CRITICAL RAYLEIGH NUMBER IN RAYLEIGH-BÉNARD CONVECTION

BY

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Dedicated to Prof. W. A. Strauss on the occasion of his 70th birthday

Abstract. The Rayleigh-Bénard convection is a classical problem in fluid dynamics. In the presence of rigid boundary condition, we identify the critical Rayleigh number \( R^* \) by a reduced variational problem. We prove nonlinear asymptotic stability for motionless steady states for \( R_a < R^* \), and their nonlinear instability for \( R_a > R^* \). The dynamic of such instability is determined by the leading growing mode(s) for the corresponding linearized system within the time interval of instability.

1. Introduction. Rayleigh-Bénard convection in a shallow horizontal layer of a fluid heated from below has been widely studied in \[2, 3, 4, 6, 10, 11, 12, 13, 16, 17\]. Assuming so-called Boussinesq approximation \[3\], we obtain the basic hydrodynamic equations governing Rayleigh-Bénard convection as

\[
\begin{align*}
\partial_t v + (v \cdot \nabla)v + \frac{1}{\rho_0} \nabla p &= \nu \Delta v + g(\alpha(T - T_0) - 1)e_z, \\
\partial_t T + (v \cdot \nabla)T &= \kappa \Delta T, \quad u|_{z=0,h} = 0, \\
T|_{z=0} &= T_1, \quad T|_{z=h} = T_2, \quad T_1 > T_2, \\
v|_{t=0} &= u_0(x,y,z), \quad T|_{t=0} = T_0(x,y,z).
\end{align*}
\]

Here \( v = (v_1, v_2, v_3) \) is the velocity field of the fluid satisfying \( \nabla \cdot v = 0 \), \( p \) the pressure, \( \nu \) the kinematic viscosity, \( \alpha \) the thermal expansion coefficient, \( e_z = (0,0,1) \) the unit upward vector, \( T \) the temperature field of the fluid, \( \kappa \) the thermal diffusivity coefficient, \( T_0 \) the properly chosen mean temperature, and \( \rho_0 \) the density at the temperature \( T_0 \).
also impose periodic boundary conditions in the horizontal directions with period $2\pi h$.
There is a motionless steady state
\[
\begin{align*}
v_s & \equiv 0, \\
p_s & = -g \rho_0 + g \alpha ((T_s - T_0)z + \frac{T_2 - T_1}{2h} z^2), \\
T_s & = T_1 + \frac{T_2 - T_1}{h} z.
\end{align*}
\] (1.1)

We denote the perturbation of such a steady state (1.1) as:
\[
\begin{align*}
v & = v_s + u, \\
p & = p_s + P, \\
T & = T_s + \theta.
\end{align*}
\]

By using the units of the layer depth $h$ as the typical length scale, $(gh)^{1/2}$ as the typical velocity, $(h/g)^{1/2}$ as the typical time, $\rho_0 gh$ as the typical pressure, and $T_1 - T_2$ as the typical temperature, we derive the nondimensional form of the Boussinesq system for the perturbation as:
\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla P & = \mu_1 \Delta u + \mu_1 \mu_2 R_a \theta \mathbf{e}_z, \\
\partial_t \theta + (u \cdot \nabla) \theta + u \cdot \mathbf{e}_z & = \mu_2 \Delta \theta,
\end{align*}
\] (1.2), (1.3)

with the following initial conditions:
\[
\begin{align*}
u|_{t=0} & = u_0(x, y, z), \\
\theta|_{t=0} & = \theta_0(x, y, z),
\end{align*}
\] (1.4)

and the boundary conditions
\[
\begin{align*}
u|_{z=0, h} & = 0, \\
\theta|_{z=0, h} & = 0, \\
u(x + 2\pi, y, z, t) & = u(x, y, z, t), \\
\theta(x + 2\pi, y, z, t) & = \theta(x, y, z, t).
\end{align*}
\] (1.5)

Here $\mu_1 = \nu g^{1/2} h^{-3/2}$ and $\mu_2 = \kappa g^{1/2} h^{-3/2}$, and the Rayleigh number is given by:
\[
R_a \equiv \frac{\alpha T_1 - T_2}{\mu_1 \mu_2} > 0.
\] (1.6)

In order to characterize the stability and instability of the motionless steady state (1.1), we introduce the critical Rayleigh number $R_a^*$ as follows. For any integer $k \geq 1$, let $\Theta_k(z)$ be the minimizer of the following variational problem:
\[
\begin{align*}
R(k) & = \min_{\Theta \in B, k^2 \int_0^1 ((\partial_z \Theta)^2 + k^2 \Theta^2)dz = 1} \int_0^1 (|\partial_z^2 \Theta - k^2 \Theta|^2)dz,
\end{align*}
\] (1.7)

where the function space $B$ is defined by
\[
\begin{align*}
\{ \Theta_k \in H^1, \\ \Theta_k|_{z=0, 1} = (\partial_z^2 - k^2) \Theta_k|_{z=0, 1} = \partial_z (\partial_z^2 - k^2) \Theta_k|_{z=0, 1} = 0 \}.
\end{align*}
\]

We define
\[
R_a^* = \min_{k \neq 0} R(k).
\] (1.8)

The stability problem in the Rayleigh-Bénard convection has been investigated in [3, 4, 6, 10, 12, 13]. The purpose of this paper is to identify the sharp Rayleigh number $R_a^*$ for stability.
In section 2, we first establish that the motionless steady state (1.1) is linearly stable for \( R_a < R^* \) while linearly unstable for \( R_a > R^* \). We carefully study appropriate variational problems and obtain a complete set of eigenfunctions for the linearized Boussinesq system, which leads to a precise formula for the linear solutions.

In section 3, we prove the nonlinear stability for \( R_a < R^* \) by a standard semigroup approach (Theorem 3). In section 4, we prove that the motionless steady state (1.1) is nonlinear unstable for \( R_a > R^* \) (Theorem 8). Moreover, the dynamics of the nonlinear instability is characterized by the fastest exponential growing mode(s) constructed in section 2 for the linearized Boussinesq system, within the instability regime \( 0 < \lambda_1 \). These growing mode(s) can exhibit interesting circular and role structures observed in experiments. The proof for the nonlinear instability is based on a general framework given in [8, 9]. The crucial step is to establish a bootstrap energy estimate (Lemma 6) in which we employ higher order anisotropic Sobolev norms in the presence of rigid boundary conditions.

We introduce the following notation: Let \((0, 2\pi)^2 \times (0, 2\pi) \) and \((E)^3 = E \times E \times E\), where \( E \) is any Banach space. The Hilbert space \( H \) denotes the completion of

\[
\{(u_1, u_2, u_3) | u_1, u_2, u_3 \in C^\infty_{per}((0, 2\pi)^2; C^2_0(0, 1)); \partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0\}
\]  

(1.9)

with respect to the norm of \((L^2(Q))^3\), and endowed with the scalar product of \((L^2(Q))^3\), where \( Q = (0, 2\pi)^2 \times (0, 1) \).

We denote

\[
V = \{(u_1, u_2, u_3) | (u_1, u_2, u_3) \in H^1(Q) \cap L^2_{per}((0, 2\pi)^2; H^1_0(0, 1)); \\
\partial_x u_1 + \partial_y u_2 + \partial_z u_3 = 0\}
\]  

(1.10)

endowed with the scalar product and the norm of \((H^1(Q))^3\).

**2. Linear stability and instability.** We study the linearized Boussinesq system around the steady state (1.1):

\[
\begin{align*}
\partial_t u + \nabla P &= \mu_1 \Delta u + R_a \mu_1 \mu_2 \theta e_2, \\
\partial_t \theta - u_3 &= \mu_2 \Delta \theta
\end{align*}
\]  

(2.1)

with the initial condition (1.3) and boundary conditions (1.5). We rewrite the equations (1.20)–(1.22) as

\[
\partial_t (u, \theta) = L(u, \theta).
\]  

(2.2)

**Lemma 1.** There exist countable eigenvalues \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \) for the eigenvalue problem:

\[
\begin{align*}
-\lambda u + \nabla P &= \mu_1 \Delta u + R_a \mu_1 \mu_2 \theta e_2, \\
-\lambda \theta - u_3 &= \mu_2 \Delta \theta
\end{align*}
\]  

(2.3)

The corresponding eigenfunctions \([u_k, \theta_k]_{k=1}^\infty\) form an orthonormal basis with respect to

\[
\langle [u, \theta], [\tilde{u}, \tilde{\theta}] \rangle = \langle u, \tilde{u} \rangle + R_a \mu_1 \mu_2 \langle \theta, \tilde{\theta} \rangle,
\]  

(2.4)
with \([u_1, P_1, \theta_1]\) smooth. Moreover, for any initial condition \([u^0, \theta^0]\) \(\in L^2\), if
\[
[u^0, \theta^0] = \sum_k \gamma_k [u_k, \theta_k],
\]
then the solution to the linearized Boussinesq system \((2.7)\) is given by
\[
e^{Lt}[u^0, \theta^0] = \sum_k \gamma_k e^{-\lambda_k t}[u_k, \theta_k].
\]
(2.5)
In particular, there exists a constant \(C > 0\) such that
\[
||e^{Lt}[u^0, \theta^0]|| \leq Ce^{-\lambda_1 t}||[u^0, \theta^0]||.
\]
(2.6)

**Proof.** Recall the inner product of \((2.4)\) with the corresponding Hilbert space \(L^2_{R_a}\), and recall \(H\) and \(V\) in \((1.9)\) and \((1.10)\). We consider an equivalent eigenvalue problem as
\[
-\lambda u = \mu_1 \mathcal{P} \Delta u + R_a \mu_1 \mu_2 \mathcal{P} \{\theta e_z\},
\]
\[
-\lambda \theta = \mu_2 \Delta \theta + u_3,
\]
(2.7)
where \(\mathcal{P}\) denotes the projection \(\{L^2(Q)\}^3 \rightarrow H\). Clearly, by the definition of \((2.4)\), the operator
\[
\left(\begin{pmatrix} \mu_1 \Delta - \lambda_0 \end{pmatrix} \mathcal{I} 
 R_a \mu_1 \mu_2 \left(\begin{array}{c} e_z \\ \mathcal{P} \{\theta e_z\} \end{array}\right) \right)^{-1}
\]
is a bounded, linear, compact, symmetric operator mapping \(L^2_{R_a}(Q) \cap \{H \times L^2\}\) into itself for \(\lambda_0\) large. The theory of compact, symmetric operators implies that all the eigenvalues of \((2.7)\) are real with finite multiplicity, \(\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots\). There are corresponding eigenfunctions \(\{[u_k, \theta_k]\}_{k=1}^\infty\) which make up an orthonormal basis of \(L^2_{R_a}(Q)\). The minimizer \((u_{\lambda_1}, \theta_{\lambda_1})\) of the following variational problem
\[
\min_{(U, \Theta) \in A} \int_Q \left(\mu_1 |\nabla U|^2 + R_a \mu_1 \mu_2 |\nabla \Theta|^2 - 2R_a \mu_1 \mu_2 U_{\theta} \right) dx dy dz
\]
is a weak solution of \((2.7)\), where the function space \(A\) is given by
\[
\left\{ U \in V, \Theta \in H^1(Q) \cap L^2_{per}((0, 2\pi)^2; H^1_0(0, 1)), \quad \|U\|^2 + R_a \mu_1 \mu_2 |\Theta|^{2} = 1 \right\}.
\]
Since \(F(U, \Theta)\) is coercive and convex, there exists at least one \((u_1, \theta_1) \in A\) solving \((2.7)\). By Lemma 1.1 in \[[5] \text{ Page 180}\], there exists a pressure field \(P_1 \in L^2_{per}((0, 2\pi)^2; L^2(0, 1))\) such that \((u_1, P_1, \theta_1)\) is the weak solution of the original \((2.3)\) with \(\lambda = \lambda_1\). Thanks to the periodic boundary condition, \((u_1, P_1, \theta_1)\) satisfies \((2.3)\) with \(\lambda = \lambda_1\) in the domain \(\Omega = \{(x, y, z)| - 2\pi < x, y < 4\pi, \quad 0 < z < 1\}\). Let \(\Omega_0\) be any bounded subdomain of \(\Omega\) with \(\partial \Omega_0 \in C^\infty\) and \(\partial \Omega_0 \cap \partial \Omega = \{(x, y, z)| - \pi \leq x, y \leq 3\pi; \quad z = 0, 1\}\). By Theorem 5.1 in \[[5] \text{ Page 218}\], we have \(u_1 \in (H^3_{per}((0, 2\pi)^2; H^3(0, 1)))^3 \cap V\) and \(P_1 \in H^2_{per}((0, 2\pi)^2; H^2(0, 1))\). By the regular theory of weak solutions to elliptic equations \([7]\), we deduce \(\theta_1 \in H^3_{per}((0, 2\pi)^2; H^3(0, 1)) \cap H^1_0(0, 1))\). By a bootstrap method, we have \(u_1 \in (H^m_{per}((0, 2\pi)^2; H^{m+1}(0, 1)))^3 \cap V, P_1 \in H^m_{per}((0, 2\pi)^2; H^m(0, 1))\). Thanks to \((2.3)\) and \((2.6)\) then follow.

**Lemma 2.** Recall the critical Rayleigh number \(R^*_a\) defined in \((1.8)\). If \(R_a < R^*_a\), then \(\lambda_1 > 0\). If \(R_a > R^*_a\), then \(\lambda_1 < 0\).
Proof. In order to construct an eigenfunction to (2.3), we first notice that it suffices to find the third component of \( u \) and \( \theta \). In fact, taking the curl of equation (2.1) and letting \( \omega = (\omega_1, \omega_2, \omega_3) = \text{curl} u = \nabla \times u \), we have

\[
\partial_t \omega = \mu_1 \Delta \omega + R_a \mu_1 \mu_2 (\nabla \times \mathbf{e}_z) \theta. \tag{2.8}
\]

Taking the curl of equation (2.1) once again, we have

\[
\partial_t (\nabla \times \omega) = -\partial_t \Delta u = -\mu_1 \Delta^2 u + R_a \mu_1 \mu_2 \left( \nabla \times (\nabla \times \mathbf{e}_z) \right) \theta. \tag{2.9}
\]

Since the horizontal components \( u_1 \) and \( u_2 \) of the velocity can be determined by \( u_3 \) and \( \omega_3 \), see [3], equations (2.3) are equivalent to the following equations:

\[
\begin{align*}
-\lambda \omega_3 &= \mu_1 \Delta \omega_3, \tag{2.10} \\
-\lambda \Delta u_3 &= \mu_1 \Delta^2 u_3 + R_a \mu_1 \mu_2 (\partial^2_x + \partial^2_y) \theta, \tag{2.11} \\
-\lambda \theta &= \mu_2 \Delta \theta + u_3.
\end{align*}
\]

We now construct an eigenfunction to (2.3) by first studying the following reduced variational problem for the third component function \( U_3(z) \) and \( \Theta(z) \). For any \( k \geq 1 \), define

\[
F_3(U_3, \Theta, R_a) \equiv \int_0^1 \{ \mu_1 |\partial^2_x U_3 - k^2 U_3|^2 + R_a \mu_1 \mu_2 k^2 (|\partial_x \Theta|^2 + k^2 |\Theta|^2) \} dz - 2R_a \mu_1 \mu_2 k^2 U_3 \Theta \}, \tag{2.12}
\]

where the function space is

\[
A_3 = \left\{ U_3 \in H^3_0, \Theta \in H^1_0, \int_0^1 (|\partial^2_x U_3|^2 + k^2 |U_3|^2 + R_a \mu_1 \mu_2 k^2 |\Theta|^2) \right\} dz = 1 \}. \tag{2.13}
\]

It is standard to show that there exists a minimizer \([U_3, \Theta]\) for such a variational problem, which satisfies the Euler-Lagrange equations

\[
\begin{align*}
-\lambda(R_a) (\partial^2_x - k^2) U_3 &= \mu_1 (\partial^2_x - k^2)^2 U_3 - R_a \mu_1 \mu_2 k^2 U_3, \tag{2.14} \\
-\lambda(R_a) \Theta &= \mu_2 (\partial^2_x - k^2) \Theta + U_3,
\end{align*}
\]

with boundary conditions: \( U_3 \big|_{z=0} = \partial_z U_3 \big|_{z=0} = 0, \quad \Theta \big|_{z=0} = 0 \). A direct computation shows that

\[
\left[ -\frac{1}{k_0} \partial_z U_3(z) \sin \{ k_0 x \}, \quad 0, \quad U_3(z) \cos \{ k_0 x \}, \quad \Theta(z) \cos \{ k_0 x \} \right] \tag{2.15}
\]

is an eigenfunction with eigenvalue \( \lambda(R_a) \) for the original equations (2.3).

We first prove that if \( R_a > R^*_a \), then \( \lambda_1 \leq \lambda(R_a) < 0 \). By (1.8), there exists \( k_0 \geq 1 \) such that

\[
R(k_0) = \min_{\Theta \in B(k_0^2 \mu_1 \mu_2)} \int_0^1 |(\partial^2_x - k_0^2)^2 \Theta|^2 dz < R_a.
\]

Since the linear operator

\[
(\partial^2_x - k_0^2)^{-1} : L^2(0, 1) \rightarrow L^2(0, 1)
\]

is bounded, compact and symmetric, the eigenvalue $0 < R(k)$ is real with finite multiplicity and there exists a minimizer $\Theta_{k_0} \in B$ of $R(k_0)$. Letting

$$U_{3,k_0} \equiv -\mu_2 (\partial_z^2 - k_0^2) \Theta_{k_0},$$

we plug such a pair $[U_{3,k_0}, \Theta_{k_0}]$ into (2.11) to get

$$\lambda(R_a) \int_0^1 (|\partial_z U_{3,k_0}|^2 + k_0^2 |U_{3,k_0}|^2 + R_a \mu_1 \mu_2 k_0^2 |\Theta|^2) dz$$

$$\leq \int_0^1 \{\mu_1 |\partial_z U_{3,k_0} - k_0^2 U_{3,k_0}|^2 + R_a \mu_1 \mu_2 k_0^2 (|\partial_z \Theta_{k_0}|^2 + k_0^2 |\Theta_{k_0}|^2) - 2 R_a \mu_1 \mu_2 k_0^2 |U_{3,k_0} \Theta_{k_0}| dz$$

$$= \mu_1 \mu_2 \int_0^1 \{|(\partial_z^2 - k_0^2)^2 \Theta_{k_0}|^2 - R_a k_0^2 (|\partial_z \Theta_{k_0}|^2 + k_0^2 |\Theta_{k_0}|^2)| dz$$

$$\leq \mu_1 \mu_2 \int_0^1 \{|(\partial_z^2 - k_0^2)^2 \Theta_{k_0}|^2 - R(k_0) k_0^2 (|\partial_z \Theta_{k_0}|^2 + k_0^2 |\Theta_{k_0}|^2)| dz = 0.$$ 

Hence $\lambda(R_a) < 0$.

We now assume that $R_a < R^*_a$ and prove $\lambda_1 > 0$ by contradiction. If not, $\lambda_1 \leq 0$, then the first equation in (2.10) implies $\omega = 0$, so that the corresponding eigenfunction $[u_3, \theta]$ satisfies the last two equations in (2.10), or the Euler-Lagrange equations (2.14). This implies that, for all $k \geq 1$,

$$\lambda(R_a) \leq \lambda_1 \leq 0.$$

By a change of $\tilde{\Theta} = \sqrt{R_a} \Theta$, we deduce that $\lambda(R_a)$ takes the form

$$\min_{\int_0^1 \{|\partial_z U_3|^2 + k^2 |U_3|^2 + \mu_1 \mu_2 k^2 |\tilde{\Theta}|^2} dz = 1 \} F_3(U_3, \tilde{\Theta}, R_a),$$

(2.16)

where

$$F_3(U_3, \tilde{\Theta}, R_a) = \int_0^1 \{\mu_1 |\partial_z U_3 - k^2 U_3|^2 + \mu_1 \mu_2 k_0^2 (|\partial_z \tilde{\Theta}|^2 + k^2 |\tilde{\Theta}|^2)$$

$$- 2 \sqrt{R_a} \mu_1 \mu_2 k^2 |\partial_z U_3 | dz.$$ (2.17)

We claim that as a function of the Rayleigh number $R_a$, $\lambda(R_a)$ is continuous in $R_a$. In fact, for any two Rayleigh numbers $R_{a_1}$ and $R_{a_2}$, we choose corresponding minimizers $[U_1, \Theta_1]$ and $[U_2, \Theta_2]$. Clearly, by (2.17), we have

$$|F_3(U_1, \tilde{\Theta}, R_{a_1}) - F_3(U_1, \tilde{\Theta}, R_{a_2})| \leq C |R_{a_1} - R_{a_2}|.$$

Letting $\tilde{\Theta} = \Theta_2$, we deduce that

$$\lambda(R_{a_1}) \leq F_3(U_1, \Theta_2, R_{a_1}) < \lambda(R_{a_2}) + C |R_{a_1} - R_{a_2}|.$$

Similarly, letting $\tilde{\Theta} = \Theta_1$, we have

$$\lambda(R_{a_2}) \leq F_3(U_2, \Theta_1, R_{a_2}) < \lambda(R_{a_1}) + C |R_{a_1} - R_{a_2}|.$$

Hence, the continuity of $\lambda(R_a)$ follows. Moreover, by letting $U_3 = \tilde{\Theta}$ in (2.17), we have

$$\lim_{R_a \to \infty} \lambda(R_a) = -\infty.$$
This implies that for any \( k \geq 1 \), there exists \( R_a^0(k) \leq R_a \) such that
\[
\lambda(R_a^0(k)) = 0.
\]
We now denote the corresponding minimizer of \( \lambda(R_a^0(k)) \) in (2.16) and (2.11) by \([U_3^0, \Theta^0]\), which satisfies (2.14) with \( \lambda(R_a^0(k)) = 0 \):
\[
0 = \mu_1(\partial_z^2 - k^2)^2 U_3^0 - R_a \mu_1 \mu_2 k^2 \Theta^0, \\
0 = \mu_2(\partial_z^2 - k^2) \Theta^0 + U_3^0.
\]
(2.18)
Equivalently, we have
\[
- (\partial_z^2 - k^2)^2 \Theta^0 - R_a^0(k) k^2 (\partial_z^2 - k^2) \Theta^0 = 0,
\]
which implies that \( R_a^0(k) \geq R(k) \). Therefore, by (1.8),
\[
R_a \geq R_a^0(k) \geq R^*_a,
\]
a contradiction. We therefore conclude the lemma.

3. Nonlinear stability for \( Ra < R^*_a \).

**Theorem 3.** If the Rayleigh number \( Ra < R^*_a \), then the motionless solution \((u_s, P_s, T_s)\) in (1.1) of system (1.2), (1.3) is nonlinear stable with respect to the norm of \( C(0, \infty; H^2(Q)) \cap W^1_\infty(0, \infty; L^2(Q)) \).

**Proof.** The unconditional nonlinear stability with respect to the norm of \( C(0, \infty; L^2(Q)) \) can be found in [6].

Now we only give the proof of nonlinear stability with respect to the norm of \( C(0, \infty; H^2(Q)) \cap W^1_\infty(0, \infty; L^2(Q)) \). Let
\[
A = \begin{pmatrix} (\mu_1 P \Delta) I & 0 \\ 0 & \mu_2 \Delta \end{pmatrix}, \\
B = \begin{pmatrix} 0 \\ \sqrt{K_0} \mu_1 \mu_2 e_z^t \end{pmatrix},
\]
where vector \( e_z = (0, 0, 1) \) and \( I \) is the 3 \( \times \) 3 identity matrix. If \( \tilde{\lambda} \) is an eigenvalue and \([\tilde{u}, \tilde{\theta}]\) is the corresponding eigenfunction of the following eigenvalue problem,
\[
(A + B)[\tilde{u}, \tilde{\theta}] = -\tilde{\lambda}[\tilde{u}, \tilde{\theta}],
\]
then \( \lambda = \tilde{\lambda} \) is the eigenvalue and \([u, \theta] = [\tilde{u}, \tilde{\theta} / \sqrt{R_a \mu_1 \mu_2}] \) is the corresponding eigenfunction of the eigenvalue problem (2.3). Similarly, for the eigenvalue \( \lambda \) and the corresponding eigenfunction \([u, \theta] \) of (2.3), then \( \tilde{\lambda} = \lambda \) is the eigenvalue and \([\tilde{u}, \tilde{\theta}] = [u, \sqrt{R_a \mu_1 \mu_2} \theta] \) is the corresponding eigenfunction of the eigenvalue problem (3.1). Thus \( \lambda_1 = \lambda_1 > 0 \).

It is well known [14] [15] that \( A + B \) is the infinitesimal generator of the analytic semigroup \( e^{t(A+B)} : H \times L^2(Q) \rightarrow H \times L^2(Q) \) with
\[
\|e^{t(A+B)}\| \leq C e^{-\lambda_1 t}, \quad \forall t > 0, \\
\|(-A - B)^{1/2} e^{t(A+B)}\| \leq C t^{-1/2} e^{-\lambda_1 t}, \quad \forall t > 0,
\]
(3.2)
where \( \lambda_1 > 0 \) is defined in Lemma 1. Since \((-A - B)^{-1} : H \times L^2(Q) \to H \times L^2(Q)\) and \(A(-A - B)^{-1} : H \times L^2(Q) \to H \times L^2(Q)\) are both selfadjoint linear bounded operators, we have

\[
\|(-A)^{1/2} e^{t(A+B)} [u, \theta]\| = \left(\|(-A)(-A - B)^{-1}(-A - B)^{-1/2} e^{t(A+B)} [u, \theta]\|, \|(-A - B)^{-1/2} e^{t(A+B)} [u, \theta]\|\right)^{1/2} \\
\leq \|(-A)(-A - B)^{-1}\|^{1/2} \|(A - B)^{-1/2} e^{t(A+B)} [u, \theta]\| \\
\leq C t^{-1/2} e^{-\lambda_1 t} \|[u, \theta]\|, \quad \forall t > 0, \quad [u, \theta] \in H \times L^2(Q).
\]

Equations (1.2) and (1.3) can be rewritten in the following equivalent form:

\[
[u(t), \sqrt{R_{a_1} \mu_2} \theta(t)] \\
= e^{t(A+B)} (u_0, \sqrt{R_{a_1} \mu_2} \theta_0) - \int_0^t e^{(t-s)(A+B)} \left((u \cdot \nabla) u(s), (u \cdot \nabla) (\sqrt{R_{a_1} \mu_2} \theta(s))\right) ds.
\]

Applying (3.2), (3.4) and (3.5), we have

\[
\|u(t)\|_{L^2} + \|\theta(t)\|_{L^2} \\
\leq C_1 e^{-\lambda_1 t} \left(\|u_0\|_{L^2} + \|\theta_0\|_{L^2}\right) \\
+ C_2 \int_0^t e^{-\lambda_1 (t-s)} \left(\|u \cdot \nabla\| u(s)\|_{L^2} + \|(u \cdot \nabla) \theta\|_{L^2}\right) ds \\
\leq C_1 \left(\|u_0\|_{L^2} + \|\theta_0\|_{L^2}\right) + C_2 \sup_{0 \leq s \leq T} \left(\|u(s)\|_{H^1}^2 + \|\theta(s)\|_{H^1}^2\right), \quad \forall T \geq t,
\]

\[
\|\Delta u(t)\|_{L^2} + \|\Delta \theta(t)\|_{L^2} \leq C_1 e^{-\lambda_1 t} \left(\|\Delta u_0\|_{L^2} + \|\Delta \theta_0\|_{L^2}\right) \\
+ C_2 \int_0^t (t-s)^{-1/2} e^{-\lambda_1 (t-s)} \left(\|\nabla \{u \cdot \nabla\} u(s)\|_{L^2} + \|\nabla \{u \cdot \nabla\} \theta(s)\|_{L^2}\right) ds \\
\leq C_1 \left(\|\Delta u_0\|_{L^2} + \|\Delta \theta_0\|_{L^2}\right) + C_2 \sup_{0 \leq s \leq T} \left(\|u(s)\|_{H^2}^2 + \|\theta(s)\|_{H^2}^2\right), \quad \forall T \geq t.
\]

Let

\[
E(T) = \sup_{0 \leq s \leq T} \left(\|u(s)\|_{H^2} + \|\theta(s)\|_{H^2}\right).
\]

Then we have

\[
E(T) \leq C_1 \left(\|u_0\|_{H^2} + \|\theta_0\|_{H^2}\right) + C_2 E^2(T), \quad \forall T \geq 0,
\]

where the positive constants \( C_1 \) and \( C_2 \) are independent of \( T \). Therefore, if \( \|u_0\|_{H^2} + \|\theta_0\|_{H^2} \) is small enough, then there exists a unique solution \((u, \theta) \in C\left([0, \infty); (H^2(Q))^4\right)\) of (1.2) and (1.3) such that

\[
\|u(t)\|_{H^2} + \|\theta(t)\|_{H^2} \leq 2C_1 \left(\|u_0\|_{H^2} + \|\theta_0\|_{H^2}\right), \quad \forall t \geq 0,
\]

and our theorem follows. \( \square \)
4. Instability of motionless state. We now turn to the nonlinear problem (1.2) and (1.3). We first recall a general framework for nonlinear instability as:

**Lemma 4 (Bootstrap Instability [8]).** Assume that \( L \) is a linear operator on a Banach space \( X \) with norm \( \| \cdot \| \), and \( e^{tL} \) generates a strongly continuous semigroup on \( X \) such that

\[
\| e^{tL} \| (X, X) \leq C_L e^{\lambda t} \tag{4.1}
\]

for some \( C_L \) and \( \lambda > 0 \). Assume a nonlinear operator \( N(y) \) on \( X \) and another norm \( \| \cdot \| \), and constant \( C_N \), such that

\[
\| N(y) \| \leq C_N \| y \| \tag{4.2}
\]

for all \( y \in X \) and \( \| y \| < \infty \). Assume for any solution \( y(t) \) to the equation

\[
y' = Ly + N(y) \tag{4.3}
\]

with \( \| y(t) \| \leq \sigma \) that there exists \( C_\sigma > 0 \) such that for any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that the following sharp energy estimate holds:

\[
\frac{d}{dt} \| y(t) \| \leq \epsilon \| y(t) \| + C_\sigma \| y(t) \| + C_\epsilon \| y(t) \|. \tag{4.4}
\]

Consider a family of initial data \( y^\delta(0) = \delta y_0 \) with \( \| y_0 \| = 1 \) and \( \| y_0 \| < \infty \) and let \( \beta_0 \) be a sufficiently small (fixed) number. Then there exists some constant \( C > 0 \) such that if

\[
0 \leq t \leq T^\delta \equiv \frac{1}{\lambda} \log \frac{\beta_0}{\delta},
\]

then we have

\[
\| y(t) - \delta e^{tL} y_0 \| \leq C (\| y_0 \|^2 + 1)\delta^2 e^{2\lambda t}. \tag{4.5}
\]

In particular, if there exists a constant \( C_p \) such that \( \| \delta e^{tL} y_0 \| \geq C_p \delta e^{\lambda t} \), then there exists an escape time \( T^{\text{esc}} \leq T^\delta \) such that

\[
\| y(T^{\text{esc}}) \| \geq \tau_0 > 0,
\]

where \( \tau_0 \) depends explicitly on \( C_L, C_N, C_\sigma, C_p, \lambda, y_0, \sigma \) and is independent of \( \delta \).

To apply such a method, we need to verify (4.4) for an appropriate Sobolev norm \( \| \cdot \| \). Let \( \| \cdot \| = \| \cdot \|_{L^2} \), \( D_{x,y}^k = \sum_{k_1 + k_2 = k} \delta_{x_1}^k \delta_{y_2}^k \) and

\[
E_0 = \| u(t) \|^2 + \| \theta(t) \|^2,
\]

\[
E_k = E_0 + \| \nabla u(t) \|^2 + \| \nabla \theta(t) \|^2 + \| D_{x,y}^k \nabla u(t) \|^2 + \| D_{x,y}^k \nabla \theta(t) \|^2.
\]

**Lemma 5.** Let \( k \geq 2 \). Then

\[
\| (u \cdot \nabla) u \| + \| (u \cdot \nabla) \theta \| \leq CE_k. \tag{4.7}
\]

**Proof.** Applying Sobolev imbedding theorems and Hölder’s inequality, we have

\[
u^2(x, y, z, t) = 2 \int_0^z u_z(x, y, s, t) ds \leq C \int_0^z \| u_z(\cdot, s, t) \|_{H^2_{x,y}} \| u(\cdot, s, t) \|_{H^2_{x,y}} ds,
\]

\[
\| u(t) \|_{L^\infty} \leq C(\| u(t) \| + (\| \partial_x^2 + \partial_y^2 \| u(t) \| + \| \partial_z u(t) \| + \| (\partial_x^2 + \partial_y^2) \partial_z u(t) \|).
\]
By using the multiplicative inequality [1, p. 323], we have
\[ \|(u \cdot \nabla)u\| + \|(u \cdot \nabla)\theta\| \leq \|u\|_{L^\infty} (\|\nabla u\| + \|\nabla \theta\|) \leq CE_k, \]
where we have used the following estimates:
\[ \|\nabla D_{x,y}^{k-l} u\| \leq C (\|\nabla u\|^l/k \|\nabla D_{x,y}^k u\|^{(k-l)/k}, \quad \forall 1 \leq l \leq k - 1. \]

**Lemma 6.** Let \( k \geq 3 \). Then we have
\[ \frac{d}{dt} E_k \leq \epsilon E_k + CE_k^2 + C \epsilon \epsilon_0, \quad \forall \epsilon > 0. \] (4.8)

**Proof.** Taking scalar products of \( u \) with equation (1.2) and \( \theta \) with equation (1.3), we have
\[ \frac{d}{dt} \|u\|^2 + 2 \mu_1 \|\nabla u\|^2 \leq 2 R_2 \|u_3\| \|\theta\|, \]
\[ \frac{d}{dt} \|\theta\|^2 + 2 \mu_2 \|\nabla \theta\|^2 \leq 2 \|u_3\| \|\theta\|. \]
Taking scalar products of \( u_t \) with equation (1.2) and \( \Delta \theta \) with equation (1.3), we have
\[ \frac{d}{dt} \mu_1 \|\nabla u\|^2 + 2 \|u_t\|^2 \leq 2 (u \cdot \nabla) u + \|\partial_t u_3\|^2 + C \|\theta\|^2, \]
\[ \frac{d}{dt} \|\nabla \theta\|^2 + 2 \mu_2 \|\Delta \theta\|^2 \leq C (u \cdot \nabla) \theta + \mu_2 \|\Delta \theta\|^2 + C \|u_3\|^2. \]
Taking scalar products of \( D_{x,y}^{2k} u_t \) with equation (1.2) and \( D_{x,y}^{2k} \Delta \theta \) with equation (1.3), we have
\[ \frac{d}{dt} \mu_1 \|\nabla D_{x,y}^k u\|^2 + 2 \|D_{x,y}^k u_t\|^2 \leq 2 \|D_{x,y}^k (u \cdot \nabla) u\|^2 + \|D_{x,y}^k u_t\|^2 + C \|D_{x,y}^k \theta\|^2, \]
\[ \frac{d}{dt} \|\nabla D_{x,y}^k \theta\|^2 + 2 \mu_2 \|\Delta D_{x,y}^k \theta\|^2 \leq C \|D_{x,y}^k (u \cdot \nabla) \theta\|^2 + \mu_2 \|\Delta D_{x,y}^k \theta\|^2 + C \|D_{x,y}^k u\|^2. \]

By Hölder’s inequality, the Sobolev imbedding theorems, the multiplicative inequality [1, p. 323] and (4.7), \( \|D_{x,y}^k (u \cdot \nabla) u\| \) is bounded by
\[ C \sum_{l=0}^{k-2} \|D_{x,y}^l u\|_{L^\infty} \|D_{x,y}^{k-l} \nabla u\| + 2 \|D_{x,y}^{k-1} u\|_{L^\infty(0,1;L^2_{x,y})} \|D_{x,y} \nabla u\|_{L^2(0,1;L^\infty_{x,y})} \]
\[ + 2 \|D_{x,y}^k u\|_{L^\infty(0,1;L^2_{x,y})} \|\nabla u\|_{L^2(0,1;L^\infty_{x,y})} \leq CE_k + C \|D_{x,y}^{k-1} u\|_{H^1(0,1;L^2_{x,y})} \|D_{x,y} \nabla u\|_{L^2(0,1;H^2_{x,y})} \]
\[ + C \|D_{x,y}^k u\|_{H^2(0,1;L^2_{x,y})} \|\nabla u\|_{L^2(0,1;H^2_{x,y})} \leq CE_k, \]
and \( \|D_{x,y}^k (u \cdot \nabla \theta)\| \) is bounded by
\[
C \sum_{l=0}^{k-2} \|D_{x,y}^l u\|_{L^\infty} \|\nabla D_{x,y}^{k-l} \theta\|_{L^2} + C \|D_{x,y}^{k-l} u\|_{L^\infty_0 (0,1;L^2_2)} \|\nabla \theta\|_{L^2_2 (0,1;L^\infty_0)}
+ C \|D_{x,y}^k u\|_{L^\infty_0 (0,1;L^2_2)} \|\nabla \theta\|_{L^2_2 (0,1;L^\infty_0)}
\leq CE_k + C \|D_{x,y}^{k-1} u\|_{H^2_2 (0,1;L^2_2)} \|\nabla \theta\|_{L^2_2 (0,1;H^2_2)}
+ C \|D_{x,y}^k u\|_{H^1_2 (0,1;L^2_2)} \|\nabla \theta\|_{L^2_2 (0,1;H^2_2)}
\leq CE_k.
\]
Here we have used the following estimates:
\[
\|\nabla D_{x,y}^{k-l} \theta\| \leq C \|\nabla \theta\|^{l/k} \|\nabla D_{x,y}^k \theta\|^{(k-l)/k}, \quad \forall 1 \leq l \leq k - 1.
\]
Since
\[
C \|D_{x,y}^k u\|^2 + C \|D_{x,y}^k \theta\|^2 \leq C \|D_{x,y}^{k+1} u\|^{2(k+1)} \|\theta\|^{2(k+1)}
+ C \|D_{x,y}^{k+1} \theta\|^{2(k+1)} \|\theta\|^{2(k+1)}
\leq \epsilon \left( \|D_{x,y}^{k+1} u\|^2 + \|D_{x,y}^{k+1} \theta\|^2 \right) + C \epsilon \left( \|u\|^2 + \|\theta\|^2 \right),
\]
by putting together all these estimates, we obtain (4.8).

Since \( \lambda_1 < 0 \) for \( Ra > Ra^* \), there exists some corresponding wave numbers \( k^* \geq 1 \) such that \( \lambda^* = \lambda(Ra) < 0 \), where \( \lambda(Ra) \) is defined in (2.12) with \( k = k^* \), and the maximal growth rate is given by \( -\lambda^* > 0 \). Let \( [u_k; \Theta_k] \) be the minimizer of (2.12).

**Definition 7.** We define a smooth generic profile for the initial perturbation as
\[
[\tilde{u}, \tilde{\theta}] = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{\theta})
= \sum_{k_1^2 + k_2^2 = k^2} \left( V_{1,k_1,k_2}, V_{2,k_1,k_2}, v_{k_1,k_2} U_k (z), \vartheta_{k_1,k_2} \Theta_k(z) \right) e^{ik_1 x + ik_2 y}
\]
such that \( k_1^2 + k_2^2 = (k^*)^2 \), then either \( v_{k_1,k_2} \) or \( \vartheta_{k_1,k_2} \) is nonzero. Moreover, \( V_{1,0,0} = V_{2,0,0} = 0 \),
\[
V_{1,k_1,k_2} = \frac{ik_1}{k^2} v_{k_1,k_2} \partial_z U_k (z), \quad k_1^2 + k_2^2 = k^2 \geq 1,
V_{2,k_1,k_2} = \frac{ik_2}{k^2} v_{k_1,k_2} \partial_z U_k (z), \quad k_1^2 + k_2^2 = k^2 \geq 1.
\]

Therefore, from Lemmas 4–6, we have

**Theorem 8.** Suppose that the Rayleigh number \( Ra > Ra^* \). Let \( [u^\delta; \theta^\delta] \) be a solution to the equations (17.2) and (17.9) with initial value
\[
[u^\delta(0); \theta^\delta(0)] = \delta[\tilde{u}, \tilde{\theta}],
\]
where \( [\tilde{u}, \tilde{\theta}] \) is a generic profile defined in (4.9) and \( \|\tilde{u}, \tilde{\theta}\| = 1 \). Then,
\[
\|\|u^\delta(t); \theta^\delta(t)\| - \sum_{k_1^2 + k_2^2 = (k^*)^2} \delta^2 e^{tk} \left( V_{1,k_1,k_2}, V_{2,k_1,k_2}, v_{k_1,k_2} U_k (z), \vartheta_{k_1,k_2} \Theta_k(z) \right) e^{ik_1 x + ik_2 y} \right) \leq C \left( 1 + \|\nabla \tilde{u}, \nabla \tilde{\theta}\|^2 + \|\partial_z^2 \nabla \tilde{u}, \partial_z^2 \nabla \tilde{\theta}\|^2 \right) \delta^2 e^{-2\lambda^* t}, \quad k \geq 3,
\]
(4.10)
for any $\delta \leq \delta_0$ sufficiently small, and $0 \leq t \leq T^d$, where $||[u, \theta]|| = ||u_1|| + ||u_2|| + ||u_3|| + ||\theta||$.

Remark 9. We note that $e^{tL}(V_1, k_1, k_2, V_2, k_1, k_2, v_{k_1, k_2}U_k(z), \vartheta_{k_1, k_2}\Theta_k(z))e^{i(k_1^2 + k_2^2)y}$ is given explicitly in (2.20). Clearly, for $k_1^2 + k_2^2 = (k^*)^2$ and $v_{k_1, k_2}$ not both zero,

$$
||e^{tL}(V_1, k_1, k_2, V_2, k_1, k_2, v_{k_1, k_2}U_k(z), \vartheta_{k_1, k_2}\Theta_k(z))e^{i(k_1^2 + k_2^2)y}|| \geq C e^{-\lambda t^d}
$$

for some positive constant $C$. Then, from Lemma 4 and Theorem 8, the motionless state $(u_s, P_s, T_s)$ of the system (1.2) and (1.3) is nonlinear unstable.

The proof of Theorem 8 follows from Lemmas 4–6 with $||\cdot|| = E^1_k$ and $||\cdot|| = E^3_0$. By (2.5), we split $\delta e^{\epsilon L}(\epsilon \hat{u}, \epsilon \hat{\theta})$ in Lemma 4 into maximal growing modes

$$
\delta e^{tL}(V_1, k_1, k_2, V_2, k_1, k_2, v_{k_1, k_2}U_k(z), \vartheta_{k_1, k_2}\Theta_k(z))e^{i(k_1^2 + k_2^2)y}, \quad \forall k_1^2 + k_2^2 = (k^*)^2
$$

and the remaining modes to conclude the proof.

References


