THE MATHEMATICS OF THE THOMSON EFFECT

BY

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Abstract. The paper deals with the elliptic system of the thermistor problem in 3 dimensions taking into account the Thomson effect. Existence and uniqueness results are presented. The proofs are based on a reduction to a two-point problem for an ordinary differential equation.

1. Introduction. The phenomenological theory of thermoelectric effects is summarized (see [7]) in the constitutive equations

\[ J = \sigma(E - \alpha \nabla u), \quad q = -\kappa \nabla u + u\alpha J, \quad E = -\nabla \varphi, \]

where \( J \) is the current density, \( q \) the heat flux, \( E \) the electric field, \( u \) the absolute temperature, \( \varphi \) the electric potential, \( \sigma \) and \( \kappa \) are the thermal and electric conductivity which in this paper are assumed to be positive functions of the temperature. \( \alpha \) is also a function of \( u \) but with no definite sign. In a stationary state, equations (1), together with the balance equations

\[ \nabla \cdot J = 0, \quad \nabla \cdot q = E \cdot J, \]

give the system of partial differential equations

\[ \nabla \cdot (\sigma(u) \nabla v) = 0, \]
\[ \nabla \cdot (\kappa(u) \nabla u) + \sigma(u) \beta(u) \nabla u \cdot \nabla v + \sigma(u) |\nabla v|^2 = 0, \]

where \( \beta(u) = u\alpha'(u) \) and \( v = \varphi + \int_{u_1}^{u} \alpha(t)dt \) is the effective potential so that

\[ J = -\sigma(u) \nabla v. \]

Let \( \Omega \) be an open and bounded subset of \( \mathbb{R}^3 \) representing a body conductor of heat and electricity with a \( C^1 \) boundary consisting of two disjoint two-dimensional surfaces \( S_1, S_2 \) which act as electrodes to which a given difference of potential \( V > 0 \) is applied. The goal of this paper is to study the following problem \( Pb_V \):

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To find \( v(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega) \), \( u(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega) \) such that equations (2) and (3) are satisfied with the boundary conditions

\[
\begin{align*}
v &= 0 \text{ on } S_1, \quad v = V \text{ on } S_2, \\
u &= u_1 \text{ on } S_1, \quad u = u_2 \text{ on } S_2,
\end{align*}
\]

where \( u_1 \) and \( u_2 \) are given positive constants. We suppose

\[
\begin{align*}
\sigma(u) &\in C^0(\mathbb{R}^1), \quad \kappa(u) \in C^0(\mathbb{R}^1), \quad \beta(u) \in C^1(\mathbb{R}^1), \quad \sigma(u) > 0, \quad \kappa(u) > 0.
\end{align*}
\]

Thus no assumption of uniform ellipticity is made. In Section 2 we prove that a functional relation \( u = \hat{u}(v) \) between the temperature and potential exists and permits the reduction of problem \( Pb_V \) to the Dirichlet problem for the Laplacian. \( \hat{u}(v) \) is a solution of the following ordinary differential equation:

\[
\frac{\kappa(u)}{\sigma(u)} \frac{du}{dv} = \gamma - v - \int_{u_1}^{u} \beta(t) dt
\]

satisfying the conditions

\[
\begin{align*}
u(0) &= u_1, \quad u(V) = u_2.
\end{align*}
\]

The method is not new (see W. Voigt [8] and H. Diesselhorst [5]). However, the question of existence and uniqueness for problem (8), (9), and in turn for problem \( Pb_V \), is not treated in [5] and [8]. The special case of the metallic conduction is discussed in Section 3. Finally, Section 4 deals with a related problem in which a multiplicity of solutions may exist. Problem \( Pb_V \) is known as the “thermistor problem” when \( \beta = 0 \). It has been extensively studied with arbitrary boundary conditions by many authors (see, among others, [6], [1], [2] and the references therein).

2. The main theorem. If \( (v(x), u(x)) \) is a regular solution of problem \( Pb_V \), we have from the maximum principle the “a priori” estimates

\[
\begin{align*}
V &\geq v(x) \geq 0 \text{ in } \bar{\Omega}, \\
u(x) &\geq u_1 \text{ in } \bar{\Omega}.
\end{align*}
\]

**Theorem 1.** If (7) holds and

\[
\int_{u_1}^{\infty} \frac{\kappa(t)}{\sigma(t)} dt = \infty,
\]

then

\[
\begin{align*}
\left| \frac{\beta(u)\sigma(u)}{\kappa(u)} \right| &\leq C, \quad \text{for all } u \geq u_1,
\end{align*}
\]

(i) there exists at least one regular solution to problem \( Pb_V \).

(ii) The problem can be reduced to the following Dirichlet problem for the Laplacian:

\[
\Delta \psi = 0 \text{ in } \Omega, \quad \psi = \psi_1 \text{ on } S_1, \quad \psi = \psi_2 \text{ on } S_2,
\]

where \( \psi_1 \) and \( \psi_2 \) are constants which can be expressed in terms of the data.
Proof. If \( v(x) \) is a solution to problem \( PbV \) we have, by (2),
\[
\nabla \cdot (v\sigma(u)\nabla v) = \sigma(u)|\nabla v|^2,
\]
\[
\nabla \cdot \left[ \sigma(u) \int_{u_1}^u \beta(t) dt \nabla v \right] = \sigma(u)\beta(u)\nabla u \cdot \nabla v.
\]

Therefore, equation (3) can be rewritten in divergence form as
\[
\nabla \cdot \left\{ \sigma(u) \left[ v\nabla v + \frac{\kappa(u)}{\sigma(u)} \nabla u + \int_{u_1}^u \beta(t) dt \nabla v \right] \right\} = 0.
\]

(15)

Let \( u = \hat{u}(v) \) be the sought for functional relation. Define
\[
\theta = \frac{1}{2} v^2 + \int_{u_1}^u \frac{\kappa(t)}{\sigma(t)} dt + \int_0^v \int_{u_1}^u \beta(t) dt d\xi.
\]

(16)

Then (3) becomes
\[
\nabla \cdot (\sigma(u)\nabla \theta) = 0.
\]

(17)

Moreover, \( \theta(x) \) satisfies the boundary conditions:
\[
\theta = 0 \text{ on } S_1, \quad \theta = \gamma V^{-1} \text{ on } S_2,
\]

(18)

where \( \gamma \) is an unknown constant. To prove that \( \theta(x) \) and \( v(x) \) are related by the functional relation
\[
\theta = \gamma v
\]

(19)

we consider
\[
\nabla \cdot (\sigma(\hat{u}(v))\nabla \theta) = 0 \text{ in } \Omega, \quad \theta = 0 \text{ on } S_1, \quad \theta = \gamma V^{-1} \text{ on } S_2,
\]

(20)

\[
\nabla \cdot (\sigma(\hat{u}(v))\nabla v) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } S_1, \quad v = V \text{ on } S_2.
\]

(21)

This system is uncoupled, and it is easy to see that the solution \( v(x) \) of (21) is unique. Since \( \theta(x) = \gamma v(x) \) solves (20), we obtain (19). From (16) we have
\[
\gamma v = \frac{1}{2} v^2 + \int_{u_1}^u \frac{\kappa(t)}{\sigma(t)} dt + \int_0^v \int_{u_1}^u \beta(t) dt d\xi.
\]

(22)

Taking the derivative of (22) with respect to \( v \) we obtain the ordinary differential equation
\[
\frac{\kappa(u)}{\sigma(u)} \frac{du}{dv} = \gamma - v - B(u),
\]

(23)

where
\[
B(u) = \int_{u_1}^u \beta(t) dt,
\]

(24)

which must be supplemented by (5) with the conditions:
\[
u(0) = u_1,
\]

(25)

\[
u(V) = u_2.
\]

(26)
The two-point boundary value problem (23), (25), (26) determines \( \hat{u}(v) \). We prove now that (23), (25), (26) has one and only one solution for arbitrary data \( u_1, u_2 \) and \( V \).

Let us define the function

\[
F : [u_1, \infty) \to [0, \infty), \quad w = F(u), \quad F(u) = \int_{u_1}^{u} \frac{k(t)}{\sigma(t)} dt. \tag{27}
\]

\( F \) applies one-to-one \([u_1, \infty)\) onto \([0, \infty)\) by (7) and (12). Under the transformation (27), problem (23), (25), (26) becomes

\[
\frac{dw}{dv} = \gamma - v - \mathcal{B}(w), \quad w(0) = 0, \quad w(V) = F(u_2), \tag{28}
\]

where

\[
\mathcal{B}(w) = B(F^{-1}(w)).
\]

Since

\[
\mathcal{B}'(w) = \beta(F^{-1}(w)) \frac{\sigma(F^{-1}(w))}{\kappa(F^{-1}(w))}, \tag{31}
\]

the solution of the Cauchy problem (28), (29) is defined in \([0, V]\) by (13). We claim that equation (30), i.e.

\[
w(V; \gamma) = F(u_2), \tag{32}
\]

is solvable with respect to \( \gamma \). By (13) and (31), we have

\[
\mathcal{B}(0) - \tau w \leq \mathcal{B}(w) \leq \mathcal{B}(0) + \tau w. \tag{33}
\]

Hence \( w(V; \gamma) \) can be estimated from above and from below in terms of the solutions of the problems:

\[
\frac{dy^+}{dv} = \gamma - v + \tau y^+, \quad y^+(0) = 0, \quad \frac{dy^-}{dv} = \gamma - v - \tau y^-, \quad y^-(0) = 0. \tag{34}
\]

We find:

\[
y^-(V; \gamma) \leq w(V; \gamma) \leq y^+(V; \gamma). \tag{35}
\]

On the other hand we have, by direct computation,

\[
\lim_{\gamma \to \infty} y^-(V; \gamma) = \infty, \quad \lim_{\gamma \to -\infty} y^+(V; \gamma) = -\infty. \tag{36}
\]

By the continuous and differentiable dependence of \( w(V; \gamma) \) on the parameter \( \gamma \), \( w_\gamma(v; \gamma) \) satisfies the Cauchy problem:

\[
\frac{dw_\gamma}{dv} = 1 - \mathcal{B}'(w)w_\gamma, \quad w_\gamma(0) = 0. \tag{37}
\]

With an easy calculation we find from (37) that

\[
w_\gamma(V; \gamma) \geq \frac{e^{-CV}}{C} \left(1 - e^{-CV}\right) > 0. \tag{38}
\]
From (38) and (36) we conclude that (32), and therefore (26), has one and only one solution \( \tilde{\gamma} \). Let \( \tilde{u}(v) = u(v, \tilde{\gamma}) \) and consider the problem:

\[
\nabla \cdot \left( \sigma(\tilde{u}(v)) \nabla v \right) = 0 \text{ in } \Omega,
\]

\[
v = 0 \text{ on } S_1, \quad v = V \text{ on } S_2.
\]

Let

\[
\psi = G(v), \quad G(v) = \int_0^v \sigma(\tilde{u}(\xi)) d\xi, \quad \psi_2 = G(V).
\]

If \( \psi(x) \) solves

\[
\Delta \psi = 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } S_1, \quad \psi = \psi_2 \text{ on } S_2,
\]

then by the maximum principle we have

\[
0 \leq \psi(x) \leq \psi_2 \text{ in } \bar{\Omega}.
\]

If we define

\[
v(x) = G^{-1}(\psi(x))
\]

and

\[
u(x) = \tilde{u}(v(x))
\]

we obtain a solution to problem \( Pb_V \).

3. Two special cases. When equation (23) has a first integral it is possible to prove that problem \( Pb_V \) not only has a solution, but also that the solution is unique. This is the case of the metallic conduction. For metals the thermal and electric conductivity are related by the Wiedemann-Franz law [7], which reads:

\[
k(u) = K \sigma(u) u,
\]

where \( K \) is a positive constant. If we assume, as in [3],

\[
\beta(u) = Cu
\]

(a linear dependence which is verified if \( u \) varies in a small interval), we have from (46) and (47),

\[
\beta(u) = \tau \frac{k(u)}{\sigma(u)} \quad \tau = \frac{C}{K}.
\]

In this case it is possible to study problem \( Pb_V \) completely, in particular to prove the uniqueness of the solution. The proof is based on the following simple observation.

**Lemma 1.** Let \( A(\theta, \psi) \in C^0(\mathbb{R}^2) \),

\[
A(\theta, \psi) > 0.
\]

If \( \theta_i, \psi_i, \ i = 1, 2 \) are given constants, the problem

\[
\nabla \cdot (A(\theta, \psi) \nabla \theta) = 0 \text{ in } \Omega, \quad \theta = \theta_1 \text{ on } S_1, \quad \theta = \theta_2 \text{ on } S_2,
\]

\[
\nabla \cdot (A(\theta, \psi) \nabla \psi) = 0 \text{ in } \Omega, \quad \psi = \psi_1 \text{ on } S_1, \quad \psi = \psi_2 \text{ on } S_2
\]

has one and only one solution \((\theta(x), \psi(x))\), which can be represented in terms of the solution of a Dirichlet problem for the Laplacian.
Proof. Let \((\theta(x), \psi(x))\) be a solution to (50), (51). Then
\[ \psi(x) = a\theta(x) + b, \tag{52} \]
where \(a = (\psi_2 - \psi_1)(\theta_2 - \theta_1)^{-1} \) and \(b = \psi_1 - (\psi_2 - \psi_1)(\theta_2 - \theta_1)^{-1}. \) If \(w(x) = \psi(x) - a\theta(x) - b\) we have \(w = 0\) on \(S_1 \cup S_2\) and \(\nabla \cdot (A(\theta, \psi)\nabla w) = 0\) in \(\Omega.\) Hence \(w(x) = 0\) in \(\Omega\) and
\[ \psi(x) = a\theta(x) + b. \tag{53} \]
Define
\[ z = L(\theta), \quad L(\theta) = \int_{\theta_1}^{\theta} A(\xi, a\xi + b)d\xi. \tag{54} \]
If \(z(x) = L(\theta(x))\), we get:
\[ \Delta z = 0 \text{ in } \Omega, \quad z = 0 \text{ on } S_1, \quad z = L(\theta_2) \text{ on } S_2. \tag{55} \]
Now, if \((\hat{\theta}, \hat{\psi})\) is a second solution to (50), (51) and \(\hat{\psi}(x)\) is obtained as before, we have
\[ \Delta \hat{z} = 0 \text{ in } \Omega, \quad \hat{z} = 0 \text{ on } S_1, \quad \hat{z} = L(\theta_2) \text{ on } S_2. \tag{56} \]
Thus, \(\hat{z}(x) = z(x)\) and \(\theta(x) = L^{-1}(\hat{z}(x)) = L^{-1}(z(x)) = \hat{\theta}(x),\) and by (52), \(\hat{\psi}(x) = \psi(x).\)

In the next theorem we assume for definiteness \(\tau > 0\) and \(u_2 = u_1 = \bar{u} > 0.\) The general case can be treated similarly.

**Theorem 2.** Let us suppose (48) to hold and define
\[ G(v, V, \tau) = \frac{V}{\tau} \frac{1 - e^{-\tau v}}{1 - e^{-\tau V}} - \frac{v}{\tau}. \tag{57} \]
If
\[ \int_{\bar{u}}^{\infty} \frac{\kappa(t)}{\sigma(t)}dt = \infty, \tag{58} \]
then there exists one and only one solution of problem \(P\beta_V,\) which can be constructed from the functional relation
\[ u = F^{-1}(G(v, V, \tau)), \] \(F(u) = \int_{\bar{u}}^{u} \frac{\kappa(t)}{\sigma(t)}dt. \tag{59} \]
Let
\[ \mu = \int_{\bar{u}}^{\infty} \frac{\kappa(t)}{\sigma(t)}dt < \infty. \tag{60} \]
If
\[ G(v^*, V, \tau) \geq \mu, \] \(v^* = -\frac{1}{\tau} \log \frac{1 - e^{-\tau V}}{\tau V}, \tag{61} \]
then problem \(P\beta_V\) has no solution. When
\[ G(v^*, V, \tau) < \mu, \tag{62} \]
there exists one and only one solution, which is again obtained from (59).
Proof. By (48), equation (23) has the integrating factor $e^{\tau v}$, which gives the first integral

$$F(u) + \frac{1}{\tau}v - \frac{1}{\tau^2} - \frac{\gamma}{\tau} = Ce^{-\tau v}.$$  \hspace{1cm} (63)

Equation (63) permits us to solve problem (23), (25), (26). After computing $C$ and $\gamma$ we find the functional relation between $u$ and $v$, i.e.

$$F(u) = G(v, V, \tau),$$  \hspace{1cm} (64)

where $G(v, V, \tau)$ is given by (57).

$G(v, V, \tau)$ is easily studied. We find $G(0, V, \tau) = 0$; moreover, $G'(v) > 0$ if $v \in (0, v^*)$ and $G'(v) < 0$ if $v \in (v^*, V)$, where $v^* \in (0, \frac{V}{2})$ is given by (61). Hence, if (68) holds, (64) is uniquely solvable with respect to $u$. When (60) holds, (64) is solvable only if (62) is satisfied. Once the functional relation $u = \hat{u}(v)$ is obtained, problem $Pb'_\tau$ is reducible to a Dirichlet problem for the Laplacian as in Theorem 1. To prove that the solution obtained in this way is unique we use the transformation

$$\theta = e^{\tau v} \left[ \tau F(u) + v - \frac{1}{\tau} + 1 \right],$$  \hspace{1cm} (65)

$$\psi = e^{\tau v} \left[ \tau F(u) + v - \frac{1}{\tau} - 1 \right].$$  \hspace{1cm} (66)

Under (65), (66) equations (2) and (3) become

$$\nabla \cdot (e^{-\tau v} \sigma(u) \nabla \theta) = 0,$$

$$\nabla \cdot (e^{-\tau v} \sigma(u) \nabla \psi) = 0,$$

where

$$v = \frac{1}{\tau} \log \frac{\theta - \psi}{\theta + \psi},$$

$$u = F^{-1} \left[ \frac{1}{\tau} \left( \frac{\theta + \psi}{\theta - \psi} - \frac{1}{\tau} \log \frac{\theta - \psi}{\theta + \psi} + 1 \right) \right].$$

Since $\theta$ and $\psi$ are constants on $S_1$ and $S_2$ all assumptions of Lemma 1 are satisfied and we can conclude that the solution is unique $\square$

Remark 1. The present method permits a precise estimate of the maximum of the temperature. More precisely we have:

$$\max_{\mathbf{x} \in \Omega} u(\mathbf{x}) = F^{-1}(G(v^*, V, \tau))$$

as an examination of the graphs of $F(u)$ and $G(v)$ immediately shows. Moreover, this maximum is assumed in an interior point of $\Omega$.

When $\beta(u) = 0$ (complete absence of Thomson effect), problem (23), (25), (26) simplifies and we have:

$$\frac{\kappa(u)}{\sigma(u)} \frac{du}{dv} = \gamma - v,$$  \hspace{1cm} (67)

$$u(0) = u_1, \ u(V) = u_2.$$  \hspace{1cm} (68)

In (67) variables separate and, solving with condition (68), we obtain

$$F(u) = H(v),$$  \hspace{1cm} (69)
where
\[ F(u) = \int_{u_1}^{u} \frac{\kappa(t)}{\sigma(t)} dt, \quad H(v) = \frac{v}{V} \left( F(u_2) + \frac{V^2}{2} \right) - \frac{v^2}{2}. \]

We have
\[ H'(v^*) = 0, \quad \text{with} \quad v^* = \frac{V}{2} + \frac{F(u_2)}{V}. \]

If
\[ \int_{u_1}^{\infty} \frac{\kappa(t)}{\sigma(t)} dt = \infty, \]
problem (67), (68), and therefore problem PbV, has one and only one solution. In particular, when \( v^* < V \), we have
\[ u(x_M) = \max_{x \in \Omega} u(x) = F^{-1}(H(v^*)), \quad x_M \in \Omega, \] (70)
whereas, if \( V^* \geq V \),
\[ u(x_M) = \max_{x \in \Omega} u(x) = u_2, \quad x_M \in S_2. \] (71)

Assume now
\[ \infty > \int_{u_1}^{\infty} \frac{\kappa(t)}{\sigma(t)} dt = \mu. \] (72)

If \( v^* < V \), equation (67) is solvable with respect to \( u \) if and only if \( H(v^*) < \mu \). Hence, in this case, problem PbV has one and only one solution. On the contrary, when \( v^* \geq V \), equation (69), and therefore problem PbV, is always solvable, since \( H(V) = F(u_2) < \mu \).

The maximum of the temperature is again given by (70) and (71). To prove uniqueness, we proceed as in Theorem 2 using this time the transformation
\[ \theta = \frac{1}{2} v^2 + \int_{u_1}^{u} \frac{\kappa(t)}{\sigma(t)} dt, \quad \psi = v, \]
which reduces (2) and (3) to a form which permits the use of Lemma 1.

4. Multiplicity of solutions. In this last section we assume that the potential \( V \) is not applied directly to the electrodes \( S_1 \) and \( S_2 \), but via a one-dimensional resistor of \( R \) ohms, a natural situation, since in practice \( R \) is always greater than zero. In this model the potential \( \Gamma \) on \( S_2 \) is an unknown constant. By (4) the total electric current \( I \) flowing in the resistor is given by
\[ I = \sigma(u_2) \int_{S_2} \frac{\partial v}{\partial n} dS, \]
where \( \frac{\partial}{\partial n} \) denotes the outer normal derivative to \( S_2 \). We have problem Pb2:
To find \( v(x), u(x) \) and \( \Gamma \in \mathbb{R}^1 \) such that

\[
\nabla \cdot (\sigma(u)\nabla v) = 0 \text{ in } \Omega,
\]

\[
v = 0 \text{ on } S_1, \quad v = \Gamma \text{ on } S_2,
\]

\[
\nabla \cdot (\kappa(u)\nabla u) + \sigma(u)\beta(u)\nabla u \cdot \nabla v + \sigma(u)|\nabla v|^2 = 0 \text{ in } \Omega,
\]

\[
u = u_1 \text{ on } S_1, \quad u = u_2 \text{ on } S_2,
\]

\[
V - \Gamma = R\sigma(u_2) \int_{S_2} \frac{\partial v}{\partial n} dS.
\]

(73)

(74)

(75)

(76)

(77)

Lemma 2. If \((v(x), u(x), \Gamma)\) is a solution to problem \(P_{b, \Gamma}\), then

\[
0 < \Gamma < V.
\]

(78)

Proof. By contradiction assume \(\Gamma \leq 0\). If \(\Gamma = 0\), then \(v(x) = 0\) and from (77) we have \(\Gamma = V\). If \(\Gamma < 0\), by the maximum principle in Hopf’s form, we have \(\frac{\partial v}{\partial n} < 0\) on \(S_2\) by (73). Hence \(V - \Gamma = R\sigma(u_2) \int_{S_2} \frac{\partial v}{\partial n} dS < 0\). Therefore \(\Gamma > 0\), and as a consequence, \(\frac{\partial v}{\partial n} > 0\) on \(S_2\). Thus (78) holds. \(\square\)

For every fixed \(\Gamma \in (0, V)\) we can reduce problem \(P_{b, \Gamma}\), reasoning as in Theorem 1, to the following two-point problem for an ordinary differential equation:

\[
\frac{\kappa(u)}{\sigma(u)} \frac{du}{dv} = \gamma - v - B(u),
\]

\[
u(0) = u_1, \quad u(\Gamma) = u_2,
\]

(79)

(80)

which has, under the assumption (13), one and only one solution

\[
u = \hat{u}(v, \Gamma).
\]

Define

\[
\psi = L(v, \Gamma), \quad L(v, \Gamma) = \int_0^v \sigma(\hat{u}(\xi, \Gamma)) d\xi, \quad v \in [0, V]
\]

and solve

\[
\Delta \psi = 0 \text{ in } \Omega,
\]

\[
\psi = 0 \text{ on } S_1, \quad \psi = L(\Gamma, \Gamma) \text{ on } S_2.
\]

If \(w(x)\) is given by

\[
\Delta w = 0 \text{ in } \Omega, \quad w = 0 \text{ on } S_1, \quad w = 1 \text{ on } S_2
\]

we have

\[
\psi(x) = L(\Gamma, \Gamma)w(x),
\]

(81)

whence \(\nu(x) = L^{-1}(\psi(x), \Gamma)\) is a solution to

\[
\nabla \cdot (\sigma(\hat{u}(v, \Gamma))\nabla v) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } S_1, \quad v = \Gamma \text{ on } S_2.
\]

Let

\[
K = \int_{S_2} \frac{\partial w}{\partial n} dS.
\]

By (81), equation (77) reads

\[
V - \Gamma = RK\psi(L(\Gamma, \Gamma)).
\]

(82)
If \( g(\Gamma) = V - \Gamma - RKL(\Gamma, \Gamma) \), we have \( g(0) = V > 0 \) and \( g(V) = -RKL(\Gamma, \Gamma) < 0 \). Therefore equation (82) has in \((0, V)\) at least one solution. However, uniqueness is not to be expected as can already be seen when \( \beta = 0 \) (see [4]). Typically there exists one or three solutions and these solutions are found in practical devices modelled after problem \( Pb_2 \). Finally, we note that the function \( g(\Gamma) \) can be written in terms of the data and that the domain \( \Omega \) enters only with the constant \( K \).

References


