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CRITICAL RAYLEIGH NUMBER IN RAYLEIGH-BÉNARD CONVECTION

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Dedicated to Prof. W. A. Strauss on the occasion of his 70th birthday

Abstract. The Rayleigh-Bénard convection is a classical problem in fluid dynamics. In the presence of rigid boundary condition, we identify the critical Rayleigh number R_a^* by a reduced variational problem. We prove nonlinear asymptotic stability for motionless steady states for $R_a < R_a^*$, and their nonlinear instability for $R_a > R_a^*$. The dynamic of such instability is determined by the leading growing mode(s) for the corresponding linearized system within the time interval of instability.

1. Introduction. Rayleigh-Bénard convection in a shallow horizontal layer of a fluid heated from below has been widely studied in [2, 3, 4, 6, 10, 11, 12, 13, 16, 17]. Assuming so-called Boussinesq approximation [3], we obtain the basic hydrodynamic equations governing Rayleigh-Bénard convection as

$$\begin{aligned}
\partial_t v + (v\cdot\nabla)v + \frac{1}{\rho_0}\nabla p &= \nu\Delta v + g[\alpha(T-T_0)-1]\mathbf{e}_z,\\
\partial_t T + (v\cdot\nabla)T &= \kappa\Delta T, u|_{z=0,h} = 0,\\
T|_{z=0} &= T_1,\ \ T|_{z=h} = T_2, \quad T_1 > T_2,\\
v|_{t=0} &= u_0(x,y,z), \quad T|_{t=0} = T_0(x,y,z).
\end{aligned}$$

Here $v = (v_1, v_2, v_3)$ is the velocity field of the fluid satisfying $\nabla\cdot v = 0$, p the pressure, ν the kinematic viscosity, α the thermal expansion coefficient, $\mathbf{e}_z = (0,0,1)$ the unit upward vector, T the temperature field of the fluid, κ the thermal diffusivity coefficient, T_0 the properly chosen mean temperature, and ρ_0 the density at the temperature T_0. We

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also impose periodic boundary conditions in the horizontal directions with period $2\pi h$. There is a motionless steady state

$$\begin{aligned} v_s &\equiv 0, \\ p_s &= -g\rho_0 + g\alpha[(T_s - T_0)z + \frac{T_2 - T_1}{2h}z^2], \\ T_s &= T_1 + \frac{T_2 - T_1}{h}z. \end{aligned} \tag{1.1}$$

We denote the perturbation of such a steady state (1.1) as:

$$v = v_s + u, \ \ p = p_s + P, \ \ T = T_s + \theta.$$

By using the units of the layer depth h as the typical length scale, $(gh)^{1/2}$ as the typical velocity, $(h/g)^{1/2}$ as the typical time, $\rho_0 gh$ as the typical pressure, and $T_1 - T_2$ as the typical temperature, we derive the nondimensional form of the Boussinesq system for the perturbation as:

$$\partial_t u + (u \cdot \nabla)u + \nabla P = \mu_1 \Delta u + \mu_1 \mu_1 R_a \theta \mathbf{e}_z, \quad \nabla \cdot u = 0, \tag{1.2}$$

$$\partial_t \theta + (u \cdot \nabla)\theta + u \cdot \mathbf{e}_z = \mu_2 \Delta \theta, \tag{1.3}$$

with the following initial conditions:

$$u|_{t=0} = u_0(x, y, z), \quad \theta|_{t=0} = \theta_0(x, y, z), \tag{1.4}$$

and the boundary conditions

$$\begin{aligned} u|_{z=0,h} &= 0, \ \ u(x + 2\pi, y, z, t) = u(x, y + 2\pi, z, t) = u(x, y, z, t), \\ \theta|_{z=0,h} &= 0, \ \ \theta(x + 2\pi, y, z, t) = \theta(x, y + 2\pi, z, t) = \theta(x, y, z, t). \end{aligned} \tag{1.5}$$

Here $\mu_1 = \frac{\nu}{g^{1/2}h^{3/2}}$ and $\mu_2 = \frac{\kappa}{g^{1/2}h^{3/2}}$, and the Rayleigh number is given by:

$$R_a \equiv \frac{\alpha(T_1 - T_2)}{\mu_1 \mu_2} > 0. \tag{1.6}$$

In order to characterize the stability and instability of the motionless steady state (1.1), we introduce the critical Rayleigh number R_a^* as follows. For any integer $k \geq 1$, let $\Theta_k(z)$ be the minimizer of the following variational problem:

$$R(k) = \min_{\Theta \in B, k^2 \int_0^1 (|\partial_z \Theta|^2 + k^2 |\Theta|^2) dz = 1,} \int_0^1 |(\partial_z^2 - k^2)^2 \Theta|^2 dz, \tag{1.7}$$

where the function space B is defined by

$$\Big\{ \Theta_k \in H^4, \ \ \Theta_k \Big|_{z=0,1} = (\partial_z^2 - k^2)\Theta_k \Big|_{z=0,1} = \partial_z(\partial_z^2 - k^2)\Theta_k \Big|_{z=0,1} = 0 \Big\}.$$

We define

$$R_a^* = \min_{k \neq 0} \{R(k)\}. \tag{1.8}$$

The stability problem in the Rayleigh-Bénard convection has been investigated in [3, 4, 6, 10, 12, 13]. The purpose of this paper is to identify the sharp Rayleigh number R_a^* for stability.

In section 2, we first establish that the motionless steady state (1.1) is linearly stable for $R_a < R_a^*$ while linearly unstable for $R_a > R_a^*$. We carefully study appropriate variational problems and obtain a complete set of eigenfunctions for the linearized Boussinesq system, which leads to a precise formula for the linear solutions.

In section 3, we prove the nonlinear stability for $R_a < R_a^*$ by a standard semigroup approach (Theorem 3). In section 4, we prove that the motionless steady state (1.1) is nonlinear unstable for $R_a > R_a^*$ (Theorem 8). Moreover, the dynamics of the nonlinear instability is characterized by the fastest exponential growing mode(s) constructed in section 2 for the linearized Boussinesq system, within the instability regime $0 \le t \le \frac{1}{-\lambda_1} \ln \frac{1}{\delta}$, where δ is the strength of the initial perturbation and $-\lambda_1 > 0$ is the largest eigenvalue (Theorem 8). These growing mode(s) can exhibit interesting circular and role structures observed in experiments. The proof for the nonlinear instability is based on a general framework given in [8, 9]. The crucial step is to establish a bootstrap energy estimate (Lemma 6) in which we employ higher order anisotropic Sobolev norms in the presence of rigid boundary conditions.

We introduce the following notation: Let $(0, 2\pi)^2 = (0, 2\pi) \times (0, 2\pi)$ and $(E)^3 = E \times E \times E$, where E is any Banach space. The Hilbert space H denotes the completion of

$$\{(u_1, u_2, u_3) \big| \, u_1, u_2, u_3 \in C_{per}^\infty((0, 2\pi)^2; C_0^\infty(0, 1)); \partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0\} \tag{1.9}$$

with respect to the norm of $(L^2(Q))^3$, and endowed with the scalar product of $(L^2(Q))^3$, where $Q = (0, 2\pi)^2 \times (0, 1)$. We denote

$$\begin{aligned} V = \{(u_1, u_2, u_3) \big| \, &(u_1, u_2, u_3) \in H^1(Q) \cap L_{per}^2((0, 2\pi)^2; H_0^1(0, 1)); \\ &\partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0\} \end{aligned} \tag{1.10}$$

endowed with the scalar product and the norm of $(H^1(Q))^3$.

2. Linear stability and instability. We study the linearized Boussinesq system around the steady state (1.1):

$$\begin{aligned} \partial_t u + \nabla P &= \mu_1 \Delta u + R_a \mu_1 \mu_2 \theta \mathbf{e}_z, \quad \nabla \cdot u = 0, \\ \partial_t \theta - u_3 &= \mu_2 \Delta \theta \end{aligned} \tag{2.1}$$

with the initial condition (1.4) and boundary conditions (1.5). We rewrite the equations (1.20)–(1.22) as

$$\partial_t (u, \theta) = L(u, \theta). \tag{2.2}$$

LEMMA 1. There exist countable eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3 \le ...$ for the eigenvalue problem:

$$\begin{aligned} -\lambda u + \nabla P &= \mu_1 \Delta u + R_a \mu_1 \mu_2 \theta \mathbf{e}_z, \quad \nabla \cdot u = 0, \\ -\lambda \theta - u_3 &= \mu_2 \Delta \theta. \end{aligned} \tag{2.3}$$

The corresponding eigenfunctions $[u_k, \theta_k]_{k=1}^\infty$ form an orthonormal basis with respect to

$$\langle [u, \theta], [\tilde{u}, \tilde{\theta}] \rangle = (u, \tilde{u}) + R_a \mu_1 \mu_2 (\theta, \tilde{\theta}), \tag{2.4}$$

with $[u_1, P_1, \theta_1]$ smooth. Moreover, for any initial condition $[u^0, \theta^0] \in L^2$, if

$$[u^0, \theta^0] = \sum_k \gamma_k [u_k, \theta_k],$$

then the solution to the linearized Boussinesq system (2.1) is given by

$$e^{Lt}[u^0, \theta^0] = \sum_k \gamma_k e^{-\lambda_k t}[u_k, \theta_k]. \tag{2.5}$$

In particular, there exists a constant $C > 0$ such that

$$||e^{Lt}[u^0, \theta^0]|| \leq Ce^{-\lambda_1 t}||[u^0, \theta^0]||. \tag{2.6}$$

Proof. Recall the inner product of (2.4) with the corresponding Hilbert space $L^2_{R_a}$, and recall H and V in (1.9) and (1.10). We consider an equivalent eigenvalue problem as

$$\begin{aligned} -\lambda u &= \mu_1 \mathcal{P}\Delta u + R_a \mu_1 \mu_2 \mathcal{P}\{\theta \mathbf{e}_z\}, \\ -\lambda\theta &= \mu_2 \Delta\theta + u_3, \end{aligned} \tag{2.7}$$

where $\mathcal{P}$ denotes the projection $\{L^2(Q)\}^3 \to H$. Clearly, by the definition of (2.4), the operator

$$\begin{pmatrix} (\mu_1\Delta - \lambda_0)I & R_a\mu_1\mu_2\mathbf{e}_z^t \\ \mathbf{e}_z & (\mu_2\Delta - \lambda_0) \end{pmatrix}^{-1}$$

is a bounded, linear, compact, symmetric operator mapping $L^2_{R_a}(Q) \cap \{H \times L^2\}$ into itself for λ_0 large. The theory of compact, symmetric operators implies that all the eigenvalues of (2.7) are real with finite multiplicity, $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$. There are corresponding eigenfunctions $\{(u_k, \theta_k)\}_{k=1}^{\infty}$ which make up an orthonormal basis of $L^2_{R_a}(Q)$. The minimizer $(u_{\lambda_1}, \theta_{\lambda_1})$ of the following variational problem

$$\min_{(U,\Theta)\in A} F(U, \Theta) = \min_{(U,\Theta)\in A} \int_Q \{\mu_1|\nabla U|^2 + R_a\mu_1\mu_2^2|\nabla\Theta|^2 - 2R_a\mu_1\mu_2 U_3\Theta\}dxdydz$$

is a weak solution of (2.7), where the function space A is given by

$$\Big\{U \in V, \Theta \in H^1(Q) \cap L^2_{per}((0, 2\pi)^2; H^1_0(0, 1)), \quad \text{and} \quad ||U||^2 + Ra\mu_1\mu_2||\Theta||^2 = 1\Big\}.$$

Since $F(U, \Theta)$ is coercive and convex, there exists at least one $(u_1, \theta_1) \in A$ solving (2.7). By Lemma 1.1 in [5, Page 180], there exists a pressure field $P_1 \in L^2_{per}((0, 2\pi)^2; L^2(0, 1))$ such that $\big(u_1, P_1, \theta_1\big)$ is the weak solution of the original (2.3) with $\lambda = \lambda_1$. Thanks to the periodic boundary condition, $\big(u_1, P_1, \theta_1\big)$ satisfies (2.3) with $\lambda = \lambda_1$ in the domain $\Omega = \{(x, y, z)| - 2\pi < x, y < 4\pi, \quad 0 < z < 1\}$. Let Ω_0 be any bounded subdomain of Ω with $\partial\Omega_0 \in C^\infty$ and $\partial\Omega_0 \cap \partial\Omega = \{(x, y, z)| - \pi \leq x, y \leq 3\pi; \quad z = 0, 1\}$. By Theorem 5.1 in [5, Page 218], we have $u_1 \in (H^3_{per}((0, 2\pi)^2; H^3(0, 1)))^3 \cap V$ and $P_1 \in H^2_{per}((0, 2\pi)^2; H^2(0, 1))$. By the regular theory of weak solutions to elliptic equations [7], we deduce $\theta_1 \in H^3_{per}((0, 2\pi)^2; H^3(0, 1) \cap H^1_0(0, 1))$. By a bootstrap method, we have $u_1 \in (H^{m+1}_{per}((0, 2\pi)^2; H^{m+1}(0, 1)))^3 \cap V$, $P_1 \in H^m_{per}((0, 2\pi)^2; H^m(0, 1))$, $\theta_1 \in H^{m+1}_{per}((0, 2\pi)^2; H^{m+1}(0, 1) \cap H^1_0(0, 1))$, $\forall m \geq 2$. Both (2.5) and (2.6) then follow.

Lemma 2. Recall the critical Rayleigh number R_a^* defined in (1.8). If $R_a < R_a^*$, then $\lambda_1 > 0$. If $R_a > R_a^*$, then $\lambda_1 < 0$.

Proof. In order to construct an eigenfunction to (2.3), we first notice that it suffices to find the third component of u and θ. In fact, taking the curl of equation (2.1) and letting $\omega = (\omega_1, \omega_2, \omega_3) = \text{curl } u = \nabla \times u$, we have

$$\partial_t \omega = \mu_1 \Delta \omega + R_a \mu_1 \mu_2 (\nabla \times \mathbf{e}_z)\theta. \tag{2.8}$$

Taking the curl of equation (2.1) once again, we have

$$\partial_t (\nabla \times \omega) = -\partial_t \Delta u = -\mu_1 \Delta^2 u + R_a \mu_1 \mu_2 \Big(\nabla \times (\nabla \times \mathbf{e}_z)\Big)\theta. \tag{2.9}$$

Since the horizontal components u_1 and u_2 of the velocity can be determined by u_3 and ω_3, see [3], equations (2.3) are equivalent to the following equations:

$$\begin{aligned} -\lambda \omega_3 &= \mu_1 \Delta \omega_3, \\ -\lambda \Delta u_3 &= \mu_1 \Delta^2 u_3 + R_a \mu_1 \mu_2 (\partial_x^2 + \partial_y^2)\theta, \\ -\lambda \theta &= \mu_2 \Delta \theta + u_3. \end{aligned} \tag{2.10}$$

We now construct an eigenfunction to (2.3) by first studying the following reduced variational problem for the third component function $U_3(z)$ and $\Theta(z)$. For any $k \geq 1$, define

$$\begin{aligned} F_3(U_3, \Theta, R_a) \equiv \int_0^1 \{&\mu_1 |\partial_z^2 U_3 - k^2 U_3|^2 + R_a \mu_1 \mu_2^2 k^2 (|\partial_z \Theta|^2 + k^2 |\Theta^2|) \\ &- 2R_a \mu_1 \mu_2 k^2 U_3 \Theta\} dz. \end{aligned} \tag{2.11}$$

We consider

$$\lambda(R_a) = \min_{(U,\Theta)\in A_3} F_3(U_3, \Theta, R_a), \tag{2.12}$$

where the function space is

$$A_3 = \Big\{U_3 \in H_0^2,\ \ \Theta \in H_0^1,\ \ \int_0^1 (|\partial_z U_3|^2 + k^2 |U_3|^2 + R_a \mu_1 \mu_2 k^2 |\Theta|^2) dz = 1\Big\}. \tag{2.13}$$

It is standard to show that there exists a minimizer $[U_3, \Theta]$ for such a variational problem, which satisfies the Euler-Lagrange equations

$$\begin{aligned} -\lambda(R_a)(\partial_z^2 - k_0^2)U_3 &= \mu_1 (\partial_z^2 - k^2)^2 U_3 - R_a \mu_1 \mu_2 k^2 \Theta, \\ -\lambda(R_a)\Theta &= \mu_2 (\partial_z^2 - k^2)\Theta + U_3, \end{aligned} \tag{2.14}$$

with boundary conditions: $U_3\Big|_{z=0,1} = \partial_z U_3\Big|_{z=0,1} = 0, \quad \Theta\Big|_{z=0,1} = 0$. A direct computation shows that

$$\Big[-\frac{1}{k_0}\partial_z U_3(z)\sin\{k_0 x\},\ \ 0,\ \ U_3(z)\cos\{k_0 x\},\ \ \Theta(z)\cos\{k_0 x\}\Big] \tag{2.15}$$

is an eigenfunction with eigenvalue $\lambda(R_a)$ for the original equations (2.3).

We first prove that if $R_a > R_a^*$, then $\lambda_1 \leq \lambda(R_a) < 0$. By (1.8), there exists $k_0 \geq 1$ such that

$$R(k_0) = \min_{\Theta\in B, k_0^2 \int_0^1 (|\partial_z \Theta|^2 + k_0^2 |\Theta|^2) dz = 1} \int_0^1 |(\partial_z^2 - k_0^2)^2 \Theta|^2 dz < R_a.$$

Since the linear operator

$$(\partial_z^2 - k_0^2)^{-1} : L^2(0,1) \to L^2(0,1)$$

is bounded, compact and symmetric, the eigenvalue $0 < R(k_0)$ is real with finite multiplicity and there exists a minimizer $\Theta_{k_0} \in B$ of $R(k_0)$. Letting

$$U_{3,k_0} \equiv -\mu_2(\partial_z^2 - k_0^2)\Theta_{k_0},$$

we plug such a pair $[U_{3,k_0}, \Theta_{k_0}]$ into (2.11) to get

$$\begin{aligned} &\lambda(R_a)\int_0^1 (|\partial_z U_{3,k_0}|^2 + k^2|U_{3,k_0}|^2 + R_a\mu_1\mu_2 k_0^2|\Theta|^2)dz \\ \leq\ & \int_0^1 \{\mu_1|\partial_z^2 U_{3,k_0} - k_0^2 U_{3,k_0}|^2 + R_a\mu_1\mu_2^2 k_0^2(|\partial_z\Theta_{k_0}|^2 + k_0^2|\Theta_{k_0}^2|) \\ &\qquad - 2R_a\mu_1\mu_2 k_0^2 U_{3,k_0}\Theta_{k_0}\}dz \\ =\ & \mu_1\mu_2^2\int_0^1 \{|(\partial_z^2 - k_0^2)^2\Theta_{k_0}|^2 - R_a k_0^2(|\partial_z\Theta_{k_0}|^2 + k_0^2|\Theta_{k_0}|^2)\}dz \\ <\ & \mu_1\mu_2^2\int_0^1 \{|(\partial_z^2 - k_0^2)^2\Theta_{k_0}|^2 - R(k_0)k_0^2(|\partial_z\Theta_{k_0}|^2 + k_0^2|\Theta_{k_0}|^2)\}dz = 0. \end{aligned}$$

Hence $\lambda(R_a) < 0$.

We now assume that $R_a < R_a^*$ and prove $\lambda_1 > 0$ by contradiction. If not, $\lambda_1 \leq 0$, then the first equation in (2.10) implies $\omega = 0$, so that the corresponding eigenfunction $[u_3, \theta]$ satisfies the last two equations in (2.10), or the Euler-Lagrange equations (2.14). This implies that, for all $k \geq 1$,

$$\lambda(R_a) \leq \lambda_1 \leq 0.$$

By a change of $\tilde{\Theta} = \sqrt{R_a}\,\Theta$, we deduce that $\lambda(R_a)$ takes the form

$$\min_{\int_0^1(|\partial_z U_3|^2 + k^2|U_3|^2 + \mu_1\mu_2 k^2|\tilde{\Theta}|^2)dz = 1} F_3(U_3, \tilde{\Theta}, R_a), \tag{2.16}$$

where

$$\begin{aligned} F_3(U_3, \tilde{\Theta}, R_a) = \int_0^1 \{&\mu_1|\partial_z^2 U_3 - k^2 U_3|^2 + \mu_1\mu_2^2 k_0^2(|\partial_z\tilde{\Theta}|^2 + k^2|\tilde{\Theta}^2|) \\ &-2\sqrt{R_a}\mu_1\mu_2 k^2 U_3\tilde{\Theta}\}dz. \end{aligned} \tag{2.17}$$

We claim that as a function of the Rayleigh number R_a, $\lambda(R_a)$ is continuous in R_a. In fact, for any two Rayleigh numbers R_{a_1} and R_{a_2}, we choose corresponding minimizers $[U_1, \Theta_1]$ and $[U_2, \Theta_2]$. Clearly, by (2.17), we have

$$|F_3(U_1, \tilde{\Theta}, R_{a_1}) - F_3(U_1, \tilde{\Theta}, R_{a_2})| \leq C|R_{a_1} - R_{a_2}|.$$

Letting $\tilde{\Theta} = \Theta_2$, we deduce that

$$\lambda(R_{a_1}) \leq F_3(U_1, \Theta_2, R_{a_1}) < \lambda(R_{a_2}) + C|R_{a_1} - R_{a_2}|.$$

Similarly, letting $\tilde{\Theta} = \Theta_1$, we have

$$\lambda(R_{a_2}) \leq F_3(U_2, \Theta_1, R_{a_2}) < \lambda(R_{a_1}) + C|R_{a_1} - R_{a_2}|.$$

Hence, the continuity of $\lambda(R_a)$ follows. Moreover, by letting $U_3 = \tilde{\Theta}$ in (2.17), we have

$$\lim_{R_a \to \infty} \lambda(R_a) = -\infty.$$

This implies that for any $k \geq 1$, there exists $R_a^0(k) \leq R_a$ such that

$$\lambda(R_a^0(k)) = 0.$$

We now denote the corresponding minimizer of $\lambda(R_a^0(k))$ in (2.16) and (2.11) by $[U_3^0, \Theta^0]$, which satisfies (2.14) with $\lambda(R_a^0(k)) = 0$:

$$\begin{aligned} 0 &= \mu_1(\partial_z^2 - k^2)^2 U_3^0 - R_a^0 \mu_1 \mu_2 k^2 \Theta^0, \\ 0 &= \mu_2(\partial_z^2 - k^2)\Theta^0 + U_3^0. \end{aligned} \tag{2.18}$$

Equivalently, we have

$$-(\partial_z^2 - k^2)^4 \Theta^0 - R_a^0(k) k^2 (\partial_z^2 - k^2)\Theta^0 = 0,$$

which implies that $R_a^0\,(k) \geq R(k)$. Therefore, by (1.8),

$$R_a \geq R_a^0\,(k) \geq R_a^*,$$

a contradiction. We therefore conclude the lemma.

3. Nonlinear stability for $Ra < Ra^*$.

THEOREM 3. If the Rayleigh number $Ra < Ra^*$, then the motionless solution (u_s, P_s, T_s) in (1.1) of system (1.2), (1.3) is nonlinear stable with respect to the norm of $C(0,\infty; H^2(Q)) \cap W_\infty^1(0,\infty; L^2(Q))$.

Proof. The unconditional nonlinear stability with respect to the norm of $C(0,\infty; L^2(Q))$ can be found in [6].

Now we only give the proof of nonlinear stability with respect to the norm of $C(0,\infty; H^2(Q)) \cap W_\infty^1(0,\infty; L^2(Q))$. Let

$$\mathcal{A} = \begin{pmatrix} (\mu_1 \mathcal{P}\Delta) I & 0 \\ 0 & \mu_2 \Delta \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 0 & \sqrt{R_a \mu_1 \mu_2}\; \mathbf{e}_z^t \\ \sqrt{R_a \mu_1 \mu_2}\; \mathbf{e}_z & 0 \end{pmatrix},$$

where vector $\mathbf{e}_z = (0,0,1)$ and I is the 3×3 identity matrix. If $\tilde{\lambda}$ is an eigenvalue and $[\tilde{u}, \tilde{\theta}]$ is the corresponding eigenfunction of the following eigenvalue problem,

$$(\mathcal{A} + \mathcal{B})[\tilde{u}, \tilde{\theta}] = -\tilde{\lambda}[\tilde{u}, \tilde{\theta}], \tag{3.1}$$

then $\lambda = \tilde{\lambda}$ is the eigenvalue and $[u, \theta] = [\tilde{u}, \tilde{\theta}/\sqrt{R_a \mu_1 \mu_2}]$ is the corresponding eigenfunction of the eigenvalue problem (2.3). Similarly, for the eigenvalue λ and the corresponding eigenfunction $[u, \theta]$ of (2.3), then $\tilde{\lambda} = \lambda$ is the eigenvalue and $[\tilde{u}, \tilde{\theta}] = [u, \sqrt{R_a \mu_1 \mu_2}\;\theta]$ is the corresponding eigenfunction of the eigenvalue problem (3.1). Thus $\tilde{\lambda}_1 = \lambda_1 > 0$.

It is well known [14, 15] that $\mathcal{A} + \mathcal{B}$ is the infinitesimal generator of the analytic semigroup $e^{t(\mathcal{A}+\mathcal{B})} : H \times L^2(Q) \to H \times L^2(Q)$ with

$$\|e^{t(\mathcal{A}+\mathcal{B})}\| \leq C e^{-\lambda_1 t}, \quad \forall t > 0, \tag{3.2}$$

$$\|(-\mathcal{A} - \mathcal{B})^{1/2} e^{t(\mathcal{A}+\mathcal{B})}\| \leq C t^{-1/2} e^{-\lambda_1 t}, \quad \forall t > 0, \tag{3.3}$$

where $\lambda_1 > 0$ is defined in Lemma 1. Since $(-\mathcal{A}-\mathcal{B})^{-1}:\ H\times L^2(Q)\to H\times L^2(Q)$ and $\mathcal{A}(-\mathcal{A}-\mathcal{B})^{-1}:\ H\times L^2(Q)\to H\times L^2(Q)$ are both selfadjoint linear bounded operators, we have

$$\begin{aligned}
&\|(-\mathcal{A})^{1/2}e^{t(\mathcal{A}+\mathcal{B})}[u,\theta]\|\\
&\quad=\Big((-\mathcal{A})(-\mathcal{A}-\mathcal{B})^{-1}(-\mathcal{A}-\mathcal{B})^{1/2}e^{t(\mathcal{A}+\mathcal{B})}[u,\theta],\ \ (-\mathcal{A}-\mathcal{B})^{1/2}e^{t(\mathcal{A}+\mathcal{B})}[u,\theta]\Big)^{1/2}\\
&\quad\le\|\mathcal{A}(-\mathcal{A}-\mathcal{B})^{-1}\|^{1/2}\ \|(-\mathcal{A}-\mathcal{B})^{1/2}e^{t(\mathcal{A}+\mathcal{B})}[u,\theta]\|\\
&\quad\le Ct^{-1/2}e^{-\lambda_1 t}\|[u,\theta]|,\qquad \forall t>0,\ \ [u,\theta]\in H\times L^2(Q).
\end{aligned}\tag{3.4}$$

Equations (1.2) and (1.3) can be rewritten in the following equivalent form:

$$\begin{aligned}
&\big[u(t),\ \ \sqrt{R_a\mu_1\mu_2}\,\theta(t)\big]\\
&=e^{t(\mathcal{A}+\mathcal{B})}(u_0,\ \sqrt{R_a\mu_1\mu_2}\,\theta_0)-\int_0^t e^{(t-s)(\mathcal{A}+\mathcal{B})}\Big((u\cdot\nabla)u(s),\ (u\cdot\nabla)(\sqrt{R_a\mu_1\mu_2}\,\theta(s))\Big)ds.
\end{aligned}\tag{3.5}$$

Applying (3.2), (3.4) and (3.5), we have

$$\begin{aligned}
&\|u(t)\|_{L^2}+\|\theta(t)\|_{L^2}\\
&\qquad\le C_1e^{-\lambda_1 t}\Big(\|u_0\|_{L^2}+\|\theta_0\|_{L^2}\Big)\\
&\qquad\quad+C_2\int_0^t e^{-\lambda_1(t-s)}\Big(\|(u\cdot\nabla)u(s)\|_{L^2}+\|(u\cdot\nabla)\theta(s)\|_{L^2}\Big)ds\\
&\qquad\le C_1\Big(\|u_0\|_{L^2}+\|\theta_0\|_{L^2}\Big)+C_2\sup_{0\le s\le T}\Big(\|u(s)\|^2_{H^2}+\|\theta(s)\|^2_{H^2}\Big),\quad \forall T\ge t,
\end{aligned}\tag{3.6}$$

$$\begin{aligned}
&\|\Delta u(t)\|_{L^2}+\|\Delta\theta(t)\|_{L^2}\le C_1e^{-\lambda_1 t}\Big(\|\Delta u_0\|_{L^2}+\|\Delta\theta_0\|_{L^2}\Big)\\
&\qquad+C_2\int_0^t(t-s)^{-1/2}e^{-\lambda_1(t-s)}\Big(\|\nabla\{(u\cdot\nabla)u(s)\}\|_{L^2}+\|\nabla\{(u\cdot\nabla)\theta(s)\}\|_{L^2}\Big)ds\\
&\qquad\le C_1\Big(\|\Delta u_0\|_{L^2}+\|\Delta\theta_0\|_{L^2}\Big)+C_2\sup_{0\le s\le T}\Big(\|u(s)\|^2_{H^2}+\|\theta(s)\|^2_{H^2}\Big),\quad \forall T\ge t.
\end{aligned}\tag{3.7}$$

Let

$$E(T)=\sup_{0\le s\le T}\Big(\|u(s)\|_{H^2}+\|\theta(s)\|_{H^2}\Big).$$

Then we have

$$E(T)\le C_1\Big(\|u_0\|_{H^2}+\|\theta_0\|_{H^2}\Big)+C_2E^2(T),\quad \forall T\ge 0,\tag{3.8}$$

where the positive constants C_1 and C_2 are independent of T. Therefore, if $\|u_0\|_{H^2}+\|\theta_0\|_{H^2}$ is small enough, then there exists a unique solution $(u,\theta)\in C\Big([0,\infty);\big(H^2(Q)\big)^4\Big)$ of (1.2) and (1.3) such that

$$\|u(t)\|_{H^2}+\|\theta(t)\|_{H^2}\le 2C_1\Big(\|u_0\|_{H^2}+\|\theta_0\|_{H^2}\Big).\quad \forall t\ge 0,\tag{3.9}$$

and our theorem follows. □

4. Instability of motionless state. We now turn to the nonlinear problem (1.2) and (1.3). We first recall a general framework for nonlinear instability as:

LEMMA 4 (Bootstrap Instability [8]). Assume that L is a linear operator on a Banach space X with norm $\|\cdot\|$, and e^{tL} generates a strongly continuous semigroup on X such that

$$\|e^{tL}\|_{(X,X)} \leq C_L e^{t\lambda} \tag{4.1}$$

for some C_L and $\lambda > 0$. Assume a nonlinear operator $N(y)$ on X and another norm $|||\cdot|||$, and constant C_N, such that

$$\|N(y)\| \leq C_N |||y|||^2 \tag{4.2}$$

for all $y \in X$ and $|||y||| < \infty$. Assume for any solution $y(t)$ to the equation

$$y' = Ly + N(y) \tag{4.3}$$

with $|||y(t)|||^2 \leq \sigma$ that there exists $C_\sigma > 0$ such that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that the following sharp energy estimate holds:

$$\frac{d}{dt}|||y(t)||| \leq \epsilon|||y(t)||| + C_\sigma|||y(t)||| + C_\epsilon\|y(t)\|. \tag{4.4}$$

Consider a family of initial data $y^\delta(0) = \delta y_0$ with $\|y_0\| = 1$ and $|||y_0||| < \infty$ and let β_0 be a sufficiently small (fixed) number. Then there exists some constant $C > 0$ such that if

$$0 \leq t \leq T^\delta \equiv \frac{1}{\lambda}\log\frac{\beta_0}{\delta},$$

then we have

$$\|y(t) - \delta e^{tL}y_0\| \leq C\big(|||y_0|||^2 + 1\big)\delta^2 e^{2\lambda t}. \tag{4.5}$$

In particular, if there exists a constant C_p such that $\|\delta e^{tL}y_0\| \geq C_p\delta e^{\lambda t}$, then there exists an escape time $T^{esc} \leq T^\delta$ such that

$$\|y(T^{esc})\| \geq \tau_0 > 0, \tag{4.6}$$

where τ_0 depends explicitly on $C_L, C_N, C_\sigma, C_p, \lambda, y_0, \sigma$ and is independent of δ.

To apply such a method, we need to verify (4.4) for an appropriate Sobolev norm $|||\cdot|||$. Let $\|\cdot\| = \|\cdot\|_{L^2}$, $D^k_{x,y} = \sum\limits_{k_1+k_2=k} \partial_x^{k_1}\partial_y^{k_2}$ and

$$E_0 = \|u(t)\|^2 + \|\theta(t)\|^2,$$
$$E_k = E_0 + \|\nabla u(t)\|^2 + \|\nabla\theta(t)\|^2 + \|D^k_{x,y}\nabla u(t)\|^2 + \|D^k_{x,y}\nabla\theta(t)\|^2.$$

LEMMA 5. Let $k \geq 2$. Then

$$\|(u\cdot\nabla)u\| + \|(u\cdot\nabla)\theta\| \leq CE_k. \tag{4.7}$$

Proof. Applying Sobolev imbedding theorems and Hölder's inequality, we have

$$\begin{aligned} u^2(x,y,z,t) &= 2\int_0^z u_z u(x,y,s,t)ds \leq C\int_0^z \|u_z(\cdot,s,t)\|_{H^2_{x,y}}\|u(\cdot,s,t)\|_{H^2_{x,y}}ds, \\ \|u(t)\|_{L^\infty} &\leq C(\|u(t)\| + \|(\partial_x^2+\partial_y^2)u(t)\| + \|\partial_z u(t)\| + \|(\partial_x^2+\partial_y^2)\partial_z u(t)\|). \end{aligned}$$

By using the multiplicative inequality [1, p. 323], we have

$$\|(u\cdot\nabla)u\| + \|(u\cdot\nabla)\theta\| \le \|u\|_{L^\infty}(\|\nabla u\| + \|\nabla\theta\|) \le CE_k,$$

where we have used the following estimates:

$$\|\nabla D^{k-l}_{x,y}u\| \le C\|\nabla u\|^{l/k}\|\nabla D^k_{x,y}u\|^{(k-l)/k},\ \ \forall 1\le l\le k-1.$$

LEMMA 6. Let $k\ge 3$. Then we have

$$\frac{d}{dt}E_k \le \epsilon E_k + CE_k^2 + C_\epsilon E_0, \qquad \forall\epsilon>0. \tag{4.8}$$

Proof. Taking scalar products of u with equation (1.2) and θ with equation (1.3), we have

$$\begin{aligned}\frac{d}{dt}\|u\|^2 + 2\mu_1\|\nabla u\|^2 &\le 2R_a\|u_3\|\|\theta\|,\\ \frac{d}{dt}\|\theta\|^2 + 2\mu_2\|\nabla\theta\|^2 &\le 2\|u_3\|\|\theta\|.\end{aligned}$$

Taking scalar products of u_t with equation (1.2) and $\Delta\theta$ with equation (1.3), we have

$$\begin{aligned}\frac{d}{dt}\mu_1\|\nabla u\|^2 + 2\|u_t\|^2 &\le 2\|(u\cdot\nabla)u\|^2 + \|\partial_t u_3\|^2 + C\|\theta\|^2,\\ \frac{d}{dt}\|\nabla\theta\|^2 + 2\mu_2\|\Delta\theta\|^2 &\le C\|(u\cdot\nabla)\theta\|^2 + \mu_2\|\Delta\theta\|^2 + C\|u_3\|^2.\end{aligned}$$

Taking scalar products of $D^{2k}_{x,y}u_t$ with equation (1.2) and $D^{2k}_{x,y}\Delta\theta$ with equation (1.3), we have

$$\begin{aligned}\frac{d}{dt}\mu_1\|\nabla D^k_{x,y}u\|^2 + 2\|D^k_{x,y}u_t\|^2 &\le 2\|D^k_{x,y}\{(u\cdot\nabla)u\}\|^2 + \|D^k_{x,y}u_t\|^2 + C\|D^k_{x,y}\theta\|^2,\\ \frac{d}{dt}\|\nabla D^k_{x,y}\theta\|^2 + 2\mu_2\|\Delta D^k_{x,y}\theta\|^2 &\le C\|D^k_{x,y}\{(u\cdot\nabla)\theta\}\|^2 + \mu_2\|\Delta D^k_{x,y}\theta\|^2 + C\|D^k_{x,y}u\|^2.\end{aligned}$$

By Hölder's inequality, the Sobolev imbedding theorems, the multiplicative inequality [1, p. 323] and (4.7) , $\|D^k_{x,y}\{(u\cdot\nabla)u\}\|$ is bounded by

$$\begin{aligned}&C\sum_{l=0}^{k-2}\|D^l_{x,y}u\|_{L^\infty}\|D^{k-l}_{x,y}\nabla u\| + 2\|D^{k-1}_{x,y}u\|_{L^\infty_z(0,1;L^2_{x,y})}\|D_{x,y}\nabla u\|_{L^2_z(0,1;L^\infty_{x,y})}\\ &\quad +2\|D^k_{x,y}u\|_{L^\infty_z(0,1;L^2_{x,y})}\|\nabla u\|_{L^2_z(0,1;L^\infty_{x,y})}\\ \le\ & CE_k + C\|D^{k-1}_{x,y}u\|_{H^1_z(0,1;L^2_{x,y})}\|D_{x,y}\nabla u\|_{L^2_z(0,1;H^2_{x,y})}\\ &\quad +C\|D^k_{x,y}u\|_{H^1_z(0,1;L^2_{x,y})}\|\nabla u\|_{L^2_z(0,1;H^2_{x,y})}\\ \le\ & CE_k,\end{aligned}$$

and $\|D^k_{x,y}\{(u\cdot\nabla)\theta\}\|$ is bounded by

$$\begin{aligned}
& C\sum_{l=0}^{k-2}\|D^l_{x,y}u\|_{L^\infty}\|\nabla D^{k-l}_{x,y}\theta\|_{L^2}+C\|D^{k-1}_{x,y}u\|_{L^\infty_z(0,1;L^2_{x,y})}\|D_{x,y}\nabla\theta\|_{L^2_z(0,1;L^\infty_{x,y})}\\
& +C\|D^k_{x,y}u\|_{L^\infty_z(0,1;L^2_{x,y})}\|\nabla\theta\|_{L^2_z(0,1;L^\infty_{x,y})}\\
\leq\ & CE_k+C\|D^{k-1}_{x,y}u\|_{H^1_z(0,1;L^2_{x,y})}\|D_{x,y}\nabla\theta\|_{L^2_z(0,1;H^2_{x,y})}\\
& +C\|D^k_{x,y}u\|_{H^1_z(0,1;L^2_{x,y})}\|\nabla\theta\|_{L^2_z(0,1;H^2_{x,y})}\\
\leq\ & CE_k.
\end{aligned}$$

Here we have used the following estimates:

$$\|\nabla D^{k-l}_{x,y}\theta\|\leq C\|\nabla\theta\|^{l/k}\|\nabla D^k_{x,y}\theta\|^{(k-l)/k},\ \ \forall 1\leq l\leq k-1.$$

Since

$$\begin{aligned}
C\|D^k_{x,y}u\|^2+C\|D^k_{x,y}\theta\|^2 &\leq C\|D^{k+1}_{x,y}u\|^{2k/(k+1)}\|u\|^{2/(k+1)}\\
&\quad +C\|D^{k+1}_{x,y}\theta\|^{2k/(k+1)}\|\theta\|^{2/(k+1)}\\
&\leq \epsilon\{\|D^{k+1}_{x,y}u\|^2+\|D^{k+1}_{x,y}\theta\|^2\}+C_\epsilon\{\|u\|^2+\|\theta\|^2\},
\end{aligned}$$

by putting together all these estimates, we obtain (4.8).

Since $\lambda_1<0$ for $R_a>R_a^*$, there exists some corresponding wave numbers $k^*\geq 1$ such that $\lambda^*=\lambda(R_a)<0$, where $\lambda(R_a)$ is defined in (2.12) with $k=k^*$, and the maximal growth rate is given by $-\lambda^*>0$. Let $[U_k,\ \Theta_k]$ be the minimizer of (2.12).

Definition 7. We define a smooth generic profile for the initial perturbation as

$$\begin{aligned}
[\tilde u,\tilde\theta] &= \big(\tilde u_1,\tilde u_2,\tilde u_3,\tilde\theta\big)\\
&= \sum_{k_1^2+k_2^2=k^2}\Big(V_{1,k_1,k_2},\ V_{2,k_1,k_2},\ v_{k_1,k_2}U_k(z),\ \vartheta_{k_1,k_2}\Theta_k(z)\Big)e^{ik_1x+ik_2y} \qquad (4.9)
\end{aligned}$$

such that if $k_1^2+k_2^2=(k^*)^2$, then either v_{k_1,k_2} or ϑ_{k_1,k_2} is nonzero. Moreover, $V_{1,0,0}=V_{2,0,0}=0$,

$$\begin{aligned}
V_{1,k_1,k_2} &= \frac{ik_1}{k^2}v_{k_1,k_2}\partial_zU_k(z),\quad \forall\, k_1^2+k_2^2=k^2\geq 1,\\
V_{2,k_1,k_2} &= \frac{ik_2}{k^2}v_{k_1,k_2}\partial_zU_k(z),\quad \forall\, k_1^2+k_2^2=k^2\geq 1.
\end{aligned}$$

Therefore, from Lemmas 4–6, we have

Theorem 8. Suppose that the Rayleigh number $R_a>Ra^*$. Let $[u^\delta,\theta^\delta]$ be a solution to the equations (1.2) and (1.3) with initial value

$$[u^\delta(0),\theta^\delta(0)]=\delta[\tilde u,\tilde\theta],$$

where $[\tilde u,\tilde\theta]$ is a generic profile defined in (4.9) and $\|[\tilde u,\tilde\theta]\|=1$. Then,

$$\begin{aligned}
&\|[u^\delta(t),\ \theta^\delta(t)]-\sum_{k_1^2+k_2^2=(k^*)^2}\delta e^{tL}\Big(V_{1,k_1,k_2},\ V_{2,k_1,k_2},\ v_{k_1,k_2}U_k(z),\ \vartheta_{k_1,k_2}\Theta_k(z)\Big)e^{ik_1x+ik_2y}\|\\
&\leq C\{1+\big(\|[\nabla\tilde u,\nabla\tilde\theta]\|^2+\|[\partial_x^k\nabla\tilde u,\partial_x^k\nabla\tilde\theta]\|^2\big)\}\delta^2e^{-2\lambda^*t},\quad k\geq 3, \qquad (4.10)
\end{aligned}$$

for any $\delta \le \delta_0$ sufficiently small, and $0 \le t \le T^\delta$, where $|||[u,\theta]||| = \|u_1\| + \|u_2\| + \|u_3\| + \|\theta\|$.

REMARK 9. We note that $e^{tL}\Big(V_{1,k_1,k_2},\ V_{2,k_1,k_2},\ v_{k_1,k_2}U_k(z),\ \vartheta_{k_1,k_2}\Theta_k(z)\Big)e^{ik_1x+ik_2y}$ is given explicitly in (2.5). Clearly, for $k_1^2+k_2^2=(k^*)^2$ and $v_{k_1,k_2},\ \vartheta_{k_1,k_2}$ not both zero,

$$\|e^{tL}\Big(V_{1,k_1,k_2},\ V_{2,k_1,k_2},\ v_{k_1,k_2}U_k(z),\ \vartheta_{k_1,k_2}\Theta_k(z)\Big)e^{ik_1x+ik_2y}\| \ge Ce^{-\lambda^* t}$$

for some positive constant C. Then, from Lemma 4 and Theorem 8, the motionless state (u_s, P_s, T_s) of the system (1.2) and (1.3) is nonlinear unstable.

The proof of Theorem 8 follows from Lemmas 4–6 with $|||\cdot||| = E_k^{1/2}$ and $\|\cdot\| = E_0^{1/2}$. By (2.5), we split $\delta e^{tL}(\tilde{u},\tilde{\theta})$ in Lemma 4 into maximal growing modes

$$\delta e^{tL}\Big(V_{1,k_1,k_2},\ V_{2,k_1,k_2},\ v_{k_1,k_2}U_k(z),\ \vartheta_{k_1,k_2}\Theta_k(z)\Big)e^{ik_1x+ik_2y}, \qquad \forall\ k_1^2+k_2^2=(k^*)^2$$

and the remaining modes to conclude the proof.

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