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# GLOBAL SOLUTIONS OF THE MEAN–FIELD, VERY HIGH TEMPERATURE CALDEIRA–LEGGETT MASTER EQUATION

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**Abstract.** In this paper, global well–posedness as well as regularity of very high temperature Caldeira–Leggett models with repulsive Poisson coupling are proved by using Green function techniques and Fokker–Planck smoothing arguments along with kinetic energy and elliptic estimates.

**1. Introduction and main result.** The mathematical analysis of quantum dissipation phenomena ruled by Fokker–Planck-type mechanisms has experienced a great impulse in past years (an extensive review of dissipative quantum models can be found in [6, 7]). In particular, this analysis has focused mainly on the problems of existence, uniqueness, regularity and long-time behaviour of solutions to different approaches of quantum mean–field Fokker–Planck models in a Wigner function context. In [2], the derivation of the so-called Wigner–Poisson–Fokker–Planck (WPFP) equation from the density matrix formalism of an electron ensemble interacting dissipatively with a thermal bath as well as its well–posedness (in the frame of Lindblad's class; see [10]) were widely discussed. Also, a local existence theory was developed for the simplest Markovian (frictionless) model which reads

$$\frac{\partial W}{\partial t} + (\xi \cdot \nabla_x) W + \Theta[V] W = \frac{D_{pp}}{m^2} \Delta_\xi W \,, \tag{1.1}$$

$$W(x, \xi, 0) = W_0(x, \xi) \,, \tag{1.2}$$

$$V(x,t) = \frac{1}{\varepsilon_0 |x|} * n(x,t) \,, \quad n(x,t) = \int_{\mathbb{R}^3} W(x, \xi, t)\, d\xi \,, \tag{1.3}$$

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where $m$ is the effective mass of electrons, $\varepsilon_0$ is the vacuum permitivity, $D_{pp} = \eta k_B T$ is a physical constant related to particle–reservoir interaction ($\eta > 0$ is the damping constant of the bath, $k_B$ the Boltzmann constant and $T$ the temperature of the bath) and where the nonlinear (pseudo–differential) term reads (see [2, 5])

$$\Theta[V]W(x,\xi,t) = H(x,\xi,t) *_\xi W(x,\xi,t)\,, \tag{1.4}$$

$$H(x,\xi,t) = 16\left(\frac{m}{\hbar}\right)^3 \mathrm{Re}\left\{ i\, e^{i\frac{2m}{\hbar}(x\cdot\xi)}\, \mathcal{F}^{-1}_{x\mapsto\xi} V\Big(\frac{2m}{\hbar}\xi, t\Big)\right\}, \tag{1.5}$$

$\hbar$ denoting the Planck constant, $\mathrm{Re}(z)$ the real part of $z$ and $\mathcal{F}^{-1}$ being the inverse Fourier transform

$$\mathcal{F}^{-1}_{x\mapsto y}[f](y) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} f(x)\, e^{ix\cdot y}\, dx\,.$$

A first approach to the existence of global (mild) solutions to quantum mean–field Fokker–Planck systems was recently performed in [5]. Indeed, the existence of a unique, regular solution to the following three–dimensional WPFP problem with nonvanishing viscosity

$$\begin{aligned} &\frac{\partial W}{\partial t} + (\xi\cdot\nabla_x)W + \Theta[V]W = \frac{D_{pp}}{m^2}\,\Delta_\xi W + 2\lambda\,\mathrm{div}_\xi(\xi W) + D_{qq}\,\Delta_x W, \\ &W(x,\xi,0) = W_0(x,\xi) \end{aligned} \tag{1.6}$$

was proved. Here, $\lambda = \frac{\eta}{2m}$ is the friction coefficient and $D_{qq} = \frac{\eta\hbar^2}{12m^2k_BT}$ is also an interaction constant. The proof makes essential use of the elliptic $x$–regularization (via the term $D_{qq}\,\Delta_x W$) and the connection between the kinetic energies of the Wigner function $W(x,\xi,t)$ and of the Husimi function

$$W^H(x,\xi,t) = W(x,\xi,t) *_{x,\xi} \left(\frac{m}{\hbar\pi}\right)^3 \exp\left\{-\frac{m}{\hbar}\Big(|x|^2 + |\xi|^2\Big)\right\}\,,$$

whose great advantage is to be pointwise nonnegative on $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$ because of the positivity of the density matrix operator (guaranteed by the Lindblad condition). A global existence theory has also been dealt with in [3] from a different perspective in a weighted $L^2$ space. In this direction, a remarkable advance is that the positivity assumption for the particle density is not required. Recently, the system of 1D hydrodynamic equations for the current and the charge density associated with the most general WPFP equation

$$\begin{aligned} &\frac{\partial W}{\partial t} + (\xi\cdot\nabla_x)W + \Theta[V]W \\ &= \frac{D_{pp}}{m^2}\,\Delta_\xi W + 2\lambda\,\mathrm{div}_\xi(\xi W) + 2\frac{D_{pq}}{m}\,\mathrm{div}_x(\nabla_\xi W) + D_{qq}\,\Delta_x W \end{aligned} \tag{1.7}$$

was also tackled in [9], where $D_{pq} = \frac{\eta\omega\hbar^2}{12\pi m k_B T}$ is the quantum Drude factor (which in some situations prevents the diffusion matrix to be positive semidefinite), with $\omega$ standing for the cut–off frequency of the reservoir oscillators. Here, the authors study the rate of time decay of solutions via the entropy dissipation method.

The long–time dynamics of (1.1)–(1.5) were also analyzed in [2] under the assumption that the solution is defined in $[0,\infty)$. Actually, it was proved that

$$\lim_{t\to\infty} \|W(t) - Q\,G_0(t)\|_{L^1(\Omega_t)} = 0\,, \tag{1.8}$$

$$\lim_{t\to\infty} \|W^H(t) - Q\,G_0(t)\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} = 0\,, \tag{1.9}$$

where $\Omega_t = \{(t^{\frac{3}{2}}x, t^{\frac{1}{2}}\xi) : (x,\xi)\in\Omega\}$, $\Omega$ being an arbitrary bounded open subset of $\mathbb{R}^6$, $G_0$ is defined below (see § 3.1), and where $Q$ denotes the total charge of the system which is an invariant of motion.

Our main goal in this paper is to prove the existence of a unique global–in–time mild solution to the 3D frictionless WPFP problem (1.1)–(1.5). To this purpose, the most important difficulties to be overcome are the nonpositivity of the Wigner function (which does not allow for the application of maximum principles and in general for the standard techniques known to work for mean–field Vlasov–Fokker–Planck systems) and the obvious lack of elliptic regularity in the position variable (because of absence of the $\Delta_x W$ term), giving rise to much worse estimates than those available for Eqs. (1.6) or (1.7), for example. A global existence theory for (1.1)–(1.5) would also fully justify the arguments in [2] leading to the long–time (linear) behaviour of the solutions (cf. (1.8)–(1.9)), for which existence in $[0,\infty)$ was assumed. Our main result is the following

Theorem 1.1. Let $W_0$ be a physically admissible initial datum (i.e. such that the density matrix operator corresponding to $W_0$ is nonnegative) belonging to $L^1(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)\cap L^1(\mathbb{R}^3_\xi; L^2(\mathbb{R}^3_x))$. Then, there exists $0 < T_{max} \le \infty$ such that the Wigner–Poisson–Fokker–Planck equation (1.1) admits a unique mild solution

$$W \in C([0,T_{max}); L^1(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)\cap L^1(\mathbb{R}^3_\xi; L^2(\mathbb{R}^3_x)))$$

when solved along with a given initial data $W_0$ in the conditions above.

Besides, if

$$\||\xi|^2 W_0\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} + \|\nabla_x W_0\|_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)} + \|W_0\|_{L^1(\mathbb{R}^3_\xi; L^{p_0}(\mathbb{R}^3_x))} < \infty$$

for some $p_0 > 3$, then $T_{max} = \infty$.

This result contributes to completing the global well–posedness analysis of physically relevant quantum Fokker–Planck models initiated in [5]. The paper is structured as follows: Section 2 is devoted to sketch the derivation and main properties of Eq. (1.1), and Section 3 concerns the proof of Theorem 1.1, which is split into several steps. First we set our problem in a mild context and analyze the more relevant properties of its fundamental solution. Then, local well–posedness and regularity of the solutions are shown by standard fixed–point arguments and the use of the smoothing properties of the Fokker–Planck kernel. Finally, global existence is deduced from the (linear) control of the total energy and an extension of Lieb–Thirring inequalities.

**2. On the very high temperature Caldeira–Leggett master equation.** The system to be dealt with throughout this paper is the quantum Fokker–Planck equation

(cf. (1.1))

$$\frac{\partial W}{\partial t} + (\xi \cdot \nabla_x)W + \Theta[V]W = \frac{D_{pp}}{m^2}\,\Delta_\xi W \tag{2.1}$$

in the Wigner phase–space representation, corresponding to the simplest Markovian approximation of open quantum systems in the high–temperature limit (set $D_{qq} = D_{pq} = 0$ in Eq. (1.7)) and frictionless case (set also $\lambda = 0$ in (1.7)). Here, the last term accounts for dissipation and fluctuation effects due to the coupling to the environment and, from a mathematical viewpoint, contributes to the regularization in the momentum variable of the solutions and macroscopic observables. This simple model was shown in [8] to yield a mathematically consistent master equation which does not take into account energy dissipation of the electron ensemble by the background. Actually, frictionless models constitute the only physically relevant Fokker–Planck quantum models which make the quantum entropy $S(R) := -\mathrm{Tr}(R\log(R))$ grow (see [2]), Tr denoting the usual trace operator and $R(t) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ standing for the density matrix operator of the electron (ensemble) for all $t \geq 0$.

If $D_{qq} = D_{pq} = 0$ in Eq. (1.7) but the friction term is still retained ($\lambda > 0$), then we find the celebrated Caldeira–Leggett model introduced in [4], which is known not to belong to the so-called Lindblad class [10]. This implies an eventual lack of positivity for the density matrix operator along the evolution. At variance, Eq. (2.1) may be cast in Lindblad form, which ensures that the system is well posed in the sense that total charge and quantum density and entropy are well defined, as well as the fact that mathematical consistency of the problem (that is, $R(t)$ is positivity preserving) holds. Albeit seemingly simple, the mathematical analysis of the frictionless WPFP equation (2.1) becomes much more complex than that carried out in [5] and [3], due to an obvious lack of *a priori* elliptic regularization in the $x$–variable.

Let us finally say a few words on the formal derivation of Eq. (2.1). Let $\rho(t) \equiv \rho(\cdot,\cdot,t) \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$ be the density matrix function, i.e. the integral kernel of $R(t)$. The model for the motion of the electron reads

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}(H_x - H_y)\rho - \frac{D_{pp}}{\hbar^2}\,|x-y|^2\rho\,, \tag{2.2}$$

where

$$H = -\frac{\hbar^2}{2m}\Delta_x + V(x,t)$$

is the electron Hamiltonian and $H_x, H_y$ stand for copies of $H$ acting on the variables $x$ and $y$, respectively. Consider now the Wigner transform

$$W(x,\xi,t) := \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_\eta} \rho\Big(x + \frac{\hbar}{2m}\eta, x - \frac{\hbar}{2m}\eta, t\Big)e^{-i\xi\cdot\eta}\,d\eta\,.$$

Wignerization of Eq. (2.2) straightforwardly yields (1.2). We remark that in [8] L. Diósi used an asymptotic expansion in the parameter $\alpha = \frac{\lambda\hbar}{k_B T}$ to derive the (whole) evolution equation for the density matrix

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\frac{i}{\hbar}(H_x - H_y)\rho - \gamma(x-y)\cdot(\nabla_x - \nabla_y)\rho \\ &\quad + \Big(D_{qq}\,|\nabla_x + \nabla_y|^2 - \frac{D_{pp}}{\hbar^2}\,|x-y|^2 + \frac{2i}{\hbar}D_{pq}\,(x-y)\cdot(\nabla_x + \nabla_y)\Big)\rho\,,\end{aligned} \tag{2.3}$$

the error of which being $O(\alpha^3)$. When the terms with coefficients $D_{qq}$ and $D_{pq}$ (which are both of order $O(\frac{1}{T})$) are neglected from Eq. (2.3) (or equivalently from Eq. (1.7)), then the Caldeira–Leggett equation is obtained. This approach, which gives a high temperature $O(\alpha^2)$–accurate model, is shown to be admissible only if the coherence length connected with the state of the electron is larger than the de Broglie wavelength $l_{dB} = \frac{\hbar}{\sqrt{4mk_BT}}$. When one goes beyond the lowest order Markovian approximation at high and medium temperatures, the dissipation term (that with coefficient $\lambda$) is also dropped and there only remains the $D_{pp}$ interaction. The so obtained master equation constitutes an $O(\alpha)$–accurate model. This is the sense in which Eq. (2.2) (or equivalently Eq. (2.1)) can be regarded as a very high temperature model (or also as a model with very weak system–reservoir coupling).

**3. Proof of Theorem 1.1.** We start by introducing some concepts and notation. Define the following quantities:

$$Q(t) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} W(x,\xi,t)\,d\xi\,dx \quad \text{total charge}\,, \tag{3.1}$$

$$E_K[W](t) := \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\xi|^2\,W(x,\xi,t)\,d\xi\,dx \quad \text{kinetic energy}\,, \tag{3.2}$$

$$E_P(t) := \frac{\varepsilon_0}{2}\int_{\mathbb{R}^3} |\nabla_x V(x,t)|^2\,dx \quad \text{potential energy}\,, \tag{3.3}$$

$$E(t) := E_K[W](t) + E_P(t) \quad \text{total energy}\,, \tag{3.4}$$

$$j(x,t) := \int_{\mathbb{R}^3_\xi} \xi\,W(x,\xi,t)\,d\xi \quad \text{current density}\,. \tag{3.5}$$

Also, denote by $L^{q,p}$ the functional spaces $L^q(\mathbb{R}^3_\xi; L^p(\mathbb{R}^3_x))$ and use $L^p$ for the particular case $q = p$. In the sequel, $C$ will denote different positive constants.

3.1. *Fundamental solution and mild formulation.* The fundamental solution of the frictionless kinetic Fokker–Planck equation is that obtained by solving the following linear initial value problem:

$$\frac{\partial G_0}{\partial t} + (\xi\cdot\nabla_x)G_0 = \frac{D_{pp}}{m^2}\,\Delta_\xi G_0\,, \tag{3.6}$$

$$G_0(x,\xi,0) = \delta_0(x,\xi)\,, \tag{3.7}$$

$\delta_0$ standing for the Dirac delta centered at $(x_0,\xi_0) = (0,0)$. Fourier transformation of (3.6)–(3.7) in both $x \mapsto y$ and $\xi \mapsto \eta$ variables leads straightforwardly to

$$\frac{\partial \hat{G}_0}{\partial t} - (y\cdot\nabla_\eta)\hat{G}_0 + \frac{D_{pp}}{m^2}\,|\eta|^2\hat{G}_0 = 0\,, \tag{3.8}$$

$$\hat{G}_0(y,\eta,0) = 1\,. \tag{3.9}$$

Then, solving (3.8)–(3.9) along the characteristic system

$$y \equiv y_0 \in \mathbb{R}^3\,,$$
$$\eta(t) = \eta_0 - y_0\,t\,,$$

we find

$$\hat{G}_0(y,\eta,t) = \exp\left\{-\frac{D_{pp}}{m^2}\Big(|\eta|^2 t + (\eta\cdot y)t^2 + \frac{|y|^2}{3}t^3\Big)\right\}.$$

Finally, by inverse Fourier transformation of $\hat{G}_0$ we have the following expression for the fundamental solution $G_0$:

$$G_0(x,\xi,t) = d(t)\exp\left\{-a(t)|x|^2 - c(t)|\xi|^2 + b(t)(x\cdot\xi)\right\}$$

with coefficients

$$a(t) = \frac{3m^2}{D_{pp}\,t^3}\,,\quad b(t) = \frac{3m^2}{D_{pp}\,t^2}\,,\quad c(t) = \frac{m^2}{D_{pp}\,t}\,,\quad d(t) = \left(\frac{\sqrt{3}\,m^2}{2\pi D_{pp}\,t^2}\right)^3.$$

Then, the solution to the (linear) initial value problem

$$\frac{\partial W}{\partial t} + (\xi\cdot\nabla_x)W = \frac{D_{pp}}{m^2}\,\Delta_\xi W\,,\quad W(0) = W_0\,,$$

is given by the action of the kinetic Fokker–Planck flux operator $G(t)$ on $W_0 \in L^1$ by means of the following pseudo–convoluted expression:

$$W(t) = G(t)[W_0] := \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_z} G(x,\xi,z,v,t)W_0(z,v)\,dz\,dv\,,$$

where (see [11, 5]) $G(x,\xi,z,v,t) = G_0(x-z-vt,\xi-v,t)$. The notion of a mild solution of the WPFP system (1.1)–(1.5) is now introduced as the solution of a (formally) equivalent integral equation.

Definition 3.1 (Mild solution). We call $W \in C([0,T];L^1\cap L^{1,2})$ a mild solution of the initial value problem (1.1)–(1.2) if it solves the integral equation

$$W(t) = G(t)[W_0] - \int_0^t G(t-s)[(H *_\xi W)(s)]\,ds \tag{3.10}$$

for all $0 \le t \le T$, where $H$ and $V$ are given by (1.5) and (1.3), respectively.

In the following result we list our main *a priori* estimates.

Proposition 3.2 (A priori estimates). The following bounds are fulfilled:

(i) $\|G_0(t)\|_{L^1} = 1$.

(ii) $\||\xi|^\alpha G_0(t)\|_{L^{q,p}} = C(q,p,\alpha)\,t^{\frac{3}{2q}+\frac{9}{2p}+\frac{\alpha}{2}-6}$ for all $1 \le q,p < \infty$ and $\alpha \ge 0$.

(iii) $\|G(t)[f]\|_{L^{q,p}} \le \|G_0\|_{L^{s,r}}\|f\|_{L^{m,l}}$, for $1 \le p,q < \infty$ and $1 \le m \le l \le \infty$ such that

$$1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{l}\,,\quad 1 + \frac{1}{q} = \frac{1}{s} + \frac{1}{m}\,.$$

(iv) $E_K[G(t)[f]] = E_K[f] + C\,t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f\,dx\,d\xi$.

(v) $\|H(t)\|_{L^1(\mathbb{R}^3_\xi)} \le C\,(\|n(t)\|_{L^1(\mathbb{R}^3_x)} + \|n(t)\|_{L^2(\mathbb{R}^3_x)})$. As consequence, we have $\|H(t)\|_{L^1(\mathbb{R}^3_\xi)} \le C\,(\|W(t)\|_{L^1} + \|W(t)\|_{L^{1,2}})$.

*Proof.* (i), (ii) and (iv) follow from direct computations involving the fundamental solution $G_0$. (iii) is a consequence of Young's inequality for convolutions. (v) follows from the inequality (cf. (1.5))

$$|H(x,\xi,t)| \le 16\left(\frac{m}{\hbar}\right)^3\Big|\mathcal{F}^{-1}_{x\mapsto\xi}V\Big(\frac{2m}{\hbar}\xi,t\Big)\Big|$$

and the identity

$$\begin{aligned}\|\mathcal{F}^{-1}_{x\mapsto y}V(t)\|_{L^1(\mathbb{R}^3)} &= \frac{1}{\varepsilon_0}\Big\|\mathcal{F}^{-1}_{x\mapsto y}\Big(\frac{1}{|x|}*n\Big)(t)\Big\|_{L^1(\mathbb{R}^3)}\\ &= \frac{1}{2\pi^2\varepsilon_0}\Big\|\frac{1}{|\cdot|^2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)\Big\|_{L^1(\mathbb{R}^3)}.\end{aligned}$$

Indeed, we first estimate the $L^1$ norm of $|\cdot|^{-2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)$ outside and inside the 3D unit ball $B$. Using Hölder's inequality we have

$$\Big\|\frac{1}{|\cdot|^2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)\Big\|_{L^1(\mathbb{R}^3\setminus B)} \le C\|\mathcal{F}^{-1}_{x\mapsto y}n(t)\|_{L^2(\mathbb{R}^3)} \le C\|n(t)\|_{L^2(\mathbb{R}^3)}\,.$$

Likewise, inside $B$ we get

$$\Big\|\frac{1}{|\cdot|^2}(\mathcal{F}^{-1}_{x\mapsto y}n)(\cdot,t)\Big\|_{L^1(B)} \le C\|\mathcal{F}^{-1}_{x\mapsto y}n(t)\|_{L^\infty(\mathbb{R}^3)} \le C\,\|n(t)\|_{L^1(\mathbb{R}^3)}.$$

The second inequality in (v) follows from the obvious estimate $\|n(t)\|_{L^1(\mathbb{R}^3_x)} \le \|W(t)\|_{L^1}$ and Minkowski's inequality applied to the integral expression of the position density

$$\begin{aligned}\|n(t)\|_{L^2(\mathbb{R}^3)} &= \Big(\int_{\mathbb{R}^3_x}\Big(\int_{\mathbb{R}^3_\xi} W(x,\xi,t)\,d\xi\Big)^2 dx\Big)^{1/2}\\ &\le \int_{\mathbb{R}^3_\xi}\Big(\int_{\mathbb{R}^3_x} W(x,\xi,t)^2\,dx\Big)^{1/2} d\xi = \|W(t)\|_{L^{1,2}}\,.\end{aligned}$$

□

3.2. *Local well–posedness.* Our aim in this section is to investigate the existence of a solution of the mild WPFP system (3.10) by means of a Banach fixed–point argument. To this purpose, we first introduce the appropriate function spaces in which we shall develop our theory.

Let $T>0$ and consider the Banach space $X_T = C([0,T]; L^1\cap L^{1,2})$ endowed with the norm

$$\|W\|_T := \sup_{0\le t\le T}\Big(\|W(t)\|_{L^1} + \|W(t)\|_{L^{1,2}}\Big)\,.$$

Also define the following closed, bounded subset of $X_T$:

$$X_T^K = \{W\in X_T:\ W(x,\xi,0) = W_0(x,\xi) \text{ a.e. }\ ,\ \|W\|_T \le K\}$$

and the map $\Gamma : X_T^K \to X_T$ by

$$\Gamma(W)(t) = G(t)[W_0] - \int_0^t G(t-s)[(H*_\xi W)(s)]\,ds\,.$$

We first note that $\Gamma$ is well defined. Indeed, from Proposition 3.2(i), (iii) and (v) we have

$$\begin{aligned}&\|G(t)[W_0]\|_{L^1} \le \|W_0\|_{L^1},\\ &\|G(t-s)[(H*_\xi W)(s)]\|_{L^1} \le C\|W\|_T\|W(s)\|_{L^1}\,,\end{aligned}$$

where we estimated

$$\|H*_\xi W\|_{L^1} \le \|H\|_{L^1(\mathbb{R}^3_\xi)}\|W\|_{L^1}$$

by Young's inequality. We observe that the same type of estimates are also true for the $L^{1,2}$ norm. Indeed, we have

$$\|G(t)[W_0]\|_{L^{1,2}} \leq \|W_0\|_{L^{1,2}}\,,$$
$$\|G(t-s)[(H *_\xi W)(s)]\|_{L^{1,2}} \leq C\|W\|_T\|W(s)\|_{L^{1,2}}\,.$$

By appropriately choosing $K > \|W_0\|_T$ and $T < 1/(2CK)$ it is clear that $\Gamma$ is a contractive map from $X_T^K$ onto itself, thus it has a unique fixed point (i.e. there exists a unique mild solution in the sense of Definition 3.1 for sufficiently small $T > 0$ only depending on $W_0$). Let us denote by $T_{max}$ the maximal existence time of the mild solution.

Under the additional hypotheses of Theorem 1.1, we shall actually prove in § 3.4 that the mild solution belongs to $C([0,\infty); L^1 \cap L^{1,2})$, so that global existence is attained. To this aim we only need to observe that the norm $\|W\|_T$ cannot blow up in a finite time $T$ (see [11] for details).

3.3. *Regularity.* This part of the proof is devoted to show some auxiliary regularity properties and (intrinsic) smoothing effects of our system that will help us to reach the global existence.

LEMMA 3.3. Let $0 < T < T_{max}$ and also let $W$ be the mild solution of the WPFP system with initial data $W_0$ fulfilling the hypotheses of Theorem 1.1. Then, the following inequalities hold:

(i) $W \in C((0,T]; L^\infty \cap L^{1,\infty})$ and $W \in C([0,T]; L^{1,p_0})$,
(ii) $\|n(t)\|_{L^1(\mathbb{R}^3_x)} = Q(t) \equiv Q = \|n(0)\|_{L^1(\mathbb{R}^3_x)}$ and $n \in L^\infty(0,T; L^{p_0}(\mathbb{R}^3_x))$,
(iii) $\|\nabla_x V(t)\|_{L^\infty(\mathbb{R}^3_x)} \leq C(Q)\|n(t)\|_{L^{p_0}(\mathbb{R}^3_x)}^{\frac{2}{3}(\frac{p_0}{p_0-1})}$, for all $0 \leq t \leq T$,
(iv) $\|\nabla_x W(t)\|_{L^{1,2}} \leq C\|\nabla_x W_0\|_{L^{1,2}}$, for all $0 \leq t \leq T$,
(v) $\|j(t)\|_{(L^1(\mathbb{R}^3_x))^3} \leq C$ for all $0 \leq t \leq T$,
(vi) $|E_K(t)| < \infty$ for all $0 \leq t \leq T$.

*Proof.* The first result is reached as a consequence of Proposition 3.2(ii) and (iv) by noting that $\|G_0(t)\|_{L^p} \leq Ct^{\frac{6}{p}-6}$ implies $\|G_0(t)\|_{L^p} \in L^1(0,T)$ for $p < \frac{6}{5}$. We prove that $W(t) \in L^\infty$ for all $0 < t \leq T$ in seven steps. The *first step* consists of estimating

$$\|W(t)\|_{L^p} \quad \leq \quad C\|W_0\|_{L^1}\, t^{\frac{6}{p}-6} + C\|W\|_T^2 \int_0^t (t-s)^{\frac{6}{p}-6}\, ds < \infty\,, \quad 0 < t \leq T\,.$$

Then, it is easily deduced that $W \in C((0,T]; L^p)$ for all $p < \frac{6}{5}$. The *second step* starts from the choice of an arbitrarily small time $\varepsilon > 0$, that is, we rewrite $W(t)$ for $t > \varepsilon$ as

$$W(t) = G(t-\varepsilon)[W(\varepsilon)] - \int_\varepsilon^t G(t-s)[(H * W)(s)]\, ds.$$

Then we can estimate $\|W(t)\|_{L^q}$ with $q < \frac{3}{2}$ as

$$\|W(t)\|_{L^q} \leq C(t-\varepsilon)^{\frac{6}{p}-6}\|W(\varepsilon)\|_{L^p} + C\|W\|_T \int_\varepsilon^t (t-s)^{\frac{6}{p}-6}\|W(s)\|_{L^p}\, ds.$$

In the *third step* we consider $r < 2$ and $t > 2\varepsilon$ and obtain

$$\|W(t)\|_{L^r} \leq C(t-2\varepsilon)^{\frac{6}{p}-6}\|W(2\varepsilon)\|_{L^q} + C\|W\|_T \int_{2\varepsilon}^t (t-s)^{\frac{6}{p}-6}\|W(s)\|_{L^q}\, ds\,.$$

Analogously, we find that $W \in C([3\varepsilon, T]; L^\sigma)$ for all $\sigma < 3$ in the *fourth step*, $W \in C([4\varepsilon, T]; L^\sigma)$ for all $\sigma < 6$ in the *fifth step* and $W \in C([5\varepsilon, T]; L^\sigma)$ for all $\sigma < \infty$ in the *sixth step*. Finally, a uniform bound for $W$ in $L^\infty$ is reached in the *seventh step*. Note that the intermediate regularities are linked by Young's relations at every step. The arbitrariness of $\varepsilon$ allows us to conclude. The $L^{1,p}$ regularity is analogously deduced in five steps by now observing that $\|G_0(t)\|_{L^{1,p}} \le Ct^{\frac{9}{2p}-\frac{9}{2}}$ belongs to $L^1(0,T)$ for $p < \frac{9}{7}$.

To end the proof of (i) we just take the $L^{1,p_0}$ norm of the Wigner function in the mild formulation and obtain

$$\|W(t)\|_{L^{1,p_0}} \le \|W_0\|_{L^{1,p_0}} + \|W\|_T \int_0^t \|W(s)\|_{L^{1,p_0}}\, ds\,.$$

Then, Gronwall's lemma allows us to conclude result (i).

The charge conservation in (ii) is straightforwardly deduced from the mild formulation (3.10) of $W$, while $\|n(t)\|_{L^{p_0}(\mathbb{R}^3_x)} \le \|W(t)\|_{L^{1,p_0}}$ is a consequence of Minkowski's inequality for integrals.

(iii) is a standard result concerning singular integrals of convolution type whose proof can be found in [12].

We now prove (iv). Differentiating in (3.10) we get

$$\begin{aligned}\|\nabla_x W(t)\|_{L^{1,2}} &\le \|\nabla_x W_0\|_{L^{1,2}} + \int_0^t \|\nabla_x (H *_\xi W)(s)\|_{L^{1,2}}\, ds \\ &\le \|\nabla_x W_0\|_{L^{1,2}} + \int_0^t \big(\|\nabla_x H *_\xi W(s)\|_{L^{1,2}} + \|H *_\xi \nabla_x W(s)\|_{L^{1,2}}\big)\, ds\,. \qquad (3.11)\end{aligned}$$

It is clear from (1.5) and the proof of Proposition 3.2(v) that

$$\begin{aligned}\|\nabla_x H\|_{L^1(\mathbb{R}^3_\xi)} &\le C\left\|\xi \mathcal{F}^{-1}_{x\mapsto\xi} V\left(\frac{2m}{\hbar}\xi, t\right)\right\|_{L^1(\mathbb{R}^3_\xi)} \\ &\le C\big(\|n(t)\|_{L^1(\mathbb{R}^3_x)} + \|\nabla_x n(t)\|_{L^2(\mathbb{R}^3_x)}\big)\,. \qquad (3.12)\end{aligned}$$

Insertion of (3.12) into (3.11) and use of the fact that $\|\nabla n(t)\|_{L^2(\mathbb{R}^3_x)} \le \|\nabla_x W(t)\|_{L^{1,2}}$ yields

$$\|\nabla_x W(t)\|_{L^{1,2}} \le \|\nabla_x W_0\|_{L^{1,2}} + t\,\|W\|_T^2 + 2\|W\|_T \int_0^t \|\nabla_x W(s)\|_{L^{1,2}}\, ds\,.$$

Thus, Gronwall's inequality ends the proof of (iv).

To prove (v) we multiply Eq. (3.10) by $\xi$ and take $L^1$ norms. Then, we obtain

$$\begin{aligned}\|\xi W(t)\|_{L^1} &\le \|\xi G_0\|_{L^1}\|W_0\|_{L^1} + \|G_0\|_{L^1}\|\xi W_0\|_{L^1} \\ &\quad + \int_0^t \|\xi G_0\|_{L^1}\|(H *_\xi W)(s)\|_{L^1}\, ds \\ &\quad + \int_0^t \|G_0\|_{L^1}\|\xi H(s)\|_{L^1(\mathbb{R}^3_\xi)}\|W(s)\|_{L^1}\, ds \\ &\quad + \int_0^t \|G_0\|_{L^1}\|H(s)\|_{L^1(\mathbb{R}^3_\xi)}\|\xi W(s)\|_{L^1}\, ds \\ &\le g(t) + C\|W\|_T \int_0^t \|\xi W(s)\|_{L^1}\, ds\,,\end{aligned}$$

with

$$\begin{aligned} g(t) &= C\|W_0\|_{L^1}\sqrt{t} + \|\xi W_0\|_{L^1} + \frac{2}{3}\|W\|_T^2\, t^{\frac{3}{2}} \\ &\quad + C\|W\|_T^2\, t + C\|W\|_T \int_0^t \|\nabla_x n(s)\|_{L^2(\mathbb{R}^3_x)}\, ds\,, \end{aligned}$$

where we estimated (cf. (3.12)) $\|\xi H\|_{L^1(\mathbb{R}^3_\xi)} \leq C(\|n(t)\|_{L^1(\mathbb{R}^3_x)} + \|\nabla_x n(t)\|_{L^2(\mathbb{R}^3_x)})$ and calculated the moments of $G_0$ via Proposition 3.2(i) and (ii). Now, Gronwall's inequality applies to yield (v).

(vi) follows from Eq. (3.10) by integrating against $|\xi|^2$ and estimating

$$\begin{aligned} E_K[W](t) &= E_K[W](0) + C\,Q\,t + \int_0^t \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_\xi} |\xi|^2 (H *_\xi W)\, d\xi\, dx\, ds \\ &= E_K[W](0) + C\,Q\,t + \int_0^t \Big(\int_{\mathbb{R}^3_x} j\cdot\nabla_x V\, dx\Big)\, ds, \end{aligned} \tag{3.13}$$

where we have used Proposition 3.2(iv). Then, (iii) and (v) allow us to conclude that (vi) holds. □

3.4. *Global well–posedness.* We first remind the reader (see [11]) that the mild solution achieved in § 3.2 is either global ($T_{max} = \infty$) or $T_{max} < \infty$, in which case

$$\lim_{T \nearrow T_{max}} \|W\|_T = \infty\,.$$

Of course, we need to show that the second option cannot occur. To control the $\|\cdot\|_T$ norm for all $T > 0$ we shall use the following estimates.

Proposition 3.4 (A posteriori estimates). Let $W$ be a mild solution of the WPFP system with initial data $W_0$ fulfilling the hypotheses of Theorem 1.1. Then, the following properties

(i) $\|n(t)\|_{L^2(\mathbb{R}^3_x)} \leq C\,Q^{\frac{1}{4}}\, E_K[W](t)^{\frac{3}{4}}$,
(ii) $E(t)$ grows linearly with time: $E(t) = E(0) + C\,Q\,t$,
(iii) $\|H\|_{L^1(\mathbb{R}^3_\xi)} \leq C(1+t)$,

hold in the maximal existence interval.

*Proof.* Assertion (i) follows from application of a particular case of a generalization of the Lieb–Thirring inequality (see [1]). To deduce (ii) we use the identity (3.13), the Poisson equation (1.2) and the continuity equation $\partial_t n + \mathrm{div}(j) = 0$ (see [2]). Finally, (iii) is a straightforward consequence of (i), (ii), Lemma 3.3(ii) and Proposition 3.2(v). □

By taking the $L^1$ and $L^{1,2}$ norms in the mild–formulated WPFP problem (3.10) we get (as in § 3.2)

$$\begin{aligned} \|W(t)\|_{L^1} + \|W(t)\|_{L^{1,2}} &\leq \|W_0\|_{L^1} + \|W_0\|_{L^{1,2}} \\ &\quad + C\int_0^t \|H(s)\|_{L^1(\mathbb{R}^3_\xi)}(\|W(s)\|_{L^1} + \|W(s)\|_{L^{1,2}})\, ds \\ &\leq \|W_0\|_{L^1} + \|W_0\|_{L^{1,2}} + C\int_0^t (1+s)(\|W(s)\|_{L^1} + \|W(s)\|_{L^{1,2}})\, ds \end{aligned}$$

thanks to Proposition 3.4. We then conclude that

$$\|W\|_T \leq (\|W_0\|_{L^1} + \|W_0\|_{L^{1,2}})\, e^{C\,T^2}$$

via Gronwall's inequality. Now we are done with the proof.

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