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UNILATERAL DYNAMIC CONTACT OF TWO VISCOELASTIC BEAMS

By

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Abstract. This work is focused on a dynamic unilateral contact problem between two viscoelastic beams. Global-in-time existence of weak solutions describing the dynamics of the system is established. In addition, asymptotic longtime behavior of weak solutions is discussed: it is shown that the energy solutions decay exponentially to zero under suitable decay properties of the memory kernels.

1. Introduction. In this paper we analyze the mechanical problem modeling the evolution of two viscoelastic beams in unilateral contact across a joint. The longitudinal axes of the beams coincide with the intervals $[0, l_0]$ and $[l_0, l]$, respectively. Let $0 < T \leq \infty$. Denoting by $u = u(x,t) : (0, l_0) \times (0,T) \to \mathbb{R}$ and $v = v(x,t) : (l_0, l) \times (0,T) \to \mathbb{R}$ the vertical deflection of the first and second beam, respectively, from their configuration at rest, under the assumption of small displacements, the motion can be described by the following equations (see, e.g., [14, 23, 37]):

$$\begin{aligned} &u_{tt}(x,t) + \gamma_1\, u_{xxxx}(x,t) - \int_0^t g_1(t-\tau)\, u_{xxxx}(x,\tau)\, d\tau = 0 \quad \text{in } (0,l_0)\times(0,T), \\ &v_{tt}(x,t) + \gamma_2\, v_{xxxx}(x,t) - \int_0^t g_2(t-\tau)\, v_{xxxx}(x,\tau)\, d\tau = 0 \quad \text{in } (l_0,l)\times(0,T). \end{aligned} \tag{1.1}$$

Here γ_i, $i = 1,2$, are positive constants while the memory kernels g_i, $i = 1,2$, are nonnegative absolutely continuous nonincreasing functions on $[0, +\infty)$ such that

$$\gamma_i - \int_0^\infty g_i(\tau)\, d\tau > 0, \qquad i = 1,2.$$

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The initial conditions read

$$\begin{aligned} u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in } (0,l_0), \\ v(x,0) &= v_0(x), \quad v_t(x,0) = v_1(x) \quad \text{in } (l_0,l), \end{aligned} \tag{1.2}$$

for some given functions $u_0, u_1 : (0,l_0) \to \mathbb{R}$ and $v_0, v_1 : (l_0,l) \to \mathbb{R}$. System (1.1) is supplemented with the clamped boundary conditions at $x = 0$ and $x = l$:

$$\begin{aligned} u(0,t) &= 0, \quad u_x(0,t) = 0 \quad \text{in } (0,T), \\ v(l,t) &= 0, \quad v_x(l,t) = 0 \quad \text{in } (0,T). \end{aligned} \tag{1.3}$$

The joint at $x = l_0$ is modeled with the classical Signorini nonpenetration condition (see, e.g., [10, 19, 21]). In particular, the joint with gap d is asymmetrical so that $d = d_1 + d_2$, where $d_1 > 0$ and $d_2 > 0$ are, respectively, the upper and lower clearance, when the system is at rest (see Fig. 1).

Then, the right end of the left beam is assumed to be within the clearance of the left end of the right beam, namely

$$v(l_0,t) - d_2 \le u(l_0,t) \le v(l_0,t) + d_1 \quad \text{in } (0,T). \tag{1.4}$$

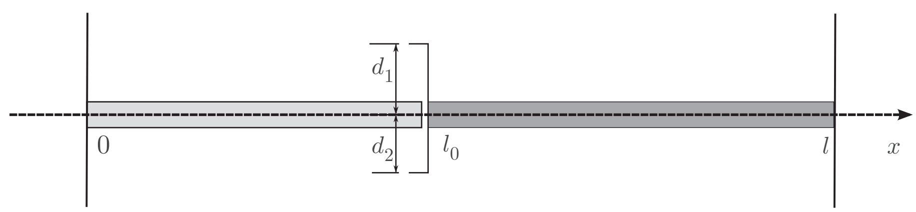

FIG. 1. The two beams and the joint at $x = l_0$ with clearance $d = d_1 + d_2$

In addition to (1.4), we assume that the stresses at the joint are equal; namely,

$$\sigma(t) := \sigma_1(l_0,t) = \sigma_2(l_0,t) \quad \text{in } (0,T), \tag{1.5}$$

where

$$\sigma_1(l_0,t) = -\gamma_1\, u_{xxx}(l_0,t) + \int_0^t g_1(t-\tau)\, u_{xxx}(l_0,\tau)\, d\tau,$$

$$\sigma_2(l_0,t) = -\gamma_2\, v_{xxx}(l_0,t) + \int_0^t g_2(t-\tau)\, v_{xxx}(l_0,\tau)\, d\tau.$$

Moreover, we prescribe

$$-\sigma(t) \in \partial\chi_{v(l_0,t)}(u(l_0,t)) \quad \text{in } (0,T), \tag{1.6}$$

where $\partial\chi_v$ denotes the subdifferential of the indicator function χ_v,

$$\chi_v(\phi) = \begin{cases} 0, & \text{if } v - d_2 \le \phi \le v + d_1, \\ +\infty, & \text{otherwise;} \end{cases}$$

namely,

$$\partial\chi_v(\phi) = \begin{cases} (-\infty, 0] & \text{if } \phi = v - d_2, \\ 0 & \text{if } v - d_2 < \phi < v + d_1, \\ [0, +\infty) & \text{if } \phi = v + d_1. \end{cases}$$

Let us explain the condition expressed by (1.6). When $v(l_0,t)-d_2 < u(l_0,t) < v(l_0,t)+d_1$ is verified, there is no contact, the ends at $x = l_0$ are free, and $\sigma(t) = 0$. On the other hand, when $v(l_0,t) - d_2 = u(l_0,t)$ or $u(l_0,t) = v(l_0,t) + d_1$, the ends at $x = l_0$ are in contact. More precisely, when contact occurs at the lower end, relations $v(l_0,t) - d_2 = u(l_0,t)$ and $\sigma(t) \geq 0$ hold; when contact takes place at the upper end, relations $u(l_0,t) = v(l_0,t) + d_1$ and $\sigma(t) \leq 0$ are satisfied.

Finally, we suppose that the ends, evaluated at $x = l_0$, do not exert moments on each other; namely,

$$\begin{aligned} &\gamma_1\, u_{xx}(l_0,t) - \int_0^t g_1(t-\tau)\, u_{xx}(l_0,\tau)\, d\tau = 0 \quad \text{in } (0,T), \\ &\gamma_2\, v_{xx}(l_0,t) - \int_0^t g_2(t-\tau)\, v_{xx}(l_0,\tau)\, d\tau = 0 \quad \text{in } (0,T). \end{aligned} \tag{1.7}$$

Questions related to the modeling, well-posedness and longtime behavior of systems in contact have drawn considerable attention in recent years. Some applications of unilateral multibody dynamics can be found in, e.g., [36, 45]. A substantial contribution to the mathematical theory of contact mechanics, which is concerned with the mathematical modeling and analysis of the many aspects of contact between deformable bodies, has been given by several authors (see, e.g., [2, 11, 15, 18, 38, 41, 42, 44] and the references therein). A first line of research is the mathematical formulation of the models leading to systems of partial differential equations that are worth analyzing also in respect to existence results, uniqueness, regularity of the solutions (see, e.g., [1, 12, 20, 22, 40]) or in respect to its numerical analysis (see, e.g., [6, 7, 8, 39]). Another area of interest concerns the study of the energy decay related to the contact system. Several works appeared over the years that dealt with the longtime behavior of the solutions in viscoelasticity (see, e.g., [9, 13, 17, 24, 25, 35]) via Laplace transform methods, semigroup techniques or direct energy estimates. The asymptotic behavior of contact problems involving only a single displacement and/or a single variation of temperature, have been studied extensively (see, e.g., [16, 26, 27, 30, 31, 32, 33, 34]). Concerning the energy decay for dynamic contact between two bodies we recall some results contained in [3, 4, 5, 28].

The focus of the present paper is on a global-in-time existence result for problem (1.1)–(1.7) and associated uniform stability questions. As is usual in mechanical problems with unilateral constraints, we cannot expect classical solutions because of the possible velocity discontinuity upon impact. So we look for weak solutions (see Definition 2.1). Therefore, we consider an approximate version of the problem (1.1)–(1.7) by introducing viscosity terms and a normal compliance condition (Remark 4.1 below) as regularization of the Signorini condition (1.6). Then, we prove a well-posedness result for the approximate problem by means of a Faedo-Galerkin method (Proposition 4.2), we derive suitable a priori estimates and we pass to the limit in the regularization parameter obtaining the existence of a solution to the original problem (Theorem 2.2). The uniqueness of the solution to the limit problem remains an open issue (Remark 5.1 below).

Once a global-in-time existence result for system (1.1)–(1.7) is established, a natural question to ask is that of asymptotic stability. The dissipative mechanism in the model is exhibited by the memory component of the system. The main goal of this paper is to show the exponential stability of a solution to the problem (1.1)–(1.7) as time goes to infinity (see Theorem 2.3) under the assumption that the memory functions g_i, $i = 1, 2$, decay exponentially as time goes to infinity. First, we work in the penalized framework: we prove the exponential decay for the penalized solution by introducing a suitable Lyapunov functional and by using the multiplier method. The weight functions in the Lyapunov functional are crucial in handing the boundary terms, as well as in controlling the norm of the solution (cf. Lemmas 6.2–6.3). Subsequently, by weak lower semicontinuity arguments, the exponential decay for a solution to the original problem is achieved. That is, we show that there exists one solution, of system (1.1)–(1.7), originating from the above-mentioned approximation procedure, that decays exponentially as time goes to infinity.

The plan of the paper is as follows. In Section 2, we enlist all of the assumptions on the problem data and state our results. Section 3 is devoted to some preliminary results that we will use in the following sections. In Section 4, we dwell on the model with the normal compliance condition, considered as a regularization of the Signorini condition, establishing the existence and the uniqueness of its solution. The proof of the existence of a weak solution to (1.1)–(1.7) and the study of its asymptotic behavior are carried out in Section 5 and Section 6, respectively.

2. Main results. In order to proceed with the exposition of our results, we introduce some notation and definitions.

To obtain a precise formulation of the problem, we introduce the following functional spaces:

$$\begin{aligned} \mathcal{V}_1 &= \{\varphi \in H^2(0, l_0) : \quad \varphi(0) = \varphi_x(0) = 0\}, \\ \mathcal{V}_2 &= \{\varphi \in H^2(l_0, l) : \quad \varphi(l) = \varphi_x(l) = 0\}, \end{aligned}$$

and the convex set of admissible pairs of displacements (u, v):

$$\mathcal{K} = \{(\varphi_1, \varphi_2) \in \mathcal{V}_1 \times \mathcal{V}_2 : \quad \varphi_2(l_0) - d_2 \leq \varphi_1(l_0) \leq \varphi_2(l_0) + d_1\},$$

incorporating the constraint (1.4).

We now enlist our assumptions on the problem data:

$$(u_0, v_0) \in \mathcal{K}, \tag{2.1}$$

$$(u_1, v_1) \in L^2(0, l_0) \times L^2(l_0, l). \tag{2.2}$$

Definition 2.1. Let $0 < T \leq \infty$. Let u_0, v_0, u_1, v_1 be given as in (2.1)–(2.2). A couple (u, v) is a weak solution to problem (1.1)–(1.7) when

$$(u, v) \in W^{1,\infty}\left(0, T; L^2(0, l_0) \times L^2(l_0, l)\right) \cap L^\infty\left(0, T; \mathcal{K}\right)$$

and satisfies the relation

$$
\begin{aligned}
&\int_0^T \int_0^{l_0} \Big\{ [-u_t(x,t)[w_t(x,t) - u_t(x,t)] \\
&\qquad + \left[\gamma_1 u_{xx}(x,t) - \int_0^t g_1(t-\tau)\, u_{xx}(x,\tau)\, d\tau\right] [w_{xx}(x,t) - u_{xx}(x,t)] \Big\}\, dx\, dt \\
&+ \int_0^T \int_{l_0}^{l} \Big\{ - v_t(x,t)[z_t(x,t) - v_t(x,t)] \\
&\qquad + \left[\gamma_2 v_{xx}(x,t) - \int_0^t g_2(t-\tau)\, v_{xx}(x,\tau)\, d\tau\right] [z_{xx}(x,t) - v_{xx}(x,t)] \Big\}\, dx\, dt \\
&\qquad \geq \int_0^{l_0} u_1(x)[w(x,0) - u_0(x)]\, dx + \int_{l_0}^{l} v_1(x)[z(x,0) - v_0(x)]\, dx, \qquad (2.3)
\end{aligned}
$$

for every $(w,z) \in W^{1,\infty}\big(0,T; L^2(0,l_0) \times L^2(l_0,l)\big) \cap L^\infty(0,T;\mathcal{K})$ such that $w(\cdot,T) = u(\cdot,T)$ and $z(\cdot,T) = v(\cdot,T)$.

Henceforth, we suppose that

(H.1) the memory kernels g_i, $i = 1,2$, are nonnegative absolutely continuous nonincreasing functions on $[0,+\infty)$ such that

$$\gamma_i - \int_0^\infty g_i(\tau)\, d\tau > 0, \qquad i = 1,2.$$

For any $t \geq 0$ and $i = 1,2$, we let

$$G_i(t) := \gamma_i - \int_0^t g_i(\tau)\, d\tau \quad \text{and} \quad G_i^\infty := \gamma_i - \int_0^\infty g_i(\tau)\, d\tau.$$

From (H.1) we have

$$0 < G_i^\infty < G_i(t) \leq \gamma_i.$$

The main result pertaining to global-in-time existence of solutions is the following:

THEOREM 2.2 (Global in time existence). Let $0 < T \leq \infty$. Under assumptions (2.1)–(2.2) and (H.1), there exists a weak solution (in the sense of Definition 2.1) of problem (1.1)–(1.7).

The proof of this result will be carried out in Section 5, by a regularization, a priori estimates, and passage to the limit procedure.

Next, in Section 6, we investigate the asymptotic behavior of the weak solutions provided by Theorem 2.2. For this purpose, we suppose that the memory kernels g_i, $i = 1,2$, satisfying assumption (H.1), decay exponentially to zero, as $t \to +\infty$:

$$-\alpha_i\, g_i(t) \leq g_i'(t) \leq -\beta_i\, g_i(t), \tag{H.2}$$

$$|g_i''(t)| \leq \kappa_i\, g_i(t), \tag{H.3}$$

for any $t \geq 0$ and with $\alpha_i, \beta_i, \kappa_i$ strictly positive constants.

We define by

$$E(t,u,v) = \frac{1}{2}\int_0^{l_0} \left[|u_t(x,t)|^2 + G_1(t)|u_{xx}(x,t)|^2 + (g_1\Box u_{xx})(x,t)\right]\,dx$$
$$+\frac{1}{2}\int_{l_0}^{l} \left[|v_t(x,t)|^2 + G_2(t)|v_{xx}(x,t)|^2 + (g_2\Box v_{xx})(x,t)\right]\,dx \tag{2.4}$$

the energy associated with system (1.1)–(1.7), with the operator $\Box$ as introduced in (3.1). We now establish in the following theorem that $E(t,u,v)$ decays exponentially to zero, as $t \to +\infty$.

THEOREM 2.3 (Exponential decay). Let $T = \infty$. Assume hypotheses (H.1)–(H.3). Let (u,v) be a weak solution of problem (1.1)–(1.7). Then, there exist two positive constants M and μ, independent of t, such that

$$\text{for all}\ \ t \geq 0, \qquad E(t,u,v) \leq M\,E(0,u,v)\,e^{-\mu t}. \tag{2.5}$$

3. Preliminaries. Before proceeding, let us collect here some properties which will be useful in that follows. Denoting by

$$(k * w)(t) := \int_0^t k(t-\tau)\,w(\tau)\,d\tau$$

the convolution product, and introducing the following notation:

$$(k\Box w)(t) := \int_0^t k(t-\tau)\,|w(t)-w(\tau)|^2\,d\tau, \tag{3.1}$$

$$(k\diamond w)(t) := \int_0^t k(t-\tau)\,\left[w(\tau)-w(t)\right]\,d\tau,$$

it is apparent that the equalities

$$\begin{aligned}(k * w)(t) &= \left[\int_0^t k(\tau)\,d\tau\right] w(t) - (k\diamond w)(t),\\ (k * w)_t(t) &= k(0)\,w(t) + (k' * w)(t) = k(t)\,w(t) - (k'\diamond w)(t)\end{aligned} \tag{3.2}$$

hold, with $(\cdot)' = d(\cdot)/dt$ and $(\cdot)_t = \partial(\cdot)/\partial t$. We now recall some lemmas used in the sequel (for more details see, e.g., [25, 29]). The former two are a consequence of the above definitions and of differentiation of the term $k\Box w$.

LEMMA 3.1. For any function $k \in C(\mathbb{R})$ and any $w \in W^{1,2}(0,T)$ we have that

$$(k * w)(t) = (k\diamond w)(t) + \left[\int_0^t k(\tau)\,d\tau\right]\,w(t).$$

LEMMA 3.2. For any function $k \in C^1(\mathbb{R})$ and any $w \in W^{1,2}(0,T)$ we have that

$$(k * w)(t)\;w_t(t) = -\,\frac{1}{2}\,k(t)\,|w(t)|^2 + \frac{1}{2}\,(k'\Box w)(t)$$
$$-\,\frac{1}{2}\frac{d}{dt}\left\{(k\Box w)(t) - \left[\int_0^t k(\tau)\,d\tau\right]|w(t)|^2\right\}.$$

LEMMA 3.3. For any function $k \in C(\mathbb{R})$ and any $w \in W^{1,2}(0,T)$ we have that

$$|(k \diamond w)(t)|^2 \leq \left[\int_0^T |k(\tau)|\,d\tau\right] (|k|\Box w)(t).$$

LEMMA 3.4. Let H be a Hilbert space. For any functions $k \in C(\mathbb{R})$, $w \in W^{1,2}(0,T)$, $f \in H$, and for any $\varepsilon > 0$, there exists a positive constant C_ε such that

$$|f(t)\ (k \diamond w)(t)| \quad \leq \quad \varepsilon |f(t)|^2 + C_\varepsilon (|k|\Box w)(t)\,.$$

In addition, we recall that, by the Sobolev embedding theorem, the continuous injections hold,

$$H^1(0,l_0) \hookrightarrow C^{0,1/2}([0,l_0]), \qquad H^1(l_0,l) \hookrightarrow C^{0,1/2}([l_0,l]),$$

and, in particular, there exists a positive constant C_S such that

$$\begin{aligned} &\|w\|_{C^0([0,l_0])} \leq C_S \|w\|_{H^1(0,l_0)}, \quad \forall w \in H^1(0,l_0), \\ &\|w\|_{C^0([l_0,l])} \leq C_S \|w\|_{H^1(l_0,l)}, \quad \forall w \in H^1(l_0,l). \end{aligned} \tag{3.3}$$

Finally, for the sake of simplicity, we will employ the same symbols C for different constants, even in the same formula.

4. Penalized problem. To show an existence result for the problem (1.1)–(1.7), we approximate (1.1)–(1.7) by a penalization procedure and we prove well-posedness for the regularized problem.

We introduce the families of initial data $\{u_0^\varepsilon\}_{\varepsilon>0}$, $\{u_1^\varepsilon\}_{\varepsilon>0}$, $\{v_0^\varepsilon\}_{\varepsilon>0}$, $\{v_1^\varepsilon\}_{\varepsilon>0}$, satisfying

$$(u_0^\varepsilon, v_0^\varepsilon) \in [H^4(0,l_0) \times H^4(l_0,l)] \cap \mathcal{K} \quad \text{and} \quad (u_1^\varepsilon, v_1^\varepsilon) \in \mathcal{V}_1 \times \mathcal{V}_2. \tag{4.1}$$

For any $\varepsilon > 0$, we consider the differential equations

$$\begin{aligned} &u_{tt}^\varepsilon(x,t) + \gamma_1 u_{xxxx}^\varepsilon(x,t) - (g_1 * u_{xxxx}^\varepsilon)(x,t) = 0 \quad \text{in } (0,l_0) \times (0,T), \\ &v_{tt}^\varepsilon(x,t) + \gamma_2 v_{xxxx}^\varepsilon(x,t) - (g_2 * v_{xxxx}^\varepsilon)(x,t) = 0 \quad \text{in } (l_0,l) \times (0,T). \end{aligned} \tag{4.2}$$

The boundary conditions at $x = 0$ and $x = l$ are

$$\begin{aligned} &u^\varepsilon(0,t) = 0, \quad u_x^\varepsilon(0,t) = 0 \quad \text{in } (0,T), \\ &v^\varepsilon(l,t) = 0, \quad v_x^\varepsilon(l,t) = 0 \quad \text{in } (0,T). \end{aligned} \tag{4.3}$$

Furthermore, at the joint $x = l_0$, for $t \in (0,T)$ we let

$$\begin{aligned} &\sigma_1^\varepsilon(l_0,t) = -\gamma_1\, u_{xxx}(l_0,t) + (g_1 * u_{xxx})(l_0,t), \\ &\sigma_2^\varepsilon(l_0,t) = -\gamma_2\, v_{xxx}(l_0,t) + (g_2 * v_{xxx})(l_0,t), \\ &\sigma^\varepsilon(t) := \sigma_1^\varepsilon(l_0,t) = \sigma_2^\varepsilon(l_0,t), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \sigma^\varepsilon(t) = &-\frac{1}{\varepsilon}\{[u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t) - d_1]^+ - [v^\varepsilon(l_0,t) - u^\varepsilon(l_0,t) - d_2]^+\} \\ &- \varepsilon[u_t^\varepsilon(l_0,t) - v_t^\varepsilon(l_0,t)], \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\gamma_1\, u^\varepsilon_{xx}(l_0,t) - (g_1 * u^\varepsilon_{xx})(l_0,t) = 0,\\ \gamma_2\, v^\varepsilon_{xx}(l_0,t) - (g_2 * v^\varepsilon_{xx})(l_0,t) = 0.\end{aligned} \tag{4.6}$$

Here and in the sequel, $[f]^+ := \max\{f,0\}$ denotes the positive part of f. The initial conditions are

$$\begin{aligned} u^\varepsilon(x,0) = u^\varepsilon_0(x), \quad u^\varepsilon_t(x,0) = u^\varepsilon_1(x) \quad \text{in } (0,l_0),\\ v^\varepsilon(x,0) = v^\varepsilon_0(x), \quad v^\varepsilon_t(x,0) = v^\varepsilon_1(x) \quad \text{in } (l_0,l).\end{aligned} \tag{4.7}$$

REMARK 4.1. Assuming (4.5), we are considering a normal compliance condition (see, e.g., [4, 21, 22, 28]) as a regularization of the Signorini contact condition (1.6). Actually, we relax the nonpenetration condition by assuming for instance that the stops at the left end of the right beam are flexible. As $\varepsilon \to 0$, we recover formally the constraint (1.4) and the condition (1.6). Moreover, let us stress that the viscosity term $-\varepsilon\,[u^\varepsilon_t(l_0,t) - v^\varepsilon_t(l_0,t)]$ (introduced in [28]) will play a crucial role in the proof of the uniqueness of the approximating solution (see Proposition 4.3 below).

We now state existence and uniqueness results related to the penalized problem (4.2)–(4.7).

PROPOSITION 4.2 (Existence of an approximating solution). Let (H.1) hold. Then, for each $\varepsilon > 0$ and $T > 0$, problem (4.2)–(4.7) has a solution

$$\begin{aligned}(u^\varepsilon, v^\varepsilon) \in W^{2,\infty}\left(0,T; L^2(0,l_0)\times L^2(l_0,l)\right) \cap W^{1,\infty}\left(0,T; H^2(0,l_0)\times H^2(l_0,l)\right)\\ \cap L^\infty\left(0,T; H^4(0,l_0)\times H^4(l_0,l)\right),\end{aligned} \tag{4.8}$$

with initial data satisfying (4.7)–(4.1) and compatible with the boundary conditions (4.3)–(4.6) for $t = 0$.

Proof. The existence of a solution to problem (4.2)–(4.7) is shown by means of the Faedo-Galerkin scheme.

Construction of Faedo-Galerkin approximations: Let $\{w_j\}_{j\in\mathbb{N}}$ and $\{z_j\}_{j\in\mathbb{N}}$ be bases of $\mathcal{V}_1, \mathcal{V}_2$ such that $u^\varepsilon_0, u^\varepsilon_1 \in \operatorname{span}\{w_1,w_2\}$ and $v^\varepsilon_0, v^\varepsilon_1 \in \operatorname{span}\{z_1,z_2\}$. For any $n\in\mathbb{N}$, let us denote by

$$u^n(x,t) = \sum_{j=1}^n h^n_j(t)\, w_j(x), \qquad v^n(x,t) = \sum_{j=1}^n k^n_j(t)\, z_j(x)$$

the solutions of the following system:

$$\int_0^{l_0} \{u^n_{tt}(x,t)w_j(x) + [\gamma_1 u^n_{xx}(x,t) - (g_1 * u^n_{xx})(x,t)]w_{jxx}(x)\}\,dx - \sigma^n(t)w_j(l_0) = 0, \tag{4.9}$$

$$\int_{l_0}^{l} \{v^n_{tt}(x,t)z_j(x) + [\gamma_2 v^n_{xx}(x,t) - (g_2 * v^n_{xx})(x,t)]z_{jxx}(x)\}\,dx + \sigma^n(t)z_j(l_0) = 0, \tag{4.10}$$

for $j = 1,\dots,n$, where

$$\begin{aligned}\sigma^n(t) = -\frac{1}{\varepsilon}\{[u^n(l_0,t) - v^n(l_0,t) - d_1]^+ - [v^n(l_0,t) - u^n(l_0,t) - d_2]^+\}\\ - \varepsilon[u^n_t(l_0,t) - v^n_t(l_0,t)]\end{aligned}$$

and

$$\begin{aligned} u^n(x,0) &= u_0^\varepsilon(x), \qquad & u_t^n(x,0) &= u_1^\varepsilon(x), \\ v^n(x,0) &= v_0^\varepsilon(x), \qquad & v_t^n(x,0) &= v_1^\varepsilon(x). \end{aligned} \tag{4.11}$$

Existence of Galerkin approximations: System (4.9)–(4.10) appended by initial conditions (4.11) admits a local solution, and the a priori estimates derived below show that this solution can be extended to $(0,T)$, for any $T>0$.
A priori estimates: In order to extend the local solution to $(0,T)$, we establish some a priori estimates independent of n. By differentiating equations (4.9) and (4.10) with respect to t, we obtain

$$\begin{aligned} \int_0^{l_0} \{u_{ttt}^n(x,t)w_j(x) + [\gamma_1 u_{txx}^n(x,t) - (g_1 * u_{txx}^n)(x,t) - g_1(t)u_{0xx}^n(x)]w_{jxx}(x)\}\, dx& \\ - {\sigma^n}_t(t)\, w_j(l_0) = 0&, \\ \int_{l_0}^{l} \{v_{ttt}^n(x,t)z_j(x) + [\gamma_2 v_{txx}^n(x,t) - (g_2 * v_{txx}^n)(x,t) - g_2(t)v_{0xx}^n(x)]z_{jxx}(x)\}\, dx& \\ + {\sigma^n}_t(t)\, z_j(l_0) = 0&, \end{aligned}$$

for $j=1,\dots,n$. We multiply the first equation by $h^n_{j\,tt}$, the second one by $k^n_{j\,tt}$ and we add the resulting equations. Thus, summing up over $j=1,\dots,n$, we have

$$\begin{aligned} &\int_0^{l_0} \{u_{ttt}^n(x,t)u_{tt}^n(x,t) + [\gamma_1 u_{txx}^n(x,t) - (g_1 * u_{txx}^n)(x,t) - g_1(t)u_{0xx}^n(x)]u_{ttxx}^n(x,t)\}\, dx \\ &+ \int_{l_0}^{l} \{v_{ttt}^n(x,t)v_{tt}^n(x,t) + [\gamma_2 v_{txx}^n(x,t) - (g_2 * v_{txx}^n)(x,t) - g_2(t)v_{0xx}^n(x)]v_{ttxx}^n(x,t)\}\, dx \\ &\qquad + \frac{1}{\varepsilon}B^n(t)[u_{tt}^n(l_0,t) - v_{tt}^n(l_0,t)] + \varepsilon[u_{tt}^n(l_0,t) - v_{tt}^n(l_0,t)]^2 = 0, \end{aligned}$$

with $B^n(t)$ defined as

$$B^n(t) := \frac{d}{dt}\{[u^n(l_0,t) - v^n(l_0,t) - d_1]^+ - [v^n(l_0,t) - u^n(l_0,t) - d_2]^+\}.$$

The energy functional introduced in (2.4) and Lemma 3.2 imply

$$\begin{aligned} &\frac{d}{dt}\left[E(t,u_t^n,v_t^n) - g_1(t)\int_0^{l_0} u_{0xx}^n(x)u_{txx}^n(x,t)\,dx - g_2(t)\int_{l_0}^{l} v_{0xx}^n(x)v_{txx}^n(x,t)\,dx\right] \\ &= \frac{1}{2}\left\{\int_0^{l_0} \left[(g_1'\Box u_{txx}^n)(x,t) - g_1(t)|u_{txx}^n(x,t)|^2\right]\, dx \right. \\ &\qquad \left. + \int_{l_0}^{l} \left[(g_2'\Box v_{txx}^n)(x,t) - g_2(t)|v_{txx}^n(x,t)|^2\right]\, dx\right\} \\ &\quad - g_1'(t)\int_0^{l_0} u_{0xx}^n(x)u_{txx}^n(x,t)\,dx - g_2'(t)\int_{l_0}^{l} v_{0xx}^n(x)v_{txx}^n(x,t)\,dx \\ &\quad - \frac{1}{\varepsilon}B^n(t)[u_{tt}^n(l_0,t) - v_{tt}^n(l_0,t)] - \varepsilon[u_{tt}^n(l_0,t) - v_{tt}^n(l_0,t)]^2. \end{aligned}$$

As in [4, Proposition 3.2], by applying Young and Sobolev inequalities, we can estimate

$$\frac{1}{\varepsilon}|B^n(t)||u^n_{tt}(l_0,t)-v^n_{tt}(l_0,t)| \le \frac{\varepsilon}{2}|u^n_{tt}(l_0,t)-v^n_{tt}(l_0,t)|^2 + C\left[\int_0^{l_0}|u^n_{txx}(x,t)|^2\,dx+\int_{l_0}^{l}|v^n_{txx}(x,t)|^2\,dx\right].$$

Denoting by

$$M(t,u^n_t,v^n_t) := E(t,u^n_t,v^n_t) - g_1(t)\int_0^{l_0}|u^n_{0xx}(x)u^n_{txx}(x,t)|\;dx - g_2(t)\int_{l_0}^{l}|v^n_{0xx}(x)v^n_{txx}(x,t)|\;dx, \tag{4.12}$$

an integration over $(0,t)$ leads to

$$\begin{aligned} M(t,u^n_t,v^n_t) \le M(0,u^n_t,v^n_t) &- \frac{1}{2}\int_0^t g_1'(\tau)\left[\int_0^{l_0}\left|u^n_{0xx}(x)\right|^2\;dx+\int_0^{l_0}\left|u^n_{txx}(x,\tau)\right|^2\;dx\right]d\tau \\ &- \frac{1}{2}\int_0^t g_2'(\tau)\left[\int_{l_0}^{l}\left|v^n_{0xx}(x)\right|^2\;dx+\int_{l_0}^{l}\left|v^n_{txx}(x,\tau)\right|^2\;dx\right]d\tau \\ &+ C\int_0^t\left[\int_0^{l_0}|u^n_{txx}(x,\tau)|^2\,dx+\int_{l_0}^{l}|v^n_{txx}(x,\tau)|^2\,dx\right]\,d\tau. \end{aligned} \tag{4.13}$$

By considering (4.12), the Hölder and Young inequalities and assumptions (H.1) yield

$$\begin{aligned} \frac{1}{2}E(t,u^n_t,v^n_t) - \frac{[g_1(t)]^2}{G_1(t)}\int_0^{l_0}|u^\varepsilon_{0xx}(x)|^2\,dx - \frac{[g_2(t)]^2}{G_2(t)}\int_{l_0}^{l}|v^\varepsilon_{0xx}(x)|^2\,dx \le M(t,u^n_t,v^n_t) \\ \le \frac{3}{2}E(t,u^n_t,v^n_t) + \frac{[g_1(t)]^2}{G_1(t)}\int_0^{l_0}|u^\varepsilon_{0xx}(x)|^2\,dx + \frac{[g_2(t)]^2}{G_2(t)}\int_{l_0}^{l}|v^\varepsilon_{0xx}(x)|^2\,dx. \end{aligned} \tag{4.14}$$

By (4.13)–(4.14), we find

$$\begin{aligned}
\frac{1}{2}E(t,u_t^n,v_t^n) &\leq M(t,u_t^n,v_t^n) + \frac{[g_1(t)]^2}{G_1(t)}\int_0^{l_0}|u^\varepsilon_{0xx}(x)|^2\,dx + \frac{[g_2(t)]^2}{G_2(t)}\int_{l_0}^{l}|v^\varepsilon_{0xx}(x)|^2\,dx \\
&\leq \frac{3}{2}E(0,u_t^n,v_t^n) + \frac{[g_1(0)]^2}{\gamma_1}\int_0^{l_0}|u^\varepsilon_{0xx}(x)|^2\,dx + \frac{[g_2(0)]^2}{\gamma_2}\int_{l_0}^{l}|v^\varepsilon_{0xx}(x)|^2\,dx \\
&\quad - \frac{1}{2}\int_0^t g_1'(\tau)\left[\int_0^{l_0}|u^n_{0xx}(x)|^2\,dx + \int_0^{l_0}|u^n_{txx}(x,\tau)|^2\,dx\right]d\tau \\
&\quad - \frac{1}{2}\int_0^t g_2'(\tau)\left[\int_{l_0}^{l}|v^n_{0xx}(x)|^2\,dx + \int_{l_0}^{l}|v^n_{txx}(x,\tau)|^2\,dx\right]d\tau \\
&\quad + \frac{[g_1(t)]^2}{G_1(t)}\int_0^{l_0}|u^\varepsilon_{0xx}(x)|^2\,dx + \frac{[g_2(t)]^2}{G_2(t)}\int_{l_0}^{l}|v^\varepsilon_{0xx}(x)|^2\,dx \\
&\quad + C\int_0^t\left[\int_0^{l_0}|u^n_{txx}(x,\tau)|^2\,dx + \int_{l_0}^{l}|v^n_{txx}(x,\tau)|^2\,dx\right]d\tau.
\end{aligned}$$

Then, there exist two positive constants C_1 and C_2 such that

$$\begin{aligned}
E(t,u_t^n,v_t^n) &\leq C_1\,E(0,u_t^n,v_t^n) - C_2\int_0^t [g_1'(\tau)+g_2'(\tau)]\,E(\tau,u_t^n,v_t^n)\,d\tau \\
&\quad + C\int_0^t\left[\int_0^{l_0}|u^n_{txx}(x,\tau)|^2\,dx + \int_{l_0}^{l}|v^n_{txx}(x,\tau)|^2\,dx\right]d\tau.
\end{aligned}$$

We now prove that the second-order energy is bounded initially, i.e. that

$$\begin{aligned}
E(0,u_t^n,v_t^n) = &\frac{1}{2}\left\{\int_0^{l_0}\left[|u^n_{tt}(x,0)|^2 + \gamma_1|u^\varepsilon_{0xx}(x)|^2\right]dx\right. \\
&\left. + \int_{l_0}^{l}\left[|v^n_{tt}(x,0)|^2 + \gamma_2|v^\varepsilon_{0xx}(x)|^2\right]dx\right\}
\end{aligned}$$

is bounded independently of n. To this aim, we will take advantage of the special bases chosen above, containing the initial data. In fact, we multiply (4.9) by h^n_{jtt}, we sum up over j and we let $t\to 0$. Hence, we have

$$\int_0^{l_0}\left[|u^n_{tt}(x,0)|^2 + \gamma_1 u^\varepsilon_{0xx}(x)u^n_{ttxx}(x,0)\right]dx - \sigma^n(0)u^n_{tt}(l_0,0) = 0.$$

Integrating by parts and owing to the compatibility conditions (4.3)–(4.6) for $t=0$ lead to

$$\int_0^{l_0}\left[|u^n_{tt}(x,0)|^2 + \gamma_1 u^\varepsilon_{0xxxx}(x)u^n_{tt}(x,0)\right]dx = 0.$$

Therefore, in view of the Hölder and Young inequalities, we obtain

$$\int_0^{l_0}|u^n_{tt}(x,0)|^2\,dx \leq C\int_0^{l_0}|u^\varepsilon_{0xxxx}(x)|^2\,dx; \tag{4.15}$$

namely, $u^n_{tt}(x,0)$ is bounded in $L^2(0,l_0)$. Similarly, we find

$$\int_{l_0}^{l} |v^n_{tt}(x,0)|^2\,dx \le C\int_{l_0}^{l} |v^\varepsilon_{0xxxx}(x)|^2\,dx, \tag{4.16}$$

which implies that $E(0,u^n_t,v^n_t)$ is bounded independently of n. Thus, by applying the Gronwall inequality, we deduce that $E(t,u^n_t,v^n_t)$ is bounded in $[0,T]$.
Passage to the limit: The boundedness of $E(t,u^n_t,v^n_t)$ guarantees that

$$\begin{aligned} &u^n \text{ is bounded in } W^{2,\infty}\left(0,T;L^2(0,l_0)\right)\cap W^{1,\infty}\left(0,T;H^2(0,l_0)\right),\\ &v^n \text{ is bounded in } W^{2,\infty}\left(0,T;L^2(l_0,l)\right)\cap W^{1,\infty}\left(0,T;H^2(l_0,l)\right). \end{aligned}$$

Therefore we deduce, up to a subsequence, the convergences

$$\begin{aligned} u^n &\overset{*}{\rightharpoonup} u^\varepsilon \qquad \text{in } W^{2,\infty}\left(0,T;L^2(0,l_0)\right)\cap W^{1,\infty}\left(0,T;H^2(0,l_0)\right),\\ v^n &\overset{*}{\rightharpoonup} v^\varepsilon \qquad \text{in } W^{2,\infty}\left(0,T;L^2(l_0,l)\right)\cap W^{1,\infty}\left(0,T;H^2(l_0,l)\right). \end{aligned}$$

In view of a generalized version of Ascoli's theorem (see, e.g., [43]), the following strong convergences hold:

$$\begin{aligned} u^n &\to u^\varepsilon \qquad \text{in } C^1\left(0,T;H^{2-s}(0,l_0)\right), \quad s>0,\\ v^n &\to v^\varepsilon \qquad \text{in } C^1\left(0,T;H^{2-s}(l_0,l)\right), \quad s>0. \end{aligned}$$

Thus, the existence of a solution is achieved, by letting $n\to\infty$. In particular, from $(4.2)_1$ it follows that

$$\gamma_1\|u^\varepsilon_{xxxx}(x,t)\|_{L^2(0,l_0)} \le \|u^\varepsilon_{tt}(x,t)\|_{L^2(0,l_0)} + \int_0^t g_1(t-\tau)\|u^\varepsilon_{xxxx}(x,\tau)\|_{L^2(0,l_0)}\,d\tau.$$

Recalling (H.1), this implies that

$$\|u^\varepsilon_{xxxx}(x,t)\|_{L^\infty(0,T;L^2(0,l_0))} \le \frac{1}{G_1^\infty}\|u^\varepsilon_{tt}(x,t)\|_{L^2(0,l_0)}.$$

Then, $u^\varepsilon_{xxxx}\in L^\infty\left(0,T;L^2(0,l_0)\right)$. Similarly, $v^\varepsilon_{xxxx}\in L^\infty\left(0,T;L^2(l_0,l)\right)$. □

PROPOSITION 4.3 (Uniqueness of the approximating solution). Let (H.1) hold. Then, given $T>0$ for each $\varepsilon>0$, the solution $(u^\varepsilon,v^\varepsilon)$ to problem (4.2)–(4.7), with initial data satisfying (4.7)–(4.1) and compatible with the boundary conditions (4.3)–(4.6) for $t=0$, is unique.

Proof. Let $(u^\varepsilon,v^\varepsilon,)$ and $(w^\varepsilon,z^\varepsilon)$ be two solutions of (4.2)–(4.7) whose regularity is specified by (4.8). Then

$$(\tilde u^\varepsilon,\tilde v^\varepsilon) := (u^\varepsilon-w^\varepsilon, v^\varepsilon-z^\varepsilon)$$

satisfies

$$\begin{aligned} &\tilde u^\varepsilon_{tt}(x,t)+\gamma_1\tilde u^\varepsilon_{xxxx}(x,t)-(g_1*\tilde u^\varepsilon_{xxxx})(x,t)=0 \quad \text{in } (0,l_0)\times(0,T),\\ &\tilde v^\varepsilon_{tt}(x,t)+\gamma_2\tilde v^\varepsilon_{xxxx}(x,t)-(g_2*\tilde v^\varepsilon_{xxxx})(x,t)=0 \quad \text{in } (l_0,l)\times(0,T), \end{aligned} \tag{4.17}$$

with

$$\begin{aligned} &\tilde{u}^\varepsilon(0,t)=0, \quad \tilde{u}^\varepsilon_x(0,t)=0 \quad \text{in } (0,T),\\ &\tilde{v}^\varepsilon(l,t)=0, \quad \tilde{v}^\varepsilon_x(l,t)=0 \quad \text{in } (0,T), \end{aligned} \tag{4.18}$$

$$\begin{aligned} &\gamma_1\,\tilde{u}^\varepsilon_{xx}(l_0,t)-(g_1*\tilde{u}^\varepsilon_{xx})(l_0,t)=0 \quad \text{in } (0,T),\\ &\gamma_2\,\tilde{v}^\varepsilon_{xx}(l_0,t)-(g_2*\tilde{v}^\varepsilon_{xx})(l_0,t)=0 \quad \text{in } (0,T), \end{aligned} \tag{4.19}$$

and

$$\tilde{\sigma}^\varepsilon(t):=\tilde{\sigma}^\varepsilon_1(l_0,t)=\tilde{\sigma}^\varepsilon_2(l_0,t) \quad \text{in } (0,T), \tag{4.20}$$

where

$$\begin{aligned} \tilde{\sigma}^\varepsilon_1(l_0,t)=&-\gamma_1\,\tilde{u}_{xxx}(l_0,t)+(g_1*\tilde{u}_{xxx})(l_0,t),\\ \tilde{\sigma}^\varepsilon_2(l_0,t)=&-\gamma_2\,\tilde{v}_{xxx}(l_0,t)+(g_2*\tilde{v}_{xxx})(l_0,t),\\ \tilde{\sigma}^\varepsilon(t)=&-\frac{1}{\varepsilon}\{[u^\varepsilon(l_0,t)-v^\varepsilon(l_0,t)-d_1]^+-[v^\varepsilon(l_0,t)-u^\varepsilon(l_0,t)-d_2]^+\}\\ &+\frac{1}{\varepsilon}\{[w^\varepsilon(l_0,t)-z^\varepsilon(l_0,t)-d_1]^+-[z^\varepsilon(l_0,t)-w^\varepsilon(l_0,t)-d_2]^+\}\\ &-\varepsilon[\tilde{u}^\varepsilon_t(l_0,t)-\tilde{v}^\varepsilon_t(l_0,t)]. \end{aligned} \tag{4.21}$$

The initial conditions are

$$\begin{aligned} &\tilde{u}^\varepsilon(x,0)=0, \quad \tilde{u}^\varepsilon_t(x,0)=0 \quad \text{in } (0,l_0),\\ &\tilde{v}^\varepsilon(x,0)=0, \quad \tilde{v}^\varepsilon_t(x,0)=0 \quad \text{in } (l_0,l). \end{aligned} \tag{4.22}$$

Multiplying $(4.17)_1$ by $\tilde{u}^\varepsilon_t$ in $L^2(0,l_0)$, $(4.17)_2$ by $\tilde{v}^\varepsilon_t$ in $L^2(l_0,l)$, respectively, and summing up, we find

$$\begin{aligned} \frac{d}{dt}\tilde{E}^\varepsilon(t)=&\frac{1}{2}\int_0^{l_0}\left[(g_1'\Box\tilde{u}^n_{xx})(x,t)-g_1(t)|\tilde{u}^n_{xx}(x,t)|^2\right]\,dx\\ &+\frac{1}{2}\int_{l_0}^{l}\left[(g_2'\Box\tilde{v}^n_{xx})(x,t)-g_2(t)|\tilde{v}^n_{xx}(x,t)|^2\right]\,dx+\tilde{\sigma}^\varepsilon(t)\,[\tilde{u}^\varepsilon_t(l_0,t)-\tilde{v}^\varepsilon_t(l_0,t)]\\ =&\frac{1}{2}\int_0^{l_0}\left[(g_1'\Box\tilde{u}^n_{xx})(x,t)-g_1(t)|\tilde{u}^n_{xx}(x,t)|^2\right]\,dx\\ &+\frac{1}{2}\int_{l_0}^{l}\left[(g_2'\Box\tilde{v}^n_{xx})(x,t)-g_2(t)|\tilde{v}^n_{xx}(x,t)|^2\right]\,dx-\varepsilon\,[\tilde{u}^\varepsilon_t(l_0,t)-\tilde{v}^\varepsilon_t(l_0,t)]^2\\ &-\frac{1}{\varepsilon}\Big\{[u^\varepsilon(l_0,t)-v^\varepsilon(l_0,t)-d_1]^+-[v^\varepsilon(l_0,t)-u^\varepsilon(l_0,t)-d_2]^+\\ &-[w^\varepsilon(l_0,t)-z^\varepsilon(l_0,t)-d_1]^++[z^\varepsilon(l_0,t)-w^\varepsilon(l_0,t)-d_2]^+\Big\}\\ &\times[\tilde{u}^\varepsilon_t(l_0,t)-\tilde{v}^\varepsilon_t(l_0,t)], \end{aligned} \tag{4.23}$$

where, according to (2.4), $\tilde{E}^\varepsilon(t) := E(t, \tilde{u}^\varepsilon, \tilde{\theta}^\varepsilon, \tilde{v}^\varepsilon, \tilde{\varphi}^\varepsilon)$. As in the proof of [4, Proposition 3.2], we now estimate the last term on the right-hand side of (4.23). Since $|f^+ - g^+| \le |f - g|$, we have

$$\begin{aligned}
&\Big| [u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t) - d_1]^+ - [v^\varepsilon(l_0,t) - u^\varepsilon(l_0,t) - d_2]^+ \\
&\qquad - [w^\varepsilon(l_0,t) - z^\varepsilon(l_0,t) - d_1]^+ + [z^\varepsilon(l_0,t) - w^\varepsilon(l_0,t) - d_2]^+ \Big| \\
&\le |u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t) - w^\varepsilon(l_0,t) + z^\varepsilon(l_0,t)| + |z^\varepsilon(l_0,t) - w^\varepsilon(l_0,t) - v^\varepsilon(l_0,t) + u^\varepsilon(l_0,t)| \\
&\le 2 \left[|\tilde{u}^\varepsilon(l_0,t)| + |\tilde{v}^\varepsilon(l_0,t)| \right]. \qquad (4.24)
\end{aligned}$$

Then, applying the Young and Poincaré inequalities and the Sobolev embedding theorem (cf. (3.3)), (4.23) becomes

$$\frac{d}{dt}\tilde{E}^\varepsilon(t) \le C\tilde{E}^\varepsilon(t). \tag{4.25}$$

By the Gronwall lemma and recalling that $\tilde{E}^\varepsilon(0) = 0$, we find that $\tilde{E}^\varepsilon(t) = 0$ on $(0, T)$. This implies that $(u^\varepsilon, v^\varepsilon) = (w^\varepsilon, z^\varepsilon)$, and our conclusion follows. □

5. Proof of Theorem 2.2. In this section we prove existence of a weak solution to problem (1.1)–(1.7) by showing that the solution to the penalized problem (4.2)–(4.7) approaches a weak solution to (1.1)–(1.7) as $\varepsilon \to 0$. For later convenience, let us introduce the following functionals:

$$\mathcal{J}^\varepsilon(t) = \frac{1}{2\varepsilon}\left\{ |[u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t) - d_1]^+|^2 + |[v^\varepsilon(l_0,t) - u^\varepsilon(l_0,t) - d_2]^+|^2 \right\}, \tag{5.1}$$

$$\mathcal{E}(t, u^\varepsilon, v^\varepsilon) = E(t, u^\varepsilon, v^\varepsilon) + \mathcal{J}^\varepsilon(t). \tag{5.2}$$

Let $(u_0, v_0) \in \mathcal{K}$, $(u_1, v_1) \in L^2(0, l_0) \times L^2(l_0, l)$, $(u_0^\varepsilon, v_0^\varepsilon)$, $(u_1^\varepsilon, v_1^\varepsilon)$ be sequences of functions such that

$$\begin{aligned}
(u_0^\varepsilon, v_0^\varepsilon) &\to (u_0, v_0) \qquad \text{in } H^2(0, l_0) \times H^2(l_0, l), \\
(u_1^\varepsilon, v_1^\varepsilon) &\to (u_1, v_1) \qquad \text{in } L^2(0, l_0) \times L^2(l_0, l).
\end{aligned} \tag{5.3}$$

We multiply equation $(4.2)_1$ by u_t^ε. An integration over $(0, l_0)$ and the boundary conditions (4.3)–(4.6) lead to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^{l_0} \left[|u_t^\varepsilon(x,t)|^2 + G_1(t)|u_{xx}^\varepsilon(x,t)|^2 + (g_1 \Box u_{xx}^\varepsilon)(x,t) \right] dx \\
&\qquad + \frac{1}{2}\int_0^{l_0} \left[(-g_1' \Box u_{xx}^\varepsilon)(x,t) + g_1(t)|u_{xx}^\varepsilon(x,t)|^2 \right] dx - \sigma^\varepsilon(t) u_t^\varepsilon(l_0,t) = 0.
\end{aligned}$$

Similarly, multiplying $(4.2)_2$ by v_t^ε and integrating over (l_0, l), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{l_0}^{l} \left[|v_t^\varepsilon(x,t)|^2 + G_2(t)|v_{xx}^\varepsilon(x,t)|^2 + (g_2 \Box v_{xx}^\varepsilon)(x,t) \right] dx \\
&\qquad + \frac{1}{2}\int_{l_0}^{l} \left[(-g_2' \Box v_{xx}^\varepsilon)(x,t) + g_2(t)|v_{xx}^\varepsilon(x,t)|^2 \right] dx + \sigma^\varepsilon(t) v_t^\varepsilon(l_0,t) = 0.
\end{aligned}$$

Adding the resulting equations, on account of (4.5), (5.1) and (5.2), we obtain

$$\frac{d}{dt}\mathcal{E}(t,u^\varepsilon,v^\varepsilon)+\frac{1}{2}\int_0^{l_0}\left[(-g_1'\Box u^\varepsilon_{xx})(x,t)+g_1(t)|u^\varepsilon_{xx}(x,t)|^2\right]\,dx$$

$$+\frac{1}{2}\int_{l_0}^{l}\left[(-g_2'\Box v^\varepsilon_{xx})(x,t)+g_2(t)|v^\varepsilon_{xx}(x,t)|^2\right]\,dx+\varepsilon|u^\varepsilon_t(l_0,t)-v^\varepsilon_t(l_0,t)|^2=0. \quad (5.4)$$

We integrate over $(0,t)$. Thus, in view of (H.1),we deduce

$$\mathcal{E}(t,u^\varepsilon,v^\varepsilon)+\varepsilon\int_0^t|u^\varepsilon_t(l_0,t)-v^\varepsilon_t(l_0,t)|^2ds\le E(0,u^\varepsilon,v^\varepsilon)\le C.$$

The boundedness of $\mathcal{E}(t,u^\varepsilon,v^\varepsilon)$ implies that there exists a subsequence, still denoted by $(u^\varepsilon,v^\varepsilon)$, satisfying the following convergences:

$$\begin{aligned} &u^\varepsilon \overset{*}{\rightharpoonup} u &&\text{in } W^{1,\infty}\big(0,T;L^2(0,l_0)\big)\cap L^\infty\big(0,T;H^2(0,l_0)\big),\\ &v^\varepsilon \overset{*}{\rightharpoonup} v &&\text{in } W^{1,\infty}\big(0,T;L^2(l_0,l)\big)\cap L^\infty\big(0,T;H^2(l_0,l)\big), \end{aligned} \quad (5.5)$$

and

$$\varepsilon\int_0^t|u^\varepsilon_t(l_0,s)-v^\varepsilon_t(l_0,s)|^2ds\to 0.$$

A generalized version of Ascoli's theorem (see, e.g., [43]) yields

$$\begin{aligned} &u^\varepsilon\to u &&\text{in } C^0\big(0,T;H^{2-s}(0,l_0)\big), && s>0,\\ &v^\varepsilon\to v &&\text{in } C^0\big(0,T;H^{2-s}(l_0,l)\big), && s>0, \end{aligned}$$

from which we conclude that $(u,v)\in W^{1,\infty}\big(0,T;L^2(0,l_0)\times L^2(l_0,l)\big)\cap L^\infty(0,T;\mathcal{K})$.

Our goal now consists in showing that (u,v) is a weak solution to (1.1)–(1.7). Let $(w,z)\in W^{1,\infty}\big(0,T;L^2(0,l_0)\times L^2(l_0,l)\big)\cap L^\infty(0,T;\mathcal{K})$ such that $w(\cdot,T)=u(\cdot,T)$ and $z(\cdot,T)=v(\cdot,T)$. We multiply $(4.2)_1$ by $w-u^\varepsilon$, $(4.2)_2$ by $z-v^\varepsilon$ and we sum up the resulting equations. By taking (4.7)–(4.6) into account, we obtain

$$\int_0^T\int_0^{l_0}[-u^\varepsilon_t(x,t)[w_t(x,t)-u^\varepsilon_t(x,t)]\,dx\,dt$$

$$+\int_0^T\int_0^{l_0}[\gamma_1 u^\varepsilon_{xx}(x,t)-(g_1*u^\varepsilon_{xx})(x,t)][w_{xx}(x,t)-u^\varepsilon_{xx}(x,t)]\,dx\,dt$$

$$+\int_0^T\int_{l_0}^{l}-v^\varepsilon_t(x,t)[z_t(x,t)-v^\varepsilon_t(x,t)]\,dx\,dt$$

$$+\int_0^T\int_{l_0}^{l}[\gamma_2 v^\varepsilon_{xx}(x,t)-(g_2*v^\varepsilon_{xx})(x,t)][z_{xx}(x,t)-v^\varepsilon_{xx}(x,t)]\,dx\,dt$$

$$\ge\int_0^{l_0}u^\varepsilon_1(x)[w(x,0)-u^\varepsilon_0(x)]\,dx+\int_{l_0}^{l}v^\varepsilon_1(x)[z(x,0)-v^\varepsilon_0(x)]\,dx.$$

We pass to $\limsup_{\varepsilon\to 0}$ in the previous inequality. By proceeding as in [22, Lemma 4.3], we can obtain

$$\limsup_{\varepsilon\to 0}\int_0^T\int_0^{l_0}\left\{|u_t^\varepsilon(x,t)|^2-[\beta_1 u_{xx}^\varepsilon(x,t)-(g_1*u_{xx}^\varepsilon)(x,t)]u_{xx}^\varepsilon(x,t)\right\}dx\,dt$$
$$\leq\int_0^T\int_0^{l_0}\left\{|u_t(x,t)|^2-[\beta_1 u_{xx}(x,t)-(g_1*u_{xx})(x,t)]u_{xx}(x,t)\right\}dx\,dt$$

and

$$\limsup_{\varepsilon\to 0}\int_0^T\int_{l_0}^{l}\left\{|v_t^\varepsilon(x,t)|^2-[\beta_2 v_{xx}^\varepsilon(x,t)-(g_2*v_{xx}^\varepsilon)(x,t)]v_{xx}^\varepsilon(x,t)\right\}dx\,dt$$
$$\leq\int_0^T\int_{l_0}^{l}\left\{|v_t(x,t)|^2-[\beta_2 v_{xx}(x,t)-(g_2*v_{xx})(x,t)]v_{xx}(x,t)\right\}dx\,dt.$$

Accordingly, in view of convergences (5.3) and (5.5), we recover (2.3).

REMARK 5.1. Concerning uniqueness, we point out that we are able to prove it only in the case of the penalized problem (4.2)–(4.7) (see Proposition 4.3). In fact, our argument for uniqueness relies on two crucial properties of the approximate problem: the presence of viscosity terms $-\varepsilon\,[\tilde{u}_t^\varepsilon(l_0,t)-\tilde{v}_t^\varepsilon(l_0,t)]$ in the approximate stress $\tilde{\sigma}^\varepsilon(t)$ and the Lipschitz continuity (of constant $1/\varepsilon$) of $\tilde{\sigma}^\varepsilon(t)$ (cf. (4.21)). In the original problem (limit problem as $\varepsilon\to 0$) these properties fall and our procedure does not apply. As mentioned in the introduction, as far as we know, the uniqueness of the solution to the limit problem remains an open issue.

6. Exponential decay. To prove that the energy related to system (1.1)–(1.7) decays exponentially as $t\to\infty$, we first show that the energy associated to the approximating problem decays exponentially, and subsequently let $\varepsilon\to 0$.

We suppose that the memory kernels g_i, $i=1,2$, satisfy assumptions (H.1)–(H.3). Here and in that follows, let $(u^\varepsilon,v^\varepsilon)$ be the solution (as found in Proposition 4.3) to problem (4.2)–(4.7), with initial data satisfying (4.7)–(4.1) and compatible with the boundary conditions (4.3)–(4.6) for $t=0$.

6.1. *Useful lemmas.* We now define the functionals

$$I_1^u(t)=\int_0^{l_0}u^\varepsilon(x,t)u_t^\varepsilon(x,t)\,dx,\qquad I_1^v(t)=\int_{l_0}^{l}v^\varepsilon(x,t)v_t^\varepsilon(x,t)\,dx,$$
$$I_2^u(t)=\int_0^{l_0}\psi_1(x)u_x^\varepsilon(x,t)u_t^\varepsilon(x,t)\,dx,\qquad I_2^v(t)=\int_{l_0}^{l}\psi_2(x)v_x^\varepsilon(x,t)v_t^\varepsilon(x,t)\,dx,$$
$$I_3^u(t)=-\int_0^{l_0}\phi(x)u_t^\varepsilon(x,t)(g_1*u^\varepsilon)_t(x,t)\,dx+\frac{1}{2}\int_0^{l_0}\phi(x)|(g_1*u_{xx}^\varepsilon)(x,t)|^2\,dx,$$
$$I_3^v(t)=\int_{l_0}^{l}\phi(x)v_t^\varepsilon(x,t)(g_2*v^\varepsilon)_t(x,t)\,dx-\frac{1}{2}\int_{l_0}^{l}\phi(x)|(g_2*v_{xx}^\varepsilon)(x,t)|^2\,dx,$$
$$I_4^u(t)=\int_0^{l_0}\phi(x)u^\varepsilon(x,t)u_t^\varepsilon(x,t)\,dx,\qquad I_4^v(t)=-\int_{l_0}^{l}\phi(x)v^\varepsilon(x,t)v_t^\varepsilon(x,t)\,dx,$$

where $\psi_1 : [0, l_0] \to \mathbb{R}$, $\psi_2 : [l_0, l] \to \mathbb{R}$, $\phi : [0, l] \to \mathbb{R}$ such that

$$\psi_1(x) = x^2 - l_0 x, \qquad \psi_2(x) = -x^2 + (l_0 + l)x - l_0 l, \qquad \phi(x) = l_0 - x. \tag{6.1}$$

The choice of the functions ψ_1, ψ_2, ϕ has been performed in order to get rid of some boundary terms as well as to obtain a control on the norm of the solution (cf. Lemmas 6.2–6.3).

Let δ_j, $j = 1, 2, 3$, be some suitable positive constants which will be specified in Section 6.2.

Lemma 6.1. The following holds:

$$\begin{aligned} &\frac{d}{dt}\left[I_1^u(t) + I_1^v(t) + \frac{\varepsilon}{2}|u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t)|^2\right] \\ &\qquad \le -\int_0^{l_0} G_1(t)|u^\varepsilon_{xx}(x,t)|^2\,dx - \int_{l_0}^{l} G_2(t)|v^\varepsilon_{xx}(x,t)|^2\,dx - 2\mathcal{J}^\varepsilon(t) \\ &\qquad + \int_0^{l_0}|u^\varepsilon_t(x,t)|^2\,dx + \delta_1\int_0^{l_0}|u^\varepsilon_{xx}(x,t)|^2\,dx + C\int_0^{l_0}(g_1\Box u^\varepsilon_{xx})(x,t)\,dx \\ &\qquad + \int_{l_0}^{l}|v^\varepsilon_t(x,t)|^2\,dx + \delta_1\int_{l_0}^{l}|v^\varepsilon_{xx}(x,t)|^2\,dx + C\int_{l_0}^{l}(g_2\Box v^\varepsilon_{xx})(x,t)\,dx. \end{aligned} \tag{6.2}$$

Proof. By means of equation $(4.2)_1$, we obtain

$$\begin{aligned} \frac{d}{dt}I_1^u(t) &= \int_0^{l_0} u^\varepsilon(x,t)u^\varepsilon_{tt}(x,t)\,dx + \int_0^{l_0}|u^\varepsilon_t(x,t)|^2\,dx \\ &= \int_0^{l_0} u^\varepsilon(x,t)[-\gamma_1 u^\varepsilon_{xxxx}(x,t) + (g_1 * u^\varepsilon_{xxxx})(x,t)]\,dx + \int_0^{l_0}|u^\varepsilon_t(x,t)|^2\,dx. \end{aligned}$$

We perform two integrations by parts by taking the boundary conditions (4.3), (4.4), (4.6) into account. Assumption (H.1), identity (3.2) and Lemma 3.4 lead to

$$\begin{aligned} \frac{d}{dt}I_1^u(t) &= \int_0^{l_0} u^\varepsilon_{xx}(x,t)[-\gamma_1 u^\varepsilon_{xx}(x,t) + (g_1 * u^\varepsilon_{xx})(x,t)]\,dx + \int_0^{l_0}|u^\varepsilon_t(x,t)|^2\,dx \\ &\quad + [-\gamma_1 u^\varepsilon_{xxx}(l_0,t) + (g_1 * u^\varepsilon_{xxx})(l_0,t)]u^\varepsilon(l_0,t) \\ &= \int_0^{l_0}\left\{-\gamma_1|u^\varepsilon_{xx}(x,t)|^2 + \left[\int_0^t g_1(\tau)\,d\tau\right]|u^\varepsilon_{xx}(x,t)|^2 - [g_1 \diamond u^\varepsilon_{xx}(x,t)]u^\varepsilon_{xx}(x,t)\right\}dx \\ &\quad + \int_0^{l_0}|u^\varepsilon_t(x,t)|^2\,dx + \sigma^\varepsilon(t)u^\varepsilon(l_0,t) \\ &\le -\int_0^{l_0} G_1(t)|u^\varepsilon_{xx}(x,t)|^2\,dx + \delta_1\int_0^{l_0}|u^\varepsilon_{xx}(x,t)|^2\,dx + C\int_0^{l_0}(g_1\Box u^\varepsilon_{xx})(x,t)\,dx \\ &\quad + \int_0^{l_0}|u^\varepsilon_t(x,t)|^2\,dx + \sigma^\varepsilon(t)u^\varepsilon(l_0,t), \end{aligned}$$

where $\delta_1 > 0$. Similarly, we have

$$\frac{d}{dt}I_1^v(t) \leq -\int_{l_0}^{l} G_2(t)|v_{xx}^\varepsilon(x,t)|^2\,dx + \delta_1\int_{l_0}^{l}|v_{xx}^\varepsilon(x,t)|^2\,dx + C\int_{l_0}^{l}(g_2\Box v_{xx}^\varepsilon)(x,t)\,dx + \int_{l_0}^{l}|v_t^\varepsilon(x,t)|^2\,dx - \sigma^\varepsilon(t)v^\varepsilon(l_0,t).$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}\left[I_1^u(t) + I_1^v(t) + \frac{\varepsilon}{2}|u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t)|^2\right] \\
&\leq -\int_0^{l_0} G_1(t)|u_{xx}^\varepsilon(x,t)|^2\,dx + \delta_1\int_0^{l_0}|u_{xx}^\varepsilon(x,t)|^2\,dx + C\int_0^{l_0}(g_1\Box u_{xx}^\varepsilon)(x,t)\,dx \\
&\quad + \int_0^{l_0}|u_t^\varepsilon(x,t)|^2\,dx - \int_{l_0}^{l} G_2(t)|v_{xx}^\varepsilon(x,t)|^2\,dx + \delta_1'\int_{l_0}^{l}|v_{xx}^\varepsilon(x,t)|^2\,dx \\
&\quad + C\int_{l_0}^{l}(g_2\Box v_{xx}^\varepsilon)(x,t)\,dx + \int_{l_0}^{l}|v_t^\varepsilon(x,t)|^2\,dx \\
&\quad + \{\sigma^\varepsilon(t) + \varepsilon[u_t^\varepsilon(l_0,t) - v_t^\varepsilon(l_0,t)]\}[u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t)].
\end{aligned}$$

In order to reach the conclusion, we have to estimate the last term in the previous inequality. By exploiting (4.5) and recalling the equality $f^+f = |f^+|^2$, we obtain

$$\begin{aligned}
&\{\sigma^\varepsilon(t) + \varepsilon[u_t^\varepsilon(l_0,t) - v_t^\varepsilon(l_0,t)]\}\,[u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t)] \\
&= -\frac{1}{\varepsilon}\{[u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t) - d_1]^+ - [v^\varepsilon(l_0,t) - u^\varepsilon(l_0,t) - d_2]^+\}[u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t)] \\
&\leq -2\mathcal{J}^\varepsilon(t),
\end{aligned}$$

where $\mathcal{J}^\varepsilon(t)$ is given in (5.1). Accordingly, (6.2) holds. □

LEMMA 6.2. The following inequalities hold:

$$\begin{aligned}\frac{d}{dt}I_2^u(t) \le & -\frac{l_0}{4}\int_{3/4l_0}^{l_0}|u_t^\varepsilon(x,t)|^2\,dx - \frac{3l_0}{4}\gamma_1\int_{3/4l_0}^{l_0}|u_{xx}^\varepsilon(x,t)|^2\,dx \\ & + \frac{l_0}{2}\int_0^{3/4l_0}|u_t^\varepsilon(x,t)|^2\,dx + \frac{3l_0}{2}\gamma_1\int_0^{3/4l_0}|u_{xx}^\varepsilon(x,t)|^2\,dx + \delta_2\int_0^{l_0}|u_{xx}^\varepsilon(x,t)|^2\,dx \\ & + C\int_0^{l_0}\left[|(g_1 * u_{xx}^\varepsilon)(x,t)|^2 + |(g_1 * u_{xxx}^\varepsilon)(x,t)|^2\right]\,dx,\end{aligned} \tag{6.3}$$

$$\begin{aligned}\frac{d}{dt}I_2^v(t) \le & -\frac{l-l_0}{4}\int_{l_0}^{(3l_0+l)/4}|v_t^\varepsilon(x,t)|^2\,dx - \frac{3(l-l_0)}{4}\gamma_2\int_{l_0}^{(3l_0+l)/4}|v_{xx}^\varepsilon(x,t)|^2\,dx \\ & + \frac{l-l_0}{2}\int_{(3l_0+l)/4}^{l_0}|v_t^\varepsilon(x,t)|^2\,dx + \frac{3(l-l_0)}{2}\gamma_2\int_{(3l_0+l)/4}^{l}|v_{xx}^\varepsilon(x,t)|^2\,dx \\ & + \delta_2\int_{l_0}^{l}|v_{xx}^\varepsilon(x,t)|^2\,dx + C\int_{l_0}^{l}\left[|(g_2 * v_{xx}^\varepsilon)(x,t)|^2 + |(g_2 * v_{xxx}^\varepsilon)(x,t)|^2\right]\,dx.\end{aligned} \tag{6.4}$$

Proof. We differentiate I_2^u with respect to t and we substitute $(4.2)_1$. By integrating by parts and taking (4.3)–(4.6), (6.1) into account, we obtain

$$\begin{aligned}\frac{d}{dt}I_2^u(t) &= \int_0^{l_0}\psi_1(x)u_{xt}^\varepsilon(x,t)u_t^\varepsilon(x,t)\,dx \\ &\quad - \int_0^{l_0}\psi_1(x)u_x^\varepsilon(x,t)[\gamma_1 u_{xxxx}^\varepsilon(x,t) - (g_1 * u_{xxxx}^\varepsilon)(x,t)]\,dx \\ &= -\frac{1}{2}\int_0^{l_0}\psi_1'(x)|u_t^\varepsilon(x,t)|^2\,dx \\ &\quad + \int_0^{l_0}\psi_1(x)u_{xx}^\varepsilon(x,t)[\gamma_1 u_{xxx}^\varepsilon(x,t) - (g_1 * u_{xxx}^\varepsilon)(x,t)]\,dx \\ &\quad - \int_0^{l_0}[\psi_1''(x)u_x^\varepsilon(x,t) + \psi_1'(x)u_{xx}^\varepsilon(x,t)][\gamma_1 u_{xx}^\varepsilon(x,t) - (g_1 * u_{xx}^\varepsilon)(x,t)]\,dx \\ &= -\frac{1}{2}\int_0^{l_0}\psi_1'(x)|u_t^\varepsilon(x,t)|^2\,dx - \frac{3}{2}\gamma_1\int_0^{l_0}\psi_1'(x)|u_{xx}^\varepsilon(x,t)|^2\,dx \\ &\quad - \int_0^{l_0}\psi_1(x)u_{xx}^\varepsilon(x,t)(g_1 * u_{xxx}^\varepsilon)(x,t)\,dx - \gamma_1|u_x(l_0,t)|^2 \\ &\quad + \int_0^{l_0}[2u_x^\varepsilon(x,t) + \psi_1'(x)u_{xx}^\varepsilon(x,t)](g_1 * u_{xx}^\varepsilon)(x,t)\,dx.\end{aligned}$$

From (6.1) it follows that

$$\psi_1'(x) \geq \begin{cases} -l_0 & \text{if } x \in \left[0, \dfrac{3}{4} l_0\right), \\ \dfrac{l_0}{2} & \text{if } x \in \left[\dfrac{3}{4} l_0, l_0\right]. \end{cases}$$

Accordingly, the Hölder and Young inequalities lead to (6.3). Similarly, by differentiating I_2^v we obtain

$$\begin{aligned} \frac{d}{dt} I_2^v(t) \quad = \quad & -\frac{1}{2}\int_{l_0}^{l} \psi_2'(x)|u_t^\varepsilon(x,t)|^2\,dx - \frac{3}{2}\gamma_2 \int_{l_0}^{l} \psi_2'(x)|v_{xx}^\varepsilon(x,t)|^2\,dx - \gamma_2|v_x(l_0,t)|^2 \\ & - \int_{l_0}^{l} \psi_2(x) v_{xx}^\varepsilon(x,t)(g_2 * v_{xxx}^\varepsilon)(x,t)\,dx \\ & + \int_{l_0}^{l} [2v_x^\varepsilon(x,t) + \psi_2'(x) v_{xx}^\varepsilon(x,t)](g_2 * v_{xx}^\varepsilon)(x,t)\,dx. \end{aligned}$$

Furthermore, in view of (6.1), we have

$$\psi_2'(x) \geq \begin{cases} \dfrac{l - l_0}{2} & \text{if } x \in \left[l_0, \dfrac{3l_0 + l}{4}\right), \\ -(l - l_0) & \text{if } x \in \left[\dfrac{3l_0 + l}{4}, l\right], \end{cases}$$

and (6.4) is proved. □

LEMMA 6.3. The following inequalities hold:

$$\begin{aligned} \frac{d}{dt}\left[4I_3^u(t) + g_1(0) I_4^u(t)\right] \leq & -g_1(0)\frac{l_0}{4}\int_0^{3/4 l_0} |u_t^\varepsilon(x,t)|^2\,dx \\ & - \frac{l_0}{8}\gamma_1 g_1(0)\int_0^{3/4 l_0} |u_{xx}^\varepsilon(x,t)|^2\,dx \\ & + \delta_3 \int_0^{l_0} |u_{xx}^\varepsilon(x,t)|^2\,dx + C g_1(t)\int_0^{l_0} |u_{xx}^\varepsilon(x,t)|^2\,dx \\ & + C\int_0^{l_0} \left[|u_x^\varepsilon(x,t)|^2 + |(g_1 * u_{xx}^\varepsilon)(x,t)|^2\,dx + (g_1 \Box u_{xx}^\varepsilon)(x,t)\right]\,dx, \end{aligned} \tag{6.5}$$

$$\frac{d}{dt}\left[4I_3^v(t)+g_2(0)I_4^v(t)\right] \le -g_2(0)\frac{l-l_0}{4}\int_{(3l_0+l)/4}^{l}|v_t^\varepsilon(x,t)|^2\,dx$$
$$-\frac{l-l_0}{8}\gamma_2 g_2(0)\int_{(3l_0+l)/4}^{l}|v_{xx}^\varepsilon(x,t)|^2\,dx$$
$$+\delta_3\int_{l_0}^{l}|v_{xx}^\varepsilon(x,t)|^2\,dx + Cg_2(t)\int_{l_0}^{l}|v_{xx}^\varepsilon(x,t)|^2\,dx$$
$$+C\int_{l_0}^{l}\left[|v_x^\varepsilon(x,t)|^2+|(g_2*v_{xx}^\varepsilon)(x,t)|^2\,dx+(g_1\Box v_{xx}^\varepsilon)(x,t)\right]\,dx. \quad (6.6)$$

Proof. By multiplying $(4.2)_1$ by $\phi(g_1*u^\varepsilon)_t$ and integrating over $(0,l_0)$, we obtain

$$\int_0^{l_0}\frac{d}{dt}\left[\phi(x)u_t^\varepsilon(x,t)(g_1*u^\varepsilon)_t(x,t)\right]\,dx-\int_0^{l_0}\phi(x)u_t^\varepsilon(x,t)(g_1*u^\varepsilon)_{tt}(x,t)\,dx$$
$$+\int_0^{l_0}[\gamma_1 u_{xxxx}^\varepsilon(x,t)-(g_1*u_{xxxx}^\varepsilon)(x,t)]\phi(x)(g_1*u^\varepsilon)_t(x,t)\,dx=0.$$

We perform two integrations by parts. Recalling function ϕ as introduced in (6.1), boundary conditions (4.4)–(4.6) yield

$$\int_0^{l_0}\frac{d}{dt}\left[\phi(x)u_t^\varepsilon(x,t)(g_1*u^\varepsilon)_t(x,t)\right]-\int_0^{l_0}\phi(x)u_t^\varepsilon(x,t)(g_1*u^\varepsilon)_{tt}(x,t)\,dx$$
$$+\int_0^{l_0}[\gamma_1 u_{xx}^\varepsilon(x,t)-(g_1*u_{xx}^\varepsilon)(x,t)][-2(g_1*u_x^\varepsilon)_t(x,t)+\phi(x)(g_1*u_{xx}^\varepsilon)_t(x,t)]\,dx=0.$$

Therefore, in view of $(3.2)_2$, we deduce

$$\frac{d}{dt}I_3^u(t)=-\int_0^{l_0}\phi(x)g_1(0)|u_t^\varepsilon(x,t)|^2\,dx-\int_0^{l_0}\phi(x)g_1'(t)u^\varepsilon(x,t)u_t^\varepsilon(x,t)\,dx$$
$$+\int_0^{l_0}\phi(x)u_t^\varepsilon(x,t)(g_1''\diamond u^\varepsilon)(x,t)\,dx+\int_0^{l_0}\gamma_1 g_1(t)\phi(x)|u_{xx}^\varepsilon(x,t)|^2\,dx$$
$$-\int_0^{l_0}\gamma_1\phi(x)u_{xx}^\varepsilon(x,t)(g_1'\diamond u_{xx}^\varepsilon)(x,t)\,dx-\gamma_1 g_1(t)|u_x^\varepsilon(l_0)|^2$$
$$+2\gamma_1\int_0^{l_0}u_{xx}^\varepsilon(x,t)(g_1'\diamond u_x^\varepsilon)(x,t)\,dx+2g_1(t)\int_0^{l_0}u_x^\varepsilon(x,t)(g_1*u_{xx}^\varepsilon)(x,t)\,dx$$
$$-2\int_0^{l_0}(g_1'\diamond u_x^\varepsilon)(x,t)(g_1*u_{xx}^\varepsilon)(x,t)\,dx. \quad (6.7)$$

Owing to the Hölder, Young, and Poincaré inequalities, Lemma 3.4, and assumption (H.3) we have

$$\begin{aligned}\frac{d}{dt}I_3^u(t) \quad &\leq \quad -\int_0^{l_0}\frac{g_1(0)}{2}\phi(x)|u_t^\varepsilon(x,t)|^2\,dx+\lambda_1\int_0^{l_0}|u_{xx}^\varepsilon(x,t)|^2\,dx\\ &\qquad +C\int_0^{l_0}\left[g_1(t)|u_{xx}^\varepsilon(x,t)|^2+(g_1\Box u_{xx}^\varepsilon)(x,t)+|(g_1*u_{xx}^\varepsilon)(x,t)|^2\right]\,dx,\end{aligned}$$

where λ_1 is a small enough positive constant.

We now differentiate I_4^u. On account of $(4.2)_1$, $(4.3)_1$, $(4.6)_1$ and (6.1), we obtain

$$\begin{aligned}\frac{d}{dt}I_4^u(t) &= \int_0^{l_0}\phi(x)|u_t^\varepsilon(x,t)|^2\,dx\\ &\quad+\int_0^{l_0}\phi(x)u^\varepsilon(x,t)[-\gamma_1 u_{xxxx}^\varepsilon(x,t)+(g_1*u_{xxxx}^\varepsilon)(x,t)]\,dx\\ &= \int_0^{l_0}\phi(x)|u_t^\varepsilon(x,t)|^2\,dx\\ &\quad+\int_0^{l_0}[2u_x^\varepsilon(x,t)-\phi(x)u_{xx}^\varepsilon(x,t)][\gamma_1 u_{xx}^\varepsilon(x,t)-(g_1*u_{xx}^\varepsilon)(x,t)]\,dx.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d}{dt}I_4^u(t) \quad &\leq \quad \int_0^{l_0}\phi(x)|u_t^\varepsilon(x,t)|^2\,dx-\frac{\gamma_1}{2}\int_0^{l_0}\phi(x)|u_{xx}^\varepsilon(x,t)|^2\,dx+\lambda_2\int_0^{l_0}|u_{xx}^\varepsilon(x,t)|^2\,dx\\ &\qquad +C\int_0^{l_0}[|u_x^\varepsilon(x,t)|^2+|(g_1*u_{xx}^\varepsilon)(x,t)|^2]\,dx,\end{aligned}$$

where λ_2 is a small enough positive constant. Accordingly,

$$\begin{aligned}&\frac{d}{dt}\left[4I_3^u(t)+g_1(0)I_4^u(t)\right]\\ &\leq -g_1(0)\int_0^{l_0}\phi(x)|u_t^\varepsilon(x,t)|^2\,dx-\frac{\gamma_1}{2}g_1(0)\int_0^{l_0}\phi(x)|u_{xx}^\varepsilon(x,t)|^2\,dx+\delta_3\int_0^{l_0}|u_{xx}^\varepsilon(x,t)|^2\,dx\\ &\quad +C\int_0^{l_0}\left[|u_x^\varepsilon(x,t)|^2+|(g_1*u_{xx}^\varepsilon)(x,t)|^2\,dx+g_1(t)|u_{xx}^\varepsilon(x,t)|^2+(g_1\Box u_{xx}^\varepsilon)(x,t)\right]\,dx,\end{aligned}$$

where $\delta_3=4\lambda_1+g_1(0)\lambda_2$. Finally, by means of (6.1), we obtain (6.5). By repeating the same arguments, we deduce (6.6) and we reach the conclusion. $\square$

In order to establish the exponential decay of the approximating problem, we have to estimate the norms of the terms $g_1*u_{xx}^\varepsilon$, $g_1*u_{xxx}^\varepsilon$, $g_2*v_{xx}^\varepsilon$, $g_2*v_{xxx}^\varepsilon$ which appear in Lemmas 6.2–6.3. Let us define

$$\begin{aligned}S(t,u^\varepsilon,v^\varepsilon) \quad &= \quad \int_0^{l_0}\left[|u_x^\varepsilon(x,t)|^2+|(g_1*u_{xx}^\varepsilon)(x,t)|^2+|(g_1*u_{xxx}^\varepsilon)(x,t)|^2\right]\,dx\\ &\qquad +\int_{l_0}^{l}\left[|v_x^\varepsilon(x,t)|^2+|(g_2*v_{xx}^\varepsilon)(x,t)|^2+|(g_2*v_{xxx}^\varepsilon)(x,t)|^2\right]\,dx,\end{aligned}$$

where $(u^\varepsilon, v^\varepsilon)$ is a weak solution to problem (4.2)–(4.7). Proposition 4.2 guarantees that S is well defined.

LEMMA 6.4. For any $\eta > 0$ there exists a suitable constant C_η such that

$$\int_0^T S(t,u^\varepsilon,v^\varepsilon)\,dt \le \eta\left\{\int_0^T \mathcal{E}(t,u^\varepsilon,v^\varepsilon)\,dt + \varepsilon\int_0^T [u_t^\varepsilon(l_0,t)-v_t^\varepsilon(l_0,t)]^2\,dt\right\}$$

$$+ C_\eta\left\{\int_0^T\int_0^{l_0}\left[(g_1\Box u_{xx}^\varepsilon)(x,t)+g_1(t)|u_{xx}^\varepsilon(x,t)|^2\right]\,dx\,dt\right.$$

$$\left.+\int_0^T\int_{l_0}^{l}\left[(g_2\Box v_{xx}^\varepsilon)(x,t)+g_2(t)|v_{xx}^\varepsilon(x,t)|^2\right]\,dx\,dt\right\}. \quad (6.8)$$

Proof. We suppose by contradiction that inequality (6.8) is not satisfied. Then, there exist a positive constant η_0 and a sequence of initial data $(u_0^n, v_0^n) \in \mathcal{K}$, $(u_1^n, v_1^n) \in L^2(0,l_0)\times L^2(l_0,l)$, such that the corresponding solutions $\{(u^n,v^n)\}_{n\in\mathbb{N}}$ to the equations

$$u_{tt}^n(x,t)+\gamma_1 u_{xxxx}^n(x,t)-(g_1 * u_{xxxx}^n)(x,t) = 0, \quad (6.9)$$

$$v_{tt}^n(x,t)+\gamma_2 v_{xxxx}^n(x,t)-(g_2 * v_{xxxx}^n)(x,t) = 0, \quad (6.10)$$

with the boundary conditions

$$\begin{aligned}
&u^n(0,t)=0, \qquad u_x^n(0,t)=0,\\
&v^n(l,t)=0, \qquad v_x^n(l,t)=0,\\
&-\gamma_1\, u_{xx}^n(l_0,t)+(g_1 * u_{xx}^n)(l_0,t)=0,\\
&-\gamma_2\, v_{xx}^n(l_0,t)+(g_2 * v_{xx}^n)(l_0,t)=0,\\
&\sigma_1^n(l_0,t)=\sigma_2^n(l_0,t)=\sigma^n(t),
\end{aligned} \quad (6.11)$$

where

$$\begin{aligned}
\sigma_1^n(l_0,t) &= -\gamma_1\, u_{xxx}(l_0,t)+(g_1 * u_{xxx})(l_0,t),\\
\sigma_2^n(l_0,t) &= -\gamma_2\, v_{xxx}(l_0,t)+(g_2 * v_{xxx})(l_0,t),\\
\sigma^n(t) &= -\frac{1}{\varepsilon}\{[u^n(l_0,t)-v^n(l_0,t)-d_1]^+ - [v^n(l_0,t)-u^n(l_0,t)-d_2]^+\}\\
&\quad -\varepsilon[u_t^n(l_0,t)-v_t^n(l_0,t)],
\end{aligned}$$

and the initial conditions

$$\begin{aligned}
u^n(x,0)=u_0^n(x), &\qquad u_t^n(x,0)=u_1^n(x),\\
v^n(x,0)=v_0^n(x), &\qquad v_t^n(x,0)=v_1^n(x)
\end{aligned}$$

satisfy the inequality

$$\begin{aligned}\int_0^T S(t,u^n,v^n)\,dt \quad > \quad & \eta_0\left\{\int_0^T \mathcal{E}(t,u^n,v^n)\,dt+\varepsilon\int_0^T [u_t^n(l_0,t)-v_t^n(l_0,t)]^2\,dt\right\}\\
& +n\left\{\int_0^T\int_0^{l_0}\left[(g_1\Box u_{xx}^n)(x,t)+g_1(t)|u_{xx}^n(x,t)|^2\right]\,dx\,dt\right.\\
& \left.\quad+\int_0^T\int_{l_0}^{l}\left[(g_2\Box v_{xx}^n)(x,t)+g_2(t)|v_{xx}^n(x,t)|^2\right]\,dx\,dt\right\}\end{aligned}$$

for every $n\in\mathbb{N}$. We now suppose that

$$\int_0^T S(t,u^n,v^n)\,dt=1.$$

Accordingly, we have

$$\begin{aligned}&\eta_0\left\{\int_0^T \mathcal{E}(t,u^n,v^n)\,dt+\varepsilon\int_0^T [u_t^n(l_0,s)-v_t^n(l_0,s)]^2 ds\right\}\\
&+n\left\{\int_0^T\int_0^{l_0}\left[(g_1\Box u_{xx}^n)(x,t)+g_1(t)|u_{xx}^n(x,t)|^2\right]\,dx\,dt\right.\\
&\qquad\left.+\int_0^T\int_{l_0}^{l}\left[(g_2\Box v_{xx}^n)(x,t)+g_2(t)|v_{xx}^n(x,t)|^2\right]\,dx\,dt\right\}<1. \qquad (6.12)\end{aligned}$$

We multiply (6.9) by u_t^n, (6.10) by v_t^n and we add the resulting equations. By the same procedure used to obtain (5.4), we find

$$\begin{aligned}\frac{d}{dt}\mathcal{E}(t,u^n,v^n)+\varepsilon[u_t^n(l_0,t)-v_t^n(l_0,t)]^2 &=\frac{1}{2}\int_0^{l_0}\left[(g_1'\Box u_{xx}^n)(x,t)-g_1(t)|u_{xx}^n(x,t)|^2\right]\,dx\\
&\quad+\frac{1}{2}\int_{l_0}^{l}\left[(g_2'\Box v_{xx}^n)(x,t)-g_2(t)|v_{xx}^n(x,t)|^2\right]\,dx;\end{aligned}$$

namely, $\mathcal{E}$ is nonincreasing. An integration from 0 to T and assumption (H.2) lead to

$$\begin{aligned}\mathcal{E}(0,u^n,v^n) &= \mathcal{E}(T,u^n,v^n)+\varepsilon\int_0^T [u_t^n(l_0,t)-v_t^n(l_0,t)]^2\,dt\\ &\quad -\frac{1}{2}\int_0^T\int_0^{l_0}\left[(g_1'\Box u_{xx}^n)(x,t)-g_1(t)|u_{xx}^n(x,t)|^2\right]\,dx\,dt\\ &\quad -\frac{1}{2}\int_0^T\int_{l_0}^{l}\left[(g_2'\Box v_{xx}^n)(x,t)-g_2(t)|v_{xx}^n(x,t)|^2\right]\,dx\,dt\\ &\le \frac{1}{T}\int_0^T \mathcal{E}(t,u^n,v^n)\,dt+\varepsilon\int_0^T [u_t^n(l_0,t)-v_t^n(l_0,t)]^2\,dt\\ &\quad +\frac{1}{2}\int_0^T\int_0^{l_0}\left[\alpha_1(g_1\Box u_{xx}^n)(x,t)+g_1(t)|u_{xx}^n(x,t)|^2\right]\,dx\,dt\\ &\quad +\frac{1}{2}\int_0^T\int_{l_0}^{l}\left[\alpha_2(g_2\Box v_{xx}^n)(x,t)+g_2(t)|v_{xx}^n(x,t)|^2\right]\,dx\,dt.\end{aligned}$$

Condition (6.12) ensures that $\mathcal{E}(0,u^n,v^n)$ is bounded. Accordingly, there exist a subsequence of (u_0^n,v_0^n) and (u_1^n,v_1^n), denoted by the same symbols, such that

$$\begin{aligned}(u_0^n,v_0^n)&\rightharpoonup(u_0^\varepsilon,v_0^\varepsilon) \quad \text{in} \quad H^2(0,l_0)\times H^2(l_0,l),\\ (u_1^n,v_1^n)&\rightharpoonup(u_1^\varepsilon,v_1^\varepsilon) \quad \text{in} \quad L^2(0,l_0)\times L^2(l_0,l).\end{aligned}$$

Since $\mathcal{E}$ is nonincreasing, $\mathcal{E}(t,u^n,v^n)$ is bounded too. Thus, we can extract subsequences still denoted by (u^n,v^n), (u_t^n,v_t^n) such that

$$\begin{aligned}(u^n,v^n)&\overset{*}{\rightharpoonup}(u^\varepsilon,v^\varepsilon) \quad \text{in} \quad L^\infty\left(0,T;H^2(0,l_0)\times H^2(l_0,l)\right),\\ (u_t^n,v_t^n)&\overset{*}{\rightharpoonup}(u_t^\varepsilon,v_t^\varepsilon) \quad \text{in} \quad L^\infty\left(0,T;L^2(0,l_0)\times L^2(l_0,l)\right).\end{aligned}$$

By [43, Corollary 8.4], we deduce that

$$\begin{aligned}\int_0^T\int_0^{l_0}|u_x^n(x,t)|\,dx\,dt &\to \int_0^T\int_0^{l_0}|u_x^\varepsilon(x,t)|\,dx\,dt,\\ \int_0^T\int_{l_0}^{l}|v_x^n(x,t)|\,dx\,dt &\to \int_0^T\int_{l_0}^{l}|v_x^\varepsilon(x,t)|\,dx\,dt.\end{aligned}$$

Moreover, inequality

$$\|(g_1*u_{xx}^\varepsilon)_t\|_{L^2(0,l_0)}\le g_1(0)\|u_{xx}^\varepsilon\|_{L^2(0,l_0)}+\alpha_1\int_0^t g_1(t-s)\|u_{xx}^\varepsilon\|_{L^2(0,l_0)}\,ds$$

and Proposition 4.2 assure that g_1*u^ε and $(g_1*u^\varepsilon)_t$ are bounded respectively in $L^\infty\left(0,T;H^4(0,l_0)\right)$ and $L^\infty\left(0,T;H^2(0,l_0)\right)$. A further application of [43, Corollary 8.4] yields

$$\int_0^T\int_0^{l_0}|(g_1*u^n_{xx})(x,t)|\,dx\,dt \quad\to\quad \int_0^T\int_0^{l_0}|(g_1*u^\varepsilon_{xx})(x,t)|\,dx\,dt,$$

$$\int_0^T\int_0^{l_0}|(g_1*u^n_{xxx})(x,t)|\,dx\,dt \quad\to\quad \int_0^T\int_0^{l_0}|(g_1*u^\varepsilon_{xxx})(x,t)|\,dx\,dt.$$

Similarly, we have

$$\int_0^T\int_{l_0}^{l}|(g_2*v^n_{xx})(x,t)|\,dx\,dt \quad\to\quad \int_0^T\int_{l_0}^{l}|(g_2*v^\varepsilon_{xx})(x,t)|\,dx\,dt,$$

$$\int_0^T\int_{l_0}^{l}|(g_2*v^n_{xxx})(x,t)|\,dx\,dt \quad\to\quad \int_0^T\int_{l_0}^{l}|(g_2*v^\varepsilon_{xxx})(x,t)|\,dx\,dt.$$

The previous convergences and condition (6.13) lead to

$$\int_0^T S(t,u^\varepsilon,v^\varepsilon)\,dt = 1. \tag{6.13}$$

In view of (6.12) we deduce

$$\int_0^T\int_0^{l_0} g_1(t)|u^n_{xx}(x,t)|^2\,dx\,dt\to 0, \qquad \int_0^T\int_{l_0}^{l} g_2(t)|v^n_{xx}(x,t)|^2\,dx\,dt\to 0;$$

that is, $u^\varepsilon_{xx}=0$ a.e. in $(0,l_0)\times(0,T)$ and $v^\varepsilon_{xx}=0$ a.e. in $(l_0,l)\times(0,T)$. The boundary conditions $u^\varepsilon(0,t)=u^\varepsilon_x(0,t)=v^\varepsilon(0,t)=v^\varepsilon_x(0,t)=0$ imply

$$\begin{aligned} u^\varepsilon &\equiv 0, \quad \text{a.e. in} \quad (0,l_0)\times(0,T),\\ v^\varepsilon &\equiv 0, \quad \text{a.e. in} \quad (l_0,l)\times(0,T), \end{aligned}$$

in contradiction with (6.13). $\square$

6.2. *Proof of Theorem* 2.3. Now, we are able to show that the energy associated to system (1.1)–(1.7) decays exponentially to zero as $t\to\infty$.

First, we show that the energy associated to the penalized problem (4.2)–(4.7) decays exponentially as $t\to\infty$. Let $a,b>0$. In view of Lemmas 6.2–6.3, we obtain the following estimates:

$$\begin{aligned} &\frac{d}{dt}\{I_2^u(t)+a\,[4I_3^u(t)+g_1(0)I_4^u(t)]\}\\ &\qquad\le \left[\frac{l_0}{2}-g_1(0)\frac{l_0}{4}a\right]\int_0^{3/4l_0}|u^\varepsilon_t(x,t)|^2\,dx-\frac{l_0}{4}\int_{3/4l_0}^{l_0}|u^\varepsilon_t(x,t)|^2\,dx\\ &\qquad\quad+\left(\frac{3l_0}{2}\gamma_1-\frac{l_0}{8}\gamma_1 g_1(0)a+\delta_2+a\delta_3\right)\int_0^{3/4l_0}|u^\varepsilon_{xx}(x,t)|^2\,dx\\ &\qquad+\left(-\frac{3l_0}{4}\gamma_1+\delta_2+a\delta_3\right)\int_{3/4l_0}^{l_0}|u^\varepsilon_{xx}(x,t)|^2\,dx+C\int_0^{l_0}g_1(t)|u^\varepsilon_{xx}(x,t)|^2\,dx\\ &+C\int_0^{l_0}\left[|u^\varepsilon_x(x,t)|^2+|(g_1*u^\varepsilon_{xx})(x,t)|^2\,dx+(g_1\Box u^\varepsilon_{xx})(x,t)+|(g_1*u^\varepsilon_{xxx})(x,t)|^2\right]\,dx \end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}\{I_2^v(t) + b\,[4I_3^v(t) + g_1(0)I_4^v(t)]\}\\
&\le -\frac{l-l_0}{4}\int_{l_0}^{(3l_0+l)/4} |v_t^\varepsilon(x,t)|^2\,dx + \left[\frac{l-l_0}{2} - g_2(0)b\frac{l-l_0}{4}\right]\int_{(3l_0+l)/4}^{l_0} |v_t^\varepsilon(x,t)|^2\,dx\\
&\quad + \left[-\frac{3(l-l_0)}{4}\gamma_2 + \delta_2 + b\delta_3\right]\int_{l_0}^{(3l_0+l)/4} |v_{xx}^\varepsilon(x,t)|^2\,dx\\
&\quad + \left[\frac{3(l-l_0)}{2}\gamma_2 - \frac{l-l_0}{48}\gamma_2 g_2(0)b + \delta_2 + b\delta_3\right]\int_{(3l_0+l)/4}^{l} |v_{xx}^\varepsilon(x,t)|^2\,dx\\
&\quad + C\int_{l_0}^{l} g_2(t)|v_{xx}^\varepsilon(x,t)|^2\,dx\\
&\quad + C\int_{l_0}^{l}\left[|v_x^\varepsilon(x,t)|^2 + |(g_2 * v_{xx}^\varepsilon)(x,t)|^2\,dx + (g_2\Box v_{xx}^\varepsilon)(x,t) + |(g_2 * v_{xxx}^\varepsilon)(x,t)|^2\right]\,dx.
\end{aligned}$$

Accordingly, by choosing δ_2, δ_3 small enough and a, b large enough, we have

$$\begin{aligned}
&\frac{d}{dt}\{I_2^u(t) + I_2^v(t) + a\,[4I_3^u(t) + g_1(0)I_4^u(t)] + b\,[4I_3^v(t) + g_1(0)I_4^v(t)]\}\\
&\le -\delta\int_0^{l_0} |u_t^\varepsilon(x,t)|^2\,dx - \delta\int_{l_0}^{l} |v_t^\varepsilon(x,t)|^2\,dx - \delta\int_0^{l_0} |u_{xx}^\varepsilon(x,t)|^2\,dx\\
&\quad - \delta\int_{l_0}^{l} |v_{xx}^\varepsilon(x,t)|^2\,dx + C\left[\int_0^{l_0} g_1(t)|u_{xx}^\varepsilon(x,t)|^2\,dx + \int_{l_0}^{l} g_2(t)|v_{xx}^\varepsilon(x,t)|^2\,dx\right]\\
&\quad + C\int_0^{l_0}\left[|u_x^\varepsilon(x,t)|^2 + |(g_1 * u_{xx}^\varepsilon)(x,t)|^2\,dx + (g_1\Box u_{xx}^\varepsilon)(x,t) + |(g_1 * u_{xxx}^\varepsilon)(x,t)|^2\right]\,dx\\
&\quad + C\int_{l_0}^{l}\left[|v_x^\varepsilon(x,t)|^2 + |(g_2 * v_{xx}^\varepsilon)(x,t)|^2\,dx + (g_2\Box v_{xx}^\varepsilon)(x,t) + |(g_2 * v_{xxx}^\varepsilon)(x,t)|^2\right]\,dx,
\end{aligned}$$

where δ is a suitable positive constant.

As shown in the proof of Proposition 4.3, we can find an analogous equation to (5.4), and then from (H.2) the following inequality holds:

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t, u^\varepsilon, v^\varepsilon) \le{}& -\frac{1}{2}\int_0^{l_0}\left[\alpha_1(g_1\Box u_{xx}^\varepsilon)(x,t) + g_1(t)|u_{xx}^\varepsilon(x,t)|^2\right]\,dx\\
&- \frac{1}{2}\int_{l_0}^{l}\left[\alpha_2(g_2\Box v_{xx}^\varepsilon)(x,t) + g_2(t)|v_{xx}^\varepsilon(x,t)|^2\right]\,dx - \varepsilon|u_t^\varepsilon(l_0,t) - v_t^\varepsilon(l_0,t)|^2.
\end{aligned}$$

We choose $\delta_1 = 1$ in Lemma 6.1 and we let

$$\begin{aligned}
\mathcal{F}(t) &= \frac{\delta}{2}\left[I_1^u(t) + I_1^v(t) + \frac{\varepsilon}{2}|u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t)|^2\right],\\
\mathcal{G}(t) &= I_2^u(t) + I_2^v(t) + a\left[4I_3^u(t) + g_1(0)I_4^u(t)\right] + b\left[4I_3^v(t) + g_1(0)I_4^v(t)\right],\\
\mathcal{L}(t) &= \mathcal{F}(t) + \mathcal{G}(t) + N\mathcal{E}(t),
\end{aligned}$$

where N, a and b are positive constants that we will choose later. We begin by noting that, by means of the Poincaré inequality, we obtain

$$\varepsilon|u^\varepsilon(l_0,t) - v^\varepsilon(l_0,t)|^2 \leq 2\left[\int_0^{l_0}|u^\varepsilon_{xx}(x,t)|^2\,dx + \int_0^{l_0}|v^\varepsilon_{xx}(x,t)|^2\,dx\right] \leq CE(t,u^\varepsilon,v^\varepsilon).$$

Similarly, one can easily check that if N is large enough, there exist two positive constants, K_1, K_2, independent of ε and t such that

$$K_1\mathcal{E}(t,u^\varepsilon,v^\varepsilon) \leq \mathcal{L}(t) \leq K_2\mathcal{E}(t,u^\varepsilon,v^\varepsilon). \tag{6.14}$$

Furthermore, by means of Lemma 6.1 we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}(t) + \frac{N}{2}\varepsilon|u_t^\varepsilon(l_0,t) - v_t^\varepsilon(l_0,t)|^2 \leq &-\frac{\delta}{2}\left[\int_0^{l_0}|u_t^\varepsilon(x,t)|^2\,dx + \int_{l_0}^{l}|v_t^\varepsilon(x,t)|^2\,dx\right]\\
&-\frac{\delta}{2}\left[\int_0^{l_0}|u^\varepsilon_{xx}(x,t)|^2\,dx + \int_{l_0}^{l}|v^\varepsilon_{xx}(x,t)|^2\,dx\right]\\
&-\frac{\delta}{2}\int_0^{l_0}G_1(t)|u^\varepsilon_{xx}(x,t)|^2\,dx - \frac{\delta}{2}\int_{l_0}^{l}G_2(t)|v^\varepsilon_{xx}(x,t)|^2\,dx - \delta\mathcal{J}^\varepsilon(t)\\
&-(\widetilde{\alpha}N - C)\left[\int_0^{l_0}g_1(t)|u^\varepsilon_{xx}(x,t)|^2\,dx + \int_{l_0}^{l}g_2(t)|v^\varepsilon_{xx}(x,t)|^2\,dx\right]\\
&-(\widetilde{\alpha}N - C)\left[\int_0^{l_0}(g_1\Box u^\varepsilon_{xx})(x,t)\,dx + \int_{l_0}^{l}(g_2\Box v^\varepsilon_{xx})(x,t)\,dx\right]\\
&+C\int_0^{l_0}\left[|u^\varepsilon_x(x,t)|^2 + |(g_1 * u^\varepsilon_{xx})(x,t)|^2\,dx + |(g_1 * u^\varepsilon_{xxx})(x,t)|^2\right]\,dx\\
&+C\int_{l_0}^{l}\left[|v^\varepsilon_x(x,t)|^2 + |(g_2 * v^\varepsilon_{xx})(x,t)|^2\,dx + |(g_2 * v^\varepsilon_{xxx})(x,t)|^2\right]\,dx,
\end{aligned}$$

where $\widetilde{\alpha} = \dfrac{1}{2}\min\{1, \alpha_1, \alpha_2\}$. We now fix $T > 0$ and integrate over $[r, T+r]$, with $r \geq 0$. By applying Lemma 6.4 over $[r, T+r]$, we have

$$\begin{aligned}
\mathcal{L}(T+r) \leq \mathcal{L}(r) - \left(\frac{N}{2} - \eta C\right) \varepsilon \int_r^{T+r} |u_t^\varepsilon(l_0,\tau) - v_t^\varepsilon(l_0,\tau)|^2 \, d\tau + \eta C \int_r^{T+r} \mathcal{E}(t, u^\varepsilon, v^\varepsilon)\, dt \\
- \frac{\delta}{2} \int_r^{T+r} \int_0^{l_0} \left[|u_t^\varepsilon(x,t)|^2 + |u_{xx}^\varepsilon(x,t)|^2 + G_1(t)|u_{xx}^\varepsilon(x,t)|^2 \right] dx\, dt \\
- \frac{\delta}{2} \int_r^{T+r} \int_{l_0}^{l} \left[|v_t^\varepsilon(x,t)|^2 + |v_{xx}^\varepsilon(x,t)|^2 + G_2(t)|v_{xx}^\varepsilon(x,t)|^2 \right] dx\, dt \\
- \delta \int_r^{T+r} \mathcal{J}^\varepsilon(t)\, dt - (\widetilde{\alpha} N - C) \int_r^{T+r} \int_0^{l_0} \left[g_1(t)|u_{xx}^\varepsilon(x,t)|^2 + (g_1 \Box u_{xx}^\varepsilon)(x,t) \right] dx\, dt \\
- (\widetilde{\alpha} N - C) \int_r^{T+r} \int_{l_0}^{l} \left[g_2(t)|v_{xx}^\varepsilon(x,t)|^2 + (g_2 \Box v_{xx}^\varepsilon)(x,t) \right] dx\, dt.
\end{aligned}$$

We choose η small enough and N large enough such that

$$c_2 := \min\{\delta, \tilde{\alpha} N - C\} - \eta C > 0 \quad \text{and} \quad \frac{N}{2} - \eta C > 0.$$

Accordingly, the estimate

$$\mathcal{L}(T+r) \leq \mathcal{L}(r) - c_2 \int_r^{T+r} \mathcal{E}(\tau, u^\varepsilon, v^\varepsilon)\, d\tau \tag{6.15}$$

holds. From (6.14), (6.15) and the nonincreasing character of $\mathcal{E}$, it follows that

$$\mathcal{L}(T+r) \leq \mathcal{L}(r) - c_2 T \mathcal{E}(T+r, u^\varepsilon, v^\varepsilon) \leq \mathcal{L}(r) - \frac{c_2 T}{K_2} \mathcal{L}(T+r),$$

which implies

$$\mathcal{L}(T+r) \leq \nu \mathcal{L}(r), \quad \text{with} \quad \nu = \left(1 + \frac{c_2 T}{K_2}\right)^{-1}. \tag{6.16}$$

Let $t > 0$. Thus, there exist $n \in \mathbb{N}, r \in (0, T)$ such that

$$t = nT + r. \tag{6.17}$$

By applying inequality (6.16) n times and recalling (6.14), we have

$$\mathcal{E}(t) \leq K_2 \mathcal{L}(nT + r) \leq K_2 \nu^n \mathcal{L}(r) \leq \frac{K_2}{K_1} \nu^n \mathcal{E}(r). \tag{6.18}$$

Since the energy $E(t)$ is decreasing and letting $\mu = -\dfrac{1}{T} \ln \nu$, $M = \dfrac{K_2}{K_1}\, \nu^{-r/T}$, we obtain

$$\mathcal{E}(t) \leq M \mathcal{E}(0) e^{-\mu t}.$$

Conditions (4.1) and (4.7) assure that $\mathcal{J}^\varepsilon(0) = 0$. Hence, the inequality

$$E(t, u^\varepsilon, v^\varepsilon) \leq \mathcal{E}(t) \leq M E(0, u^\varepsilon, v^\varepsilon) e^{-\mu t}$$

holds. We reach the conclusion by means of lower weak semicontinuity arguments.

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