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L^q-APPROACH OF WEAK SOLUTIONS TO STATIONARY ROTATING OSEEN EQUATIONS IN EXTERIOR DOMAINS

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Abstract. We establish the existence and uniqueness of a weak solution of the three-dimensional nonhomogeneous stationary Oseen flow around a rotating body in an exterior domain D. We mainly use the localization procedure (see Kozono and Sohr (1991)) to combine our previous results (see Kračmar, Nečasová, and Penel (2007, 2008)) with classical results in an appropriate bounded domain. We study the case of a nonintegrable right-hand side, where f is given in $(\widehat{W}^{-1,q}(D))^3$ for certain values of q.

1. Introduction. The study of Navier–Stokes fluid flows past a rigid body translating with a constant velocity (or past a rotating obstacle with a prescribed constant velocity) is one of the most fundamental questions in theoretical and applied Fluid Dynamics. A systematic and rigorous mathematical study was initiated by the fundamental pioneering works of Oseen (1927), Leray (1933, 1934) and then developed by several other mathematicians with significant contributions.

In the last decade much effort has been made on the analysis of solutions to different problems: stationary as well as nonstationary, linear models as well as nonlinear ones, in the whole space as well as in exterior domains. We refer to [6, 7, 8, 9, 10, 11, 13, 14, 15, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30].

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In the present paper we mainly investigate the existence and uniqueness of weak solutions to the linear stationary rotating Oseen system in exterior domains in the case of a nonintegrable right-hand side.

Let D be an exterior domain in $\mathbb{R}^3$ with a boundary ∂D regular enough, say of class C^2. We consider the motion of a viscous fluid filling the domain D when the "obstacle" $\Omega = \mathbb{R}^3 \setminus D$, which consists of a finite number of rigid bodies, is rotating about an axis with constant angular velocity ω and moving in the direction of this axis. We assume the fluid with a nonzero velocity $v_\infty = ke_3$ at infinity, and that $\omega = |\omega| e_3 = (0, 0, |\omega|)^T$:

Our aim is to solve the time-periodic Oseen system of equations for the velocity field $v = v(y, t)$ and the associated pressure $q = q(y, t)$:

$$\begin{array}{rlll} \partial_t v - \nu\Delta v + (v\cdot\nabla)v + \nabla q & = \tilde{f} & \text{in} & D(t),\ t>0, \\ \operatorname{div} v & = \tilde{g} & \text{in} & D(t),\ t>0, \\ v(y,t) & = \omega\wedge y & \text{on} & \partial D(t),\ t>0, \\ v(y,t) & \to v_\infty & \text{as} & |y|\to\infty\,. \end{array} \tag{1.1}$$

The time-dependent exterior domain is

$$D(t) = \{y \in \mathbb{R}^3 : y = O(|\omega|t)x, x \in D\},$$

where

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|\,t & -\sin|\omega|\,t & 0 \\ \sin|\omega|\,t & \cos|\omega|\,t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.2}$$

The coefficient of viscosity is $\nu > 0$ and we will assume that nonnecessary integrable external forces $\tilde{f} = \tilde{f}(y, t)$ are given.

Introducing the change of variables

$$x = O_\omega(t)^T y \tag{1.3}$$

and the new functions

$$u(x,t) = O_\omega^T(t)(v(y,t) - v_\infty), \quad p(x,t) = q(y,t), \tag{1.4}$$

as well as the force term $f_1(x, t) = O_\omega(t)^T \tilde{f}(y, t)$, we arrive at the linear system of equations in $D \times (0, \infty)$:

$$\begin{array}{rcl} \partial_t u - \nu\Delta u + k\partial_3 u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u + \nabla p & = & f_1 \\ \operatorname{div} u & = & g. \end{array} \tag{1.5}$$

In the case $\omega = 0$, the system of equations (1.5) is a nonhomogeneous Stokes system when $k = 0$ and it is a classical nonhomogeneous Oseen system when $k > 0$.

We are interested in the stationary flow in D (and, therefore, the time-periodic solution to (1.5) as well as the periodic solution to the initial model (1.1)), and for simplicity we will consider $\nu = 1$, $k = 1$. So, for given f and g, our system of equations is the following:

$$\left.\begin{array}{rcl} -\Delta u + \partial_3 u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u + \nabla p & = & f \\ \operatorname{div} u & = & g \end{array}\right\} \text{ in D.} \tag{1.6}$$

We assume that the system of equations (1.6) is now complemented by a homogeneous condition at infinity

$$u \to 0 \text{ as } |x| \to \infty \tag{1.7}$$

and Dirichlet boundary conditions on $\partial\Omega$, either

$$u|_{\partial\Omega} = 0 \tag{1.8}$$

or

$$u|_{\partial\Omega} = \omega \wedge x - e_3, \ x \in \partial\Omega. \tag{1.9}$$

If $D = \mathbb{R}^3$, of course $\{u, p\}$ is described by equations (1.6) and condition (1.7) only. Through the relation div $(((\omega\wedge x)\cdot\nabla)u - \omega\wedge u) = ((\omega\wedge x)\cdot\nabla)$ div $u =$ div $((\omega\wedge x)$ div $u)$, we define p. The strong solution of the corresponding Cauchy problem (1.6), (1.7) has been analyzed in L^q-spaces, $1 < q < \infty$, in [8] proving the *a priori* estimates

$$\|\nabla^2 u\|_q + \|\nabla p\|_q \le c(\|f\|_q + \|\nabla g + (\omega \wedge x) \cdot g - g e_3\|_q), \tag{1.10}$$

$$\|\partial_3 u\|_q + \| - ((\omega \wedge x) \cdot \nabla)u + \omega \wedge u\|_q \le c(1 + |\omega|^{-2})\|f\|_q \tag{1.11}$$

with the constant $c > 0$ independent of $|\omega|$, the second estimate being written with $g = 0$ just to simplify. Further, these results were improved in [6] in weighted spaces, obtaining the following *a priori* estimates (always written with $g = 0$ to simplify):

$$\|\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \le c\|f\|_{q,w}, \tag{1.12}$$

$$\|\partial_3 u\|_{q,w} + \| - ((\omega \wedge x) \cdot \nabla)u + \omega \wedge u\|_{q,w} \le c(1 + |\omega|^{-5/2})\|f\|_{q,w} \tag{1.13}$$

with the constant $c > 0$ independent of $|\omega|$, and where the weights w belong to the more general Muckenhoupt class $\tilde{A}_q^-$. A weak solution to the same Cauchy problem (1.6), (1.7) in the L^q setting, $1 < q < \infty$, was investigated in [21] and the following *a priori* estimates were proved (always written with $g = 0$):

$$\|\nabla u\|_q + \|p\|_q + \| - ((\omega \wedge x) \cdot \nabla)u + \omega \wedge u\|_{-1,q} \ \le \ C\|f\|_{-1,q}, \tag{1.14}$$

where data belong to the dual of nonhomogeneous Sobolev spaces (see at the end of this section).

In the work of Galdi [14], pointwise estimates for Navier-Stokes equations with rotating terms were proved. He obtained that

$$|x|^{-1}|u(x)| \le c, \qquad |x|^{-2}(\|\nabla u(x)\| + \|p(x)\|) \le c.$$

Another outlook on the above pointwise estimates in a differential framework by use of functional spaces has been recently proved by Farwig and Hishida [11]. Further, Galdi and Silvestre [16] have proved a stability of solution u. A generalization in the $L_{3,\infty}$ setting was done by Hishida and Shibata [27].

We will study the boundary value problem (1.6), (1.7), (1.8). By applying the so-called localization technique [19], we immediately observe that it combines both systems in the whole space and in a bounded domain. Indeed, choose $\rho > \rho_0 > 0$ so large that $\Omega \subset B_{\rho_0} = \{x \in \mathbb{R}^3 : |x| < \rho_0\}$ and take a cut-off function $\psi \in C_0^\infty(B_\rho; [0, 1])$ such that

$\psi = 1$ on B_{ρ_0} and $\mathrm{supp}(\nabla\psi) \subset \{x : \rho_0 < |x| < \rho\}$; introducing now $U = (1-\psi)u, V = \psi u, \pi = (1-\psi)p$ and $\tau = \psi p$, we get

$$u = U + V \tag{1.15}$$

$$p = \pi + \tau \tag{1.16}$$

within the whole space

$$\left.\begin{aligned} -\Delta U + \partial_3 U - ((\omega \wedge x)\cdot\nabla)U + \omega \wedge U + \nabla\pi &= F_1(u,p) \\ \mathrm{div} U &= G_1(u) \\ U \to 0 \text{ as } |x| \to \infty \end{aligned}\right\}, \tag{1.17}$$

where $G_1(u) = (1-\psi)g - \nabla\psi \cdot u$, and in the bounded domain $D_\rho = D \cap B_\rho$,

$$\left.\begin{aligned} -\Delta V + \partial_3 V + \nabla\tau &= F_2(u,p) \\ \mathrm{div} V &= G_2(u) \\ V|_{\partial D_\rho} &= 0 \end{aligned}\right\}, \tag{1.18}$$

where $G_2(u) = \psi g + \nabla\psi \cdot u$ and

$$\begin{aligned} F_1(u,p) =& (1-\psi)f + 2(\nabla\psi\cdot\nabla)u + [\Delta\psi + ((\omega\wedge x)\nabla)\psi]u \\ & - (\nabla\psi)p + (\partial_3\psi)u, \\ F_2(u,p) =& \psi f + \psi[((\omega\wedge x)\cdot\nabla)u - \omega\wedge u] - 2(\nabla\psi\cdot\nabla)u \\ & - (\Delta\psi)u + (\nabla\psi)p + (\partial_3\psi)u. \end{aligned} \tag{1.19}$$

Let us observe that, in the bounded domain D_ρ, we can equivalently write the following nonhomogeneous Stokes problem:

$$\left.\begin{aligned} -\Delta V + \nabla\tau &= F_2 \\ \mathrm{div} V &= G_2(u) \\ V|_{\partial D_\rho} &= 0 \end{aligned}\right\}, \tag{1.20}$$

modifying $F_2 = \psi f + \psi[((\omega\wedge x)\cdot\nabla)u - \omega\wedge u] - 2(\nabla\psi\cdot\nabla)u - (\Delta\psi)u + (\nabla\psi)p + \partial_3(\psi u)$.

Let us also observe that, in order to prove the existence of solution in an exterior domain even if $g = 0$ in (1.6), we need to study the nonhomogeneous case (1.17) in the whole space.

In Section 2, we will give the definition of a weak solution to problem (1.6), (1.7), (1.8) and our main result, existence and uniqueness of its solution. Sections 4 and 5 are devoted to the proof of the main result. We need intermediate results for both problems (1.17) and (1.18); Section 3 is devoted to them. Two appendices complete the paper, the first one with the general results by Bogovskii, Farwig and Sohr [1, 2, 3], Kozono and Sohr [19] and generalization in negative Sobolev spaces by Geissert, Heck, and Hieber [18], and a second appendix with the technical treatment of $\partial_3 u$ in negative Sobolev spaces.

Let us fix the notations: $C_0^\infty(\mathbb{R}^3)$ consists of functions of the class C^∞ with compact supports contained in $\mathbb{R}^3$. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue spaces with norm

$\|\cdot\|_q$. We define the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_q} = \{v \in L^q_{\mathrm{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\}/\mathbb{R}, \tag{1.21}$$

$$\begin{aligned}\widehat{W}^{1,q}(D) &= \overline{C_0^\infty(D)}^{\|\nabla\cdot\|_{q,D}}\\ &= \{v \in L^q_{\mathrm{loc}}(\overline{D});\ \nabla v \in L^q(D)^3, v|_{\partial D} = 0\} \ \text{ for } 3 \le q < \infty, \qquad (1.22)\\ &= \{v \in L^{3q/(3-q)}(\overline{D});\ \nabla v \in L^q(D)^3, v|_{\partial D} = 0\} \text{ for } 1 \le q < 3, \qquad (1.23)\end{aligned}$$

and their dual spaces

$$\widehat{W}^{-1,q}(\mathbb{R}^3) = (\widehat{W}^{1,q/(q-1)}(\mathbb{R}^3))',$$
$$\widehat{W}^{-1,q}(D) = (\widehat{W}^{1,q/(q-1)}(D))'.$$

$\langle .,. \rangle$ denotes either different duality pairings or the inner product in L^2.

2. The main result. We consider the problem (1.6), (1.7), (1.8) with $g = 0$ in the exterior domain D. Let f be given in $(\widehat{W}^{-1,q}(D))^3$, $1 < q < \infty$.

Definition 2.1. We call $\{u, p\}$ a weak solution to $(1.6)_{g=0}$, (1.7), (1.8) if

(1) $\{u, p\} \in (\widehat{W}^{1,q}(D))^3 \times L^q(D)$,
(2) $\mathrm{div} u = 0 \quad \text{in} \quad L^q(D)$,
(3) $\partial_3 u - ((\omega \wedge x)\cdot\nabla)u + \omega \wedge u \quad \text{in} \quad \widehat{W}^{-1,q}(D)^3$,
(4) $\langle \nabla u, \nabla\varphi\rangle + \langle \partial_3 u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u, \varphi\rangle = \langle p, \mathrm{div}\varphi\rangle + \langle f, \varphi\rangle$ for all φ in $C_0^\infty(D)^3$.

Our main result is

Theorem 2.1. Let $3/2 < q < 3$, and suppose f is as given in $(\widehat{W}^{-1,q}(D))^3$. Then there exists a unique weak solution $\{u, p\}$ to $(1.6)_{g=0}$, (1.7), (1.8), which satisfies the estimate

$$\begin{aligned}\|\nabla u\|_{q,D} + \|p\|_{q,D} + \|\partial_3 u\|_{-1,q,D} + \|-((\omega\wedge x)\cdot\nabla)u + \omega\wedge u\|_{-1,q,D}\\ \le c_q\|f\|_{-1,q,D},\end{aligned} \tag{2.1}$$

with some constant $c_q > 0$ independent of $|\omega|$.

Remark 2.2. • Similar results were obtained by Hishida for the Stokes problem [24].

- As a corollary of Theorem 2.1, we also obtain existence and uniqueness of the solution to the nonhomogeneous Dirichlet problem $(1.6)_{g=0}$, (1.7), (1.9); see Theorem 5.1.
- It is possible to avoid some of the restrictions $3/2 < q < 3$ (see [30]). In this way, we can consider the null space of the problem

$$\begin{aligned}K = \Big\{&u \in \widehat{W}^{1,q}(D) |\, \mathrm{div}\ u = 0, u|_{\partial\Omega} = 0,\\ &(u, p) \text{ is a solution of (1.6) for some } p \in L^q(D)\Big\}.\end{aligned}$$

  Then the solution u will be unique in $W^{1,q}(D)/K$ for $3/2 < q < \infty$.
- We can improve the result from Theorem 2.1 admitting nonsolenoidal solutions; this will be the partial subject of a forthcoming paper. In the present paper, we have decided to simply use the approach based on the Lax-Milgram theorem (see Section 4, Step 1).

3. Intermediate results. We can start with the analysis of the homogeneous problem in the whole space, then precisely the problem (1.17) for given F_1 and $G_1 = 0$. So we use the notations $\{U_0, \pi_0\}$, and the preliminary results from [21, 23] are:

DEFINITION 3.1. Let $1 < q < \infty$. Given $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we call $\{U_0, \pi_0\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ a weak solution to (1.17) with $G_1 = 0$ if

$$\begin{aligned}
&(1) \qquad \mathrm{div} U_0 = 0 \quad \text{in} \quad L^q(\mathbb{R}^3),\\
&(2) \qquad \partial_3 U_0 - ((\omega \wedge x)\cdot\nabla)U_0 + \omega \wedge U_0 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3,\\
&(3) \qquad \langle \nabla U_0, \nabla\varphi\rangle + \langle \partial_3 U_0 - ((\omega\wedge x)\cdot\nabla)U_0 + \omega\wedge U_0, \varphi\rangle\\
&\qquad\qquad\qquad -\langle \pi_0, \mathrm{div}\varphi\rangle = \langle F_1, \varphi\rangle \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbb{R}^3)^3.
\end{aligned}$$

THEOREM 3.1 ([21]). Let $1 < q < \infty$ and let $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ be given. Then the problem (1.17) with $G_1 = 0$ possesses a weak solution $\{U_0, \pi_0\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ which satisfies

$$\|\nabla U_0\|_q + \|\pi_0\|_q + \|\partial_3 U_0 - ((\omega\wedge x)\cdot\nabla)U_0 + \omega\wedge U_0\|_{-1,q} \;\le\; c\|F_1\|_{-1,q}, \tag{3.1}$$

with some $c > 0$ depending on q.

REMARK. Without a special treatment of the third component of the gradient of U_0, we only have:

$$\begin{aligned}
&\|\partial_3 U_0 - ((\omega\wedge x)\cdot\nabla)U_0 + \omega\wedge U_0\|_{-1,q,\mathbb{R}^3}\\
&= \|F_1 + \Delta U_0\|_{-1,q,\mathbb{R}^3} \le c(\|F_1\|_{-1,q,\mathbb{R}^3} + \|\nabla U_0\|_{q,\mathbb{R}^3}) \le c\|F_1\|_{-1,q,\mathbb{R}^3},
\end{aligned}$$

with some $c > 0$ depending on q.

PROPOSITION 3.2. Let $1 < q < \infty$ and let $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ be given. Then the problem (1.17) with $G_1 = 0$ possesses a weak solution $\{U_0, \pi_0\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ which satisfies 1.14 and, moreover,

$$\|\partial_3 U_0\|_{-1,q} \le c\|F_1\|_{-1,q}$$

with some $c > 0$ depending on q.

COROLLARY 3.3. Let $1 < q < \infty$ and let $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ be given. Then the problem (1.17) with $G_1 = 0$ possesses a weak solution $\{U_0, \pi_0\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ which satisfies:

$$\|\nabla U_0\|_q + \|\pi_0\|_q + \|\partial_3 U_0\|_{-1,q} + \| - ((\omega\wedge x)\cdot\nabla)U_0 + \omega\wedge U_0\|_{-1,q} \;\le\; c\|F_1\|_{-1,q}.$$

Proof of Proposition 3.2. We need to estimate the quantity $\sup_{\|\nabla\phi\|\le 1}\langle \partial_3 U_0,\, \phi\rangle$. Farwig [9] estimated $\partial_3 U_0$ in L^q-norms in the case $\|F_1\|_q$ bounded and then, in a similar study, we also used the multiplier theory starting from the integral representation of $\partial_3 U_0$ induced in [21].

We know that $F_1 = \mathrm{Div}\, H$ with $H_{\cdot k} = \mathcal{B}F_{1_{,k}} \in L^q(\mathbb{R}^3)^3$ (see Appendix 1, and using the fact that the space $\{F_1 \,|\, F_1 = \mathrm{Div}\, H, H \in C_0^\infty(\mathbb{R}^3)^{3\times 3}\}$ is dense in $\widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we can assume first that $H \in C_0^\infty(\mathbb{R}^3)^{3\times 3}$). U_0 can be expressed in dependence of H as

$$U_0(x) = \int_{\mathbb{R}^3} \Gamma\,(x,y)\cdot\ \mathrm{Div}\ H\,(y)\ dy,$$

where $\Gamma(x,y) = \int_0^\infty E_t\,(O_\omega(t)x - y - te_3)\; O_\omega^T(t)\,dt$ and $E_t = \frac{1}{(4\pi t)^{3/2}} e^{-|\xi|^2/4t}$. Applying the Fourier transformation, we get the following form:

$$\widehat{U_{0,k}}(\xi) = \frac{1}{D(\xi)} \int_0^{2\pi/|\omega|} e^{-(|\xi|^2+i\xi_3)t} O_\omega^T(t)\,\xi_j \widehat{H}_{jk}(O_\omega(t)\xi)\,dt,$$

denoting

$$D(\xi) = 1 - e^{-2\pi(|\xi|^2+i\xi_3)/|\omega|}.$$

So we have

$$\widehat{\partial_3 U_{0,k}}(\xi) = \frac{i\xi_3\xi_j}{D(\xi)|\omega|} \int_0^{2\pi} e^{-(|\xi|^2+i\xi_3)t/|\omega|} O_\omega^T(t) \widehat{H_{jk}}(O_\omega(t)\xi)\,dt,$$

and also

$$\frac{\widehat{\partial_3 U_{0,k}}(\xi)}{|\xi|} = \int_0^{2\pi} m_t(\xi)\cdot O_\omega^T(t)\,\widehat{H_{\cdot k}}(O_\omega(t).-te_3/|\omega|)(\xi)\,dt,$$

where the multiplier $m_t(\cdot)$ has the form

$$m_t(\xi) = \frac{i\xi_3\xi}{|\omega||\xi|D(\xi)} e^{-|\xi|^2 t/|\omega|}.$$

In fact, we decide to rewrite on purpose

$$\langle \partial_3 U_{0,k}, \phi_k\rangle = i\int_{\mathbb{R}^3} \frac{\xi_3 \widehat{U_{0,k}}(\xi)}{|\xi|}\cdot |\xi|\widehat{\phi_k}(\xi)d\xi, \tag{3.2}$$

which reads

$$\int_{\mathbb{R}^3} \widehat{V}(\xi)\cdot\widehat{\Psi}(\xi)d\xi$$

in duality L^q–$L^{q'}$. Indeed, Ψ can be taken in $L^{q'}$ and the term $\widehat{V}(\xi) = \frac{\xi_3\widehat{U_{0,k}}(\xi)}{|\xi|}$ can be estimated in L^q. To this end, observing that $\|H(O_\omega(t)\cdot - te_3)\|_q = \|H\|_q$ for all $t\in(0,2\pi)$, and knowing that $e^{-it\xi_3}\widehat{H}_{jk}(\cdot)$ is the Fourier transform of $H_{jk}(\cdot - te_3)$, one can show (see Appendix 2) that $\widehat{V}(\xi)$ satisfies the Marcinkiewicz multiplier theorem which implies the boundedness of V in L^q by $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$. Therefore we obtain

$$\|\partial_3 U_0\|_{-1,q} \le \|H\|_q \le \|F_1\|_{-1,q}. \qquad \square$$

Remark 3.3. • Let $1<q<3$. Then $U_0 \in L^{3q/(3-q)}(\mathbb{R}^3)$ and following Theorems 1.1 and 1.2 from [21], the solution $\{U_0,\pi_0\}$ is unique.

- In the same conditions, following Corollary 1.2 from [21] (see also [12], pp. 59–62), we have $\||x|^{-1}U_0\|_q \le c\|F_1\|_{-1,q}$ with a positive constant $c = c(q,\omega)$.

Concerning now the nonhomogeneous problem in the whole space, i.e. problem (1.17) for given F_1 and G_1, we recall the following result.

Theorem 3.4 ([23]). Let $1<q<\infty$ and let $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ be given. Suppose $G_1 \in L^q(\mathbb{R}^3)$ such that $(\omega\wedge x)\cdot G_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)$. Then problem (1.17) possesses a weak solution $\{U,\pi\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ which satisfies

$$\begin{aligned}\|\nabla U\|_q + \|\pi\|_q + \|\partial_3 U - ((\omega\wedge x)\cdot\nabla)U + \omega\wedge U\|_{-1,q} \\ \le\; c(\|F_1\|_{-1,q} + \|G_1\|_q + \|(\omega\wedge x)\cdot G_1\|_{-1,q}),\end{aligned} \tag{3.3}$$

with some $c>0$ depending on q.

REMARK 3.5. Due to Proposition 3.2 we are able to improve the previous estimate; now we have:

$$\begin{aligned}\|\nabla U\|_q + \|\pi\|_q + \|\partial_3 U\|_{-1,q} + \| - ((\omega\wedge x)\cdot\nabla)U + \omega\wedge U\|_{-1,q} \\ \leq\ c(\|F_1\|_{-1,q} + \|G_1\|_q + \|(\omega\wedge x)\cdot G_1\|_{-1,q}).\end{aligned} \tag{3.4}$$

We finally recall the well-known result, e.g. from [19], about the nonhomogeneous Stokes problem in bounded domains, solving problem (1.20) in the domain D_ρ. So, with the previously used notations $\{V,\tau\}$, the theorem reads

THEOREM 3.5. Let $1 < q < \infty$. Suppose that

$$F_2 \in W^{-1,q}(D_\rho)^3,\ G_2 \in L^q_0(D_\rho),\ \text{(so, we assume } \int_{D_\rho} G_2(x)dx = 0).$$

Then problem (1.20) possesses a unique (up to an additive constant for τ) weak solution $\{V,\tau\} \in W^{1,q}_0(D_\rho)^3 \times L^q(D_\rho)$, which satisfies the estimate

$$\|\nabla V\|_{q,D_\rho} + \|\tau - \bar\tau\|_{q,D_\rho} \leq C(\|F_2\|_{-1,q,D_\rho} + \|G_2\|_{q,D_\rho}), \tag{3.5}$$

where $\bar\tau = \frac{1}{|D_\rho|}\int_{D_\rho}\tau(x)dx$.

4. Proof of the main theorem. The proof is presented in four steps.

STEP 1 (Existence (homogeneous divergence case)). Let $f = \operatorname{Div} F$ with $F \in C^\infty_0(D)^9$. In the domain D_R according to the support of F, we apply the classical approach to solve

$$-\Delta u + \partial_3 u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u + \nabla p = f = \operatorname{Div} F$$

with $\operatorname{div} u = 0$ and with homogeneous Dirichlet boundary conditions on ∂D_R.

The bilinear form $b(u,\varphi) = \langle\nabla u,\nabla\varphi\rangle + \langle\partial_3 u,\varphi\rangle - \langle((\omega\wedge x)\cdot\nabla)u - \omega\wedge u,\varphi\rangle$ is coercive on $(\widehat{W}^{1,2}_\sigma(D_R))^3 \times (\widehat{W}^{1,2}_\sigma(D_R))^3$; $\langle .,.\rangle$ here stands for the L^2-inner product. One can easily verify that $b(u_R,u_R) = \|\nabla u_R\|^2_{2,\ D_R} = \langle\operatorname{Div} F, u_R\rangle$.

Using the Lax-Milgram theorem we justify the existence of a unique solution $u_R \in (\widehat{W}^{1,2}_\sigma(D_R))^3$, which satisfies the estimate

$$\|\nabla u_R\|_{2,D_R} \leq \|F\|_{2,D_R} = \|F\|_{2,D}.$$

We can extend u_R by zero in $D\setminus D_R$. Then we obtain $\widetilde{u_R} \in (\widehat{W}^{1,2}_\sigma(D))^3$ satisfying the same estimate, uniform as $R \to +\infty$.

We now choose a sequence of numbers $\{R_n\}_n$, tending to infinity, so that $\widetilde{u_{R_n}}$ converge weakly in $(\widehat{W}^{1,2}_\sigma(D))^3$. The limit u is unique and such that

$$\langle\nabla u,\nabla\varphi\rangle + \langle\partial_3 u,\varphi\rangle - \langle((\omega\wedge x)\cdot\nabla)u - \omega\wedge u,\varphi\rangle - \langle f,\varphi\rangle = 0 \tag{4.1}$$

for all $\varphi \in C^\infty_0(D)^3$, then for all $\varphi \in (\widehat{W}^{1,2}(D))^3$.

Therefore there exists $p \in L^2_{loc}(D)$ (unique up to an additive constant) such that

$$-\Delta u + \partial_3 u - ((\omega\wedge x)\cdot\nabla)u + \omega\wedge u - f = -\nabla p.$$

Applying in the next step the localization technique will imply that u and p satisfy

$$u \in \widehat{W}^{1,q}(D)^3,\ p \in L^q(D) \text{ for } 3/2 < q \leq 6.$$

STEP 2 (localization procedure). Let $\phi \in (C_0^\infty(\mathbb{R}^3))^3$, and let ψ be always the same cut-off function as in Section 1. We can successively choose the following test functions φ:

- $\varphi = (1-\psi)\phi$, so we can read over $\mathbb{R}^3$ all integrals from equation (4.4) and interpret the solved problem in the whole space by $\{U, \pi\} = \{(1-\psi)u, (1-\psi)p\}$ as in (1.17); the formulas describing $F_1 = F_1(u,p)$ and $G_1 = G_1(u)$ are given in Section 1.
- $\varphi = \psi\phi$, so we can interpret in D_ρ the solved problem by $\{V, \tau\} = \{\psi u, \psi p\}$ as in (1.18) or (1.20); see in Section 1 the detailed formulas for $F_2 = F_2(u,p)$ and $G_2 = G_2(u)$.

Theorem 3.6 and Theorem 3.7 respectively solve these problems under the following hypothesis:

$$F_1 \in (\widehat{W}^{-1,q}(\mathbb{R}^3))^3, G_1 \in L^q(\mathbb{R}^3), (\omega \wedge x) \cdot G_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3),$$
$$F_2 \in (\widehat{W}^{-1,q}(D_\rho))^3, G_2 \in L_0^q(D_\rho),$$

with estimates (3.3) resp. (3.5).

To exploit these estimates, it remains essentially to control all terms we have from formulas (1.19) in $\|F_j(u,p)\|_{-1,q,\mathbb{R}^3 or D_\rho}$, for appropriate q and $j = 1, 2$. In this way, we recall that $\nabla\psi$ and $\Delta\psi$ have compact support in D_ρ at the most (precisely in the annulus $\{x : \rho_0 < |x| < \rho\}$ closed to the "obstacle" Ω), so we have

- either $\|\phi\|_{q/(q-1),D_\rho} \le c(|D_\rho|)\|\nabla\phi\|_{q/(q-1),D_\rho}$

  by Friedrichs-Poincaré inequality,

  or $\|\phi\|_{q/(q-1),\ D_\rho} \le |D_\rho|^{1/3}\|\phi\|_{r,D_\rho}$

  by Hölder inequality, with $\frac{1}{r} + \frac{1}{3} = \frac{q-1}{q}$, so it is necessary that $\frac{q-1}{q} > \frac{1}{3}$ and $q > \frac{3}{2}$.

  Thus $\|\phi\|_{q/(q-1),D_\rho} \le |D_\rho|^{1/3}\|\phi\|_{r,\mathbb{R}^3} \le c(|D_\rho|)\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $|\langle(1-\psi)f, \phi\rangle| \le c\|f\|_{-1,q,D}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $|\langle 2(\nabla\psi\cdot\nabla)u + [\Delta\psi + ((\omega\wedge x)\cdot\nabla)\psi]u, \phi\rangle|$
  $\le |\langle(\Delta\psi)\phi, u\rangle| + |\langle 2(\nabla\psi\cdot\nabla)\phi, u\rangle| + |\langle(((\omega\wedge x)\cdot\nabla)\psi)\phi, u\rangle|$
  $\le c\|u|_{D_\rho}\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $|\langle(\nabla\psi)p, \phi\rangle| \le c\|p|_{D_\rho}\|_{-1,q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $|\langle(\partial_3\psi)u, \phi\rangle| \le c\|u|_{D_\rho}\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $\|(\omega\wedge x)(u\cdot\nabla\psi)\|_{-1,q,\mathbb{R}^3} \le C\|u|_{D_\rho}\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$.

Now, since $G_1 = -\nabla\psi u$ and $G_2 = \nabla\psi u$, we also have

$$|\langle \nabla\psi u, \phi\rangle| \le C\|u|_{D_\rho}\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}.$$

Then applying the estimates (3.3), (3.4) together with previous estimates, we get

$$\|\nabla U\|_{q,\mathbb{R}^3} + \|\pi\|_{q,\mathbb{R}^3} \le c(\|f\|_{-1,q} + \|u|_{D_\rho}\|_{q,D_\rho} + \|p|_{D_\rho}\|_{-1,q,D_\rho}), \tag{4.2}$$

$$\begin{aligned}\|\partial_3 U\|_{-1,q,\mathbb{R}^3} &+ \|((\omega\wedge x)\cdot\nabla)U - \omega\wedge U\|_{-1,q,\mathbb{R}^3} \\ &\le c(\|f\|_{-1,q} + \|u|_{D_\rho}\|_{q,D_\rho} + \|p|_{D_\rho}\|_{-1,q,D_\rho}),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\|\nabla V\|_{q,D_\rho} + \|\tau\|_{q,D_\rho} &\le c\big(\|f\|_{-1,q} + \|u|_{D_\rho}\|_{q,D_\rho} + \|p|_{D_\rho}\|_{-1,q,D_\rho} \\ &\quad + \Big|\int_{D_\rho}\psi(x)p(x)dx\Big|\big),\end{aligned} \tag{4.4}$$

and we can conclude with the estimate for $\|\partial_3 u\|_{-1,q,D}$, observing that

$$\begin{aligned}\|\partial_3 u\|_{-1,q,D} = \|\partial_3 (U+V)\|_{-1,q,D} &\le \|\partial_3 U\|_{-1,q,\mathbb{R}^3} + \|\partial_3 V\|_{-1,q,D_\rho} \\ &\le c\left(\|\partial_3 U\|_{-1,q,\mathbb{R}^3} + \|\partial_3 V\|_{q,D_\rho}\right).\end{aligned} \tag{4.5}$$

We know that $u|_{D_\rho} \in W_0^{1,2}(D_\rho)$. Then, by means of the embedding $W_0^{1,2}(D_\rho) \subset L^q(D_\rho)$ where $\frac{1}{6} \le \frac{1}{q} \le \frac{2}{3}$, we obtain $\|u|_{D_\rho}\|_{q,D_\rho} \le c$. Therefore, all inequalities make sense and $\|F_j(u,p)\|_{-1,q,\mathbb{R}^3 or D_\rho}$ is bounded in terms of $\|f\|_{-1,q}$, $\|u|_{D_\rho}\|_{q,D_\rho}$ and $\|p|_{D_\rho}\|_{-1,q,D_\rho}$.
Finally ($u = U + V$, $p = \pi + \tau$),

$$\begin{aligned}\|\nabla u\|_{q,D} + \|p\|_{q,D} \le\ & c\Big(\|f\|_{-1,q} + \|u|_{D_\rho}\|_{q,D_\rho} \\ &+ \|p|_{D_\rho}\|_{-1,q,D_\rho} + \Big|\int_{D_\rho}\psi(x)p(x)dx\Big|\Big),\end{aligned} \tag{4.6}$$

$$\begin{aligned}\|\partial_3 u\|_{-1,q,D} &+ \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,D} \\ &\le c\Big(\|f\|_{-1,q} + \|\nabla u|_{D_\rho}\|_q + \|p|_{D_\rho}\|_q\Big).\end{aligned} \tag{4.7}$$

From the estimates (4.6), (4.7) we immediately get

$$\begin{aligned}&\|\nabla u\|_{q,D} + \|p\|_{q,D} + \|\partial_3 u\|_{-1,q,D} + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,D} \\ &\le c\Big(\|f\|_{-1,q} + \|u|_{D_\rho}\|_{q,D_\rho} + \|p|_{D_\rho}\|_{-1,q,D_\rho} + \Big|\textstyle\int_{D_\rho}\psi(x)p(x)dx\Big|\Big),\end{aligned} \tag{4.8}$$

where the norms of u and p are computed only on the bounded set D_ρ, and then from step 1 all norms on the right-hand side are finite. Therefore, for $f = \operatorname{Div} F$, $F \in (C_0^\infty(D))^{3\times 3}$ we get that the problem $(1.6)_{g=0}$, (1.7), (1.8) has a solution $\{u,p\}$, $u \in (\widehat{W}^{1,q}(D))^3$, $\partial_3 u \in (\widehat{W}^{-1,q}(D))^3$, $(\omega\wedge x)\cdot\nabla u - \omega\wedge u \in \left(W^{-1,q}(D)\right)^3$, $p \in L^q(D)$, for all $3/2 < q \le 6$. In this case

$$\langle \nabla u, \nabla\varphi\rangle + \langle \partial_3 u, \varphi\rangle - \langle ((\omega\wedge x)\cdot\nabla)u - \omega\wedge u, \varphi\rangle = \langle \operatorname{Div} F, \varphi\rangle, \tag{4.9}$$

where the duality pairings all have meaning for $\varphi \in (\widehat{W}_\sigma^{1,q'}(D))^3$.

If $3/2 < q < 3$, then also $u \in (L^{3q/(3-q)}(D))^3$. We also obtain the same results for the adjoint problem.

STEP 3 (Uniqueness). We shall state the following uniqueness result: Let $\{u_0, p_0\}$ be a weak solution to problem $(1.6)_{f=0,\,g=0}$, (1.7), (1.8) in the sense of Definition 2.1 such that $u \in (\widehat{W}^{1,q}(D))^3$, $\partial_3 u \in (\widehat{W}^{-1,q}(D))^3$, $(\omega \wedge x) \cdot \nabla u - \omega \wedge u \in \left(W^{-1,q}(D)\right)^3$, $p \in L^q(D)$ for some $3/2 < q < 3$. Then $u_0 = 0$ and $p_0 = 0$.

Indeed, for $\{u_0, p_0\}$ we have

$$\left.\begin{aligned} -\Delta u_0 + \partial_3 u_0 - ((\omega \wedge x) \cdot \nabla) u_0 + \omega \wedge u_0 = -\nabla p_0 \\ \operatorname{div} u_0 = 0 \end{aligned}\right\} \text{ in } D. \tag{4.10}$$

The adjoint model admits a weak solution, say $\{u^*, p^*\}$, so given any $F \in (C_0^\infty(D))^{3\times 3}$ we have

$$\left.\begin{aligned} -\Delta u^* - \partial_3 u^* + ((\omega \wedge x) \cdot \nabla) u^* - \omega \wedge u^* = \operatorname{Div} F - \nabla p^* \\ \operatorname{div} u^* = 0 \end{aligned}\right\} \text{ in } D \tag{4.11}$$

with $u^* \in (\widehat{W}^{1,r}_\sigma(D))^3$, $p^* \in L^r(D)$, $3/2 < r \le 6$.

Taking u^* as a test function in (4.10) (here we use that $q/(q-1) > 3/2$, i.e., $q < 3$) and similarly u_0 in the dual problem (4.11) we get, in accordance with (4.9),

$$\begin{aligned} &\langle \nabla u_0, \nabla u^* \rangle + \langle \partial_3 u_0, u^* \rangle - \langle (\omega \wedge x) \cdot \nabla u_0 - \omega \wedge u_0, u^* \rangle = 0 \\ &\langle \nabla u^*, \nabla u_0 \rangle - \langle \partial_3 u^*, u_0 \rangle + \langle (\omega \wedge x) \cdot \nabla u^* - \omega \wedge u^*, u_0 \rangle = \langle \operatorname{Div} F, u_0 \rangle. \end{aligned}$$

From both equalities it follows that

$$\langle \operatorname{Div} F, u_0 \rangle = 0 \quad \text{for all } F \in (C_0^\infty(D))^{3\times 3}.$$

By Lemma 6.2 it implies that $\langle f, u_0 \rangle = 0$ for all $f \in \widehat{W}^{-1,q/(q-1)}(D)$, which gives $u_0 = 0$ in $(\widehat{W}^{1,q}(D))^3$ and $p_0 = 0$ in $L^q(D)$. This completes the proof of uniqueness.

STEP 4 (Proof of estimate (2.1) for $\{u, p\}$). Estimate (2.1) reads

$$\begin{aligned} &\|\nabla u\|_{q,D} + \|p\|_{q,D} + \|\partial_3 u\|_{-1,q,D} + \| - ((\omega \wedge x) \cdot \nabla) u + \omega \wedge u\|_{-1,q,D} \\ &\quad \le c_q(\|\operatorname{Div} F\|_{-1,q,D}). \end{aligned}$$

Let $3/2 < q < 3$. Suppose on the contrary the existence of a sequence $\{\operatorname{Div} F_k\}$ in $\widehat{W}^{-1,q}(D)^3$ tending to $f_\infty = 0$ as k tends to infinity such that, for the corresponding sequence of solutions $\{u_k, p_k\}$ in $\widehat{W}^{1,q}_0(D)^3 \times L^q(D)$,

$$\begin{aligned} &\|\nabla u_k\|_{q,D} + \|p_k\|_{q,D} + \|\partial_3 u_k\|_{-1,q,D} \\ &\qquad + \| - ((\omega \wedge x) \cdot \nabla) u_k + \omega \wedge u_k\|_{-1,q,D} = 1, \end{aligned} \tag{4.12}$$

for all k. We know from Step 2 that

$$\begin{aligned} \|\nabla u_k\|_{q,D} + \|p_k\|_{q,D} \le c\Big(&\|\operatorname{Div} F_k\|_{-1,q} + \|u_k|_{D_\rho}\|_{q,D_\rho} \\ &+ \|p_k|_{D_\rho}\|_{-1,q,D_\rho} + \Big|\int_{D_\rho} \psi(x)\, p_k(x) dx\Big|\Big), \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\|\partial_3 u_k\|_{-1,q,D} + \| - ((\omega \wedge x) \cdot \nabla) u_k + \omega \wedge u_k\|_{-1,q,D} \\ &\quad \le \|\operatorname{Div} F_k\|_{-1,q} + \|\nabla u_k|_{D_\rho}\|_{q,D_\rho} + \|p_k|_{D_\rho}\|_{q,D_\rho}, \end{aligned} \tag{4.14}$$

and also from (4.8) that

$$\begin{aligned}
&\|\nabla u_k - \nabla u_l\|_{q,D} + \|p_k - p_l\|_{q,D} + \|\partial_3 u_k - \partial_3 u_l\|_{-1,q,D} \\
&\qquad + \|(\omega \wedge x)\cdot\nabla(u_k - u_l) - \omega \wedge (u_k - u_l)\|_{-1,q,D} \\
&\le c\Big(\|\mathrm{Div}\,(F_k - F_l)\|_{-1,q} + \|u_k - u_l|_{D_\rho}\|_{q,D_\rho} + \|p_k - p_l|_{D_\rho}\|_{-1,q,D_\rho} \\
&\qquad + \Big|\int_{D_\rho} \psi(x)(p_k(x) - p_l(x))dx\Big|\Big).
\end{aligned} \tag{4.15}$$

On the other hand, we have (with $q < 3$)

$$\begin{aligned}
&\|u_k|_{D_\rho}\|_{1,q,D_\rho} \le \|\nabla u_k|_{D_\rho}\|_{q,D_\rho} + c\|u_k|_{D_\rho}\|_{3q/(3-q),D_\rho} \le c\|\nabla u_k\|_{q,D} \\
&\|p_k|_{D_\rho}\|_{q,D_\rho} \le 1;
\end{aligned}$$

thus we can extract subsequences $\{u_k'\}$ and $\{p_k'\}$ weakly convergent in $W^{1,q}(D_\rho)^3$ and $L^q(D_\rho)$, strongly convergent in $L^q(D_\rho)^3$ and $W^{-1,q}(D_\rho)$ (by Rellich's theorem), with $\{u_\infty, p_\infty\}$ the limit.

From (4.15) we can deduce that $\{u_k'\}$ and $\{p_k'\}$ are Cauchy sequences in $W^{1,q}(D_\rho)^3$ and $L^q(D_\rho)$. Then $\{u_\infty, p_\infty\} \in \widehat{W}^{1,q}(D)^3 \times L^q(D)$, and we obtain

$$\langle \nabla u_\infty, \nabla\varphi\rangle + \langle \partial_3 u_\infty, \varphi\rangle - \langle ((\omega\wedge x)\cdot\nabla)u_\infty - \omega\wedge u_\infty, \varphi\rangle = \langle f_\infty, \varphi\rangle = 0 \tag{4.16}$$

as in (4.9) for $\varphi \in \widehat{W}^{1,q'}_\sigma(D)^3$ with p_∞ as the associated pressure. From Step 3, it is clear that $u_\infty = 0$ and $p_\infty = 0$; $\{u_\infty, p_\infty\}$ is the unique weak solution to problem $(1.6)_{f=0,\,g=0}$, (1.7), (1.8) with $f_\infty = 0$. From the obtained strong convergence of $\{u_k'\}$ and $\{p_k'\}$, it is also clear that (4.12) holds for $\{u_\infty, p_\infty\}$, leading to a contradiction.
We have completed the proof of Theorem 2.1.

5. Nonhomogeneous boundary conditions. If we replace the homogeneous Dirichlet boundary conditions by nonhomogeneous ones in the form of (1.9), we also have the following theorem:

THEOREM 5.1. Let $3/2 < q < 3$, and suppose f and g as previously given. Then there exists a unique weak solution $\{u, p\}$ to $(1.6)_{g=0}$, (1.7), (1.9) (uniqueness up to a constant multiple of ω for u), which satisfies the estimate

$$\left.\begin{aligned}
&\|\nabla u\|_{q,D} + \|p\|_{q,D} + \|\partial_3 u\|_{-1,q,D} + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,D} \\
&\le c_q(\|f\|_{-1,q,D} + |\omega| + |\omega|^2 + 1),
\end{aligned}\right\} \tag{5.1}$$

with some constant $c_q > 0$ independent of $|\omega|$.

Proof. The result is a corollary of Theorem 2.1. Choose a cut-off function $\xi \in C_0^\infty(R^3; [0,1])$ satisfying $\xi = 1$ near the boundary $\partial\Omega$ and set

$$b(x) = \tfrac{1}{2}\mathrm{curl}\,(\xi(x)|x|^2\omega - \tfrac{1}{2}e_3 \wedge \nabla|x|^2)$$

$$b|_{\partial\Omega}(x) = \omega\wedge x - e_3.$$

Let $v = u - b$, $\mathrm{div} v = 0$ since $\mathrm{div} u = 0$ and $\mathrm{div} b = 0$. So we obtain

$$\begin{array}{rcll} -\Delta v + \partial_3 v - ((\omega \wedge x)\cdot\nabla)v + \omega \wedge v + \nabla p & = & f + f_b & \text{in } D, \\ \mathrm{div} v & = & 0 & \text{in } D, \\ v & = & 0 & \text{on } \partial\Omega, \\ v & \to & 0 & \text{at } \infty, \end{array} \tag{5.2}$$

where

$$f_b = -\Delta b + \partial_3 b - ((\omega \wedge x)\nabla)b + (\omega \wedge b).$$

Applying Theorem 2.1 we get the existence of the unique weak solution (v, p) with the following estimate

$$\begin{aligned} \|\nabla v\|_{q,D} + \|p\|_{q,D} + \| - ((\omega \wedge x)\nabla)v + \omega \wedge v\|_{-1,q,D} & \\ \leq c(\|f\|_{-1,q,D} + \|f_b\|_{-1,q,D}) & \\ \leq c(\|f\|_{-1,q,D} + |\omega| + |\omega|^2 + 1). & \end{aligned} \tag{5.3}$$

□

Appendix 1 - Bogovskii operator. Let us formulate the geometrical assumptions and the properties we will use to take into account a nonzero divergence vectorial field. We refer, e.g., to [1, 2, 12, 19] for the details.

Geometrical assumptions:

Let $1 < q < +\infty$. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a domain with boundary $\partial\Omega \in C^{1,1}$ and suppose one of the following two cases:

(i) Ω is bounded.

(ii) Ω is an exterior domain, i.e., a domain having a compact nonempty complement. In the bounded situation, Bogovskii [1, 2] has constructed a bounded linear operator $\mathcal{B} : L^q_0(\Omega) \to W^{1,q}_0(\Omega)^N$ such that $u = \mathcal{B}g$ is a solution to

$$\begin{array}{rcll} \mathrm{div} u & = & g & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \tag{5.4}$$

satisfying $\|\mathcal{B}g\|_{W^{1,q}(\Omega)} \leq c\|g\|_q$. The problem (5.4) is not uniquely solved, given $g \in L^q(\Omega)$, $\int_\Omega g(x)dx = 0$ is always assumed.

The generalization of the Bogovskii operator in the case of star-shaped domains has been solved by Galdi; see [12]. Additionally, $\mathcal{B}$ maps $W^{1,q}_0(\Omega) \cap L^q_0(\Omega)$ into $W^{2,q}_0(\Omega)$; see [1]. There are many situations in Fluid Dynamics that use the solution of Bogovskii's operator in Sobolev spaces with negative order (precisely, $\mathcal{B}$ is a bounded linear operator from $W^{r,q}_0(\Omega)$ in $W^{(r+1),q}_0(\Omega)^N$, $r + 2 > \frac{1}{q}$). To solve this type of problem we define Sobolev spaces in the following way:

$$W^{s,p}_0 = \overline{C_c(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}},$$

and for $s < 0$ we define

$$W^{s,p}(\Omega) := (W^{-s,p'}_0(\Omega))', W^{s,p}_0(\Omega) := (W^{-s,p'}(\Omega))',$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

For more details, see [18]. Also we would like to mention that comments concerning Sobolev spaces of negative order appear in the work of Galdi [12] and Farwig, Sohr [4].

Farwig and Sohr [3] have observed $\mathcal{B}$ as a bounded linear operator in a domain satisfying one of the assumptions (i), (ii).

- from $\widehat{W}^{1,q}(\Omega)\cap L^q_0(\Omega)$ in $W^{1,q}_0(\Omega)^N\cap W^{2,q}(\Omega)^N$, if Ω is bounded,
- from $\widehat{W}^{-1,q}(\Omega)$ in $L^q(\Omega)^N$, also if Ω is bounded,
- from $W^{1,q}(\Omega)\cap\widehat{W}^{-1,q}(\Omega)$ in $W^{1,q}_0(\Omega)^N\cap W^{2,q}(\Omega)^N$, if Ω is unbounded,

and $u=\mathcal{B}g$ always solves $\mathrm{div}\mathcal{B}g=g$ with $\mathcal{B}g|_{\partial\Omega}=0$ under the condition $\int_\Omega g(x)dx=0$, and satisfies the estimates

$$\begin{aligned}\|u\|_q &\le c\|g\|_{-1,q},\\ \|u\|_{2,q} &\le c(\|\nabla g\|_q+\|g\|_{-1,q}),\end{aligned}$$

where $c=c(\Omega,q)>0$ is a constant.

To complete our discussion regarding the denseness property, we have the following lemma:

Lemma 5.2 (Kozono-Sohr [19], Lemma 2.2, Corollary 2.3). Let $\Omega\subset\mathbb{R}^n$, $n\ge 2$, be any domain and let $1<q<\infty$. For all $f\in\widehat{W}^{-1,q}(\Omega)$, there is $F\in L^q(\Omega)^n$ such that

$$\mathrm{div}F=f,\quad \|F\|_{q,\Omega}\le C\|f\|_{-1,q,\Omega}$$

with some $C>0$. As a result, the space $\{\mathrm{div}\,F; F\in C^\infty_0(\Omega)^n\}$ is dense in $\widehat{W}^{-1,q}(\Omega)$.

Appendix 2 - Boundedness of $\partial_3 U_0$**.** We recall that $\widehat{V}(\xi)$ is proportional to the multiplier $m_t(\xi)=\frac{i\xi_3\xi}{|\omega||\xi|D(\xi)}e^{-|\xi|^2t/|\omega|}$.
Let us estimate $m_t(\xi)$ and $\xi_\alpha\partial_\alpha m_t(\xi)$, $\alpha=1,2,3$.

- In the case $|\xi|^2t/|\omega|\le 1$, we can neglect the exponential term and we have to estimate $m_0(\xi)$. So we decompose the ball $\frac{|\xi|^2}{|\omega|}\le 1$ into many (but a finite number of) slices $S_n=\{\xi\in\mathbb{R}^3:\frac{|\xi|^2}{|\omega|}\le 1,\ \left|\frac{\xi_3}{|\omega|}-n\right|\le\frac14\}$, with appropriate values $n_*\in\mathbb{Z}$, and the remaining part S' (inside the ball, where dist $(\frac{\xi_3}{|\omega|},n_*)\ge\frac14$). It is easy to verify that $|D(\xi)|\ge 1$ for $\xi\in S'$. Then $|m_0(\xi)|\le\frac{|\xi|}{|\omega|}\le\frac{1}{\sqrt{|\omega|}}$ for $\xi\in S'$.

  Now for $\xi\in S_{n_*}$, using a Taylor expansion of $1-e^{-z}$ we get the lower bound $|D(\xi)|\ge c_0|(\frac{|\xi|^2}{|\omega|}+i(\frac{\xi_3}{|\omega|}-n_*)|$. One can deduce that $|m_0(\xi)|\le c_1$. If $n_*\ne 0$, we precisely have

$$|m_0(\xi)|\le\frac{|\xi_3||\xi_j|}{|\omega||\xi||D(\xi)|}\le\frac{|\xi_3||\xi_j|}{|\omega||\xi||(\frac{|\xi|^2}{|\omega|}+i(\frac{\xi_3}{|\omega|}-n_*))|}\le\frac{1}{|\xi|}\le\frac{1}{|\omega|(n_*-1/4)}.$$

- In the case $|\xi|^2t/|\omega|>1$, $|D(\xi)|\ge c_2$. Then $|m_t(\xi)|\le\frac{|\xi_3|}{c_2|\xi|^2}\le\frac{1}{c_2\sqrt{|\omega|}}$.

We now compute

$$\xi_\alpha\partial_\alpha m_t(\xi)=m_t(\xi)-\frac{2\xi_\alpha^2|\xi|^2t}{|\omega|}m_t(\xi)-\xi_\alpha\frac{\partial_\alpha D(\xi)}{D(\xi)}m_t(\xi)-\frac{\xi_\alpha^2}{|\xi|^2}m_t(\xi),$$

with ($\delta_{3\alpha}$ denotes Kronecker's symbol)

$$\frac{\partial_\alpha D(\xi)}{D(\xi)}=\frac{2\pi}{D(\xi)}\cdot\frac{(2\xi_\alpha^2+i\xi_\alpha\delta_{3\alpha})}{|\omega|}.$$

All terms can be either uniformly bounded or estimated as m_t. Then after some elementary calculations, for each α, we get

$$\max_\beta \sup_{\xi\neq 0} |\xi_\alpha^\beta \partial_\alpha^\beta \xi m_t(\xi)| \leq C(|\omega|),$$

where β runs through the set of all multi-indices in $\{0,1\}^3$.

We now come back to $\langle \partial_3 U_0, \phi\rangle$: There exists a sequence of functions $H^{(l)} \in C_0^\infty$ such that $\|H^{(l)} - H\|_q \to 0$ as $l \to \infty$. Let $U_0^{(l)}$ be solutions corresponding to $H^{(l)}$; we apply all the previous considerations to this sequence $U_0^{(l)}$ and then to $\langle \partial_3 U_0^{(l)}, \phi\rangle$.

Let us denote $\mathcal{H}_k^{(l)}(t,\xi) = O_\omega^T(t)\, \widehat{H_{\cdot k}^{(l)}}(O_\omega(t). - te_3/|\omega|)(\xi)$. Marcinkiewicz's multiplier theorem implies that the functions $x \to \mathcal{F}^{-1}[m_t(\xi)\cdot \mathcal{H}_k^{(l)}(t,\xi)](x)$ are bounded in L^q for all t, $0 < t < 2\pi$ by $C(|\omega|)\|H_{\cdot k}(O_\omega(t)\cdot -te_3/|\omega|)\|_q$, i.e., by $C(|\omega|)\|H_{\cdot k}\|_q$.

Then we finally get

$$\langle \partial_3 U_{0,k}^{(l)}, \phi_k\rangle = i \int_{\mathbb{R}^3} \frac{\xi_3 \widehat{U_{0,k}^{(l)}}(\xi)}{|\xi|} |\xi| \widehat{\phi_k}(\xi) d\xi \tag{5.5}$$

in the form

$$\langle \int_0^{2\pi} \mathcal{F}^{-1}[m_t(\xi)\cdot \mathcal{H}_k^{(l)}(t,\xi)]dt, \Psi\rangle, \tag{5.6}$$

where we have denoted the relation by $\langle V^{(l)}, \Psi\rangle$. Passing to the limit as $l \to \infty$, and taking the supremum for $\|\Psi\|_{q'} \leq 1$, we obtain the control of $\|\partial_3 U_{0,k}\|_{-1,q}$ by $\|H_{\cdot k}\|_q$.

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