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EXISTENCE AND UNIQUENESS FOR A TWO-POINT PROBLEM WITH AN APPLICATION TO THE ELECTRICAL HEATING IN AN ELECTROLYTE

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Abstract. We study a two-point boundary value problem for a first-order nonlinear differential equation depending on a parameter. The results of existence and uniqueness obtained are applied to the determination of the electric potential and temperature in an electrolytic solution contained in a vessel of arbitrary shape.

1. Introduction. We study here the following nonlinear two-point problem (TPP)

$$\frac{dn}{d\phi} = \gamma\sigma(\phi, n, \gamma), \tag{1.1}$$

$$n(0) = n_1,\ n(\bar{\phi}) = n_2, \tag{1.2}$$

where $n(\phi)$ and the real constant γ are the unknowns. We assume

$$n_2 \geq n_1 \geq 0,\ \bar{\phi} > 0. \tag{1.3}$$

The other cases can be dealt with similarly. In Section 2 we prove that (TPP) has at least one solution if $\sigma(\phi, n, \gamma)$ is continuous and

$$\sigma_1 \geq \sigma(\phi, n, \gamma) \geq \sigma_0 > 0 \text{ for all } \phi \in [0, \bar{\phi}],\ n \in [0, \infty],\ \gamma \in [0, \infty] \tag{1.4}$$

and that the solution is unique if, in addition to (1.4), we have

$$\sigma(\phi, n, \gamma) + \gamma\sigma_\gamma(\phi, n, \gamma) \neq 0 \text{ for all } \phi \in [0, \bar{\phi}],\ n \in [0, \infty],\ \gamma \in [0, \infty]. \tag{1.5}$$

The theory is applied in Section 3 to find the electric potential and the temperature in an electrolytic solution under the assumptions of ambipolar and electroneutral diffusion of ions.

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2. Existence and uniqueness for problem (TPP).

LEMMA 2.1. If (1.4) holds, problem (TPP) has at least one solution.

Proof. It is easily seen that problem (TPP) can be rewritten in a weak form as the following Volterra-Fredholm integral equation:

$$n(\phi) = n_1 + (n_2 - n_1)\frac{\int_0^\phi \sigma(t, n(t), \gamma)dt}{\int_0^{\bar\phi} \sigma(t, n(t), \gamma)dt} \tag{2.1}$$

and that any solution $n(\phi)$ of (2.1) satisfies the a priori estimates

$$0 \le n(\phi) \le n_1 + (n_1 - n_2)\frac{\sigma_1}{\sigma_0}, \tag{2.2}$$

$$0 \le \frac{dn}{d\phi} \le (n_2 - n_1)\frac{\sigma_1}{\sigma_0\bar\phi}. \tag{2.3}$$

Moreover, we have

$$0 \le \gamma \le (n_2 - n_1)\frac{1}{\sigma_0\bar\phi}. \tag{2.4}$$

We will use the Schauder fixed-point theorem. Define

$$K_C = \left\{n(\phi) \in C^0([0, \bar\phi]),\ 0 \le n_1 + (n_2 - n_1)\frac{\sigma_1}{\sigma_0}\right\}$$

and

$$K_R = \left\{\gamma \in \mathbf{R}^1,\ 0 \le \gamma \le (n_2 - n_1)\frac{1}{\sigma_0\bar\phi}\right\}.$$

Let $\mathcal{B} = C^0([0, \bar\phi]) \times \mathbf{R}^1$. Then

$$K = \{(n(\phi), \gamma) \in \mathcal{B}, n(\phi) \in K_C,\ \gamma \in K_R\}$$

is a closed and convex subset of the Banach space $\mathcal{B}$. Define the operator

$$\mathcal{T} : K \subset \mathcal{B} \to \mathcal{B},\ (\tilde n(\phi), \tilde\gamma) = \mathcal{T}((n(\phi), \gamma)) = (\mathcal{T}_1((n(\phi), \gamma)), \mathcal{T}_2((n(\phi), \gamma)))$$

with

$$\tilde n(\phi) = \mathcal{T}_1(n(\phi), \gamma) = n_1 + (n_2 - n_1)\frac{\int_0^\phi \sigma(t, n(t), \gamma)dt}{\int_0^{\bar\phi} \sigma(t, n(t), \gamma)dt}, \tag{2.5}$$

$$\tilde\gamma = \mathcal{T}_2(n(\phi), \gamma) = \frac{(n_2 - n_1)}{\int_0^{\bar\phi} \sigma(t, n(t), \gamma)dt}. \tag{2.6}$$

We have $\mathcal{T}(K) \subset K$. Moreover $\mathcal{T}$ is compact. For, from (2.5) we obtain

$$\frac{d\tilde n}{d\phi}(\phi) = (n_2 - n_1)\frac{\sigma(\phi, n(\phi), \gamma)}{\int_0^{\bar\phi} \sigma(t, n(t), \gamma)dt} \tag{2.7}$$

and by (1.4),

$$0 \le \frac{d\tilde n}{d\phi}(\phi) \le \frac{(n_2 - n_1)\sigma_1}{\sigma_0\bar\phi}. \tag{2.8}$$

Hence the functions of $\mathcal{T}_1(K)$ are equibounded and equicontinuous and, by Arzelà's theorem, $\mathcal{T}_1(K)$ is compact in $C^0([0,\phi])$. By (2.6) we have

$$0 \le \tilde{\gamma} \le \frac{(n_2 - n_1)}{\sigma_0 \bar{\phi}}. \tag{2.9}$$

Thus $\mathcal{T}$ is compact. On the other hand, it is easily seen that

$$(n_k(\phi), \gamma_k) \to (n^*(\phi), \gamma^*) \text{ in } \mathcal{B} \text{ as } k \to \infty$$

implies

$$\mathcal{T}((n_k(\phi), \gamma_k)) \to \mathcal{T}((n^*(\phi), \gamma^*)) \text{ in } \mathcal{B} \text{ as } k \to \infty.$$

Thus $\mathcal{T}$ is continuous and, by Schauder's theorem, (2.1) has at least one solution which is absolutely continuous and, therefore, a solution to (TPP) as well. □

Remark 2.2. If condition (1.4) is not satisfied, problem (TPP) may have no solutions. For example,

$$\frac{dn}{d\phi} = \frac{\gamma}{1+\gamma^2},\ n(0) = 0,\ n(1) = k$$

has no solution if $k > 1/2$. On the other hand, even if (1.4) is satisfied, the solution might not be unique. Consider

$$\frac{dn}{d\phi} = \gamma(2 - \sin\gamma),\ n(0) = 0,\ n(1) = 18. \tag{2.10}$$

We have $n(\phi,\gamma) = \gamma(2-\sin\gamma)\phi$. The equation $n(1,\gamma) = 18$ has three real solutions $\gamma_k,\ k = 1,2,3$. Correspondingly problem (2.10) has three solutions $n(\phi,\gamma_k) = \gamma_k(2 - \sin\gamma_k)\phi,\ k = 1,2,3$. It is therefore interesting to give a result of existence and uniqueness for problem (TPP). We use the following special case of the Levy-Caccioppoli global inversion theorem; see [4], [2] and [1].

We recall that a map $F : X \to Y$ is proper if for every compact set $K \subset Y$, $F^{-1}(K)$ is compact in X.

Theorem 2.3. Let $X = \{n(\phi) \in C^1([0,\bar{\phi}]),\ n(0) = n_1,\ n(\bar{\phi}) = n_2\} \times \mathbf{R}^1$, $Y = C^0([0,\bar{\phi}])$. Assume $F(n,\gamma) \in C^1(X,Y)$ to be locally invertible in all of X and proper. Then F is a diffeomorphism from X onto Y.

We prove for later use the following elementary

Lemma 2.4. If $P(\phi),\ Q_1(\phi)$ and $Q_2(\phi)$ are continuous functions defined in $[0,\bar{\phi}]$ and

$$Q_1(\phi) \ne 0 \text{ in } [0,\bar{\phi}], \tag{2.11}$$

then the linear two-point problem

$$\frac{dN}{d\phi} + P(\phi)N = \Gamma Q_1(\phi) + Q_2(\phi), \tag{2.12}$$

$$N(0) = 0, \tag{2.13}$$

$$N(\bar{\phi}) = 0 \tag{2.14}$$

has one and only one solution $(N(\phi), \Gamma)$.

Proof. The solution of the Cauchy problem (2.12), (2.13) is given by

$$N(\phi,\Gamma) = e^{-\int_0^\phi P(t)dt} \int_0^\phi e^{\int_0^\tau P(t)dt}(\Gamma Q_1(\tau) + Q_2(\tau))d\tau. \tag{2.15}$$

By (2.11) the equation $N(\bar\phi,\Gamma) = 0$ has one and only one solution Γ^*. Hence $(N(\phi,\Gamma^*),\Gamma^*)$ is the only solution of the two-point problem. □

LEMMA 2.5. If $\sigma(\phi,n,\gamma) \in C^1([0,\bar\phi] \times [0,\infty) \times [0,\infty))$,

$$\sigma_1 \geq \sigma(\phi,n,\gamma) \geq \sigma_0 > 0, \text{ for all } \phi \in [0,\bar\phi],\ n \in [0,\infty],\ \gamma \in [0,\infty] \tag{2.16}$$

and

$$\sigma(\phi,n,\gamma) + \gamma\sigma_\gamma(\phi,n,\gamma) \neq 0 \tag{2.17}$$

for ϕ, n and γ as in (2.16) hold, then the problem

$$\frac{dn}{d\phi} = \gamma\sigma(\phi,n,\gamma),\ n(0) = n_1,\ n(\bar\phi) = n_2 \tag{2.18}$$

has one and only one solution.

Proof. Let X and Y be as in Theorem 2.3 and define

$$F : X \to Y,\ F(n\gamma) = \frac{dn}{d\phi} - \gamma\sigma(\phi,n,\gamma).$$

We use the implicit function theorem to prove that F is locally invertible. To this end we compute the differential $dF(n,\gamma)[N,\Gamma]$, $(n,\gamma) \in X$, $N(\phi) \in C^1([0,\bar\phi])$, $N(0) = 0$, $N(\bar\phi) = 0$. We find

$$dF(n,\gamma)[N,\Gamma] = \frac{dN}{d\phi} - \big[\sigma(\phi,n,\gamma) + \gamma\sigma_\gamma(\phi,n,\gamma)\big]\Gamma - \gamma\sigma_n(\phi,n,\gamma)N. \tag{2.19}$$

F is locally invertible since the linear two-point problem

$$\frac{dN}{d\phi} + P(\phi)N = \Gamma Q_1(\phi) + Q_2(\phi), \tag{2.20}$$

$$N(0) = 0,\ N(\bar\phi) = 0, \tag{2.21}$$

where

$$P(\phi) = -\gamma\sigma_n(\phi,n,\gamma),$$
$$Q_1(\phi) = \sigma(\phi,n,\gamma) + \gamma\sigma_\gamma(\phi,n,\gamma),$$
$$Q_2(\phi) = G(\phi)$$

and $G(\phi) \in X$ is uniquely solvable by Lemma 2.4 in view of (1.5). It remains to show that F is a proper map. Let K be a compact subset of $Y = C^0([0,\bar\phi])$. By Arzelà's theorem, the functions $\tilde f(\phi) \in K$ are equibounded and equicontinuous. Let $(\tilde n(\phi),\tilde\gamma) \in F^{-1}(K)$. We have

$$\frac{d\tilde n}{d\phi} = \tilde\gamma\sigma(\phi,\tilde n,\tilde\gamma) + \tilde f(\phi), \tag{2.22}$$

$$\tilde n(0) = n_1,\ \tilde n(\bar\phi) = n_2. \tag{2.23}$$

From (2.22) we obtain

$$\tilde{\gamma} = \Big(\int_0^{\bar{\phi}} \sigma(t, \tilde{n}(t), \tilde{\gamma})dt\Big)^{-1}\Big(n_2 - n_1 - \int_0^{\bar{\phi}} \tilde{f}(t)dt\Big). \tag{2.24}$$

Thus, by (1.4) there exist two constants C_1 and C_2 such that

$$|\tilde{\gamma}| \leq C_1, \tag{2.25}$$

$$\Big|\frac{d\tilde{n}}{d\phi}\Big| \leq C_2. \tag{2.26}$$

Hence the functions $\tilde{n}(\phi)$ are equibounded and equicontinuous. Let

$$|\tilde{n}(\phi)| \leq C_3. \tag{2.27}$$

We claim that the set $\{\frac{d\tilde{n}}{d\phi}\}$ is also made up of equicontinuous functions. For, letting $\phi_1,\ \phi_2 \in [0, \bar{\phi}]$, we have, by (2.22),

$$\frac{d\tilde{n}}{d\phi}(\phi_2) - \frac{d\tilde{n}}{d\phi}(\phi_1) = I_1 + I_2 = I_3, \tag{2.28}$$

where

$$\begin{aligned} I_1 &= \tilde{\gamma}(\sigma(\phi_2, \tilde{n}(\phi_2), \tilde{\gamma}) - \sigma(\phi_2, \tilde{n}(\phi_1), \tilde{\gamma})), \\ I_2 &= \tilde{\gamma}(\sigma(\phi_2, \tilde{n}(\phi_1), \tilde{\gamma}) - \sigma(\phi_1, \tilde{n}(\phi_1), \tilde{\gamma})), \\ I_3 &= \tilde{f}(\phi_2) - \tilde{f}(\phi_1). \end{aligned}$$

To estimate I_1 we set

$$\alpha = \sup\Big\{\Big|\frac{\partial\sigma}{\partial n}(\phi, n, \tilde{\gamma})\Big| + \Big|\frac{\partial\sigma}{\partial\phi}(\phi, n, \tilde{\gamma})\Big|,\ \phi \in [0, \bar{\phi}],\ n \in [-C_3, C_3],\ \tilde{\gamma} \in [-C_2, C_2]\Big\}.$$

Thus

$$|\sigma(\phi_2, \tilde{n}(\phi_2), \tilde{\gamma}) - \sigma(\phi_1, \tilde{n}(\phi_1), \tilde{\gamma})| \leq \alpha\big(|\phi_2 - \phi_1| + \tilde{n}(\phi_2) - \tilde{n}(\phi_1)|\big). \tag{2.29}$$

Since $\tilde{\gamma}$, σ and $\tilde{f}$ are bounded, we obtain from (2.28) that

$$\Big|\frac{d\tilde{n}}{d\phi}(\phi_2) - \frac{d\tilde{n}}{d\phi}(\phi_1)\Big| \leq C\big(|\phi_2 - \phi_1| + |\tilde{n}(\phi_2) - \tilde{n}(\phi_1)| + |\tilde{f}(p_2) - \tilde{f}(\phi_1)|\big). \tag{2.30}$$

By the equicontinuity of $\{\tilde{f}(\phi)\}$ and $\{\tilde{n}(\phi)\}$ we conclude that the functions in $\{\frac{d\tilde{n}}{d\phi}(\phi)\}$ are also equicontinuous. Therefore F^{-1} is compact and F is a proper map. Hence, by Theorem 2.3, problem (2.18) has one and only one solution. □

Two-point problems for systems of n first-order differential equations depending on n parameters are treated with similar methods in [3].

3. Functional solutions for the ambipolar diffusion of ions with production of heat. Let $\mathbf{J}_+,\ \mathbf{J}_-$ be the densities of flow of anions and cations in an electrolytic solution, where $p(\mathbf{x})$, $n(\mathbf{x})$ are the corresponding concentrations. z_+ and z_- are integer numbers: $z_+ > 0$ is the charge number of anions and $z_- < 0$ the charge number of cations. In the approximation of ambipolar and electroneutral diffusion, $\mathbf{J}_+$ and $\mathbf{J}_-$ are assumed to be equal to a common value $\mathbf{I}$ and the relation $z_+p + z_-n = 0$ is assumed to hold. These hypotheses lead to a considerable simplification in the Nernst-Planck equations. For, they give

$$\mathbf{I} = -D\nabla n, \tag{3.1}$$

where D is the ambipolar diffusion coefficient [5] which depends on both n and on the temperature u. By Faraday's law ([5], [6]) we have for the total electric current density $\mathbf{J}$,

$$\mathbf{J} = z_+\mathbf{J}_+ + z_-\mathbf{J}_- = (z_+ + z_-)\mathbf{I}.$$

Thus, by (3.1) we have

$$\mathbf{J} = -(z_+ + z_-)D\nabla n. \tag{3.2}$$

Moreover, the density of heat flow is given by [7]

$$\mathbf{q} = -\kappa\nabla u + 2K_B(u + T_0)\mathbf{I} + \phi\mathbf{J}, \tag{3.3}$$

where $u+T_0$ is the absolute temperature, $\kappa > 0$ the constant thermal conductivity, ϕ the electric potential and K_B the Boltzmann constant . The second term on the right-hand side of (3.3) corresponds to the heat generated by compression and attrition in the fluid, whereas the third term is related to the Joule heating. In view of (3.1), and (3.2), we have

$$\mathbf{q} = -\big[\kappa\nabla u + 2K_B(u + T_0)D(n,u)\nabla n + \phi(z_+ + z_-)D(n,u)\nabla n\big]. \tag{3.4}$$

Under stationary conditions we have

$$\nabla \cdot \mathbf{q} = 0,\ \ \nabla \cdot \mathbf{J} = 0. \tag{3.5}$$

The fixed region of space occupied by the fluid is represented by a bounded and open subset of $\mathbf{R}^3$. The boundary Γ of Ω is of class C^2 and consists of three parts Γ_0, Γ_1 and Γ_2. Γ_0 is the part of the boundary which is thermally and electrically insulated, whereas Γ_1 and Γ_2 represent the two disjoint electrodes to which a difference of potential $\bar{\phi}$ is applied. Thus we arrive, for the determination of $n(\mathbf{x})$ and $u(\mathbf{x})$, $\mathbf{x} = (x_1, x_2, x_3)$, at the following boundary value problem (P):

$$\nabla \cdot (D(n,u)\nabla n) = 0 \text{ in } \Omega, \tag{3.6}$$

$$n = 0 \text{ on } \Gamma_1,\ \ n = \bar{n} \text{ on } \Gamma_2,\ \ \frac{\partial n}{\partial \nu} = 0 \text{ on } \Gamma_0, \tag{3.7}$$

$$\nabla \cdot \big[\nabla u + K(u + T_0)D(n,u)\nabla n + \tau\phi D(n,u)\nabla n\big] = 0 \text{ in } \Omega, \tag{3.8}$$

$$u = 0 \text{ on } \Gamma_1,\ \ u = \bar{u} \text{ on } \Gamma_2,\ \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0, \tag{3.9}$$

$$\Delta\phi = 0 \text{ in } \Omega, \tag{3.10}$$

$$\phi = 0 \text{ on } \Gamma_1,\ \ \phi = \bar{\phi} \text{ on } \Gamma_2,\ \ \frac{\partial \phi}{\partial \nu} = 0 \text{ on } \Gamma_0, \tag{3.11}$$

where $K = \frac{2K_B}{\kappa}$, $\tau = \frac{z_+ + z_-}{\kappa}$. Moreover $\bar{n}$, $\bar{u}$ and $\bar{\phi}$ are given positive constants and $\frac{\partial}{\partial \nu}$ denotes the normal derivative to Γ_0. We assume hereafter that

$$0 < D_0 \le D(n,u) \le D_1. \tag{3.12}$$

The following a priori estimates for the regular solutions of problem (P) are immediate consequences of the maximum principle and of (3.12):

$$0 \leq n(\mathbf{x}) \leq \bar{n}, \quad 0 \leq \phi(\mathbf{x}) \leq \bar{\phi}. \tag{3.13}$$

The special, but physically acceptable, boundary conditions (3.7), (3.9), (3.11) and the autonomous nature of problem (P) suggest the search of functional solutions according to the following

Definition 3.1. If there exist two functions $N(\phi)$ and $U(\phi)$ of class $C^2([0, \bar{\phi}])$ such that $(n(\mathbf{x}), u(\mathbf{x}), \phi(\mathbf{x})) = (N(\phi(\mathbf{x})), U(\phi(\mathbf{x})), \phi(\mathbf{x}))$ is a solution of problem (P), we say that $(n(\mathbf{x}), u(\mathbf{x}), \phi(\mathbf{x}))$ is a functional solution.

Let $(n(\mathbf{x}), u(\mathbf{x}), \phi(\mathbf{x}))$ be a functional solution of problem (P). Define

$$\theta = \int_0^{\phi} D(N(t), U(t)) \frac{dN}{d\phi} dt, \tag{3.14}$$

$$\psi = \int_0^{\phi} \frac{dU}{d\phi} + K(U(t) + T_0) D(N(t), U(t)) \frac{dN}{d\phi} + \tau t D(N(t), U(t)) \frac{dN}{d\phi} dt. \tag{3.15}$$

Set

$$\Theta(\mathbf{x}) = \theta(\phi(\mathbf{x})), \tag{3.16}$$

$$\Psi(\mathbf{x}) = \psi(\phi(\mathbf{x})). \tag{3.17}$$

We have

$$\nabla \Theta = D(n(\mathbf{x}), u(\mathbf{x})) \nabla n, \tag{3.18}$$

$$\nabla \Psi(\mathbf{x}) = \nabla u(\mathbf{x}) + K(u(\mathbf{x}) + T_0) D(n(\mathbf{x}), u(\mathbf{x})) \nabla n + \tau \phi D(n(\mathbf{x}), u(\mathbf{x})) \nabla n. \tag{3.19}$$

Hence, in view of (3.6)–(3.9), we have

$$\Delta \Theta = 0 \text{ in } \Omega, \tag{3.20}$$

$$\Delta \Psi = 0 \text{ in } \Omega, \tag{3.21}$$

$$\Theta = 0 \text{ on } \Gamma_1, \ \Theta = C_1 \text{ on } \Gamma_2, \ \frac{\partial \Theta}{\partial \nu} = 0 \text{ on } \Gamma_0, \tag{3.22}$$

where from (3.14),

$$C_1 = \int_0^{\bar{\phi}} D(N(t), U(t)) \frac{dN}{d\phi} dt.$$

Moreover,

$$(e^{-K\gamma\bar{\phi}} - 1)^2 \Psi = 0 \text{ on } \Gamma_1, \ \Psi = C_2 \text{ on } \Gamma_2, \ \frac{\partial \Psi}{\partial \nu} = 0 \text{ on } \Gamma_0, \tag{3.23}$$

where from (3.15),

$$C_2 = \int_0^{\bar{\phi}} \frac{dU}{d\phi} + K(U(t) + T_0) D(N(t), U(t)) \frac{dN}{d\phi} + \tau t D(N(t), U(t)) \frac{dN}{d\phi} dt.$$

By (3.7) we have

$$\frac{\partial \Theta}{\partial \nu} = 0 \text{ on } \Gamma_0 \tag{3.24}$$

and by (3.9)

$$\frac{\partial \Psi}{\partial \nu} = 0 \text{ on } \Gamma_0. \tag{3.25}$$

Let $\phi(\mathbf{x})$ be the solution of the problem

$$\Delta\phi = 0 \text{ in } \Omega,\ \phi = 0 \text{ on } \Gamma_1,\ \phi = \bar{\phi} \text{ on } \Gamma_2,\ \frac{\partial\phi}{\partial\nu} = 0 \text{ on } \Gamma_0. \tag{3.26}$$

Define

$$\gamma = \frac{C_1}{\bar{\phi}},\ \mu = \frac{C_2}{\bar{\phi}}. \tag{3.27}$$

We have

$$\Theta(\mathbf{x}) = \gamma\phi(\mathbf{x}), \tag{3.28}$$

$$\Psi(\mathbf{x}) = \mu\phi(\mathbf{x}). \tag{3.29}$$

Moreover, by (3.14) and (3.28) we obtain

$$\int_0^\phi D(N(t), U(t))\frac{dN}{d\phi}(t)dt = \gamma\phi \tag{3.30}$$

and, by (3.15) and (3.29),

$$\int_0^\phi \frac{dU}{d\phi} + K(U(t) + T_0)D(N(t), N(t))\frac{dN}{d\phi} + \tau t D(N(t), N(t))\frac{dN}{d\phi}dt = \mu\phi. \tag{3.31}$$

Taking the derivative of (3.30) and (3.31) with respect to ϕ we obtain the system of ordinary differential equations,

$$D(N, U)\frac{dN}{d\phi} = \gamma, \tag{3.32}$$

$$\frac{dU}{d\phi} + K(U + T_0)D(N, U)\frac{dN}{d\phi}(t) + \tau\phi D(N, U)\frac{dN}{d\phi}dt = \mu. \tag{3.33}$$

As boundary conditions for $N(\phi)$, $U(\phi)$, recalling (3.7) and (3.9), we have

$$N(0) = 0,\ N(\bar{\phi}) = \bar{n}, \tag{3.34}$$

$$U(0) = 0,\ U(\bar{\phi}) = \bar{u}. \tag{3.35}$$

The previous considerations justify the next lemmas. For a more formal proof we refer to [3].

Lemma 3.2. The functional solutions of problem (P) are in a one-to-one correspondence with the solutions of the two-point problem (TPP).

Lemma 3.3. If the solutions of the two-point problem (TPP) can be found explicitly and the solution of the Dirichlet problem

$$\Delta z = 0 \text{ in } \Omega,\ z = 0 \text{ on } \Gamma_1,\ z = 1 \text{ on } \Gamma_2,\ \frac{\partial z}{\partial\nu} = 0 \text{ on } \Gamma_0 \tag{3.36}$$

is known, the corresponding functional solution of problem (P) is also explicitly known and is given by $(n(\mathbf{x}), u(\mathbf{x}), \phi(\mathbf{x})) = (N(\bar{\phi}z(\mathbf{x})), U(\bar{\phi}z(\mathbf{x})), \phi(\mathbf{x}))$.

Remark 3.4. The two-point problem (TPP) does not depend on Ω and contains so-to-speak the nonlinear part of problem (P). On the other hand, the Dirichlet problem (3.36) reflects the peculiarities of the domain Ω.

Remark 3.5. A question naturally arises. Can problem (P) have solutions which are not functional solutions? A partial answer is given in [3].

We apply hereafter the results of Section 2 to prove that the boundary value problem (P) has a solution which, in certain cases, is unique in the class of functional solutions. To this end we study the two-point problem (3.32)–(3.35). Substituting (3.32) in (3.33) we obtain

$$\frac{dU}{d\phi} + K(U + T_0)\gamma + \tau\phi\gamma = \mu, \tag{3.37}$$

$$U(0) = 0,\ U(\bar{\phi}) = \bar{u}. \tag{3.38}$$

The solution of (3.37), (3.38), retaining γ as a parameter, is easily computed and is given by

$$\tilde{U}(\phi,\gamma) = \frac{1}{K(1 - e^{-K\gamma\bar{\phi}})}\Big[\tau(\bar{\phi} - \phi) + \tau(\phi e^{-K\gamma\bar{\phi}} - \bar{\phi}e^{-K\gamma\phi}) + K\bar{u}(1 - e^{-K\gamma\phi})\Big]. \tag{3.39}$$

Substituting in (3.32) we obtain

$$D(N, \tilde{U}(\phi,\gamma))\frac{dN}{d\phi} = \gamma, \tag{3.40}$$

$$N(0) = 0,\ N(\bar{\phi}) = \bar{n}. \tag{3.41}$$

We apply Lemma 2.1 and Lemma 2.5 to the problem (3.40) and (3.41) in the following slightly different form.

LEMMA 3.6. If $A(\phi, n, \gamma) \in C^1([0,\bar{\phi}] \times [0,\infty) \times [0,\infty))$ satisfies

$$A_1 \geq A(\phi, n, \gamma) \geq A_0 > 0 \text{ for all } \phi \in [0,\bar{\phi}],\ n \in [0,\infty),\ \gamma \in [0,\infty), \tag{3.42}$$

then the problem

$$A(\phi, n, \gamma)\frac{dn}{d\phi} = \gamma,\ n(0) = 0,\ n(\bar{\phi}) = \bar{n} \tag{3.43}$$

has at least one solution. If, in addition to (3.42), we have

$$A(\phi, n, \gamma) - \gamma A_\gamma(\phi, n, \gamma) \neq 0 \text{ for all } \phi \in [0,\bar{\phi}],\ n \in [0,\infty),\ \gamma \in [0,\infty), \tag{3.44}$$

then the solution is unique.

Proof. Simply note that, by setting $\sigma(\phi, n, \gamma) = \frac{1}{A(\phi,n,\gamma)}$, condition (2.17) becomes (3.44). □

Setting

$$A(\phi, N, \gamma) = D(N, \tilde{U}(\phi,\gamma)),$$

we arrive at a problem of the form (3.43). Thus, we have

THEOREM 3.7. If

$$D_1 \geq D(n, u) \geq D_0 > 0,$$

then problem (P) has at least one functional solution.

To present a result of uniqueness for problem (P), using Lemma 3.6, we need to evaluate the expression on the right-hand side of (3.44) which, for the case at hand, takes the form

$$A(\phi, N, \gamma) - \gamma A_\gamma(\phi, N, \gamma) = D(N, \tilde{U}(\phi,\gamma)) - \gamma D_u(N, \tilde{U}(\phi,\gamma))\frac{\partial\tilde{U}}{\partial\gamma}. \tag{3.45}$$

Substituting

$$\frac{\partial \tilde{U}}{\partial \gamma} = \frac{\tau\bar{\phi} + K\bar{u}}{(e^{-K\gamma\bar{\phi}} - 1)^2}\Big[\phi e^{-K\gamma\phi}\Big(1 - e^{-K\gamma\bar{\phi}}\Big) + \bar{\phi}e^{-K\gamma\bar{\phi}}\Big(e^{-K\gamma\phi} - 1\Big)\Big]$$

in (3.45) we have

$$A(\phi, N, \gamma) - \gamma A_\gamma(\phi, N, \gamma) = D(N, \tilde{U}(\phi, \gamma)) - \gamma D_u(N, \tilde{U}(\phi, \gamma))(\tau\bar{\phi} + K\bar{u})f(\phi, \bar{\phi}, \gamma, K), \tag{3.46}$$

where

$$f(\phi, \bar{\phi}, \gamma, K) = \frac{\phi e^{-K\gamma\phi}\Big(1 - e^{-K\gamma\bar{\phi}}\Big) - \bar{\phi}e^{-K\gamma\bar{\phi}}\Big(1 - e^{-K\gamma\phi}\Big)}{(e^{-K\gamma\bar{\phi}} - 1)^2}.$$

We note that

$$f(\phi, \bar{\phi}, \gamma, K) > 0 \text{ for } \phi \in (0, \bar{\phi}) \text{ and } \gamma > 0,\ K > 0. \tag{3.47}$$

Therefore the condition

$$A(\phi, N, \gamma) - \gamma A_\gamma(\phi, N, \gamma) \neq 0 \tag{3.48}$$

is satisfied when

$$D_u(n, u)(\tau\bar{\phi} + K\bar{u}) \leq 0.$$

Hence, we obtain

Theorem 3.8. If

$$D_1 \geq D(n, u) \geq D_0 > 0$$

and

$$D_u(n, u)(\tau\bar{\phi} + K\bar{u}) \leq 0,$$

then problem (P) has one and only one functional solution.

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