_1, C_2) \cdot \|u_o\|_{\mathbf{L^1}},
\end{aligned}$$

which gives the desired estimate (2.13), thanks to (3.36). □

Finally, we prove Corollary 2.6.

Proof of Corollary 2.6. Fix $\varepsilon > 0$ and $u_o \in \mathcal{D}$. By (3.41), we can apply Proposition 2.1 with $S = S^\varepsilon$ and $w(t) = S_t^{B_\varepsilon} w_o$. As a result, using (2.15), (3.44)–(3.45) and (2.11) we obtain

$$\begin{aligned}
\left\|S_t^\varepsilon u_o - S_t^{B_\varepsilon} u_o\right\|_{\mathbf{L^1}} \; & \leq \; L \cdot \int_0^t \liminf_{h\to 0} \frac{1}{h} \left\|S_{\tau+h}^{B_\varepsilon} u_o - S_h^\varepsilon S_\tau^{B_\varepsilon} u_o\right\|_{\mathbf{L^1}} d\tau \\
& \leq \; L \cdot \int_0^t \|B_\varepsilon - \varepsilon\, \mathrm{Id}\| \cdot \left\|S_\tau^{B_\varepsilon} u_o\right\|_{\mathbf{W^{2,1}}} d\tau \\
& \leq \; L \cdot \|B_\varepsilon - \varepsilon\, \mathrm{Id}\| \\
& \quad \cdot \int_0^t \left(\frac{K(\beta_1^\varepsilon)}{\sqrt{t}} + L_2\left(t, \|u_o\|_{\mathbf{L}^\infty}; \beta_1^\varepsilon\right)\right) d\tau \cdot \mathrm{TV}(u_o)\,.
\end{aligned}$$

The proof is now completed through the triangle inequality. □

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