ERRATUM TO “THIN ELASTIC FILMS: 
THE IMPACT OF HIGHER ORDER PERTURBATIONS”

BY

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1. Introduction. In the paper [2] the main objective was to identify

\[ E^\gamma(u, b; A) := \inf \left\{ \liminf_{\varepsilon \to 0^+} E^\varepsilon(u_\varepsilon; A) : \right. \]

\[ u_\varepsilon \in W^{2,2}(\Omega; \mathbb{R}^3), \]

\[ u_\varepsilon \rightharpoonup u \text{ in } W^{1,q}(\Omega; \mathbb{R}^3), \quad \frac{1}{\varepsilon} D_3 u_\varepsilon \to b \text{ in } L^q(\Omega; \mathbb{R}^3) \}

where

\[ E^\varepsilon(u; A) := \int_{A \times I} W \left( D_p u \left[ \frac{1}{\varepsilon} D_3 u \right] \right) dx \]

\[ + \varepsilon^\gamma \int_{A \times I} \left( |D_p^2 u|^2 + \frac{1}{\varepsilon^2} |D_p^4 u|^2 + \frac{1}{\varepsilon^4} |D^3 u|^2 \right) dx \]

if \( u \in W^{2,2}(\Omega; \mathbb{R}^3) \), and \( E^\varepsilon(u; A) := \infty \) otherwise. Here \( q > 1, \Omega = \omega \times I, A(\omega) \) is the family of open subsets of \( \omega \), and \( W \) satisfies the condition

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\((H_1)'\) \(W : \mathbb{R}^{3 \times 3} \to [0, \infty)\) is continuous and there exists \(C > 0\) such that
\[
\frac{1}{C} |F|^q - C \leq W(F) \leq C (1 + |F|^q)
\]
for all \(F \in \mathbb{R}^{3 \times 3}\).

We recall that
\[
V^\gamma := \{(u, b) \in W^{1, q}(\Omega; \mathbb{R}^3) \times L^q(\Omega; \mathbb{R}^3) : D_3 u = 0 \ \text{a.e. in} \ \Omega, \quad D_3 b = 0 \ \text{a.e. in} \ \Omega \ \text{if} \ \gamma < 2, \ D_3 b \in L^2(\Omega; \mathbb{R}^3) \ \text{if} \ \gamma = 2\}
\]
and for \((u, b) \in V^\gamma\) and \(A \in \mathcal{A}(\omega)\),
\[
H^\gamma(u, b, A) := \inf \left\{ \liminf_{\varepsilon \to 0^+} \int_{A \times I} W(D_\varepsilon u_\varepsilon(x) | b_\varepsilon(x)) \, dx : \{u_\varepsilon\} \subset W^{1, q}(\Omega; \mathbb{R}^3), \ \{b_\varepsilon\} \subset L^q(\Omega; \mathbb{R}^3), \ u_\varepsilon \rightharpoonup u \ \text{in} \ W^{1, q}(\Omega; \mathbb{R}^3), \ b_\varepsilon \to b \ \text{in} \ L^q(\Omega; \mathbb{R}^3) \right\}
\]
if \(\gamma \neq 2\), and
\[
H^2(u, b, A) := \inf \left\{ \liminf_{\varepsilon \to 0^+} \int_{A \times I} \left[ W(D_\varepsilon u_\varepsilon(x) | b_\varepsilon(x)) + |D_3 b_\varepsilon(x)|^2 \right] \, dx : \{u_\varepsilon\} \subset W^{1, q}(\Omega; \mathbb{R}^3), \ \{b_\varepsilon\} \subset L^q(\Omega; \mathbb{R}^3) \ \text{with} \ D_3 b_\varepsilon \in L^2(\Omega; \mathbb{R}^3), \ u_\varepsilon \rightharpoonup u \ \text{in} \ W^{1, q}(\Omega; \mathbb{R}^3), \ b_\varepsilon \to b \ \text{in} \ L^q(\Omega; \mathbb{R}^3) \right\}
\]
In this erratum we slightly change notation and denote \(H^\gamma\) and \(H^2\) by \(H^\gamma_{3,3}\) and \(H^2_{3,3}\), respectively. We also introduce \(H^2_{2,3}\) and \(H^2_{3,3}\), which we define analogously except for the fact that the approximating sequences \(\{u_\varepsilon\}\) belong to \(W^{1, q}(\omega; \mathbb{R}^3)\).

The proof of Theorem 3.1 in [2] has an error in the upper bound. To date this upper bound can only be established if \(\gamma < 2\) or if \(D_3 b = 0 \ \text{a.e. in} \ \Omega\). Therefore, Theorem 3.1 in [2] should be replaced by

**Theorem A** (New Theorem 3.1). Assume that condition \((H_1)'\) is satisfied. Then,
\[
H^\gamma_{3,3}(u, b, A) \leq E^\gamma_{3,3}(u, b, A) \leq H^\gamma_{2,3}(u, b, A)
\]
for all \((u, b) \in V^\gamma\) and \(A \in \mathcal{A}(\omega)\). If \(\gamma < 2\) or \(\gamma \geq 2\) and \(D_3 b = 0 \ \text{a.e. in} \ \Omega\), then
\[
E^\gamma_{3,3}(u, b, A) = H^\gamma_{3,3}(u, b, A) = H^\gamma_{2,3}(u, b, A) = H^\gamma_{2,2}(u, b, A)
\]
\[
= \int_A (Q_2 \times C_2) \left| W \right| (D_p u(x_\alpha) | b(x_\alpha)) \, dx_\alpha.
\]

The lower bound in the original Theorem 3.1 in [2] is correct. The mistake in the proof of Theorem 3.1 was in trying to show that the upper bound is still \(H^\gamma_{3,3}\). The correct upper bound is \(H^\gamma_{2,3}\), as justified next, and in general there may be a gap between \(H^\gamma_{3,3}\) and \(H^\gamma_{2,3}\). However, when \(D_3 b = 0 \ \text{a.e. in} \ \Omega\) (a condition that is automatically satisfied when \(\gamma < 2\) in the finite energy regime), then indeed
\[
H^\gamma_{2,3}(u, b, A) = H^\gamma_{3,3}(u, b, A).
\]
To see this, we refer the reader to the proof of Theorem 4.1 in [2], which still works unchanged for any \(\gamma > 0\) when \(D_3 b = 0 \ \text{a.e. in} \ \Omega\).
To prove the second inequality in (1.2) we remark that the estimates involving \{u * \varphi_j, b * \varphi_j\} led to (3.2) and (3.3) for \((u, b, A)\), and this analysis relied heavily on the fact that \(D_3u = 0\) \(L^3\) a.e. in \(\Omega\). Therefore, given \((u, b) \in \mathcal{V}, A \in \mathcal{A}(\omega)\), if \(\{u_j\} \subset W^{1, q}(\omega; \mathbb{R}^3)\) converges weakly to \(u\) in \(W^{1, q}(\omega; \mathbb{R}^3)\) and \(\{b_j\} \subset L^q(\Omega; \mathbb{R}^3)\) converges weakly to \(b\) in \(L^q(\Omega; \mathbb{R}^3)\), then we may apply (3.2) and (3.3) to \((u_j, b, A)\), and we deduce that
\[
E^*_\gamma(u, b; A) \leq \liminf_{j \to \infty} E^*_\gamma(u_j, b_j; A) \leq \liminf_{j \to \infty} \int_{A \times I} W(D_p u_j(x_\alpha) | b_j(x)) \, dx
\]
if \(\gamma \neq 2\), and if \(\gamma = 2\) that
\[
E^*_\gamma(u, b; A) \leq \liminf_{j \to \infty} E^*_\gamma(u_j, b_j; A)
\leq \liminf_{j \to \infty} \int_{A \times I} \left[ W(D_p u_j(x_\alpha) | b_j(x)) + |D_3 b_j(x)|^2 \right] \, dx.
\]
Taking the infimum over all such sequences \(\{u_j\}\) and \(\{b_j\}\) yields
\[
E^*_\gamma(u, b; A) \leq H^*_\gamma(u, b; A).
\]

Theorem \(A\) covers completely the case \(\gamma < 2\) but leaves (partially) open the case \(\gamma \geq 2\) when \(D_3 b \neq 0\) \(L^3\) a.e. in \(\Omega\). We close the gap in this erratum. To be precise, if \(\gamma = 2\) and if \(W\) satisfies the additional \(q\)-Lipschitz condition
\[
|W(F) - W(G)| \leq C \left( 1 + |F|^q - 1 + |G|^q - 1 \right) |F - G|\tag{1.3}
\]
for all \(F, G \in \mathbb{R}^{3 \times 3}\), then we characterize the \(\Gamma\)-limit in Theorem \(B\) while for \(\gamma > 2\) we refer to Theorem \(\mathcal{G}\).

When \(\gamma = 2\), we introduce the functional
\[
\overline{W}_2 : \mathbb{R}^{2 \times 3} \times W^{1, 2}(I; \mathbb{R}^3) \to [0, \infty)
\]
defined for \(F \in \mathbb{R}^{2 \times 3}\) and \(b \in W^{1, 2}(I; \mathbb{R}^3)\) by
\[
\overline{W}_2(F, b) := \inf_{\varphi \in L} \left\{ \int_Q \left( W(F + D_p \varphi(x) | b(x_3) + LD_3 \varphi(x)) + \frac{1}{L} D^2_p \varphi(x) \right)^2 \right.
\left. + |D p_3 \varphi(x)|^2 + |b'(x_3) + LD_3 \varphi(x)|^2 \right\} dx : \right.
\left. L > 0, \varphi \in W^{2, 2}(Q; \mathbb{R}^3), \varphi(\cdot, x_3) \mathcal{Q}^\prime-\text{periodic for} \mathcal{L}^1 \text{a.e. } x_3, \right.
\left. \int_Q D_3 \varphi(x_\alpha, x_3) \, dx_\alpha = 0 \text{ for } \mathcal{L}^1 \text{a.e. } x_3 \right\}.
\]

We have the following representation.

**Theorem B.** Assume that \(\gamma = 2\) and that conditions \((H_1)\)' and \((13)\) are satisfied. Then for all \((u, b) \in \mathcal{V}^2\) and \(A \in \mathcal{A}(\omega)\),
\[
E^2(u, b; A) = \int_A \overline{W}_2(D_p u(x_\alpha) | b(x_\alpha, \cdot)) \, dx_\alpha,
\]
where \(\overline{W}_2\) is defined in (14).
To prove the theorem above, we start by showing that under the $q$-Lipschitz condition (1.3), minimizing sequences for $W^{2}_{2}$ prefer scales $L$ diverging to infinity.

**Proposition C.** Assume that $\gamma = 2$ and that conditions $(H_{1})'$ and (1.3) are satisfied. Then for all $\mathbf{F} \in \mathbb{R}^{2 \times 3}$ and $b \in W^{1,2}(I; \mathbb{R}^{3})$ the $\inf_{\varphi, L}$ in the definition of $\overline{W}_{2}(\mathbf{F}|b)$ may be replaced by $\lim_{L \to \infty} \inf_{\varphi}$. 

**Proof.** For $\mathbf{F} \in \mathbb{R}^{2 \times 3}$, $b \in W^{1,2}(I; \mathbb{R}^{3})$, and $L > 0$ let

$$H_{L}(\mathbf{F}|b) := \inf_{\varphi} \left\{ \int_{Q} \left( W(\mathbf{F} + \frac{D_{p} \varphi(x)}{L} | b(x_{3}) + L D_{3} \varphi(x)) + \frac{1}{L} \frac{D_{p}^{2} \varphi(x)}{L} \right)^{2} \right. \left. + |D_{p} \varphi(x)|^{2} + |b'(x_{3}) + L D_{33} \varphi(x)|^{2} \right\} dx : \varphi \in W^{2,2}(Q; \mathbb{R}^{3}), \varphi(\cdot, x_{3}) Q'-\text{periodic for } L^{1} \text{ a.e. } x_{3},$$

$$\int_{Q'} D_{3} \varphi(x_{\alpha}, x_{3}) dx_{\alpha} = 0 \text{ for } L^{1} \text{ a.e. } x_{3} \}.$$ 

Clearly,

$$\overline{W}_{2}(\mathbf{F}|b) \leq \liminf_{L \to \infty} H_{L}(\mathbf{F}|b).$$

We now prove that

$$\limsup_{L \to \infty} H_{L}(\mathbf{F}|b) \leq \overline{W}_{2}(\mathbf{F}|b).$$

Consider a sequence $\{L_{n}\}$ converging to infinity such that

$$\limsup_{L \to \infty} H_{L}(\mathbf{F}|b) = \lim_{n \to \infty} H_{L_{n}}(\mathbf{F}|b).$$

Let $\varphi \in W^{2,\infty}(Q; \mathbb{R}^{3})$ and $L > 0$ be admissible for $\overline{W}_{2}(\mathbf{F}|b)$, and define

$$\varphi_{n}(x) := \frac{1}{m_{n}} \varphi(m_{n} x_{\alpha}, x_{3}),$$

where

$$m_{n} := \left\lfloor \frac{L_{n}}{L} \right\rfloor.$$
with \( \lfloor L_n/L \rfloor \) the integer part of \( L_n/L \). Note that \( \varphi_n \) is admissible for \( \mathcal{H}_{L_n} \), and so

\[
\lim_{n \to \infty} \mathcal{H}_{L_n}(F|b) 
\leq \liminf_{n \to \infty} \left\{ \int_Q \left( W(F + D_p \varphi_n(x)) b(x_3) + L_n D_3 \varphi_n(x) \right) + \frac{1}{L_n} D_p^2 \varphi_n(x) \right\}^2 
\stackrel{1,3}{=} \liminf_{n \to \infty} \left\{ \int_Q \left( W(F + D_p \varphi(m_n x_\alpha, x_3)) b(x_3) + \frac{L_n}{m_n} D_3 \varphi(m_n x_\alpha, x_3) \right) 
\frac{m_n}{L_n} D_p^2 \varphi(m_n x_\alpha, x_3) \right\}^2 
\left\{ |D_p \varphi(m_n x_\alpha, x_3)|^2 + |D_3 \varphi(m_n x_\alpha, x_3)|^2 
\right\} dx 
\left\{ |b'(x_3) + \frac{L_n}{m_n} D_3 \varphi(m_n x_\alpha, x_3)|^2 \right\} dx.
\]

By \( 1,3 \), together with the facts that \( L_n \to \infty \) and that \( \varphi \in W^{2,\infty}(Q; \mathbb{R}^3) \), we may rewrite the right-hand side of the previous inequality as

\[
\liminf_{n \to \infty} \left\{ \int_Q \left( W(F + D_p \varphi(m_n x_\alpha, x_3)) b(x_3) + L D_3 \varphi(m_n x_\alpha, x_3) \right) 
\frac{1}{L} D_p^2 \varphi(m_n x_\alpha, x_3) \right\}^2 
\left\{ |D_p \varphi(m_n x_\alpha, x_3)|^2 
\right\} dx 
+ o(1).
\]

Since \( \varphi(\cdot, x_3) \) is \( Q^r \)-periodic for \( L^1 \) a.e. \( x_3 \in I \), it follows from Lebesgue’s Dominated Convergence Theorem, Fubini’s Theorem, and the Riemann-Lebesgue Lemma that

\[
\lim_{n \to \infty} \mathcal{H}_{L_n}(F|b) \leq \int_Q \left( W(F + D_p \varphi(x_\alpha, x_3)) b(x_3) + L D_3 \varphi(x_\alpha, x_3) \right) 
\frac{1}{L} D_p^2 \varphi(x_\alpha, x_3) \right\}^2 
\left\{ |D_p \varphi(x_\alpha, x_3)|^2 + |b'(x_3) + L D_3 \varphi(x_\alpha, x_3)|^2 \right\} dx.
\]

Using the arbitrariness of \( \varphi \) and \( L \), the density of smooth functions in the set of test functions for \( 1,3 \), and the growth hypothesis \( (H_1)' \), we conclude that

\[
\lim_{n \to \infty} \mathcal{H}_{L_n}(F|b) \leq W_2(F|b). \quad \square
\]

To prove Theorem \( 1 \) it is enough to show that for any given sequence \( \{\varepsilon_n\} \), with \( \varepsilon_n \to 0^+ \), there exists a subsequence \( \{\varepsilon_{n_k}\} \) of \( \{\varepsilon_n\} \) such that the \( \Gamma \)-lower limit, defined
for all \((u, b) \in \mathcal{V}^{2}\) and \(A \in \mathcal{A}(\omega)\).

To choose the subsequence \(\{\varepsilon_{n_{k}}\}\), let \(\mathcal{R}(\omega)\) be the countable subfamily of \(\mathcal{A}(\omega)\) obtained by taking all finite unions of open squares in \(\omega\) with faces parallel to the axes, centered at \(x_{n} \in \omega \cap \mathbb{Q}^{2}\) and with rational edge length. Since \(L^{1}(\Omega; \mathbb{R}^{3})\) is a separable metric space, using Kuratowski’s Compactness Theorem and a diagonal argument, we may find a subsequence \(\{\varepsilon_{n_{k}}\}\) of \(\{\varepsilon_{n}\}\) such that

\[\Gamma-\lim_{k \to \infty} E^{2}_{\varepsilon_{n_{k}}} (u, b; A) \text{ exists for all } (u, b) \in \mathcal{V}^{2} \text{ and for all } A' \in \mathcal{R}(\omega).\] (1.7)

**Theorem D.** Assume that condition \((H_{1})'\) is satisfied and that \(\gamma = 2\). Then for every \((u, b) \in \mathcal{V}^{2}\) the set function \(E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; \cdot)\) is the trace of a Radon measure absolutely continuous with respect to \(\mathcal{L}^{2}\mid_{\omega}\).

**Proof:** The proof is very similar to that of Theorem 4.2 in [3], and thus we only indicate the main changes.

**Step 1:** Fix \((u, b) \in \mathcal{V}^{2}\). We claim that

\[E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; A_{1}) \leq E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; A_{2}) + E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; A_{1} \setminus \Omega_{3})\] (1.8)

for all \(A_{1}, A_{2}, A_{3} \in \mathcal{A}(\omega)\), with \(A_{3} \subseteq A_{2} \subseteq A_{1}\).

Without loss of generality we may assume that the right-hand side of the previous inequality is finite.

Fix \(\eta > 0\) and find \(\{u_{k}\} \subset W^{2, 2}(\Omega; \mathbb{R}^{3})\) converging weakly to \(u\) in \(W^{1, \gamma}(\Omega; \mathbb{R}^{3})\) and such that \(\frac{1}{\varepsilon_{n_{k}}} D_{3} u_{k} \rightharpoonup b\) in \(L^{\gamma}(\Omega; \mathbb{R}^{3})\), and

\[\lim_{k \to \infty} \inf E^{2}_{\varepsilon_{n_{k}}} (u_{k}; (A_{1} \setminus \Omega_{3})) \leq E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; A_{1} \setminus \Omega_{3}) + \eta.\]

Extract a subsequence \(\{n_{k_{j}}\}\) for which

\[\lim_{j \to \infty} E^{2}_{\varepsilon_{n_{k_{j}}}} (u_{k_{j}}; (A_{1} \setminus \Omega_{3})) \leq E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; A_{1} \setminus \Omega_{3}) + \eta.\] (1.9)

Let \(A' \in \mathcal{R}(\omega)\) be such that \(A_{3} \subseteq A' \subseteq A_{2}\). By (1.7) there exists a sequence \(\{v_{k}\} \subset W^{2, 2}(\Omega; \mathbb{R}^{3})\) converging weakly to \(u\) in \(W^{1, \gamma}(\Omega; \mathbb{R}^{3})\) and such that \(\frac{1}{\varepsilon_{n_{k}}} D_{3} v_{k} \rightharpoonup b\) in \(L^{\gamma}(\Omega; \mathbb{R}^{3})\), and

\[E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; A') = \lim_{k \to \infty} E^{2}_{\varepsilon_{n_{k}}} (v_{k}; A').\]

In particular,

\[E^{\varepsilon}_{\{\varepsilon_{n_{k}}\}} (u, b; A') = \lim_{j \to \infty} E^{2}_{\varepsilon_{n_{k_{j}}}} (v_{k_{j}}; A').\] (1.10)
For every $v \in W^{2,2}(\Omega; \mathbb{R}^3)$, for every Borel set $E \subset \omega$, and for every $j \in \mathbb{N}$ define

$$
G_j(v; E) := \int_{E \times I} \left( 1 + |D_p v|^q + \frac{1}{\varepsilon_{nk_j}} |D_3 v|^q \right) \, dx \\
+ \int_{E \times I} \left( \frac{\varepsilon_{nk_j}^2}{\varepsilon_{nk_j}^4} |D_p^2 v|^2 + |D_p v|^2 + \frac{1}{\varepsilon_{nk_j}^2} |D_3 v|^2 \right) \, dx.
$$

Due to the coercivity hypothesis $(H_1)'$ we may extract a bounded subsequence from the sequence of measures $\nu_j := G_j(u_{nk_j}; \cdot) + G_j(v_{nk_j}; \cdot)$ restricted to $A' \setminus \overline{A_3}$ converging $\ast$-weakly to some Radon measure $\nu$ defined on $A' \setminus \overline{A_3}$.

Find $t > 0$ such that $\nu(S_t) = 0$, where

$$
S_t := \{ x_\alpha \in A' : \text{dist}(x_\alpha, \partial A_3) = t \}.
$$

For $\delta > 0$ define

$$
L_\delta := \{ x_\alpha \in A' : \text{dist}(x_\alpha, S_t) < \delta \}.
$$

Choose $\delta$ so small that $L_\delta \subset A' \setminus \overline{A_3}$. Consider a smooth cut-off function $\varphi_\delta \in C_0^\infty(A_2; [0, 1])$ such that $\varphi_\delta = 1$ in

$$
\{ x_\alpha \in A' : \text{dist}(x_\alpha, \partial A_3) < t - \delta \}
$$

and $\varphi_\delta = 0$ in

$$
\{ x_\alpha \in A' : \text{dist}(x_\alpha, \partial A_3) > t + \delta \},
$$

with

$$
\| D_p \varphi_\delta \|_{L^\infty(\omega)} \leq C/\delta, \quad \| D^2_p \varphi_\delta \|_{L^\infty(\omega)} \leq C/\delta^2.
$$

Define

$$
\tilde{u}_k(x) := \left\{ \begin{array}{ll}
(1 - \varphi_\delta(x_\alpha))u_{kj}(x) + \varphi_\delta(x_\alpha)v_{kj}(x) & \text{if } k = kj \text{ for some } j \in \mathbb{N}, \\
u(x_\alpha) + \varepsilon_{nk_j} \int_0^\delta (b * \psi_{kj})(x_\alpha, s) \, ds & \text{otherwise,}
\end{array} \right.
$$

where $\psi_{kj} = \psi_k(x_\alpha)$ is a standard mollifier. Then $\tilde{u}_k \rightharpoonup u$ in $W^{1,q}(\Omega; \mathbb{R}^3)$ and since $\varphi_\delta$ does not depend on $x_\alpha$, we also have that $\frac{1}{\varepsilon_{nk_j}} D_3 \tilde{u}_k \rightharpoonup b$ in $L^q(\Omega; \mathbb{R}^3)$ as $k \to \infty$.

Hence

$$
E^-_{\varepsilon_{nk_j}}(u, b; A_1) \leq \liminf_{k \to \infty} E^2_{\varepsilon_{nk_j}}(u_{kj}; A_1) \leq \liminf_{j \to \infty} E^2_{\varepsilon_{nk_j}}(\tilde{u}_{kj}; A_1). \quad (1.11)
$$

Thus it remains to estimate the right-hand side of the previous inequality. By the growth condition $(H_1)'$, we have the estimate

$$
E^2_{\varepsilon_{nk_j}}(\tilde{u}_{kj}; A_1) \leq E^2_{\varepsilon_{nk_j}}(u_{kj}; A_1 \setminus \overline{A_3}) + E^2_{\varepsilon_{nk_j}}(v_{kj}; A')
$$

$$
+ C \left( G_j(u_{kj}; \cdot) + G_j(v_{kj}; \cdot) \right) \left( \frac{C\varepsilon_{nk_j}^2}{\delta^4} \int_{L_\delta \times I} |u_{kj} - v_{kj}|^q \, dx + \frac{C\varepsilon_{nk_j}^2}{\delta^4} \int_{L_\delta \times I} |u_{kj} - v_{kj}|^2 \, dx \right) \quad (1.12)
$$

$$
+ \frac{C\varepsilon_{nk_j}^2}{\delta^2} \int_{L_\delta \times I} |D_p u_{kj} - D_p v_{kj}|^2 \, dx + \frac{C}{\delta^2} \int_{L_\delta \times I} |D_3 u_{kj} - D_3 v_{kj}|^2 \, dx.
$$

Since

$$
\sup_{j \in \mathbb{N}} \int_{L_\delta \times I} |\varepsilon_{nk_j} D^2 u_{kj}|^2 \, dx < \infty,
$$
by Poincaré’s inequality we have that
\[
\int_{L_\delta \times I} |\varepsilon_{n_k} D u_{k_j} - c_j|^2 \, dx \leq C_\delta \int_{L_\delta \times I} |\varepsilon_{n_k} D^2 u_{k_j}|^2 \, dx,
\]
where
\[
c_j := \frac{1}{L^2(L_\delta)} \int_{L_\delta \times I} \varepsilon_{n_k} D u_{k_j} \, dx \to 0.
\]
Hence
\[
\sup_{j \in \mathbb{N}} \int_{L_\delta \times I} |\varepsilon_{n_k} D u_{k_j}|^2 \, dx < \infty.
\]
It follows by the Rellich-Kondrachov Theorem and the fact that \( u_{k_j} \to u \) in \( W^{1,q}(\Omega; \mathbb{R}^3) \) that \( \varepsilon_{n_k} D u_{k_j} \to 0 \) in \( L^2(L_\delta \times I; \mathbb{R}^{3 \times 3}) \). Again by Poincaré’s inequality we have that
\[
\int_{L_\delta \times I} |\varepsilon_{n_k} u_{k_j} - d_j|^2 \, dx \leq C_\delta \int_{L_\delta \times I} |\varepsilon_{n_k} D u_{k_j}|^2 \, dx,
\]
where
\[
d_j := \frac{1}{L^2(L_\delta)} \int_{L_\delta \times I} \varepsilon_{n_k} u_{k_j} \, dx \to 0,
\]
and so \( \varepsilon_{n_k} u_{k_j} \to 0 \) in \( L^2(L_\delta \times I; \mathbb{R}^3) \). Similar conclusions hold for \( v_{k_j} \). Hence, letting \( j \to \infty \) in \( (1.12) \) and using \( (1.9) \) and \( (1.10) \), we have
\[
E_{\{\varepsilon_{n_k}\}}(u, b; A) \leq E_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \bar{A}_3) + E_{\{\varepsilon_{n_k}\}}(u, b; A') + \eta + C \nu(L_\delta)
\]
\[
\leq E_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \bar{A}_3) + E_{\{\varepsilon_{n_k}\}}(u, b; A_2) + \eta + C \nu(L_\delta)
\]
and letting \( \delta \) go to zero we obtain
\[
E_{\{\varepsilon_{n_k}\}}(u, b; A_1) \leq E_{\{\varepsilon_{n_k}\}}(u, b; A_2) + E_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \bar{A}_3) + 2\eta + C \nu(S_\delta)
\]
\[
= E_{\{\varepsilon_{n_k}\}}(u, b; A_2) + E_{\{\varepsilon_{n_k}\}}(u, b; A_1 \setminus \bar{A}_3) + 2\eta.
\]
It suffices to let \( \eta \to 0^+ \). \( \square \)

As an immediate consequence of the previous theorem we have
\[
E_{\{\varepsilon_{n_k}\}}(u, b; A) = \int_A \frac{dE_{\{\varepsilon_{n_k}\}}(u, b; \cdot)}{dL^3} (x_\alpha) \, dx_\alpha,
\]
where \( \frac{dE_{\{\varepsilon_{n_k}\}}(u, b; \cdot)}{dL^3} \) is the Radon-Nikodým derivative of \( E_{\{\varepsilon_{n_k}\}}(u, b; \cdot) \) with respect to the Lebesgue measure in \( \mathbb{R}^3 \).

**Remark E.** A proof almost identical to that of Remark 4.3 in [2] shows that \( \bar{W}_2(\cdot, \cdot) \) is upper semi-continuous on \( \mathbb{R}^{2 \times 3} \times W^{1,2}(I; \mathbb{R}^3) \) equipped with its strong topology.

We now turn to the proof of Theorem B. The argument is very similar to that of Theorem 4.4 in [2] with the exception that in the proof of the lower bound the additional hypothesis \( (1.3) \) allows us to avoid the use of equi-integrable sequences.

**Proof.** Fix \((u, b) \in \mathcal{V}^2\) and \( A \in \mathcal{A}(\omega) \). As usual, we identify \( u \) with a function in \( W^{1,q}(\omega; \mathbb{R}^3) \). Also, for simplicity of notation, from now on we write \( \varepsilon_k \) in place of \( \varepsilon_{n_k} \).

**Lower bound.** We claim that
\[
E_{\{\varepsilon_k\}}(u, b; A) \geq \int_A \bar{W}_2(D_p u(x_\alpha))b(x_\alpha, \cdot) \, dx_\alpha.
\] (1.13)
Consider any sequence \( \{u_k\} \subset W^{2,2}(\Omega; \mathbb{R}^3) \) such that \( u_k \rightharpoonup u \) in \( W^{1,q}(\Omega; \mathbb{R}^3) \), \( \frac{1}{\varepsilon_k} D_3 u_k \rightharpoonup b \) in \( L^q(\Omega; \mathbb{R}^3) \). Extracting a subsequence, if necessary, we may assume, without loss of generality, that
\[
\liminf_{k \to \infty} E_{\varepsilon_k}^2(u_k; A) = \lim_{k \to \infty} E_{\varepsilon_k}^2(u_k; A)
\tag{1.14}
\]
and that the bounded sequence
\[
\mu_k := \left( W \left( D_p u_k \frac{1}{\varepsilon_k} D_3 u_k \right) + \varepsilon_k^2 |D_p^2 u_k|^2 + |D_p D_3 u_k|^2 + \frac{1}{\varepsilon_k} |D_3^3 u_k|^2 \right) \mathcal{E}^3((A \times I)
\right)
\]
satisfies
\[
\mu_k \rightharpoonup^* \mu \text{ in } \mathcal{M}(A \times I)
\]
for some nonnegative finite Radon measure \( \mu \) on \( A \times I \). Denote by \( \hat{\mu} \) the finite Radon measure on \( A \) defined by
\[
\hat{\mu}(B) := \mu(B \times I),
\]
for all Borel sets \( B \subset A \). We will show below that the Radon-Nikodým derivative of \( \hat{\mu} \) with respect to the Lebesgue measure on \( \mathbb{R}^2 \) satisfies
\[
\frac{d\hat{\mu}}{d\mathcal{L}^2}(x) \geq \mathcal{W}_2(D_p u(x) | b(x, \cdot))
\tag{1.15}
\]
for \( \mathcal{L}^2 \) a.e. every point \( x \in A \).

Note that if (1.15) holds, then from (1.14),
\[
\lim_{k \to \infty} E_{\varepsilon_k}^2(u_k; A) = \lim_{k \to \infty} \mu_k(A \times I) \geq \hat{\mu}(A)
\]
\[
\geq \int_A \frac{d\hat{\mu}}{d\mathcal{L}^2}(x) \, dx = \int_A \mathcal{W}_2(D_p u(x) | b(x, \cdot)) \, dx.
\]
Taking the infimum over all admissible sequences \( \{u_k\} \) we obtain (1.13).

**Step 2:** As in Lemma 5.1 in [2], it can be shown, that, up to the extraction of a subsequence,
\[
\frac{1}{\varepsilon_k} D_3 u_k(\cdot, x_3) \rightharpoonup b(\cdot, x_3) \text{ in } L^q(A; \mathbb{R}^3) \text{ for all } x_3 \in I
\tag{1.16}
\]
and that for any Borel subset \( B \subset A \) and for all \( x_3 \in I, \)
\[
\sup_{k \in \mathbb{N}} \int_B \frac{1}{\varepsilon_k} D_3 u_k(x, x_3) \, dx < \infty.
\tag{1.17}
\]

We now address the proof of (1.15). Since \( u \in W^{1,q}(\omega; \mathbb{R}^3) \), for \( \mathcal{L}^2 \) a.e. \( x^0_\alpha \in A \) we have
\[
\lim_{\delta \to 0^+} \frac{1}{\delta^{q+q}} \int_{Q'(x_\alpha, \delta)} |u(x_\alpha) - u(x^0_\alpha) - D_p u(x^0_\alpha)(x_\alpha - x^0_\alpha)|^q \, dx_\alpha = 0.
\tag{1.18}
\]
Moreover, viewing \( b \) as a Bochner integrable function, that is, an element of
\[
L^q(A; L^q(\mathbb{R}^3)),
\]
for \( \mathcal{L}^2 \) a.e. \( x^0_\alpha \in A \) we have
\[
\lim_{\delta \to 0^+} \int_{\frac{1}{\delta}} \frac{1}{\delta^q} \int_{Q'(x^0_\alpha, \delta)} b(x_\alpha, x_3) \, dx_\alpha - b(x^0_\alpha, x_3) \, dx_3 = 0.
\tag{1.19}
\]
Fix a point \( x_0^0 \in A \) that satisfies (1.18), (1.19), and such that
\[ b(x_0^0, \cdot) \in W^{1,2}(I; \mathbb{R}^3) \]
and
\[ \frac{d\mu}{d\mathcal{L}^2}(x_0^0) \text{ exists and is finite.} \]

We claim that (1.15) holds at \( x_0^0 \).
Consider a sequence \( \{\delta_i\} \), with \( \delta_i \to 0^+ \) such that
\[ \lim_{i \to \infty} \frac{1}{(\delta_i)^2} \int_{Q'(x_0^0, \delta_i)} b(x_\alpha, x_3) \, dx_\alpha = b(x_0^0, x_3) \]
for \( \mathcal{L}^1 \) a.e. \( x_3 \in I \), and
\[ \mu(\partial (Q'(x_0^0, \delta_i) \times \bar{I}) = 0. \]

From the definition of \( \hat{\mu} \) together with that of the Radon-Nikodým derivative, we obtain
\[ \frac{d\hat{\mu}}{d\mathcal{L}^2}(x_0^0) = \lim_{i \to \infty} \frac{\hat{\mu}(Q'(x_0^0, \delta_i))}{(\delta_i)^2} = \lim_{i \to \infty} \frac{\mu(Q'(x_0^0, \delta_i) \times I)}{(\delta_i)^2} \]
\[ = \lim_{i \to \infty} \lim_{k \to \infty} \frac{1}{(\delta_i)^2} \left( \int_{Q'(x_0^0, \delta_i) \times I} W \left( D_p u_k(x) \right) \frac{1}{\varepsilon_k} D_3 u_k(x) \right) \right) \, dx 
\[ + \int_{Q'(x_0^0, \delta_i) \times I} \left( \varepsilon_k^2 |D_p^2 u_k(x)|^2 + |D_p u_k(x)|^2 + \frac{1}{\varepsilon_k} |D_3 u_k(x)|^2 \right) \, dx 
\[ = \lim_{i \to \infty} \lim_{k \to \infty} \left( \int_{Q} W \left( D_p v_{k,i}(y) \right) \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y) \right) \, dy 
\[ + \int_{Q} \left( \frac{\delta_i}{\varepsilon_k} \right)^2 |D_p^2 v_{k,i}(y)|^2 + |D_p v_{k,i}(y)|^2 + \frac{\delta_i}{\varepsilon_k} |D_3 v_{k,i}(y)|^2 \right) \, dy, \]

where for \( y \in Q \),
\[ v_{k,i}(y) := \frac{u_k(x_0^0 + \delta_i y_0, y_3) - u(x_0^0)}{\delta_i}, \]
\[ u_0(y_0) := D_p u(x_0^0) \cdot y_0, \quad b_0(y_3) := b(x_0^0, y_3). \]

Note that, since \( u_k \to u \) in \( L^q(\Omega; \mathbb{R}^3) \) and by (1.18), we have
\[ \lim_{i \to \infty} \lim_{k \to \infty} \int_{Q} |v_{k,i}(y) - u_0(y_0)|^q \, dy \]
\[ = \lim_{i \to \infty} \lim_{k \to \infty} \frac{1}{(\delta_i)^{2+q}} \int_{Q'(x_0^0, \delta_i) \times I} |u_k(x) - u(x_0^0) - D_p u(x_0^0) \cdot (x_\alpha - x_0^0)|^q \, dx 
\[ = \lim_{i \to \infty} \frac{1}{(\delta_i)^{2+q}} \int_{Q'(x_0^0, \delta_i)} |u(x_\alpha) - u(x_0^0) - D_p u(x_0^0) \cdot (x_\alpha - x_0^0)|^q \, dx_\alpha = 0. \]

On the other hand, in view of (1.16), for all \( y_3 \in I \),
\[ \lim_{k \to \infty} \int_{Q'} \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y_0, y_3) \, dy_3 = \lim_{k \to \infty} \frac{1}{(\delta_i)^2} \int_{Q'(x_0^0, \delta_i)} \frac{1}{\varepsilon_k} D_3 u_k(x_\alpha, y_3) \, dx_\alpha 
\[ = \frac{1}{\delta_i^2} \int_{Q'(x_0^0, \delta_i)} b(x_\alpha, y_3) \, dx_\alpha, \]
and so by (1.17) it follows from Lebesgue’s Dominated Convergence Theorem that

\[
\lim_{k \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \int_{Q'} \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y_\alpha, y_3) \, dy_\alpha - b_0(y_3) \right|^q \, dy_3
\]

\[
= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{1}{(\delta_i)^2} \int_{Q'(x_0^i, \delta_i)} b(x_\alpha, x_3) \, dx_\alpha - b(x_0^i, x_3) \right|^q \, dx_3.
\]

By (1.19) we have

\[
\lim_{i \to \infty} \lim_{k \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \int_{Q'} \frac{\delta_i}{\varepsilon_k} D_3 v_{k,i}(y_\alpha, y_3) \, dy_\alpha - b_0(y_3) \right|^q \, dy_3 = 0.
\]

By a standard diagonalization argument, we may extract subsequences \( v_i := v_{k,i} \) and \( \varepsilon_i := \frac{\delta_{k,i}}{\delta_i} \to 0^+ \) such that

\[
\left( \int_Q W \left( D_p v_i \left| \frac{1}{\varepsilon_i} D_3 v_i \right| \right) + \int_Q \left( \varepsilon_i^2 |D_p^2 v_i|^2 + |D_{p3} v_i|^2 + \frac{1}{\varepsilon_i^2} |D_{33} v_i|^2 \right) \, dy \right)^{\frac{1}{2}} \to 0
\]

\[
\lim_{i \to \infty} \int_Q |v_i - u_0|^q \, dy = 0, \quad (1.23)
\]

\[
\lim_{i \to \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \int_{Q'} \frac{1}{\varepsilon_i} D_3 v_i(y_\alpha, y_3) \, dy_\alpha - b_0(y_3) \right|^q \, dy_3 = 0. \quad (1.24)
\]

Reasoning as in Theorem \( D \) we can assume, without loss of generality, that \( v_i = u_0 \) in a neighborhood of \( \partial Q' \times I \).

For \( y \in Q \) define

\[
\varphi_i(y) := v_i(y) - u_0(y_\alpha) - \int_{Q'} (v_i - u_0)(w_\alpha, y_3) \, dw_\alpha.
\]

Then

\[
D_p \varphi_i(y) = D_p v_i(y) - D_p u(x_0^i), \quad (1.25)
\]

\[
\frac{1}{\varepsilon_i} D_{33} \varphi_i(y) = \frac{1}{\varepsilon_i} D_{33} v_i(y) - \int_{Q'} \frac{1}{\varepsilon_i} D_{33} v_i(w_\alpha, y_3) \, dw_\alpha, \quad (1.26)
\]

\[
\frac{1}{\varepsilon_i} D_{33} \varphi_i(y) = \frac{1}{\varepsilon_i} D_{33} v_i(y) - \int_{Q'} \frac{1}{\varepsilon_i} D_{33} v_i(w_\alpha, y_3) \, dw_\alpha, \quad (1.27)
\]

and note that

\[
\int_{Q'} D_3 \varphi_i(y_\alpha, y_3) \, dy_\alpha = 0 \text{ for all } y_3 \in I. \quad (1.28)
\]
Since \( \varphi_1(\cdot, y_3) \) is \( Q' \)-periodic for \( L^1 \) a.e. \( y_3 \), it follows that \( \varphi_1 \) is admissible in the definition of \( \nabla_2 (D_p u(x_{\alpha}^0)) b(x_{\alpha}^0, \cdot)) \). Moreover,

\[
\int_Q W \left( D_p v_i (y) \left[ \frac{1}{\varepsilon_i} D_{33} v_i (y) \right] \right) dy \\
+ \int_Q \left( \varepsilon_i^2 |D_{33}^2 v_i (y)|^2 + |D_{33} v_i (y)|^2 + \frac{1}{\varepsilon_i} |D_{33} v_i (y)|^2 \right) dy \\
= \int_Q W \left( D_p u(x_{\alpha}^0) + D_p \varphi_i (y) \right) b_0 (y_3) + \frac{1}{\varepsilon_i} D_{33} \varphi_i (y) + z_i (y_3) \\
+ \int_Q \left( \varepsilon_i^2 |D_{33}^2 \varphi_i (y)|^2 + |D_{33} \varphi_i (y)|^2 + \left| b_0' (y_3) + \frac{1}{\varepsilon_i} D_{33} \varphi_i (y) + z_i' (y_3) \right|^2 \right) dy,
\]

where

\[
z_i (y_3) := \int_{Q'} \frac{1}{\varepsilon_i} D_{33} v_i (w_{\alpha}, y_3) dw_{\alpha} - b_0 (y_3).
\]

Since by (1.28),

\[
\int_{Q'} D_{33} \varphi_i (y_3, y_3) dy_{\alpha} = D_3 \left( \int_{Q'} D_{33} \varphi_i (y_3, y_3) dy_{\alpha} \right) = 0 \text{ for all } y_3 \in I,
\]

it follows that

\[
\int_Q \left| b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i + z_i' \right|^2 dy \geq \int_Q \left| b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i \right|^2 dy + 2 \int_Q \left( b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i \right) \cdot z_i' dy \\
= \int_Q \left| b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i \right|^2 dy + 2 \int_Q b_0' \cdot z_i' dy.
\]

We claim that

\[
z_i \to 0 \text{ in } W^{1,2} (I; \mathbb{R}^3).
\]

If the claim holds, then letting \( i \to \infty \) in the previous inequality yields

\[
\limsup \sup_i \int_Q \left| b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i + z_i' \right|^2 dy \geq \limsup \sup_i \int_Q \left| b_0' + \frac{1}{\varepsilon_i} D_{33} \varphi_i \right|^2 dy.
\]

To prove (1.32) note that, up to a subsequence, from (1.22) and (1.24) we may assume that \( z_i (y_3) \to 0 \) for \( L^1 \) a.e. \( y_3 \in I \) and that

\[
\sup_i \int_Q \left| \frac{1}{\varepsilon_i} D_{33} v_i \right|^2 dy \leq \infty.
\]

Hence by Hölder’s Inequality,

\[
\int_{Q} \left| z_i' (y_3) \right|^2 dy_3 \leq 2 \int_{Q} \left[ \left( \int_{Q'} \frac{1}{\varepsilon_i} D_{33} v_i (w_{\alpha}, y_3) dw_{\alpha} \right)^2 + \left| b_0' (y_3) \right|^2 \right] dy_3 \\
\leq 2 \int_{Q} \left[ \left( \int_{Q'} \frac{1}{\varepsilon_i} D_{33} v_i (w_{\alpha}, y_3) \right)^2 dw_{\alpha} + \left| b_0' (y_3) \right|^2 \right] dy_3,
\]
and so also by (1.24),
\[
\sup_i \int_{-\frac{1}{2}}^{\frac{1}{2}} |z_i'(y_3)|^2 \, dy_3 < \infty.
\]

By extracting a further subsequence, if necessary, and appealing to (1.24) we have shown (1.32). In particular, \(z_i \to 0\) uniformly. Moreover, by the coercivity hypothesis \((H_1)’\) and (1.22),
\[
\sup_i \int Q \left( |D_p \varphi_i(y)|^q + \left| \frac{1}{\varepsilon_i} D_3 \varphi_i(y) \right|^q \right) \, dy < \infty.
\]

Hence, using the \(q\)-Lipschitz condition (1.3) we obtain
\[
\int_Q W \left( D_p u(x_0^0) + D_p \varphi_i \bigg| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i + z_i \right) \, dy \\
\geq \int_Q W \left( D_p u(x_0^0) + D_p \varphi_i \bigg| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i \right) \, dy + o(1).
\]

In turn, using also (1.22), (1.33), we have that
\[
\frac{d \bar{\mu}}{dE^2}(x_0^0) \geq \lim_{i \to \infty} \sup_i \left( \int_Q \left[ W \left( D_p u(x_0^0) + D_p \varphi_i \bigg| b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i \right) \right] \, dy \\
\quad + \int_Q \left( \varepsilon_i^2 |D_p^2 \varphi_i|^2 + |D_p D_3 \varphi_i|^2 + |b_0 + \frac{1}{\varepsilon_i} D_3 \varphi_i|^2 \right) \, dy \right).
\]

Since by construction \(\varphi_i\) are admissible functions in the definition of \(\mathcal{W}(D_p u(x_0^0) | b(x_0^0, \cdot))\) (see (1.23)), it follows that
\[
\frac{d \bar{\mu}}{dE^2}(x_0^0) \geq \mathcal{W}_2(D_p u(x_0^0) | b(x_0^0, \cdot)),
\]
and the proof of (1.15) is complete.

Upper bound. We first prove the upper bound
\[
E_{(\varepsilon_k)}^-(u, b; A) \leq \int_A \mathcal{W}_2(D_p u(x_0^0) | b(x_0^0, \cdot)) \, dx_0 \tag{1.34}
\]

when \(u(x_0) = \mathcal{F} x_0 + c\) for all \(x_0 \in \Omega'\) and for some \(\mathcal{F} \in \mathbb{R}^{3 \times 2}, c \in \mathbb{R}^3,\) and \(b \in W^{1,2} (I; \mathbb{R}^3)\).

For \(\eta > 0\) fixed, choose \(\varphi \in W^{2,\infty}(Q; \mathbb{R}^3)\), and \(L \geq 0\), with \(\varphi(\cdot, x_3)\) \(Q\)-periodic and \(\int_Q D_3 \varphi(x_0, x_3) \, dx_0 = 0\) for \(L^1\) a.e. \(x_3 \in I\), such that
\[
\int_Q \left[ W \left( F + D_p \varphi(x) | b(x_3) + \frac{1}{L} D_p^2 \varphi(x) \right) \right] \, dx \\
\quad + |b'(x_3) + L D_3 \varphi(x)|^2 \, dx \tag{1.35}
\]

\[
\leq \mathcal{W}_2(F|b) + \eta.
\]

Extend \(\varphi(\cdot, x_3)\) periodically with period \(Q'\) and for \(x \in \Omega\) define
\[
u_k(x_0, x_3) := \mathcal{F} x_0 + c + \varepsilon_k \int_0^{x_3} b(s) \, ds + L \varepsilon_k \varphi \left( \frac{x_0}{L \varepsilon_k}, x_3 \right). \tag{1.36}
\]
Since \( b \) and \( \varphi \) are bounded, it follows that \( \{u_k\} \) converges uniformly to \( u \). In addition, for \( x \in \Omega \) we have that
\[
D_p u_k (x) = \mathcal{F} + D_p \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right),
\]
\[
\frac{1}{\varepsilon_k} D_3 u_k (x) = b (x_3) + LD_3 \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right).
\]
Hence \( \{D_p u_k\} \) is bounded in \( L^\infty \) while \( \{D_3 u_k\} \) goes to 0 in \( L^q \), and so \( u_k \rightharpoonup u \) in \( W^{1,q} (\Omega; \mathbb{R}^3) \), while by the Riemann-Lebesgue Lemma, Fubini’s Theorem, and the fact that
\[
\int_{Q'} D_3 \varphi (x_\alpha , x_3) \, dx_\alpha = 0
\]
for \( L^1 \) a.e. \( x_3 \in I \), we have that \( \frac{1}{\varepsilon_k} D_3 u_k \rightharpoonup b \) in \( L^q (\Omega; \mathbb{R}^3) \). This proves that the sequence \( \{u_k\} \) is admissible for \( E^-_{\{u_k\}} (u; b; A) \), and we have
\[
E^-_{\{u_k\}} (u, b; A) \leq \liminf_{k \to \infty} E^2_{\varepsilon_k} (u_k; A)
\]
\[
= \liminf_{k \to \infty} \left( \int_{\Omega \times I} W \left( D_p u_k \left| \frac{1}{\varepsilon_k} D_3 u_k \right| \right) \, dx + \int_{\Omega \times I} \varepsilon_k^2 \left( |D_p^2 u_k|^2 + \frac{1}{\varepsilon_k^2} |D_p^3 u_k|^2 + \frac{1}{\varepsilon_k^4} |D_3^3 u_k|^2 \right) \, dx \right). \tag{1.37}
\]
We have that
\[
\int_{\Omega \times I} W \left( D_p u_k (x) \left| \frac{1}{\varepsilon_k} D_3 u_k (x) \right| \right) \, dx = \int_{\Omega \times I} W \left( \mathcal{F} + D_p \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right), b (x_3) + LD_3 \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right) \right) \, dx. \tag{1.38}
\]
On the other hand, for \( x \in \Omega \) we have that
\[
D_p^2 u_k (x) = \frac{1}{L \varepsilon_k} D_p \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right),
\]
\[
D_p^3 u_k (x) = D_p^3 \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right),
\]
\[
D_3^3 u_k (x) = \varepsilon_k b' (x_3) + LD_3 \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right),
\]
and so
\[
\int_{\Omega \times I} \varepsilon_k^2 \left( |D_p^2 u_k|^2 + \frac{1}{\varepsilon_k^2} |D_p^3 u_k|^2 + \frac{1}{\varepsilon_k^4} |D_3^3 u_k|^2 \right) \, dx \tag{1.39}
\]
\[
= \int_{\Omega \times I} \left( \frac{1}{L^2} |D_p^2 \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right)|^2 + |D_p^3 \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right)|^2 + b' (x_3) + LD_3 \varphi \left( \frac{x_\alpha}{L \varepsilon_k} , x_3 \right) \right) \, dx. \tag{1.40}
\]
Since, for $\mathcal{L}^1$ a.e. $x_3 \in I$ the function
\[
W \left( F + D_{p}\varphi (\cdot, x_3) | b (x_3) + LD_{3}\varphi (\cdot, x_3) \right) + \frac{1}{L^2} \left| D_{p}^2\varphi (\cdot, x_3) \right|^2 + \left| D_{p}\varphi (\cdot, x_3) \right|^2 + \left| D_{3}\varphi (\cdot, x_3) \right|^2
\]
is $Q'$-periodic, it converges weakly in $L^1(A)$ to its mean, that is, to
\[
\int_{Q'} \left( W \left( F + D_{p}\varphi (x_3, x_3) | b (x_3) + LD_{3}\varphi (x_3, x_3) \right) + \frac{1}{L^2} \left| D_{p}^2\varphi (x_3, x_3) \right|^2 + \left| D_{p}\varphi (x_3, x_3) \right|^2 + \left| D_{3}\varphi (x_3, x_3) \right|^2 \right) dx.
\]
Lebesgue’s Dominated Convergence Theorem and Fubini’s Theorem imply that
\[
\lim_{k \to \infty} \int_{A \times I} W \left( F + D_{p}\varphi \left( \frac{x_3}{L_k}, x_3 \right) | b (x_3) + LD_{3}\varphi \left( \frac{x_3}{L_k}, x_3 \right) \right) dx = L^2(A) \int_{Q'} W \left( F + D_{p}\varphi (x_3, x_3) | b (x_3) + LD_{3}\varphi (x_3, x_3) \right) dx
\]
and
\[
\lim_{k \to \infty} \int_{A \times I} \left( L^2 \left| D_{p}^2\varphi \left( \frac{x_3}{L_k}, x_3 \right) \right|^2 + \left| D_{p}\varphi \left( \frac{x_3}{L_k}, x_3 \right) \right|^2 \right) dx \]
\[
+ \left| b' (x_3) + LD_{3}\varphi \left( \frac{x_3}{L_k}, x_3 \right) \right|^2
\]
\[
= L^2(A) \int_{Q'} \left( \frac{1}{L^2} \left| D_{p}^2\varphi (x_3, x_3) \right|^2 + \left| D_{p}\varphi (x_3, x_3) \right|^2 + \left| b' (x_3) + LD_{3}\varphi (x_3, x_3) \right|^2 \right) dx,
\]
which, in view of (1.33), (1.37), (1.38), and (1.39), finally yields
\[
E_{\{\varepsilon \to 0\}} \left( u, b; A \right) \leq L^2(A)[\overline{W}_2(F | b) + \eta].
\]
Letting $\eta$ tend to 0, we conclude
\[
E_{\{\varepsilon \to 0\}} \left( u, b; A \right) \leq L^2(A)[\overline{W}_2(F | b)].
\]
This proves (1.34) when $u(x_3) = F x_3 + c$ for all $x_3 \in Q'$ and $b \in W^{1,2} (I; \mathbb{R}^3)$. The general case follows as in Steps 2 and 3 of Theorem 4.4 in [2]. We omit the details. □

In order to address the case $\gamma > 2$ we first recall the result obtained in [1], where
\[
\mathcal{I}_\varepsilon (u; A) := \int_{A \times I} W \left( D_{p}u \left| \frac{1}{\varepsilon} D_{3}u \right| \right) dx
\]
if $u \in W^{1,q} (\Omega; \mathbb{R}^3)$, and $\mathcal{I}_\varepsilon (u; A) := \infty$ otherwise. Fix a countable dense family $\{\theta_i\}_{i \in \mathbb{N}}$ in $L^{q'} (I; \mathbb{R}^3)$, where $q'$ is the conjugate exponent of $q$. For $F \in \mathbb{R}^{2 \times 3}$, $b \in L^q (I; \mathbb{R}^3)$ we define
\[
Q \infty W (F | b) := \sup_n Q_n W (F | b) = \lim_{n \to \infty} Q_n W (F | b),
\]
where

$$Q_n W(F|b) := \inf_{\varphi \in \mathbb{K}} \left\{ \int_Q W(F + D_p \varphi(x))|b(x) + LD_3 \varphi(x)\right\} dx : L > 0, \varphi \in W^{1,q}(Q; \mathbb{R}^3), \varphi(\cdot, x_3) Q_1\text{-periodic for } L^1 \text{ a.e. } x_3, \quad (1.43)$$

$$\left| \int_Q LD_3 \varphi(x_\alpha, x_3) \theta_i(x_3) \right| dx \leq \frac{1}{n}, \quad i = 1, \ldots, n \right\}.$$  \hfill (1.44)

**Remark F.** If \( W \) satisfies \((H_1)'\), then a density argument shows that in the definition above it is possible to restrict admissible functions \( \varphi \in W^{1,q}(Q; \mathbb{R}^3) \) to functions \( \varphi \in C^\infty(Q; \mathbb{R}^3) \).

The main result in \([1]\) is that under condition \((H_1)'\),

$$I(u, b; A) := \inf \left\{ \liminf_{\varepsilon \to 0^+} I_\varepsilon(u_\varepsilon; A) : u_\varepsilon \in W^{1,q}(A \times I; \mathbb{R}^3), \quad u_\varepsilon \rightharpoonup u \text{ in } W^{1,q}(A \times I; \mathbb{R}^3), \quad \frac{1}{\varepsilon} D_3 u_\varepsilon \rightharpoonup b \text{ in } L^q(A \times I; \mathbb{R}^3) \right\}.$$  \hfill (1.44)

for all \((u, b) \in W^{1,q}(A; \mathbb{R}^3) \times L^q(A \times I; \mathbb{R}^3) \) and \( A \in A(\omega) \).

Next we show that the \( \Gamma^0 \)-liminf \( E_\gamma^- (u, b; A) \) coincides with \( I(u, b; A) \). This establishes that in the case \( \gamma > 2 \) the second-order perturbation plays no role.

**Theorem G.** Assume that \( \gamma > 2 \) and that condition \((H_1)'\) is satisfied. Then for all \((u, b) \in V^\gamma \) and \( A \in A(\omega) \),

$$E_\gamma^- (u, b; A) = \int_A Q_\infty W(D_p u(x_\alpha)|b(x_\alpha, \cdot)) \, dx_\alpha.$$  \hfill (1.44)

**Proof.** For any given sequence \( \{\varepsilon_n\} \), with \( \varepsilon_n \to 0^+ \), we extract a subsequence \( \{\varepsilon_{nk}\} \) such that

$$\Gamma^- \lim_{k \to \infty} E_{\varepsilon_{nk}}^\gamma (u, b; A) \text{ exists for all } (u, b) \in V^\gamma \text{ and for all } A' \in \mathcal{R}(\omega),$$

and we define the \( \Gamma^- \)lower limit

$$E_{\{\varepsilon_{nk}\}}^- (u, b; A) := \inf \left\{ \liminf_{k \to \infty} E_{\varepsilon_{nk}}^\gamma (u_k; A) : u_k \in W^{2,2}(\Omega; \mathbb{R}^3), \quad u_k \rightharpoonup u \text{ in } W^{1,q}(\Omega; \mathbb{R}^3), \quad \frac{1}{\varepsilon_{nk}} D_3 u_k \rightharpoonup b \text{ in } L^q(\Omega; \mathbb{R}^3) \right\}.$$  \hfill (1.44)

As in the case \( \gamma = 2 \), it suffices to show that

$$E_{\{\varepsilon_{nk}\}}^- (u, b; A) = \int_A Q_\infty W(D_p u(x_\alpha)|b(x_\alpha, \cdot)) \, dx_\alpha.$$  \hfill (1.44)

In the sequel, to simplify the notation, we write \( \varepsilon_k \) in place of \( \varepsilon_{nk} \).

**Lower bound.** In view of \( (1.44) \) we deduce immediately that

$$E_{\{\varepsilon_k\}}^- (u, b; A) \geq I(u, b; A) = \int_A Q_\infty W(D_p u(x_\alpha)|b(x_\alpha, \cdot)) \, dx_\alpha.$$
Upper bound. We first prove the upper bound
\[ E_{\mu_{
abla\iota}}(u, b; A) \leq \int_A Q_{\infty} W(D_p u(x_\alpha)|b(x_\alpha, \cdot)) \, dx_\alpha \]  \tag{1.45}
when \( u(x_\alpha) = F x_\alpha + c \) for all \( x_\alpha \in Q' \) and for some \( F \in \mathbb{R}^{3 \times 2}, c \in \mathbb{R}^3, \) and \( b \in L^q(I; \mathbb{R}^3). \)
In this case the right-hand side reduces to \( Q_{\infty} W(F|b(\cdot)|) L^2(A) \).
In view of \((H_1)'\) we have that
\[ Q_{\infty} W(F|b(\cdot)|) < \infty. \]  \tag{1.46}
By definition of \( Q_{\infty} W(F|b(\cdot)|) \) (see (1.43) and Remark [E]) we may find admissible \( \varphi_n \in C^\infty(Q; \mathbb{R}^3) \) and \( L_n > 0 \) such that
\[ \int_Q W(F + D_p \varphi_n(x)|b(x_3) + L_n D_3 \varphi_n(x)) \, dx \leq Q_{\infty} W(F|b(\cdot)| + \frac{1}{n}). \]  \tag{1.47}
Note that in view of \((1.46)\) and \((H_1)'\) we have that
\[ \sup_n \| (D_p \varphi_n|L_n D_3 \varphi_n) \|_{L^q(Q;\mathbb{R}^{3 \times 3})} < \infty. \]  \tag{1.48}
Extend \( \varphi(\cdot, x_3) \) to \( \mathbb{R}^2 \) periodically with period \( Q' \) and extend \( b \) to \( \mathbb{R} \) by zero, and for \( x \in \Omega \) define
\[ u_{k,n}(x_\alpha, x_3) := F x_\alpha + c + \varepsilon_k \int_0^{x_3} (b \ast \rho_k)(s) \, ds + L_n \varepsilon_k \varphi_n \left( \frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right), \]
where
\[ \rho_k(t) := \frac{1}{\delta_k} \rho \left( \frac{t}{\delta_k} \right), \quad \delta_k := \varepsilon_k^{-\frac{3}{2}}, \]
and \( \rho \in C^\infty_c(\mathbb{R}), \rho \geq 0, \) and \( \int_\mathbb{R} \rho(t) \, dt = 1. \) Note that
\[ \lim_{n \to \infty} \lim_{k \to \infty} \| u_{k,n} - u \|_{L^q(\Omega;\mathbb{R}^3)} = 0. \]  \tag{1.49}
Moreover, for \( x \in \Omega \) we have that
\[ D_p u_{k,n}(x) = F + D_p \varphi_n \left( \frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right), \]
\[ \frac{1}{\varepsilon_k} D_3 u_{k,n}(x) = (b \ast \rho_k)(x_3) + L_n D_3 \varphi_n \left( \frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right). \]
Since \( \varphi_n(\cdot, x_3) \) is \( Q' \)-periodic for \( L^1 \) a.e. \( x_3, \) by \((1.48)\) it follows that
\[ \sup_{k,n \in \mathbb{N}} \left\| \left( D_p u_{k,n}, \frac{1}{\varepsilon_k} D_3 u_{k,n} \right) \right\|_{L^q(\Omega;\mathbb{R}^{3 \times 3})} < \infty. \]  \tag{1.50}
For every \( i \in \mathbb{N}, \) by \((1.43),\)
\[ \lim_{n \to \infty} \lim_{k \to \infty} \left| \int_\Omega \left( \frac{1}{\varepsilon_k} D_3 u_{k,n}(x) - b(x_3) \right) \, \theta_i(x_3) \, dx \right| \]
\[ = \lim_{n \to \infty} \lim_{k \to \infty} \left| \int_\Omega \left( (b \ast \rho_k)(x_3) - b(x_3) + L_n D_3 \varphi_n \left( \frac{x_\alpha}{L_n \varepsilon_k}, x_3 \right) \right) \, \theta_i(x_3) \, dx \right| \]
\[ = \lim_{n \to \infty} \left| \int_{\omega \times Q'} L_n D_3 \varphi_n(y_\alpha, x_3) \, dy_\alpha \theta_i(x_3) \, dx \right| = 0, \]  \tag{1.51}
where we have used the Riemann–Lebesgue Lemma and the fact that for \( n \geq i, \)
\[
\left| \int_Q L_n D_3 \varphi_n (x, x_3) \theta_i (x_3) \, dx \right| \leq \frac{1}{n}.
\]
Also by (1.37), \((H_1)^\prime\), the Lebesgue Dominated Convergence Theorem, and the Riemann-Lebesgue Lemma we have that
\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_{A \times I} W \left( D_p u_{k,n} (x) \left| \frac{1}{\varepsilon_k} D_3 u_{k,n} (x) \right| \right) \, dx = 0.
\]

Finally,
\[
\lim_{n \to \infty} \lim_{k \to \infty} \int_{Q^I} \varepsilon_k^2 \left( \left| D_p^2 u_{k,n} \right|^2 + \frac{1}{\varepsilon_k} \left| D_3 u_{k,n} \right|^2 \right) \, dx = 0,
\]
where we have used the facts that \( \gamma > 2, \)
\[
\lim_{k \to \infty} \int_{Q^I} \left\| D^2 \varphi_n \left( \frac{x_3}{\varepsilon_k} \right) \right\|^2 \, dx = \mathcal{L}^2 (A) \int_{Q^I} \left\| D^2 \varphi_n (y, x_3) \right\|^2 \, dy \, dx,
\]
and
\[
\lim_{k \to \infty} \int_{Q^I} \varepsilon_k^{\gamma-2} \left| b \ast \rho_k (x_3) \right|^2 \, dx = 0,
\]
because
\[
\left\| b \ast \rho_k \right\|_{L^\infty (I; \mathbb{R}^3)} \leq C \left\| \rho_k \right\|_{L^\infty (I)} \left\| b \right\|_{L^q (I; \mathbb{R}^3)} \leq \frac{C}{\varepsilon_k^2} \left\| b \right\|_{L^q (I; \mathbb{R}^3)}.
\]
Hence, recalling (1.49), (1.50), (1.51), (1.52) and (1.53), we may find a diagonal sequence
\[
u_n := u_{k_n, n}
\]
such that \( u_n \to u \) in \( W^{1,q} (\Omega; \mathbb{R}^3) \) and \( \frac{1}{\varepsilon_{k_n}} D_3 u_n \to b \) in \( L^q (\Omega; \mathbb{R}^3) \) and
\[
\lim_{n \to \infty} E_{\varepsilon_{k_n}}^\gamma (u_n; A) \leq \mathcal{L}^2 (A) \mathcal{Q}_\infty \mathcal{W} (\mathcal{F} | b (\cdot)).
\]
Define
\[
\tilde{u}_k (x) := \begin{cases} u_{k_n, n} (x) \\ \mathcal{F} x_\alpha + c \ast k \int_0^{x_3} (b \ast \rho_k) (s) \, ds \end{cases} \quad \text{if } k = k_n \text{ for some } n \in \mathbb{N},
\]
otherwise.
Since the sequence \( \{ \tilde{u}_k \} \) is admissible for \( E_{\{\varepsilon_k\}}^{-}(u, b; A) \), we have
\[
E_{\{\varepsilon_k\}}^{-}(u, b; A) \leq \liminf_{n \to \infty} E_{\{\varepsilon_k\}}^{-}(u_n; A) \leq \mathcal{L}^2(\cdot) Q_{\infty} W(\mathcal{F}|b(\cdot)).
\]

Just as in the proof of Theorem \([\text{D}]\), it can be shown that for every \((u, b) \in \mathcal{V}^\gamma\), \( E_{\{\varepsilon_k\}}^{-}(u, b; \cdot) \) is the trace of a Radon measure absolutely continuous with respect to \( \mathcal{L}^2(\cdot) \). Therefore, to establish the inequality (1.45) for arbitrary \((u, b) \in \mathcal{V}^\gamma\) we may proceed as in the proof of Steps 2 and 3 of Theorem 4.4 in \([2]\). We omit the details. \( \Box \)

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References