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DECAY OF MASS FOR FRACTIONAL EVOLUTION EQUATION WITH MEMORY TERM

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Abstract. We investigate the decay properties of the mass $M(t) = \int_{\mathbb{R}^N} u(\cdot, t)dx$ of the solutions of a fractional diffusion equation with nonlinear memory term. We show, using a suitable class of initial data and a restriction on the diffusion and nonlinear term, that the memory term determines the large time asymptotics; that is, $M(t)$ tends to zero as $t \to \infty$.

1. Introduction and main result. In recent years, researchers have shown a considerable interest in the so-called *fractional calculus* which allows us to consider integration and differentiation of any order, not necessarily integer. This interest is due to the applications of this theory to problems in different areas of physics and engineering. Important applications of this theory can be found in various fields such as viscoelastic materials and heat conduction in materials with memory [18, 21, 27].

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The purpose of this paper is to study the large time behavior of solutions of the fractional diffusion equation with a nonlinear memory term

$$\begin{cases} u_t + (-\Delta)^{\beta/2} u = -\dfrac{1}{\Gamma(1-\gamma)} \displaystyle\int_0^t (t-s)^{-\gamma} |u|^{p-1} u(s)\, ds, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u_0 \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $N \geq 1$, $0 < \beta \leq 2$, $\gamma \in (0,1)$, $p > 1$ and the pseudo-differential operator $(-\Delta)^{\beta/2}$ is defined by the Fourier transformation:

$$\left[(-\Delta)^{\beta/2} v(x) := \mathcal{F}^{-1}\left(|\xi|^\beta \mathcal{F}(v)(\xi)\right)(x)\right], \quad v \in D((-\Delta)^{\beta/2}) = H^\beta(\mathbb{R}^N).$$

Here $H^\beta(\mathbb{R}^N)$ is the homogeneous Sobolev space of order β defined by

$$H^\beta(\mathbb{R}^N) = \left\{ u \in \mathcal{S}' \ :\ (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N) \right\}, \qquad \text{if } \beta \notin \mathbb{N},$$

$$H^\beta(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) \ :\ (-\Delta)^{\beta/2} u \in L^2(\mathbb{R}^N) \right\}, \quad \text{if } \beta \in \mathbb{N},$$

where $\mathcal{S}'$ is the space of Schwartz distributions. The terms $\mathcal{F}$ and $\mathcal{F}^{-1}$ are respectively the Fourier transform and its inverse. Γ is the Euler gamma function and $C_0(\mathbb{R}^N)$ denotes the space of all continuous functions tending to zero at infinity.

As a physical motivation, the right-hand side of (1.1) might be interpreted as the effect of a classically diffusive medium that is nonlinearly linked to a super-diffusive medium. Such a link might come in the form of a porous material with reactive properties that is partially insulated by contact with a classically diffusive material. More details are given in a recent paper of Roberts and Olmstead [23]. Furthermore, nonlinear evolution problems involving the fractional Laplacian describing *the anomalous diffusion* (or β-stable Lévy diffusion) have been extensively studied in the mathematical and physical literature [2, 6, 13]. One of the possible ways to understand the interaction between the anomalous diffusion operator (given by $(-\Delta)^{\beta/2}$ or, more generally, by the Lévy diffusion operator) and the nonlinearity in the equation (1.1) is the study of the large time asymptotics of solutions to such equations.

In this paper, we aim to give a contribution to this theory by generalizing a recent work of Fino and Karch in [8] for the approximate equation (1.1). The nonlinear memory term can be considered as an approximation of the nonlinear term of the classical semilinear parabolic equation

$$u_t + (-\Delta)^{\beta/2} u = \lambda |u|^{p-1} u, \tag{1.2}$$

with $\lambda = -1$ since the limit

$$\lim_{\gamma \to 1} \frac{1}{\Gamma(1-\gamma)} s_+^{-\gamma} = \delta(s)$$

exists in the distributional sense where $s_+ := \max(0, s)$.

In [8], Fino and Karch have proved that, for (1.2) with $\lambda = -1$, $\lim_{t\to\infty} M(t) = M_\infty > 0$ for $p > 1 + \beta/N$, where $M(t) := \int_{\mathbb{R}^N} u(\cdot, t) dx$, while $M(t)$ tends to zero as $t \to \infty$ if $1 < p \leq 1 + \beta/N$. The exponent $1 + \beta/N$ is called the Fujita critical exponent [10].

The approach which allows us to express the competition between diffusive and nonlinear terms in an evolution equation by studying the large time behavior of the space integral of a solution was introduced by Ben-Artzi and Koch [1], who considered the viscous Hamilton-Jacobi equation $u_t = \Delta u - |\nabla u|^p$ (see also Pinsky [22]). An analogous result for the equation $u_t = \Delta u + |\nabla u|^p$ (with the growing-in-time mass of solutions) was proved by Laurençot and Souplet [17]. Such questions concerning the asymptotic behavior of solutions to the Hamilton-Jacobi equation with the Lévy diffusion operator were answered by Karch and Woyczyński in [13].

For $\lambda = +1$, global existence and nonexistence of solutions of equation (1.2) have been considered, using probabilistic approaches, by Nagasawa and Sirao [20] and Sugitani [24] respectively. Using properties of semigroups, local and global existence were studied by Weissler [29]. For the blowup of solutions we refer the reader to Kirane et al. [11, 15] and Fino and Karch [8] where the test function method was used.

1.1. *The main result.* We assume that $u = u(x,t)$ is the nonnegative mild solution (see Definition 2.1) of problem (1.1) corresponding to the nonnegative initial datum

$$u_0 \in C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N). \tag{1.3}$$

In fact, using Theorem 2.3 below, one can obtain a unique mild solution $u \in C([0,\infty), C_0(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$ of (1.1). Moreover, a maximum principle (see Proposition 3.2) shows that the solution u is nonnegative if the corresponding initial datum is so.

In this paper, we deal with (1.1) containing the absorbing nonlinearity memory term and we study the decay of the "mass"

$$M(t) \equiv \int_{\mathbb{R}^N} u(x,t)\,dx = \int_{\mathbb{R}^N} u_0(x)\,dx - \int_0^t \int_{\mathbb{R}^N} J^{\alpha}_{0|s}(|u|^{p-1}u(x,\cdot))(s)\,dx\,ds, \tag{1.4}$$

where $J^{\alpha}_{0|s}$ is given by (2.7) with $\alpha = 1-\gamma \in (0,1)$. The above equality can be obtained by integrating (with respect to x) the mild solution of (1.1) using Fubini's theorem, and by using (2.3) below.

Since we limit ourselves to nonnegative solutions, the function $M(t)$ defined in (1.4) is nonnegative and nonincreasing. Hence, the limit $M_\infty = \lim_{t\to\infty} M(t)$ exists and then we have either $M_\infty = 0$ or $M_\infty > 0$.

We show that on some range of p, the mass $M(t)$ converges to zero and this can be interpreted as the domination of nonlinear effects in the large time asymptotic of solutions of (1.1). One should notice here that in the case where $u_t + (-\Delta)^{\beta/2}u = 0$, we get back the conservation of mass.

The main result can now be formulated as follows: letting

$$p_\gamma = 1 + \frac{\beta(2-\gamma)}{N} \qquad \text{and} \qquad p^* = \max\{p_\gamma; 2-\gamma\}, \tag{1.5}$$

we have:

THEOREM 1.1. Let $\beta \in (0,2]$, $\gamma \in (0,1)$, $N \geq 1$ and $p > 1$. Assume furthermore that $u_0 \geq 0$ is nonnegative and satisfies (1.3). If u is the mild solution of (1.1) and if $p < p^*$, then

$$\lim_{t\to\infty} M(t) = 0.$$

In the case $\beta = 2$, this result is extended to the case $p = p_\gamma$.

The proof of this theorem is based on the so-called rescaled test function method used in [7, 8, 9, 11, 19, 26] to prove the nonexistence of solutions to nonlinear elliptic, parabolic and hyperbolic equations.

1.2. *Organization of the paper.* This work continues in three sections: Section 2 introduces some definitions and terminology. Section 3 is devoted to the proof of Theorem 1.1. Finally, conclusions and perspective results are given in Section 4.

2. Preliminaries. Here, we present some definitions and results concerning the fractional Laplacian and fractional integrals. The linear fractional diffusion equation

$$u_t + (-\Delta)^{\beta/2} u = 0, \quad \beta \in (0,2],\ x \in \mathbb{R}^N,\ t > 0, \tag{2.1}$$

has a fundamental solution S_β given by

$$S_\beta(t)(x) := S_\beta(t,x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{ix\cdot\xi - t|\xi|^\beta}\, d\xi. \tag{2.2}$$

It is well known that S_β satisfies the following properties:

$$S_\beta(1) \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N), \quad S_\beta(t,x) \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^N} S_\beta(t,x)\, dx = 1, \tag{2.3}$$

for all $x \in \mathbb{R}^N$ and $t > 0$. Let $G(t) := e^{-t(-\Delta)^{\beta/2}}$. Then since $(-\Delta)^{\beta/2}$ is a positive definite selfadjoint operator on $L^2(\mathbb{R}^N)$ one can deduce that $G(t)$ is a strongly continuous semigroup on $L^2(\mathbb{R}^N)$ generated by the fractional power $-(-\Delta)^{\beta/2}$ (see Yosida [25]). Moreover $G(t)v = S_\beta(t) * v$. Knowing that $(-\Delta)^{\beta/2}$ is a selfadjoint operator, we have

$$\int_{\mathbb{R}^N} u(x)(-\Delta)^{\beta/2}(v(x))\, dx = \int_{\mathbb{R}^N} (-\Delta)^{\beta/2}(u(x))v(x)\, dx, \quad u, v \in H^\beta(\mathbb{R}^N). \tag{2.4}$$

We now present the definitions of mild and weak solutions.

Definition 2.1 (Mild solution). Let $u_0 \in C_0(\mathbb{R}^N)$, $0 < \beta \leq 2$, $p > 1$ and $T > 0$. We say that $u \in C([0,T], C_0(\mathbb{R}^N))$ is a mild solution of problem (1.1) if u satisfies the following integral equation:

$$u(t) = G(t)u_0 - \int_0^t G(t-s) J^\alpha_{0|s}(|u|^{p-1}u)(s)\, ds, \quad t \in [0,T]. \tag{2.5}$$

Definition 2.2 (Weak solution). Let $u_0 \in L^\infty_{Loc}(\mathbb{R}^N)$, $0 < \beta \leq 2$ and $T > 0$. We say that u is a weak solution of problem (1.1) if $u \in L^p((0,T), L^\infty_{Loc}(\mathbb{R}^N))$ and satisfies the following formulation:

$$\int_{\mathbb{R}^N} u_0(\cdot)\varphi(\cdot\,,0) - \int_0^T \int_{\mathbb{R}^N} J^\alpha_{0|t}(|u|^{p-1}u)\varphi = \int_0^T \int_{\mathbb{R}^N} u[(-\Delta)^{\beta/2}\varphi - \varphi_t], \tag{2.6}$$

for all compactly supported $\varphi \in C^1([0,T], H^\beta(\mathbb{R}^N))$ such that $\varphi(\cdot\,,T) = 0$.

The existence of solutions relies on the following result:

Theorem 2.3 (Local existence [9]). Given $u_0 \in C_0(\mathbb{R}^N)$ and $p > 1$, there exists a maximal time $T_{\max} > 0$ and a unique mild solution $u \in C([0,T_{\max}), C_0(\mathbb{R}^N))$ to problem (1.1). Furthermore, either $T_{\max} = \infty$, or else $T_{\max} < \infty$ and $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $t \to T_{\max}$. Moreover, if $u_0 \in C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ for some $1 \leq r < \infty$, then $u \in C([0,T_{\max}), C_0(\mathbb{R}^N) \cap L^r(\mathbb{R}^N))$.

The forthcoming definitions and technical arguments will be used in the proof of Theorem 1.1. For all $f \in L^q(0,T)$ $(1 \leq q \leq \infty)$, the left-handed and right-handed Riemann-Liouville fractional integrals $J^{\alpha}_{0|t}f(t)$ and $J^{\alpha}_{t|T}f(t)$ of order $\alpha \in (0,1)$ are defined respectively by (see [14])

$$J^{\alpha}_{0|t}f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds \tag{2.7}$$

and

$$J^{\alpha}_{t|T}f(t) := \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)\, ds, \tag{2.8}$$

for all $t \in [0,T]$.

Moreover, for every $f \in L^p(0,T)$, $g \in L^q(0,T)$ such that $p,q \geq 1$ and $\frac{1}{p} + \frac{1}{q} \leq 1+\alpha$ ($p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1+\alpha$), we have the following integration by parts formula for fractional integrals (see [14, (2.1.50) p. 76]):

$$\int_0^T \left(J^{\alpha}_{0|t}\varphi\right)(t)\psi(t)\, dt \;=\; \int_0^T \varphi(t) \left(J^{\alpha}_{t|T}\psi\right)(t)\, dt. \tag{2.9}$$

An important result concerning right-handed Riemann-Liouville fractional integrals is the following (see [14, (2.1.18) p.71]): if $w_1(t) = (1-t/T)^{\sigma}_{+}$, $t \geq 0$, $T > 0$ and $\sigma \gg 1$, then

$$J^{\alpha}_{t|T} w_1(t) = C\, T^{\alpha} \left(1 - \frac{t}{T}\right)^{\sigma+\alpha}_{+}. \tag{2.10}$$

3. Proof of Theorem 1.1. This section is devoted to the proof of the decay properties of the mass of the mild solutions of (1.1). We start with the following lemma:

LEMMA 3.1. Let $0 < \beta \leq 2$ and $T > 0$. Consider $u_0 \in C_0(\mathbb{R}^N)$, and let $u \in C([0,T], C_0(\mathbb{R}^N))$ be a mild solution of (1.1). Then u is also a weak solution of (1.1).

Proof. Let $T > 0$, $0 < \beta \leq 2$, $u_0 \in C_0(\mathbb{R}^N)$ and let $u \in C([0,T], C_0(\mathbb{R}^N))$ be a solution of (2.5). Given $\varphi \in C^1([0,T], H^{\beta}(\mathbb{R}^N))$ such that suppφ is compact with $\varphi(\cdot\,, T) = 0$, then by multiplying (2.5) by φ and integrating over $\mathbb{R}^N$, we get

$$\begin{aligned}\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) \;&=\; \int_{\mathbb{R}^N} G(t)u_0(x)\varphi(x,t) \\ &\quad - \int_{\mathbb{R}^N} \left(\int_0^t G(t-s) J^{\alpha}_{0|s}\left(|u|^{p-1}u\right)(x,t)\, ds\right)\varphi(x,t).\end{aligned}$$

So

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) \;&=\; \int_{\mathbb{R}^N} \frac{d}{dt}\left(G(t)u_0(x)\varphi(x,t)\right) \\ &\quad - \int_{\mathbb{R}^N} \frac{d}{dt}\int_0^t G(t-s) J^{\alpha}_{0|s}\left(|u|^{p-1}u\right)(x,s)\, ds\varphi(x,t) \\ &=:\; \mathcal{A}_1 - \mathcal{A}_2.\end{aligned} \tag{3.1}$$

Now, using (2.4) together with the following property of the semigroup $G(t)$ ([5, Chapter 3]),

$$\frac{d}{dt}(G(t)u_0(x)) = -(-\Delta)^{\beta/2}(G(t)u_0(x)),$$

we have

$$\begin{aligned}\mathcal{A}_1 &= \int_{\mathbb{R}^N} A\left(G(t)u_0(x)\right)\varphi(x,t) + \int_{\mathbb{R}^N} G(t)u_0(x)\varphi_t(x,t)\\ &= \int_{\mathbb{R}^N} G(t)u_0(x)A\varphi(x,t) + \int_{\mathbb{R}^N} G(t)u_0(x)\varphi_t(x,t) \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}_2 &= \int_{\mathbb{R}^N} f(x,t)\varphi(x,t) + \int_{\mathbb{R}^N}\int_0^t A\left(G(t-s)f(x,s)\right)\,ds\varphi(x,t)\\ &\quad + \int_{\mathbb{R}^N}\int_0^t G(t-s)f(x,s)\,ds\varphi_t(x,t)\\ &= \int_{\mathbb{R}^N} f(x,t)\varphi(x,t) + \int_{\mathbb{R}^N}\int_0^t G(t-s)f(x,s)\,dsA\varphi(x,t)\\ &\quad + \int_{\mathbb{R}^N}\int_0^t G(t-s)f(x,s)\,ds\varphi_t(x,t), \qquad (3.3)\end{aligned}$$

where $A = -(-\Delta)^{\beta/2}$ and $f := J^{\alpha}_{0|t}\left(|u|^{p-1}u\right) \in C([0,T], L^2(\mathbb{R}^N))$. Thus, using (2.5), (3.2) and (3.3), we conclude that (3.1) implies

$$\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) = \int_{\mathbb{R}^N} u(x,t)A\varphi(x,t) + \int_{\mathbb{R}^N} u(x,t)\varphi_t(x,t) - \int_{\mathbb{R}^N} f(x,t)\varphi(x,t).$$

We finally arrive at the result by integrating in time over $[0,T]$ and using the fact that $\varphi(\cdot\,,T) = 0$. □

The nonnegativity of the solutions is ensured by the next proposition.

PROPOSITION 3.2 (Maximum principle). Let $u_0 \in C_0(\mathbb{R}^N)$ and let $u \in C([0,T], C_0(\mathbb{R}^N))$ be the mild solution of (1.1). If $u_0 \geq 0$, then $u \geq 0$.

Proof. It is clear that $f := J^{\alpha}_{0|t}\left(|u|^{p-1}u\right) \in L^2(0,T;L^2(\mathbb{R}^N))$. It follows, using the maximal regularity (see for instance [16, Theorem 1]), that

$$u \in L^2(0,T;H^{\beta}(\mathbb{R}^N)) \cap W^{1,2}(0,T;L^2(\mathbb{R}^N))$$

and hence

$$u^- \in L^2(0,T;H^{\beta}(\mathbb{R}^N)) \cap W^{1,2}(0,T;L^2(\mathbb{R}^N)),$$

where $u = u^+ - u^-$ with $u^+ := \max(u,0)$ and $u^- := \max(-u;0)$. Multiplying (2.5) by u^-, and using the same steps as in Lemma 3.1, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)u^-(x,t) &= -\int_{\mathbb{R}^N} u(x,t)(-\Delta)^{\beta/2}u^-(x,t)\\ &\quad + \int_{\mathbb{R}^N} u(x,t)(u^-)_t(x,t) - \int_{\mathbb{R}^N} J^{\alpha}_{0|t}\left(|u|^{p-1}u\right)(t)u^-(x,t),\end{aligned}$$

which implies that

$$\begin{aligned} -\frac{d}{dt}\int_{\mathbb{R}^N}(u^-(x,t))^2 &= +\int_{\mathbb{R}^N} u^-(x,t)(-\Delta)^{\beta/2}u^-(x,t) \\ &\quad -\int_{\mathbb{R}^N} u^-(x,t)(u^-)_t(x,t)+\int_{\mathbb{R}^N} J^{\alpha}_{0|t}\left(|u|^{p-1}u^-\right)(t)u^-(x,t). \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N}(u^-(x,t))^2 &= -\int_{\mathbb{R}^N}|(-\Delta)^{\beta/4}(u^-(x,t))|^2 \\ &\quad +\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}(u^-(x,t))^2-\int_{\mathbb{R}^N} J^{\alpha}_{0|t}\left(|u|^{p-1}u^-\right)(t)u^-(x,t); \end{aligned}$$

hence

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}(u^-(x,t))^2=-\int_{\mathbb{R}^N}|(-\Delta)^{\beta/4}(u^-(x,t))|^2-\int_{\mathbb{R}^N} J^{\alpha}_{0|t}\left(|u|^{p-1}u^-\right)(t)u^-(x,t)\le 0.$$

The integration over $[0,t]$ leads to

$$0\le\int_{\mathbb{R}^N}(u^-(x,t))^2\le\int_{\mathbb{R}^N}(u^-(x,0))^2=0;$$

thus $u^-(x,t)=0$ for a.e. $x\in\mathbb{R}^N$, $t\ge 0$ and therefore $u(x,t)\ge 0$ for a.e. $x\in\mathbb{R}^N$, $t\ge 0$. □

We are now ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let u be the nonnegative global mild solution of (1.1). Then u is a solution of (2.5) in $C([0,T],C_0(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))$ for all $T\gg 1$ with $u(t)\ge 0$ for all $t\in[0,T]$.

Step 1 (**Choosing of the test function**). Take

$$\varphi(x,t)=(\varphi_1(x))^{\ell}\,\varphi_2(t)\quad\text{with}\quad \varphi_1(x):=\Phi\left(|x|/B\right),\ \ \varphi_2(t):=(1-t/T)^{\eta}_{+},$$

where $\ell,\eta\gg 1$ and Φ is a smooth nonnegative and nonincreasing cutoff function defined by

$$\Phi(r)=\begin{cases}1 & \text{if } 0\le r\le 1,\\ 0 & \text{if } r\ge 2,\end{cases}$$

$0\le\Phi\le 1$, $|\Phi'(r)|\le C_1/r$, for all $r>0$. The constant $B>0$ which appears in the definition of φ_1 is fixed and will be determined later.

Step 2 (**Estimation**). Using Lemma 3.1, we have

$$\int_{\Omega_B}u_0\,\varphi_1^{\ell}-\int_{\Omega(T,B)}J^{\alpha}_{0|t}(u^p)\varphi=\int_0^T\int_{\mathbb{R}^N}u(-\Delta)^{\beta/2}(\varphi_1^{\ell})\varphi_2-\int_{\Omega(T,B)}u\,\varphi_t,$$

where

$$\Omega(T,B):=[0,T]\times\Omega_B\quad\text{for}\quad \Omega_B=\left\{x\in\mathbb{R}^N\ :\ |x|\le 2B\right\}.$$

Moreover, the following inequality (see Appendix)

$$(-\Delta)^{\beta/2}\left(\varphi_1^{\ell}\right)\le\ell\varphi_1^{\ell-1}(-\Delta)^{\beta/2}\left(\varphi_1\right)$$

allows us to write

$$\begin{aligned}\int_{\Omega_B} u_0\,\varphi_1^\ell - \int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi &\leq C\int_{\Omega(T,B)} u\,\varphi_1^{\ell-1}|(-\Delta)^{\beta/2}\varphi_1|\varphi_2 + \int_{\Omega(T,B)} u|\varphi_t| \\ &=: I_1 + I_2. \end{aligned} \tag{3.4}$$

To estimate I_1, we compute:

$$\begin{aligned} I_1 &= C\int_{\Omega(T,B)} u(J^\alpha_{t|T}\varphi)^{1/p}(J^\alpha_{t|T}\varphi)^{-1/p}\varphi_1^{\ell-1}|(-\Delta)^{\beta/2}\varphi_1|\varphi_2 \\ &\leq \varepsilon\int_{\Omega(T,B)} u^p J^\alpha_{t|T}\varphi + C(\varepsilon)\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell-\tilde{p}}|(-\Delta)^{\beta/2}\varphi_1|^{\tilde{p}}\varphi_2^{\tilde{p}} \\ &= \varepsilon\int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi + C(\varepsilon)\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell-\tilde{p}}|(-\Delta)^{\beta/2}\varphi_1|^{\tilde{p}}\varphi_2^{\tilde{p}}, \end{aligned} \tag{3.5}$$

where we have used (2.9) and ε-Young's inequality

$$ab \leq \varepsilon a^p + C(\varepsilon)b^{\tilde{p}}, \quad \text{where} \quad p\tilde{p} = p + \tilde{p}, \;\; a > 0, b > 0, \;\; p > 1, \tilde{p} > 1, \tag{3.6}$$

with

$$\begin{cases} a = u(J^\alpha_{t|T}\varphi)^{1/p}, \\ b = C(J^\alpha_{t|T}\varphi)^{-1/p}\varphi_1^{\ell-1}|(-\Delta)^{\beta/2}\varphi_1|\varphi_2. \end{cases}$$

Similarly

$$\begin{aligned} I_2 &= C\int_{\Omega(T,B)} u(J^\alpha_{t|T}\varphi)^{1/p}(J^\alpha_{t|T}\varphi)^{-1/p}\varphi_1^{\ell}|(\varphi_2)_t| \\ &\leq \varepsilon\int_{\Omega(T,B)} u^p J^\alpha_{0|t}\varphi + C(\varepsilon)\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell}|(\varphi_2)_t|^{\tilde{p}} \\ &= \varepsilon\int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi + C(\varepsilon)\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell}|(\varphi_2)_t|^{\tilde{p}}. \end{aligned} \tag{3.7}$$

Combining (3.4), (3.5) and (3.7), we conclude that

$$\begin{aligned} &\int_{\Omega_B} u_0\,\varphi_1^\ell - (1+2\varepsilon)\int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi \\ &\leq C(\varepsilon)\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell-\tilde{p}}|(-\Delta)^{\beta/2}\varphi_1|^{\tilde{p}}\varphi_2^{\tilde{p}} \\ &+ C(\varepsilon)\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell}|(\varphi_2)_t|^{\tilde{p}}. \end{aligned} \tag{3.8}$$

STEP 3 (**Rescaling and passing to the limit**). At this stage, we have to distinguish three cases:

• $p < p_\gamma$: Here, we take $B = T^{1/\beta}$. Hence, using (2.10) and the change of variables: $s = T^{-1}t$, $\xi = T^{-1/\beta}x$, we get, from (3.8), that

$$\begin{aligned}
&\int_{\Omega_B} u_0\, \varphi_1^\ell - (1+2\varepsilon)\int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi \\
&\le C(\varepsilon)T^{-\delta}\left(\int_0^1 (1-s)^{\eta-\frac{\alpha}{p-1}}\,ds\right)\left(\int_{|\xi|\le 2}\Phi(|\xi|)^{\ell-\tilde p}|(-\Delta)^{\beta/2}\Phi(|\xi|)|^{\tilde p}\,d\xi\right) \\
&+ C(\varepsilon)T^{-\delta}\left(\int_0^1 (1-s)^{\eta-\frac{\alpha+p}{p-1}}\,ds\right)\left(\int_{|\xi|\le 2}\Phi(|\xi|)^{\ell}\,d\xi\right) \\
&\le CT^{-\delta},
\end{aligned} \tag{3.9}$$

where $\delta := (\alpha+p)/(p-1) - 1 - (N/\beta)$. As ($p < p_\gamma \Leftrightarrow \delta > 0$) we pass to the limit in (3.9) as T goes to ∞, and we use the Lebesgue dominated convergence theorem to get

$$\int_{\mathbb{R}^N} u_0\,dx - (1+2\varepsilon)\int_0^\infty\int_{\mathbb{R}^N} J^\alpha_{0|t}(u^p(x,\cdot)dx\,dt \le 0,$$

which implies using (1.4) that

$$\begin{aligned}
0 \le \lim_{T\to\infty} M(T) &= \int_{\mathbb{R}^N} u_0\,dx - \int_0^\infty\int_{\mathbb{R}^N} J^\alpha_{0|t}(u^p(x,\cdot)dx\,dt \\
&\le 2\varepsilon\int_0^\infty\int_{\mathbb{R}^N} J^\alpha_{0|t}(u^p(x,\cdot)dx\,dt \le 2\varepsilon\|u_0\|_{L^1(\mathbb{R}^N)}.
\end{aligned}$$

But $\varepsilon > 0$ can be chosen arbitrarily small and this leads simply to $\lim_{t\to\infty} M(t) = 0$.

• $p < 2-\gamma$: Let $B = R$, where $1 \ll R < T$ is chosen such that both T and R do not simultaneously tend to ∞. We then obtain from (3.8) and the change of variables: $s = T^{-1}t$, $\xi = R^{-1}x$ that

$$\begin{aligned}
&\int_{\Omega_R} u_0\, \varphi_1^\ell - (1+2\varepsilon)\int_{\Omega(T,R)} J^\alpha_{0|t}(u^p)\varphi \\
&\le C(\varepsilon)T^{\alpha_1}R^{N-\beta\tilde p}\left(\int_0^1 (1-s)^{\eta-\frac{\alpha}{p-1}}\,ds\right)\left(\int_{|\xi|\le 2}\Phi(|\xi|)^{\ell-\tilde p}|(-\Delta)^{\beta/2}\Phi(|\xi|)|^{\tilde p}\,d\xi\right) \\
&+ C(\varepsilon)T^{1-(\alpha+p)/(p-1)}R^N\left(\int_0^1 (1-s)^{\eta-\frac{\alpha+p}{p-1}}\,ds\right)\left(\int_{|\xi|\le 2}\Phi(|\xi|)^{\ell}\,d\xi\right) \\
&\le C\,T^{\alpha_1}R^{N-\beta\tilde p} + C\,T^{1-(\alpha+p)/(p-1)}R^N,
\end{aligned}$$

where $\alpha_1 := 1 - \alpha/(p-1)$. Taking the limit as $T\to\infty$, we infer, as $p < 2-\gamma \Leftrightarrow \alpha_1 < 0$, that

$$\int_{\Omega_R} u_0\, \varphi_1^\ell - (1+2\varepsilon)\int_0^\infty\int_{\Omega_R} J^\alpha_{0|t}(u^p)\varphi_1 \le 0. \tag{3.10}$$

Finally, computing the limit as $R\to\infty$ in (3.10) we infer that $\lim_{t\to\infty} M(t) = 0$ because $\varepsilon > 0$ can be taken arbitrarily small.

• $p = p_\gamma$ and $\beta = 2$: let $B = R^{-1/2}T^{1/2}$ with the same R introduced in the previous case. From (3.4), we have

$$\int_{\Omega_B} u_0\ \varphi_1^\ell - \int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi \le C\int_{\Delta(T,R)} u\ \varphi_1^{\ell-1}|\Delta\varphi_1|\varphi_2 + \int_{\Omega(T,B)} u|\varphi_t| =: J + I_2, \tag{3.11}$$

where

$$\Delta(T,R) = [0,T]\times\left\{x\in\mathbb{R}^N\ :\ R^{-1/2}T^{1/2}\le |x|\le 2R^{-1/2}T^{1/2}\right\}\subset\Omega(T,B).$$

Moreover, we consider the same estimate on I_2 as in the beginning of the proof. But to estimate J, we use the following Hölder inequality:

$$\int_{\Delta(T,R)} ab \le \left(\int_{\Delta(T,R)} a^{\,p}\right)^{1/p}\left(\int_{\Delta(T,R)} b^{\,\tilde p}\right)^{1/\tilde p},$$

with

$$\begin{cases} a = u(J^\alpha_{t|T}\varphi)^{1/p}, \\ b = (J^\alpha_{t|T}\varphi)^{-1/p}\varphi_1^{\ell-1}|\Delta\varphi_1|\varphi_2. \end{cases}$$

Then (using (2.9)):

$$\begin{aligned} J &= C\int_{\Delta(T,R)} u(J^\alpha_{t|T}\varphi)^{1/p}(J^\alpha_{t|T}\varphi)^{-1/p}\varphi_1^{\ell-1}|\Delta\varphi_1|\varphi_2 \\ &\le C\left(\int_{\Delta(T,R)} u^p J^\alpha_{t|T}\varphi\right)^{1/p}\left(\int_{\Delta(T,R)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell-\tilde p}|\Delta\varphi_1|^{\tilde p}\varphi_2^{\tilde p}\right)^{1/\tilde p} \\ &\le C\left(\int_{\Delta(T,R)} J^\alpha_{0|t}(u^p)\varphi\right)^{1/p}\left(\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell-\tilde p}|\Delta\varphi_1|^{\tilde p}\varphi_2^{\tilde p}\right)^{1/\tilde p}. \end{aligned} \tag{3.12}$$

Combining (3.4) and (3.5), we get from (3.11) that

$$\begin{aligned} &\int_{\Omega_B} u_0\ \varphi_1^\ell - (1+\varepsilon)\int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi \\ &\le C\left(\int_{\Delta(T,R)} J^\alpha_{0|t}(u^p)\varphi\right)^{1/p}\left(\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^{\ell-\tilde p}|\Delta\varphi_1|^{\tilde p}\varphi_2^{\tilde p}\right)^{1/\tilde p} \\ &+ C(\varepsilon)\int_{\Omega(T,B)} (J^\alpha_{t|T}\varphi_2)^{-\frac{1}{p-1}}\varphi_1^\ell|(\varphi_2)_t|^{\tilde p}. \end{aligned}$$

Using the scaled variables: $\tau = T^{-1}t$ $\xi = B^{-1}x = (T/R)^{-1/2}x$ and the fact that $(p = p_\gamma, \beta = 2) \Rightarrow (\delta = 0)$, we get

$$\begin{aligned}
&\int_{\Omega_B} u_0\, \varphi_1^\ell - (1+\varepsilon)\int_{\Omega(T,B)} J^\alpha_{0|t}(u^p)\varphi \\
&\le C\, T^{\frac{-\delta}{\tilde p}} R^{1-N/2\tilde p}\left(\int_{\Delta(T,R)} J^\alpha_{0|t}(u^p)\varphi\right)^{1/p}\left(\int_0^1 (1-s)^{\eta-\frac{\alpha}{p-1}}\,ds\right)^{1/\tilde p} \\
&\qquad\qquad \times\left(\int_{|\xi|\le 2}\Phi(|\xi|)^{\ell-\tilde p}|(-\Delta)^{\beta/2}\Phi(|\xi|)|^{\tilde p}\,d\xi\right)^{1/\tilde p} \\
&+ C(\varepsilon)T^{-\delta}R^{-N/2}\left(\int_0^1 (1-s)^{\eta-\frac{\alpha+p}{p-1}}\,ds\right)\left(\int_{|\xi|\le 2}\Phi(|\xi|)^{\ell}\,d\xi\right) \\
&\le C\, R^{1-N/2\tilde p}\left(\int_{\Delta(T,R)} J^\alpha_{0|t}(u^p)\varphi\right)^{1/p} + C\, R^{-N/2}. \qquad (3.13)
\end{aligned}$$

Now, from (1.4), we have

$$\int_0^\infty\int_{\mathbb{R}^N} J^\alpha_{0|t}(u^p(x,\cdot\,))(t)\,dx\,dt \le \int_{\mathbb{R}^N} u_0(x)\,dx < \infty,$$

which implies that

$$\lim_{T\to\infty}\left(\int_{\Delta(T,R)} J^\alpha_{0|t}(u^p)\,\tilde\varphi\right)^{1/p} = \int_0^\infty\int_{\mathbb{R}^N} J^\alpha_{0|t}(u^p)\,dx\,dt - \int_0^\infty\int_{\mathbb{R}^N} J^\alpha_{0|t}(u^p)\,dx\,dt = 0,$$

where we have used Lebesgue's dominated convergence theorem. Consequently, passing to the limit in (3.13), as $T\to\infty$, we get

$$\int_{\mathbb{R}^N} u_0\,dx - (1+\varepsilon)\int_0^\infty\int_{\mathbb{R}^N} J^\alpha_{0|t}(u^p)\,dx\,dt \le\ C\, R^{-N/2}.$$

Finally, we conclude that $\lim_{t\to\infty} M(t) = 0$ by taking the limit when R goes to infinity as, once again, $\varepsilon > 0$ can be chosen arbitrarily small. □

4. Conclusion and perspective results. Consider the problem (1.1) with the sign "-" replaced by "+" in the right-hand side, i.e.

$$u_t + (-\Delta)^{\beta/2}u = +\frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma}|u|^{p-1}u(s)\,ds. \qquad (4.1)$$

Global existence, nonexistence and blow-up rate of solutions for the problem (4.1) can be found in a recent paper of Fino and Kirane [9].

It is easy to check that if $u(t,x)$ is a solution of (4.1) with initial value u_0, then for all $\lambda > 0$, $\lambda^{\frac{\beta(2-\gamma)}{p-1}}u(\lambda^\beta t, \lambda x)$ is also a solution with initial value $\lambda^{\frac{\beta(2-\gamma)}{p-1}}u_0(\lambda x)$. Since

$$\|\lambda^{\frac{\beta(2-\gamma)}{p-1}}u_0(\lambda\cdot\,)\|_{L^q} = \lambda^{\frac{\beta(2-\gamma)}{p-1}-\frac{N}{q}}\|u_0\|_{L^q},$$

it follows that the invariant Lebesgue norm of (4.1) is given by

$$q_{sc} = \frac{N(p-1)}{\beta(2-\gamma)}.$$

One would therefore expect (as in Fujita [10], Weissler [28] and the references therein) that if $q_{sc} > 1$, i.e. $p > p_\gamma$ (where the scaling exponent p_γ is given in (1.5)), and if $\|u_0\|_{L^{q_{sc}}}$ is sufficiently small, then the solution is global. Unfortunately, this does not work for the nonlocal equation (4.1). To be more accurate, Fino and Kirane [9] (see also [4] for the case $\beta = 2$) have proved that the Fujita critical exponent for (4.1) is not the one predicted by scaling considerations and is given by

$$p_1 := \max\left\{1 + \frac{\beta(2-\gamma)}{(N-\beta+\beta\gamma)_+}; \frac{1}{\gamma}\right\} > p_\gamma.$$

On the other hand, it is expected (see for instance Ben-Artzi and Koch [1] and Fino and Karch [8]) that the Fujita critical exponent p_1 for (4.1) is going to prove the decaying properties of the mass of solutions, i.e. $\lim_{t\to\infty} M(t) = M_\infty > 0$ for $p > p_1$ while $\lim_{t\to\infty} M(t) = 0$ if $1 < p \le p_1$.

Note that p^* and p_1 both converge to $1 + \beta/N$ when $\gamma \to 1$, where the term $1 + \beta/N$ is the critical exponent given by Fino and Karch [8]. Furthermore, it is not clear if p^* or p_1 is the Fujita critical exponent for the decaying properties of the mass, because in the study of the global existence and nonexistence of solutions, the predicted exponent p_γ is not the critical one.

REMARK 4.1. It is still an open problem to find the Fujita critical exponent $p^{**}(\gamma, N, \beta) =: p^{**} \ge p^*$ to (1.1), i.e. to prove that $\lim_{t\to\infty} M(t) = M_\infty > 0$ for $p > p^{**}$ and that $\lim_{t\to\infty} M(t) = 0$ if $p^* \le p \le p^{**}$. Moreover, it is predicted that $p^{**} \to 1+\beta/N$ as $\gamma \to 1$.

Appendix. In this appendix, we give a proof of Ju's inequality (see [12, Proposition 3.3]), in dimension $N \ge 1$ where $\delta \in [0, 2]$ and $q \ge 1$, for all nonnegative Schwartz functions ψ (in the general case) such that

$$(-\Delta)^{\delta/2}\psi^q \le q\psi^{q-1}(-\Delta)^{\delta/2}\psi.$$

The cases $\delta = 0$ and $\delta = 2$ are obvious, as well as $q = 1$. If $\delta \in (0, 2)$ and $q > 1$, using [3, Definition 3.2], we have

$$(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{\psi(x+z) - \psi(x)}{|z|^{N+\delta}}\, dz, \quad \text{for all } x \in \mathbb{R}^N,$$

where $c_N(\delta) = 2^\delta\Gamma((N+\delta)/2)/(\pi^{N/2}\Gamma(1-\delta/2))$. Then

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta)\int_{\mathbb{R}^N} \frac{(\psi(x))^{q-1}\psi(x+z) - (\psi(x))^q}{|z|^{N+\delta}}\, dz.$$

By Young's inequality we have

$$(\psi(x))^{q-1}\psi(x+z) \le \frac{q-1}{q}(\psi(x))^q + \frac{1}{q}(\psi(x+z))^q.$$

Therefore,

$$(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) \ge \frac{-c_N(\delta)}{q}\int_{\mathbb{R}^N} \frac{(\psi(x+z))^q - (\psi(x))^q}{|z|^{N+\delta}}\, dz = \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.$$

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