BOUNDED SOLUTIONS FOR THE BOLTZMANN EQUATION

BY

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Dedicated to Professor W. A. Strauss on the occasion of his 70th birthday

Abstract. In either a periodic box \( T^d \) or \( \mathbb{R}^d \) (1 \( \leq \) \( d \) \( \leq \) 3), we establish a unified \( L^\infty \) estimate for solutions near Maxwellians for the Boltzmann equation, in terms of natural mass, momentum, energy conservation and the entropy inequality.

We study \( L^\infty \) estimates for the Boltzmann equation

\[
\partial_t F + v \cdot \nabla_x F = \frac{1}{\kappa} Q(F, F), \quad F(0, x, v) = F_0(x, v),
\]

where \( F(t, x, v) \geq 0 \) is the density of particles of velocity \( v \in \mathbb{R}^3 \), and position \( x \in \Omega = \mathbb{R}^d \) or \( T^d \), a periodic box, for 1 \( \leq d \) \( \leq \) 3. The Knudsen number \( \kappa \) is a bounded constant.

For simplicity we assume a hard-sphere interaction for the collision kernel \( Q \). We define a global Maxwellian given by

\[
\mu = \frac{\rho}{(2\pi T)^{3/2}} \exp \left\{ -\frac{|v - u|^2}{2T} \right\},
\]

where \( \rho, u, T \) are independent of \( t \) and \( x \). Our main result is

\textbf{Theorem 1.} Assume that the excess conservations of mass, momentum and energy,

\[
\begin{align*}
\int \int \{ F(t, x, v) - \mu \} dvdx &= \int \int \{ F_0(x, v) - \mu \} dvdx \equiv M_0, \\
\int \int v \{ F(t, x, v) - \mu \} dvdx &= \int \int v F_0(x, v) dvdx \equiv J_0, \\
\int \int |v|^2 \{ F(t, x, v) - \mu \} dvdx &= \int \int |v|^2 \{ F_0(x, v) - \mu \} dvdx \equiv E_0,
\end{align*}
\]

as well as the excess entropy inequality, hold:

\[
\mathcal{H}(F(t)) - \mathcal{H}(\mu) \leq \mathcal{H}(F_0) - \mathcal{H}(\mu),
\]

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where $\mathcal{H}(g) \equiv \int \int g \ln gdvdx$. Then for any $\beta \geq 0$, there exists $C > 0$ such that
\begin{equation}
\sup_{0 \leq t \leq \infty} \left\| \frac{1 + |v|^2}{\sqrt{\mu}} \frac{\beta}{\sqrt{\mu}} (F(t) - \mu) \right\|_{\infty} \leq C \left\| \frac{1 + |v|^2}{\sqrt{\mu}} (\mu - \mu) \right\|_{\infty} \tag{5}
\end{equation}
provided the right-hand side is sufficiently small.

Both the excess conservation laws (3) and the excess entropy inequality (4) are clearly valid when $\Omega$ is a periodic box. We remark that in the case $\Omega = \mathbb{R}^d$, local-in-time solutions satisfying both (3) and (4) can be constructed via the following approximate Boltzmann equation with finite propagation of speed in the physical space:
\[ \partial_t F_n + v \mathbf{1}_{\{|v| \leq n\}} \cdot \nabla_x F_n = \frac{1}{\kappa} Q(F_n, F_n) \]
as $n \to \infty$.

It is well known that the pointwise control of $F$ is crucial for uniqueness when $\beta$ is large. The new $L^\infty$ estimate (5) is solely based on the most natural a priori estimates in the Boltzmann theory. Even though no time decay rate is obtained, the proof is direct and robust.

**Proof.** Denote the weight function by $w(v) = \{1 + |v|^2\}^\beta$ and define
\[ h = w(v) \times \frac{F - \mu}{\sqrt{\mu}}. \]
Recall the standard linearized Boltzmann operator as
\[ L_g = -\frac{1}{\sqrt{\mu}} \{Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu)\} = \{\nu(v) + K\}g, \]
where the collision frequency $\nu(v) = c \int |v - u| \mu dv \sim |v| + 1$. Letting $K_wg \equiv wK(\frac{g}{w})$, we obtain
\[ \partial_t h + v \cdot \nabla_x h + \frac{\nu}{\kappa} h = \frac{1}{\kappa} wG\left(\frac{h}{w}, \frac{\bar{v}}{w}\right), \]
where $G(g_1, g_2) = \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu}g_1, \sqrt{\mu}g_2)$.

For any $(t, x, v)$, we denote $\bar{v} = (v_1, ..., v_d)$. Integrating along its backward trajectory $\frac{dX(s)}{ds} = V(s), \frac{dV(s)}{ds} = 0$, we express $h(t, x, v)$ as
\begin{align*}
\exp\left\{-\frac{\nu t}{\kappa}\right\} h(0, x - \bar{v}t, v) + \int_0^t \exp\left\{-\frac{\nu (t-s)}{\kappa}\right\} \left(\frac{1}{\kappa} K_w h\right)(s, x - \bar{v}(t-s), v) ds \\
+ \int_0^t \exp\left\{-\frac{\nu (t-s)}{\kappa}\right\} \frac{w}{\kappa} \Gamma\left(\frac{h}{w}, \frac{\bar{v}}{w}\right)(s, x - \bar{v}(t-s), v) ds. \tag{6}
\end{align*}
Since $\left|\frac{w}{\kappa} \Gamma\left(\frac{h}{w}, \frac{\bar{v}}{w}\right)\right| \leq C\nu(v)||h||^2_{\infty}$ from Lemma 10 of [2], and since
\[ \int_0^t \exp\left\{-\frac{\nu (t-s)}{\kappa}\right\} \nu ds \leq O(\kappa), \]
the last term in (6) is bounded by \((\nu(v)\) is bounded from below)

\[
\frac{C}{\kappa} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \{\nu(v)|h(s, x-\bar{v}(t-s), v)| + \|h(s)\|_\infty\} \|h(s)\|_\infty ds \leq C \sup_{0 \leq s \leq t} \|h(s)\|_\infty^2.
\]

(7)

We shall mainly concentrate on the second term in (6). Let \(k(v, v')\) be the corresponding kernel associated with \(K\). We now use (6) again to evaluate \(\{k_w h\}(s, x-(t-s)\bar{v}) = \int k_w(v, v')h(s, x-\bar{v}(t-s), v')dv'.\) By (7), we can bound the above by

\[
\frac{1}{\kappa} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \int \mathbb{R}^3 |k_w(v', v')\exp\left\{-\frac{\nu_s}{\kappa}\right\} h(0, x-\bar{v}(t-s) - \bar{v}' s, v') dv'ds
\]

\[
+ \frac{1}{\kappa^2} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \int \mathbb{R}^3 \times \mathbb{R}^3 |k_w(v', v')k_w(v', v'')| \times \int_0^s \exp\left\{-\frac{\nu' s}{\kappa}\right\} |h(s_1, x-\bar{v}(t-s) - \bar{v}'(s-s_1), v'')| dv'dv'' ds_1 ds
\]

\[
+ \frac{C}{\kappa} \int_0^t \exp\left\{-\frac{\nu(t-s)}{\kappa}\right\} \times \sup_v \int \mathbb{R}^3 |k_w(v, v')| dv' \times \{\sup_{0 \leq s \leq t} \|h(s)\|_\infty^2\},
\]

(8)

where \(k_w(\cdot) = w k(\cdot)\) and \(\bar{v}' = (v'_1, ..., v'_d).\) Since \(\sup_v \int \mathbb{R}^3 k_w(|v, v'|) dv' < +\infty\) from Lemma 7 of [2], the first and the third terms above are bounded by \(C\|h(0)\|_\infty + C\{\sup_{0 \leq s \leq t} \|h(s)\|_\infty\}^2\).

We now concentrate on the second term in (8), which will be estimated as in the proof of Theorem 21 in [2].

**Case 1.** For \(|v| \geq N.\) By Lemma 7 in [2],

\[
\int \int |k_w(v, v')k_w(v', v'')| dv' dv'' \leq \frac{C}{1 + |v|} \leq \frac{C}{N}.
\]

We therefore can find an upper bound for the second term in this case as

\[
\frac{C}{\kappa^2 N} \int_0^t \exp\left\{-\frac{\nu(v')(t-s)}{\kappa}\right\} \times \int_0^s \exp\left\{-\frac{\nu(v')(s-s_1)}{\kappa}\right\} \|h(s_1)\|_\infty ds_1 ds
\]

\[
\leq \frac{C}{N} \sup_{0 \leq s \leq t} \|h(s)\|_\infty.
\]

**Case 2.** For \(|v| \leq N, |v'| \geq 2N,\) or \(|v'| \leq 2N, |v''| \geq 2N.\) Notice that we have either \(|v' - v| \geq N\) or \(|v'' - v'| \geq N,\) and either one of the following are valid correspondingly for some \(\eta > 0:\)

\[
|k_w(v, v')| \leq e^{-\frac{\eta}{N^2}} |k_w(v, v')e^{\frac{\eta}{N^2}|v''|^2}|, \quad |k_w(v', v'')| \leq e^{-\frac{\eta}{N^2}} |k_w(v', v'')e^{\frac{\eta}{N^2}|v''|^2}|.
\]

(9)
From Lemma 8 in [2], both \( \int |k_w(v, v')e^{\frac{N}{2}v - v'|^2} | \) and \( \int |k_w(v', v'')e^{\frac{N}{2}v' - v''|^2} | \) are still finite. We use (9) to combine the cases of \(|v' - v| \geq N\) or \(|v' - v''| \geq N\) as:

\[
\int_0^t \int_{v_1}^{v_2} \ldots \left\{ \int_{|v| \leq N, |v'| \geq 2N} + \int_{|v'| \leq 2N, |v''| \geq 3N} \right\}
\]

\[
\leq C \int_0^t \int_{v_1}^{v_2} \ldots \left\{ \int_{|v| \leq N, |v'| \geq 2N} |k_w(v, v')|dv' + \sup_{v'} \int_{|v'| \leq 2N, |v''| \geq 3N} |k_w(v', v'')|dv'' \right\}
\]

\[
\leq C_0 e^{-\frac{2N^2}{k^2}} \int_0^t \int_{v_1}^{v_2} \exp\left\{-\frac{\nu(t - s)}{\kappa}\right\} \exp\left\{-\frac{v(s - s_1)}{\kappa}\right\} ||h(s1)||_\infty ds_1 ds
\]

\[
\leq C_0 e^{-\frac{2N^2}{k^2}} \sup_{0 \leq s \leq t} \{|h(s)|\infty \}. \quad (10)
\]

**Case 3.** \( s - s_1 \leq \varepsilon\kappa \), for \( \varepsilon > 0 \) small. We now can simply bound the second term in (8) by

\[
\frac{1}{\kappa^2} \int_0^t \int_{s - \kappa}^{s} C \exp\left\{-\frac{\nu(t - s)}{\kappa}\right\} \exp\left\{-\frac{v(s - s_1)}{\kappa}\right\} ||h(s1)||_\infty ds_1 ds
\]

\[
\leq C \sup_{0 \leq s \leq t} \{|h(s)|\infty \} \times \frac{1}{\kappa} \int_0^t \exp\left\{-\frac{\nu(t - s)}{\kappa}\right\} ds \times \int_{s - \kappa}^{s} \frac{1}{\kappa} ds_1
\]

\[
\leq \varepsilon C \sup_{0 \leq s \leq t} \{|h(s)|\infty \}. \quad (11)
\]

**Case 4.** \( s - s_1 \geq \varepsilon\kappa \), and \(|v| \leq N, |v'| \leq 2N, |v''| \leq 3N\). This is the last remaining case because if \(|v'| > 2N\), it is included in Case 2; while if \(|v''| > 3N\), either \(|v'| \leq 2N\) or \(|v'| \geq 2N\) are also included in Case 2. We now can bound the second term in (8) by

\[
C \int_0^t \int_{B} \int_{s - \kappa}^{s - \kappa} e^{-\frac{\nu(t - s)}{\kappa}} e^{-\frac{\nu(s - s_1)}{\kappa}} |k_w(v, v')k_w(v', v'')h(s, x_1 - (s - s_1)v', v'')|,
\]

where \( B = \{|v'| \leq 2N, |v''| \leq 3N\} \) and \( x_1 = x - (t - s)v \). Notice that \( k_w(v, v') \) has a possible integrable singularity of \( \frac{1}{v - v''} \). We can choose \( k_N(v, v') \) smooth with compact support such that

\[
\sup_{|p| \leq 3N} \int_{v' \mid v' \leq 3N} |k_N(p, v') - k_w(p, v')| dv' \leq \frac{1}{N}, \quad (12)
\]

Splitting

\[
k_w(v, v')k_w(v', v'') = \{k_w(v, v') - k_N(v, v')\}k_w(v', v'') + \{k_w(v', v'') - k_N(v', v'')\}k_N(v, v') + k_N(v, v')k_N(v', v''),
\]

we can use such an approximation (12) to bound the above \( s_1, s \) integration by

\[
\frac{C}{N} \sup_{0 \leq s \leq t} \{|h(s)|\infty \} \times \left\{ \sup_{|v'| \leq 2N} \int |k_w(v', v'')|dv'' + \sup_{|v| \leq 2N} \int |k_w(v, v')|dv' \right\}
\]

\[
+ C \int_0^t \int_{B} \int_{s - \kappa}^{s - \kappa} e^{-\frac{\nu(t - s)}{\kappa}} e^{-\frac{\nu(s - s_1)}{\kappa}} |k_N(v, v')k_N(v', v'')h(s, x_1 - (s - s_1)v', v'')|, \quad (13)
\]
We now make use of the conservation laws \(^{[3]}\) and the entropy inequality \(^{[4]}\) to estimate the last term. Recall from the Taylor expansion,
\[
\mathcal{H}(F(t)) - \mathcal{H}(\mu) = \int \int \{\ln \mu + 1\} \{F - \mu\} + \int \int \frac{(F(t) - \mu)^2}{2F} \leq \mathcal{H}(F_0) - \mathcal{H}(\mu),
\]
where \(\tilde{F}\) is between \(F(t)\) and \(\mu\). Since \(\mu = \frac{\rho}{2\pi T^{3/2}}\exp \left\{-\frac{|v-u|^2}{2T}\right\}\), \(\ln \mu = \ln \left(\frac{\rho}{2\pi T^{3/2}}\right) - \frac{|v-u|^2}{2T}\). Therefore, from the conservations of mass, momentum and energy \(^{[3]}\), we get
\[
\int \int \frac{(F(t) - \mu)^2}{2F} \leq \mathcal{H}(F_0) - \mathcal{H}(\mu) + C_{\rho,u,T}\{|M_0| + |J_0| + |E_0|\}.
\]
The key is to estimate \(\frac{(F(t) - \mu)^2}{2F}\) in the case of \(|F(t) - \mu| \geq \delta\mu\) for a small parameter \(\delta\). Notice that either \(F(t) \leq 1 - \delta\mu\) or \(F(t) - \mu \geq \delta\mu\) in this case. If \(F(t) \leq 1 - \delta\mu\),
\[
\frac{|F(t) - \mu|}{F(t)} \geq |F(t) - \mu| \geq 1 - F(t) \geq 1 - (1 - \delta) = \delta.
\]
On the other hand, if \(F(t) \geq 1 + \delta\mu\),
\[
\frac{|F(t) - \mu|}{F(t)} \geq |F(t) - \mu| \geq 1 - \frac{\mu}{F(t)} \geq 1 - \frac{1}{1 + \delta} = \frac{\delta}{1 + \delta}.
\]
In summary, we have
\[
\mathcal{H}(F_0) - \mathcal{H}(\mu) + C\{|M_0| + |J_0| + |E_0|\} \tag{14}
\]
\[
\int \int \frac{(F(t) - \mu)^2}{2F} \mathbf{1}_{|F(t) - \mu| \leq \delta\mu} + \int \int \frac{(F(t) - \mu)^2}{2F} \mathbf{1}_{|F(t) - \mu| \geq \delta\mu}
\]
\[
\geq \int \int \frac{(F(t) - \mu)^2}{2(1 + \delta)\mu} \mathbf{1}_{|F(t) - \mu| \leq \delta\mu} + \frac{1}{2} \frac{\delta}{1 + \delta} \int |F(t) - \mu| \mathbf{1}_{|F(t) - \mu| \geq \delta\mu}.
\]
Since \(k_N(v', v'')\) is bounded, we first integrate over \(v'\) (bounded) to get
\[
C_N \int_{|v'| \leq 2N} |h(s_1, x_1 - (s - s_1)v', v'')| \mathbf{1}_{|F(s_1, x_1 - (s - s_1)v', v'') - \mu| \leq \delta\mu} dv'
\]
\[
+ C_N \int_{|v'| \leq 2N} |h(s_1, x_1 - (s - s_1)v', v'')| \mathbf{1}_{|F(s_1, x_1 - (s - s_1)v', v'') - \mu| \geq \delta\mu} dv'
\]
\[
\leq C_N \delta + C_N \left\{ \int_{|v'| \leq 2N} \mathbf{1}_{|F(s_1, x_1 - (s - s_1)v', v'')| \leq \delta\mu} dv' \right\}
\]
\[
\leq C_N \delta + C_N \left\{ \frac{(s - s_1)^d + 1}{\kappa^d \varepsilon^d} \int \int_{|y - x_1| \leq (s - s_1)\delta\mu} |h(s_1, y, v'')| \mathbf{1}_{|F(s_1, y, v'') - \mu| \geq \delta\mu} dy \right\}
\]
Here we have made a change of variable \(y = x_1 - (s - s_1)v''\), and for \(s_1 \geq \varepsilon\kappa\), \(\frac{dy}{dv''} \geq \frac{1}{\varepsilon\kappa^{d-1}}\). In the case of \(\Omega = \mathbb{R}^d\), the factor \(\{(s_1 - s)^d + 1\}\) is not needed. By further
integrating over \( v'' \) (bounded), we then control the last term in (13) by (14):

\[
\frac{C_{N,\varepsilon}}{\kappa^2} \int_0^t \int_0^s \frac{e^{-\frac{1}{\kappa^2} |v'(s-x)|}}{e^{-\frac{1}{\kappa^2} |v''(s-x)|}} \\
x(\delta + \left\{ \frac{(s-s_1)^d + 1}{\kappa d} \right\} \int_{|v''| \leq 3\kappa} \int_{\Omega} \|F - \mu\| ds ds \\
\leq \frac{C_{N,\varepsilon}}{\kappa^2} \int_0^t \int_0^s \frac{e^{-\frac{1}{\kappa^2} |v'(s-x)|}}{e^{-\frac{1}{\kappa^2} |v''(s-x)|}} \left\{ (s-s_1)^d + 1 \right\} ds ds \\
\times \left[ \delta + \frac{1}{\kappa^d \delta} (|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |\epsilon_0|) \right] \\
\leq C_{N,\varepsilon} \left[ \delta + \frac{1}{\kappa^d \delta} (|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |\epsilon_0|) \right] \\
\leq \frac{C_{N,\varepsilon}}{\kappa^{d/2}} \sqrt{|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |\epsilon_0|}.
\]

We have optimized \( \delta \) such that (for sufficiently small \( |\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |\epsilon_0|), \)

\[
\delta = \frac{1}{\kappa^d \delta} \left\{ |\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |\epsilon_0| \right\}.
\]

In summary, we have established, for any \( \varepsilon > 0 \) and large \( N > 0, \)

\[
\sup_{0 \leq s \leq t} \|h(s)\|_\infty \leq \{ \varepsilon + \frac{C_{N,\varepsilon}}{N} \} \sup_{0 \leq s \leq t} \|h(s)\|_\infty + C_0 \|\|h(s)\|_\infty\|_\infty^2 \\
+ \frac{C_{N,\varepsilon}}{\kappa^{d/2}} \sqrt{|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |\epsilon_0|}.
\]

First choosing \( \varepsilon \) small, then \( N \) sufficiently large so that \( \{ \varepsilon + \frac{C_{N,\varepsilon}}{N} \} < \frac{1}{2} \),

\[
\sup_{0 \leq s \leq t} \|h(s)\|_\infty \leq C \left\{ \|h(0)\|_\infty + \frac{1}{\kappa^{d/2}} \sqrt{|\mathcal{H}(F_0) - \mathcal{H}(\mu)| + |M_0| + |\epsilon_0|} \right\},
\]

and we conclude our proof provided the right-hand side is sufficiently small. \( \square \)

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