REGULARITY CONDITIONS FOR THE 3D NAVIER-STOKES EQUATIONS

By

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Abstract. We obtain logarithmic improvements for conditions of regularity in the 3D Navier-Stokes equations.

1. Introduction. In this paper we consider the 3D Navier-Stokes system:

\begin{align*}
    u_t + u \cdot \nabla u + \nabla \pi - \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, T), \\
    \text{div} u &= 0 \quad \text{in} \quad \mathbb{R}^3 \times (0, T), \\
    u|_{t=0} &= u_0(x) \quad \text{in} \quad \mathbb{R}^3,
\end{align*}

where \( T \in (0, +\infty) \). The vector field \( u \) (the velocity) and the scalar field \( \pi \) (the pressure) are the unknowns of the problem. Taking the curl of (1.1), we obtain the following vorticity equation:

\begin{align*}
    \omega_t + (u \cdot \nabla) \omega - \Delta \omega &= (\omega \cdot \nabla) u,
\end{align*}

where the vorticity \( \omega \) is defined by

\begin{align*}
    \omega &= \text{curl} u.
\end{align*}

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The incompressibility condition \((1.2)\) combined with \((1.5)\) implies Biot Savart’s law,

\[
 u(x, t) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{y \times \omega(x + y, t)}{|y|^3} dy
\]

for sufficiently rapidly decaying vorticity near infinity. After the pioneering work by J. Leray \([1]\) there are many comprehensive literatures on the existence theory of the weak solutions of the NS equations \((2, 3)\). The regularity of this weak solution is known as one of the most challenging problems in mathematical fluid mechanics. The first result in this direction is the one by Prodi \([4]\), which states that if a weak solution

\[
 u \in L^s(0, T; L^r(\mathbb{R}^3)), \quad \frac{3}{r} + \frac{2}{s} = 1,
\]

for \(3 < r \leq \infty\), then \(u(x, t)\) is smooth. After that there are further developments and refinements by Serrin \([5]\), Fabes-Jones-Riviere \([6]\), Kozono-Taniuchi \([7]\), and Escauriaza-Seregin-Sverak \([8]\). In particular, Beirão da Veiga \([9]\) obtained a regularity condition in terms of \(\nabla u\), which is equivalent to the one in terms of the vorticity due to the Calderon-Zygmund inequality. This states that if the vorticity \(\omega\) satisfies

\[
 \omega \in L^s(0, T; L^r(\mathbb{R}^3)), \quad \frac{3}{r} + \frac{2}{s} = 2,
\]

for \(\frac{3}{2} < r \leq \infty\), then \(u\) remains regular. Montgomery-Smith \([10]\) proved that if

\[
 \int_0^T \frac{\|u(t)\|_r^p}{1 + \log^+ \|u(t)\|_L^p} dt < \infty, \quad \frac{3}{r} + \frac{2}{s} = 1,
\]

for \(3 < r < \infty\), then \(u\) is regular.

Now we are in a position to state the main result in this paper.

**Theorem 1.1.** If \(u\) is a solution to \((1.1)-(1.3)\) satisfying one of the following two conditions:

\[
 (i) \quad \int_0^T \frac{\|\omega(t)\|_L^p}{1 + \log^+ \|\omega(t)\|_L^p} dt < \infty, \quad \frac{3}{p} + \frac{2}{s} = 2, \quad 2 \leq p < \infty,
\]

\[
 (ii) \quad \int_0^T \frac{\|\nabla \omega(t)\|_L^p}{1 + \log^+ \|\nabla \omega(t)\|_L^p} dt < \infty, \quad \frac{3}{p} + \frac{2}{s} = 3, \quad 2 \leq p < 3,
\]

then \(u\) is regular.

**Remark 1.2.** Note that this proof can easily be adapted to show that a sufficient condition for regularity is that

\[
 \int_0^T \frac{\|\omega(t)\|_L^p}{\theta(\|\omega(t)\|_L^p)} dt < \infty
\]

or

\[
 \int_0^T \frac{\|\nabla \omega(t)\|_L^p}{\theta(\|\nabla \omega(t)\|_L^p)} dt < \infty,
\]

where \(\theta\) is any increasing function for which

\[
 \int_1^\infty \frac{dx}{x\theta(x)} = \infty.
\]
2. Proof of Theorem 1.1

First, we assume that (1.9) holds true. We multiply (1.9) by $|\omega|^{p-2}\omega$ and we perform suitable integration by parts to obtain
\[
\frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p + \frac{1}{2} \int |\nabla \omega|^2 |\omega|^{p-2} dx + \frac{4(p-2)}{p^2} \int |\nabla |\omega||^2 dx
\]
\[
= -\int (\omega \cdot \nabla) |\omega| \cdot |D\omega|^{p-2} dx + \int (\omega \cdot \nabla |\omega|) \cdot |D\omega|^{p-2} dx
\]
\[
\leq \|\omega\|_{L^{p+1}} \|\nabla u\|_{L^{p+1}} \text{ (by H"older's inequality)}
\]
\[
\leq C \|\omega\|_{L^{p+1}}^{p+1}
\]
(2.1)

by the Calderon-Zygmund's inequality
\[
\|\nabla u\|_{L^{p+1}} \leq C \|\omega\|_{L^{p+1}}.
\]
(2.2)

Since $\|\omega\|_{L^{p+1}}^\frac{p}{p+1} = \| |\omega|^{\frac{p}{p+1}} \|_{L^{\frac{2(p+1)}{p}}}$ we apply the Gagliardo-Nirenberg inequality
\[
\|f\|_{L^{\frac{2(p+1)}{p}}} \leq C \|f\|_{L^2}^{\frac{1-\theta}{\theta}} \|\nabla f\|_{L^2}^{\frac{\theta}{1-\theta}}, \quad \text{with } \theta = \frac{3}{2(p+1)},
\]
to the function $|\omega|^{\frac{p}{p+1}}$. We obtain
\[
\|\omega\|_{L^{p+1}}^{p+1} \leq C \|\omega\|_{L^p}^\frac{2p}{2p-3} \|\nabla |\omega||^\frac{3}{p}.
\]

Then the Young's inequality with exponents $\frac{2p}{2p-3}$ and $\frac{2p}{3}$ finally gives
\[
\frac{1}{p} \frac{d}{dt} \|\omega\|_{L^p}^p + \frac{1}{2} \int |\nabla \omega|^2 |\omega|^{p-2} dx + \frac{2(p-2)}{p^2} \int |\nabla |\omega||^2 dx \leq C \|\omega\|_{L^p}^{p+1} \frac{2p}{2p-3}
\]
which gives
\[
\frac{d}{dt} \|\omega\|_{L^p} \leq C \|\omega\|_{L^p} \log^{\frac{1}{2p-3}} \|\omega\|_{L^p} \cdot \frac{\|\omega\|_{L^p}^{\frac{2p}{2p-3}}}{1 + \log^{\frac{1}{2p-3}} \|\omega\|_{L^p}}
\]
and hence
\[
\sup_{0 \leq t \leq T} \|\omega(t)\|_{L^p} \leq C
\]
by (1.9) with $s = \frac{2p}{2p-3}$.

Next we assume that (1.10) holds true. We take $D = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$ to (1.4), and then take the inner product of it with $D\omega|D\omega|^{p-2}$. After integration by parts we have
\[
\frac{1}{p} \frac{d}{dt} \|D\omega\|_{L^p}^p + \frac{1}{2} \int |\nabla D\omega|^2 |D\omega|^{p-2} dx + \frac{4(p-2)}{p^2} \int |\nabla |D\omega||^2 dx
\]
\[
= -\int D(\omega \cdot \nabla) D\omega |D\omega|^{p-2} dx + \int D(\omega \cdot \nabla D\omega) |D\omega|^{p-2} dx
\]
\[
= : I + J.
\]
(2.3)

We estimate $I, J$ below.
\[
I = -\int D\omega \cdot \nabla \omega \cdot D\omega |D\omega|^{p-2} dx - \int (u \cdot \nabla) D\omega \cdot D\omega |D\omega|^{p-2} dx
\]
\[
= : I_1 + I_2.
\]
Integrating by parts, and using the fact that \( \text{div} \, u = 0 \), we get
\[
I_2 = -\frac{1}{p} \int u \cdot \nabla |D\omega|^p \, dx = \frac{1}{p} \int |D\omega|^p \text{div} \, u \, dx = 0.
\]

By Hölder’s inequality and Sobolev inequality together with the Calderon-Zygmund inequality we estimate,
\[
I_1 \leq \int |Du| \cdot |D\omega|^p \, dx \leq \|Du\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \frac{1}{p_1} + \frac{p}{p_2} = 1 \right)
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p
\]
\[
\leq C \|D\omega\|_{L^{p_2}}^{p+1} \left( \frac{1}{p_1} = \frac{1}{p_2} - \frac{1}{3}, p_2 = \frac{3}{4}(p+1) \right)
\]
\[
= C \|D\omega\|_{L^{p_2}}^{2(p+1)/p}.
\]

We apply the Gagliardo-Nirenberg inequality
\[
\|f\|_{L^q} \leq C \|f\|_{L^p}^\theta \|
abla f\|_{L^{p_2}}^{1-\theta}, \quad \theta = \frac{3}{q} - \frac{1}{2}, \quad (2.4)
\]
to the function \( |D\omega|^2 \). We obtain
\[
I_1 \leq \epsilon \int |\nabla |D\omega|^2|^2 \, dx + C \|D\omega\|_{L^p}^{p+2/\theta}
\]
for any \( \epsilon > 0 \) by Young’s inequality.

In order to estimate \( J \) we first decompose it into two terms as follows:
\[
J = \int D\omega \cdot \nabla u \cdot D\omega |D\omega|^{p-2} \, dx + \int (\omega \cdot \nabla) Du \cdot D\omega |D\omega|^{p-2} \, dx
\]
\[
= : J_1 + J_2.
\]

Since
\[
J_1 \leq \|\nabla u\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \text{by Hölder’s inequality}, \quad \frac{1}{p_1} + \frac{p}{p_2} = 1 \right)
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \quad \left( \text{by Calderon-Zygmund inequality} \right)
\]
\[
\leq C \|D\omega\|_{L^{p_2}}^{p+1} \left( \frac{1}{p_1} = \frac{1}{p_2} - \frac{1}{3}, p_2 = \frac{3}{4}(p+1) \right),
\]
the estimate of \( J_1 \) is the same as that of \( I_1 \). On the other hand, by the Hölder, Sobolev and Calderon-Zygmund inequalities,
\[
J_2 \leq \|\omega\|_{L^{p_1}}, \|\nabla Du\|_{L^{p_2}} \|D\omega\|_{L^{p_2}}^{p-1} \left( \frac{1}{p_1} + \frac{p}{p_2} = 1 \right)
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|
abla u\|_{L^{p_2}} \|D\omega\|_{L^{p_2}}^{p-1}
\]
\[
\leq C \|\omega\|_{L^{p_1}} \|D\omega\|_{L^{p_2}}^p \left( \frac{1}{p_1} = \frac{1}{p_2} - \frac{1}{3}, p_2 = \frac{3}{4}(p+1) \right)
\]
\[
\leq C \|D\omega\|_{L^{p_2}}^{p+1}.
\]
Hence, the estimate of \( J_2 \) is also the same as that of \( I_1 \). Combining the above estimates \( I \) and \( J \) and taking \( \epsilon \) small enough, we have

\[
\frac{d}{dt} \|D\omega\|_{L^p}^p + C \int |\nabla |D\omega|^{\frac{4}{3}}|^2 \, dx \leq C \|D\omega\|_{L^p}^{p + \frac{2p}{3p-3}}
\]

which implies

\[
\sup_{0 \leq t \leq T} \|D\omega\|_{L^p} \leq C
\]

by (1.10) with \( s = \frac{2p}{3p-3} \).

This completes the proof. \( \square \)

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