ON THE UPPER SEMICONTINUITY OF PULLBACK ATTRACTORS FOR MULTI-VALUED PROCESSES

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Abstract. This paper is concerned with the asymptotical behavior of multi-valued processes. First, we establish some stability results of pullback attractors for multi-valued processes and display new methods to check the continuity condition. Then we consider the effects of small time delays on the asymptotic stability of multi-valued nonautonomous functional parabolic equations. Finally, we give some new estimates of solutions and prove the existence of minimal pullback attractors in $H_0^1(\Omega)$ for nonautonomous nonclassical diffusion equations with polynomial growth nonlinearity of arbitrary order and without the uniqueness of solutions. In particular, the upper semicontinuity of pullback attractors for nonclassical diffusion equations with singular and nonautonomous perturbations is addressed.

1. Introduction. The stability of attractors for single-valued infinite-dimensional dynamical systems has been extensively studied in the mathematical literature (see, for example, [2, 5, 11, 29, 30, 35, 36, 37] and the references cited therein). In recent years, there is an increasing interest in the study of multi-valued systems; see, e.g., [4, 7, 11, 19, 22, 23, 34, 38, 39]. In the nonautonomous case, the pullback attractor is the suitable way to describe the dynamics of nonautonomous dynamical systems. Recently, some results on the stability of trajectory attractors with respect to small perturbation are presented in [11] and the references therein. More recently, we show [23] that if a nonautonomous set-valued dynamical system has a compact uniformly attracting set, then we establish some stability results for pullback attractors and the effects of small delays to the dynamical behavior for the nonautonomous systems on $\mathbb{R}^n$. In this present
work, we are mainly concerned with the upper semicontinuity of pullback attractors for nonautonomous multi-valued infinite-dimensional dynamical systems.

First, we establish the general results for the upper semicontinuity of pullback attractors of multi-valued systems under the reasonable upper semicontinuity assumptions. Furthermore, we give a simple and direct method to verify the continuity condition. It is worth mentioning that in our method the upper semicontinuity problem of the infinite-dimensional multi-valued systems reduces to the finite-dimensional case in some sense. In addition, we show that for each $M > 0$, the kernel sections for multi-valued (autonomous and nonautonomous) infinite-dimensional dynamical systems are compact in $C([-M, M]; X)$ and attract every bounded subset of $C([-M, M]; X)$, where $X$ is a Banach space.

Second, we are interested in the effects of small delays on the asymptotic behavior of multi-valued nonautonomous functional parabolic equations. For finite-dimensional multi-valued systems, the related results can be found in [22, 23] and the references therein. However, in the infinite-dimensional case, as far as we know not many results exist in the literature. Furthermore, we use some of the ideas from our work [40] to obtain the upper semicontinuity of pullback attractors in higher regularity space. We also construct a global attractor in $C([-\varepsilon_0, 0]; V)$ for the autonomous multi-valued semi-dynamical system $\{F(t)\}$ defined in Theorem 4.2.

Finally, we investigate the nonclassical diffusion equations with polynomial growth nonlinearity of arbitrary order and small perturbations. For autonomous nonclassical diffusion equations, we refer the reader to [32, 35] and the references cited therein. For the existence and stability of pullback attractors for nonautonomous nonclassical diffusion equations with proper small perturbation, see [2, 36], etc. However, there is little reference on nonautonomous nonclassical diffusion equations with polynomial growth nonlinearity of arbitrary order, even in the single-valued case. We discuss the nonclassical diffusion equations with polynomial growth nonlinearity of arbitrary order and without the uniqueness of solutions, and prove the existence of minimal pullback attractors in space $H^1_0(\Omega)$. Moreover, we also show the upper semicontinuity of pullback attractors in $L^p(\Omega)$ ($p \geq 2$ is arbitrary).

This paper is organized as follows. In Section 2 we give some preliminary results and definitions. In Section 3 we state and prove our results on the upper semicontinuity of pullback attractors for multi-valued nonautonomous infinite-dimensional dynamical systems. Finally, in Sections 4 and 5, we illustrate our results in Section 3 by studying parabolic differential equations with small delays and nonautonomous nonclassical diffusion equations with singular and nonautonomous perturbations.

2. Preliminaries. Let $X$ be a complete metric space with metric $d_X(\cdot, \cdot)$, and let $2^X$ be the set of all subsets of $X$. Denote by $H^*_X(\cdot, \cdot)$ and $H_X(\cdot, \cdot)$, respectively, the Hausdorff semidistance and Hausdorff distance between two nonempty subsets of a complete metric space $X$, which are defined by

$$H^*_X(A, B) = \sup_{a \in A} d_{X}(a, B),$$
where \( \text{dist}_X(a, B) = \inf_{b \in B} d_X(a, b) \) and
\[
H_X(A, B) = \max\{H^*_X(A, B), H^*_X(B, A)\}.
\]

Finally, denote by \( \mathcal{N}(A, r) \) the open neighborhood \( \{y \in X \mid \text{dist}_X(y, A) < r\} \) of radius \( r > 0 \) of a subset \( A \) of a complete metric space \( X \).

**Definition 2.1.** A family of mappings \( F(t) : X \to 2^X, t \in \mathbb{R}^+ \), is said to be an (autonomous) multi-valued semidynamical system (MVSS for short) if the following axioms hold:

(1) \( F(0)x = \{x\}, \forall x \in X \);

(2) \( F(s)F(t)x = F(s + t)x, \forall s, t \in \mathbb{R}^+, x \in X \).

**Definition 2.2.** Let \( \{F(t)\} \) be a multi-valued semidynamical system on \( X \). We say that \( \{F(t)\} \) is

(1) dissipative, if there exists a bounded subset \( U \) of \( X \) so that for any bounded set \( B \subset X \), there exists a \( T_0 = T_0(B) \in \mathbb{R}^+ \) such that
\[
F(t)B \subset U, \forall t \geq T_0;
\]

(2) \( \omega \)-limit compact, if for any bounded subset \( B \) of \( X \) and \( \varepsilon > 0 \), there exists a \( T_1 = T_1(B, \varepsilon) \in \mathbb{R}^+ \) such that
\[
k\left( \bigcup_{t \geq T_1} F(t)B \right) \leq \varepsilon,
\]
where \( k \) is the Kuratowski measure of noncompactness.

**Definition 2.3.** A nonempty compact subset \( A \) of \( X \) is called a global attractor for the multi-valued semidynamical system \( \{F(t)\} \) if it satisfies

(1) \( A \) is an invariant set, i.e.,
\[
F(t)A = A, \forall t \in \mathbb{R}^+;
\]

(2) \( A \) attracts each bounded subset \( B \) of \( X \), i.e.,
\[
\lim_{t \to +\infty} H^*_X(F(t)B, A) = 0.
\]

Let \( A \) be a subset of Banach space \( X \). The weakly sequential closure \( \bar{A}^{WS} \) of \( A \) is defined by
\[
\bar{A}^{WS} = \{x \in X \mid \exists \{x_n\} \subset A, \text{ s.t. } x_n \rightharpoonup x, \text{ that is, } x_n \text{ converges weakly to } x\}.
\]

Let \( \mathcal{K} \) be a kernel of the MVSS \( \{F(t)\} \) on a Banach space \( X \). The kernel \( \mathcal{K} \) consists of all bounded complete trajectories of the MVSS \( \{F(t)\} \), i.e.,
\[
\mathcal{K} = \left\{ u(\cdot) \mid \sup_{t \in (-\infty, +\infty)} \|u(t)\|_X \leq C_u, \ u(t + \tau) \in F(\tau)u(t), \forall t \in \mathbb{R}, \tau \in \mathbb{R}^+ \right\}.
\]

\( \mathcal{K}(t) \) denotes the kernel section at a time moment \( t \in \mathbb{R} \):
\[
\mathcal{K}(t) = \{u(t) \mid u(\cdot) \in \mathcal{K}, \mathcal{K}(t) \subset X \}.
\]
2.4 (38, 39). Let \( \{F(t)\} \) be an (autonomous) multi-valued semidynamical system on a Banach space \( X \), and let \( F(t)x \) be norm-to-weak upper-semicontinuous in \( x \) for fixed \( t \in \mathbb{R}^+ \) (i.e., if \( x_n \to x \) in \( X \), then for any \( y_n \in F(t)x_n \), there exist a \( y \in F(t)x \) and a subsequence \( y_{n_k} \) such that \( y_{n_k} \to y \) (weak convergence)). Then \( \{F(t)\} \) is

(1) dissipative, i.e., there exists a bounded subset \( U \) of \( X \) so that for any bounded set \( B \subset X \), there exists a \( T_0 = T_0(B) \in \mathbb{R}^+ \) such that

\[
F(t)B \subset U, \forall t \geq T_0,
\]

moreover, \( U \) coincides with the kernel section at time \( \tau \), i.e., \( U = K(\tau) \) for any \( \tau \in \mathbb{R} \).

Definition 2.5. A family of mappings \( U(t,\tau) : X \to 2^X, t \geq \tau, \tau \in \mathbb{R} \), is called a multi-valued process (MVP in short) if it satisfies:

(1) \( U(\tau, \tau)x = \{x\}, \forall \tau \in \mathbb{R}, x \in X \);

(2) \( U(t, s)U(s, \tau)x = U(t, \tau)x, \forall t \geq s \geq \tau, \tau \in \mathbb{R}, x \in X \).

Following 9 and the papers cited therein, we have the following definitions and results.

Definition 2.6. Let \( \{U(t, \tau)\} \) be a multi-valued process on \( X \). We say that \( \{U(t, \tau)\} \) is

(1) pullback bounded dissipative, if there exists a family of bounded sets \( Q = \{Q(t)\}_{t \in \mathbb{R}} \) in \( X \) so that for any bounded subset \( B \) of \( X \) and every \( t \in \mathbb{R} \), there exists a \( t_0 = t_0(B, t) \in \mathbb{R}^+ \) such that

\[
U(t, t-s)B \subset Q(t), \forall s \geq t_0;
\]

(2) pullback strongly bounded dissipative, if there exists a family of bounded sets \( Q = \{Q(t)\}_{t \in \mathbb{R}} \) in \( X \) so that for any fixed \( t \in \mathbb{R} \), every bounded subset \( B \) of \( X \) and each \( \tau \leq t \), there exists a \( t_1 = t_1(B, \tau) \in \mathbb{R}^+ \) such that

\[
U(\tau, \tau-s)B \subset Q(t), \forall s \geq t_1;
\]

(3) pullback \( \omega \)-limit compact with respect to each \( t \in \mathbb{R} \), if for any bounded subset \( B \) of \( X \) and \( \varepsilon > 0 \), there exists a \( t_2 = t_2(B, t, \varepsilon) \in \mathbb{R}^+ \) such that

\[
k \left( \bigcup_{s \geq t_2} U(t, t-s)B \right) \leq \varepsilon.
\]

Definition 2.7. A family of nonempty compact subsets \( A = \{A(t)\}_{t \in \mathbb{R}} \) of \( X \) is called a pullback attractor for the multi-valued process \( \{U(t, \tau)\} \) if it satisfies

(1) \( A = \{A(t)\}_{t \in \mathbb{R}} \) is invariant, i.e.,

\[
U(t, \tau)A(\tau) = A(t), \forall t \geq \tau, \tau \in \mathbb{R};
\]

(2) \( A \) pullback attracts every bounded subset \( B \) of \( X \), i.e., for any fixed \( t \in \mathbb{R} \),

\[
\lim_{s \to +\infty} H^*_X(U(t, t-s)B, A(t)) = 0.
\]
We call $\mathcal{A}$ minimal if there is another family of nonempty closed sets $\mathcal{C} = \{C(t)\}_{t \in \mathbb{R}}$ to which pullback attracts bounded subsets of $X$; then $A(t) \subset C(t)$ for all $t \in \mathbb{R}$.

**Theorem 2.8.** Let $X$ be a Banach space, $\{U(t, \tau)\}$ be a multi-valued process on $X$, and let $U(t, \tau)x$ be norm-to-weak upper-semicontinuous in $x$ for fixed $t \geq \tau$, $\tau \in \mathbb{R}$ (i.e., if $x_n \to x$ in $X$, then for any $y_n \in U(t, \tau)x_n$, there exist a $y \in U(t, \tau)x$ and a subsequence $y_{n_k}$ such that $y_{n_k} \to y$ (weak convergence)). If $\{U(t, \tau)\}$ is

1. pullback strongly bounded dissipative, i.e., there exists a family of bounded subsets $Q = \{Q(t)\}_{t \in \mathbb{R}}$ of $X$ so that for any fixed $t \in \mathbb{R}$, every bounded subset $B$ of $X$ and each $\tau \leq t$, there exists a $t_1 = t_1(B, \tau) \in \mathbb{R}^+$ such that

   $$U(\tau, \tau - s)B \subset Q(t), \quad \forall s \geq t_1;$$

2. pullback $\omega$-limit compact with respect to each $t \in \mathbb{R}$, then the MVP $\{U(t, \tau)\}$ has a minimal pullback attractor $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ in $X$ given by

   $$A(t) = \omega_t(Q(t)) = \bigcap_{T \in \mathbb{R}^+} \bigcup_{s \geq T} U(t, t - s)Q(t) \subset Q(t),$$

and $\bigcup_{s \leq t} A(s)$ is bounded for each $t \in \mathbb{R}$.

**Proof.** Since the MVP $\{U(t, \tau)\}$ is pullback $\omega$-limit compact and $Q$ is a family of bounded sets, for each fixed $t \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $t_\varepsilon = t_\varepsilon(t, Q(t), \varepsilon) > 0$ such that

$$k \left( \bigcup_{s \geq t_\varepsilon} U(t, t - s)Q(t) \right) \leq \varepsilon.$$

By the properties of the Kuratowski measure of noncompactness and slightly modifying the proof of Theorem 3.4 in [39], we can deduce that

$$A(t) = \omega_t(Q(t)) = \bigcap_{\tau \in \mathbb{R}^+} \bigcup_{s \geq \tau} U(t, t - s)Q(t)$$

is a nonempty compact set for each $t \in \mathbb{R}$.

Let us show that $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ is invariant. Let $y \in A(t)$; then by (2.1) we can find sequences $s_n \in \mathbb{R}^+$, $s_n \to +\infty(n \to \infty)$, $x_n \in Q(t)$, and $y_n \in U(t, t - s_n)x_n$ such that $y_n \to y$ as $n \to \infty$. Since $\{U(t, \tau)\}$ is pullback strongly bounded dissipative, given a sequence $\tau_n \in \mathbb{R}^+$ with $\tau_n \to +\infty(n \to \infty)$, there exists a sequence $\sigma_n \in \mathbb{R}^+$ with $\sigma_n \geq t + \tau_n - \tau$ such that $U(\tau - \tau_n, t - s)Q(t) \subset Q(\tau)$ for all $s \geq \sigma_n$. Observing that $s_n \to +\infty(n \to \infty)$, then for each $\tau_n$, there exists $k_n \geq n$ such that $U(\tau - \tau_n, t - s_{k_n})x_{k_n} \subset Q(\tau)$. Clearly,

$$y_{k_n} \in U(t, t - s_{k_n})x_{k_n} = U(t, t)U(\tau, \tau - \tau_n)U(\tau - \tau_n, t - s_{k_n})x_{k_n} \subset U(t, t)U(\tau, \tau - \tau_n)Q(\tau).$$

Thus there exists a sequence $\tilde{x}_n \in U(\tau, \tau - \tau_n)U(\tau - \tau_n, t - s_{k_n})x_{k_n} \subset U(\tau, \tau - \tau_n)Q(\tau)$ such that $y_{k_n} \in U(t, t)\tilde{x}_n$.

It follows from the pullback $\omega$-limit compactness that $\{\tilde{x}_n\}$ is relatively compact. Thus there is a subsequence of $\{\tilde{x}_n\}$ which we still relabel as $\{\tilde{x}_n\}$ such that $\tilde{x}_n \to x$ as $n \to \infty$, and by the definition of $A(\tau)$, we can deduce that $x \in \omega_\tau(Q(\tau)) = A(\tau)$. By the
Recall that norm-to-weak upper-semicontinuity of the MVP \( \{U(t, \tau)\} \), we can conclude that there exist a subsequence \( y_{k_{n_m}} \) of \( y_{k_n} \) and a \( y' \in U(t, \tau)x \) such that
\[
y_{k_{n_m}} \to y' \quad \text{as} \quad m \to \infty.
\]

Recall that
\[
y_n \to y \quad \text{as} \quad n \to \infty.
\]

Hence \( y = y' \in U(t, \tau)x \subset U(t, \tau)A(\tau) \) and \( A(t) \subset U(t, \tau)A(\tau) \).

On the other hand, since \( \{U(t, \tau)\} \) is pullback strongly bounded dissipative, in view of (2.1), we can deduce that for any fixed \( t \in \mathbb{R} \), \( A(\tau) \subset Q(t) \) for all \( \tau \leq t \). Then analogous to the proof of Lemma 3.8 in [39], we can conclude that \( U(t, \tau)A(\tau) = A(t), \forall t \geq \tau, \tau \in \mathbb{R} \).

Now, given a family of nonempty closed sets \( C = \{C(t)\}_{t \in \mathbb{R}} \) which pullback attracts bounded subsets of \( X \), then by the invariance of \( A = \{A(t)\}_{t \in \mathbb{R}} \) and \( \bigcup_{\tau \leq t} A(\tau) \subset Q(t) \) for each fixed \( t \in \mathbb{R} \), we can deduce that
\[
H_X^*(A(t), C(t)) 
\leq H_X^*(U(t, t-s)A(t-s), U(t, t-s)Q(t)) + H_X^*(U(t, t-s)Q(t), C(t)) \to 0
\]
as \( s \to +\infty \), and consequently, \( A(t) \subset C(t) \).

Finally, it suffices to prove that for every bounded subset \( B \) of \( X \) and each fixed \( t \in \mathbb{R} \),
\[
\lim_{s \to +\infty} H_X^*(U(t, t-s)B, A(t)) = 0.
\]

Suppose not. Then there exist a bounded subset \( B_0 \) of \( X \) and \( \tilde{t}_0 \in \mathbb{R} \) such that
\[
H_X^*(U(\tilde{t}_0, \tilde{t}_0 - s)B_0, A(\tilde{t}_0)) \not\to 0 \quad (s \to +\infty).
\]

Thus there exist \( \varepsilon' > 0 \) and sequences \( s_n \in \mathbb{R}^+, \; s_n \to +\infty(n \to \infty), \; x_n \in B_0, \) and \( y_n \in U(\tilde{t}_0, \tilde{t}_0 - s_n)x_n \) such that
\[
\text{dist}_X(y_n, A(\tilde{t}_0)) \geq \varepsilon' > 0, \; \forall n \in \mathbb{N}.
\]

Noticing that \( \{U(t, \tau)\} \) is pullback strongly bounded dissipative, similar to the proof above, for each \( \tau_n \), there exists \( j_n \geq n \) such that \( U(\tilde{t}_0 - \tau_n, \tilde{t}_0 - s_{j_n})x_{j_n} \subset Q(\tilde{t}_0) \), and thus
\[
y_{j_n} \in U(\tilde{t}_0, \tilde{t}_0 - s_{j_n})x_{j_n} = U(\tilde{t}_0, \tilde{t}_0 - \tau_n)U(\tilde{t}_0 - \tau_n, \tilde{t}_0 - s_{j_n})x_{j_n} \subset U(\tilde{t}_0, \tilde{t}_0 - \tau_n)Q(\tilde{t}_0).
\]

Due to the pullback \( \omega \)-limit compactness, we can see that \( y_{j_n} \) is relatively compact and possesses at least one cluster point \( y_0 \). Hence \( y_0 \) belongs to \( A(\tilde{t}_0) = \omega_{\tilde{t}_0}(Q(\tilde{t}_0)) \), and this contradicts (2.3). Thus the proof of Theorem 2.8 is completed.

\[\square\]

**Remark 2.9.** (1) Under the assumption that the MVP \( \{U(t, \tau)\} \) is pullback \( \omega \)-limit compact with respect to each \( t \in \mathbb{R} \), we can show that
\[
A(t) = \bigcap_{T \geq 0} \bigcup_{s \geq T} U(t, t-s)Q(t)^{WS} = \bigcap_{T \geq 0} \bigcup_{s \geq T} U(t, t-s)Q(t).
\]

(2) Let \( K \) be a semibounded kernel of the MVP \( \{U(t, \tau)\} \) defined by
\[
K = \left\{ u(\cdot) \mid \sup_{t \in (-\infty, b]} \|u(t)\|_X \leq C_{u,b}, \; \forall b \in \mathbb{R}, \; u(t) \in U(t, \tau)u(\tau), \; \forall t \geq \tau, \; \tau \in \mathbb{R} \right\},
\]
and let $\mathcal{K}(t)$ is the kernel section at a time moment $t \in \mathbb{R}$:

$$\mathcal{K}(t) = \{ u(t) \mid u(\cdot) \in \mathcal{K} \}, \quad \mathcal{K}(t) \subset X.$$ 

Then by Theorem 2.8, we can conclude from the similar proof of Theorem 3.4 in [39] that the kernel section

$$\mathcal{K}(t) = A(t), \quad \forall t \in \mathbb{R},$$

where $A(t)$ is the section of the pullback attractor $\{ A(t) \}_{t \in \mathbb{R}}$ for the MVP $\{ U(t, \tau) \}$ given in Theorem 2.8.

(3) For any fixed $t \in \mathbb{R}$, $A(t)$ given in Theorem 2.8 is compact and pullback attracts every bounded set at time $t$, so

$$\bigcup \{ \omega_t(B) : B \subset X, B \text{ bounded} \} \subset A(t).$$

On the other hand, since $Q(t)$ is a bounded subset of $X$ for each $t \in \mathbb{R}$,

$$A(t) = \omega_t(Q(t)) \subset \bigcup \{ \omega_t(B) : B \subset X, B \text{ bounded} \}.$$ 

Therefore,

$$A(t) = \omega_t(Q(t)) = \bigcup \{ \omega_t(B) : B \subset X, B \text{ bounded} \}.$$ 

(4) Let $\{ U(t, \tau) \}$ be a multi-valued process on a Banach space $X$, and let $U(t, \tau)x$ be norm-to-weak upper-semicontinuous in $x$ for fixed $t \geq \tau$, $\tau \in \mathbb{R}$. If $\{ U(t, \tau) \}$ is pullback bounded dissipative, and pullback $\omega$-limit compact with respect to each $t \in \mathbb{R}$, then $A = \{ A(t) \}_{t \in \mathbb{R}}$ defined by

$$A(t) = \bigcup \{ \omega_t(B) : B \subset X, B \text{ bounded} \}$$

is minimal, closed, invariant, and pullback attracts bounded subsets of $X$ (see [39] and the references cited therein).

**Definition 2.10.** A multi-valued process $\{ U(t, \tau) \}$ on a Banach space $X$ is said to be pullback flattening if for each $t \in \mathbb{R}$, any bounded subset $B$ of $X$ and $\varepsilon > 0$, there exist $\tau_0 = \tau_0(t, B, \varepsilon) > 0$ and a finite-dimensional subspace $X_1$ of $X$ such that

1. $\{ P \left( \bigcup_{s \geq \tau_0} U(t, t-s)B \right) \}$ is bounded,

2. $\left\| (I - P) \left( \bigcup_{s \geq \tau_0} U(t, t-s)B \right) \right\|_X < \varepsilon,$

where $P : X \to X_1$ is the canonical projector.

As a simple consequence of Theorem 2.8 and the proof of Theorem 4.10 in [38], we have

**Theorem 2.11.** Let $X$ be a uniformly convex Banach space, in particular, $X$ be a Hilbert space. Let $\{ U(t, \tau) \}$ be a multi-valued process on $X$, and $U(t, \tau)x$ is norm-to-weak upper-semicontinuous in $x$ for fixed $t \geq \tau$, $\tau \in \mathbb{R}$. If $\{ U(t, \tau) \}$ is pullback strongly bounded dissipative and pullback flattening, then the MVP $\{ U(t, \tau) \}$ has a minimal pullback attractor $A = \{ A(t) \}_{t \in \mathbb{R}}$ in $X$ given in Theorem 2.8 and $\bigcup_{s \leq t} A(s)$ is bounded for each $t \in \mathbb{R}$.
3. Robustness of pullback attractors. In this section, we discuss the upper semi-continuity of pullback attractors for multi-valued systems with small perturbations, and related work may be found in, for example, [3, 16] and the references therein.

Theorem 3.1. Let \( \{U^\lambda(t, \tau)\} \), \( \lambda > 0 \), be a family of multi-valued processes on \( X \). Suppose that the following conditions are satisfied:

(H1) Existence of attractors for \( \lambda > 0 \); for \( \lambda \in (0, \lambda_0] \) there exists a pullback attractor \( \{A^\lambda(t)\}_{t \in \mathbb{R}} \) for the MVP \( \{U^\lambda(t, \tau)\} \).

(H2) Pullback attractors are asymptotically semuniformly equi-bounded: for any fixed \( t \in \mathbb{R} \), there exist a \( s' > 0 \) and a \( \delta_0 \) with \( 0 < \delta_0 < \lambda_0 \) such that

\[
\bigcup_{\lambda \in (0, \delta_0)} \bigcup_{s \geq s'} A^\lambda(t - s)
\]

is a bounded set.

(H3) Upper semi-continuity condition: for any \( \varepsilon > 0 \), each \( t \in \mathbb{R} \), and every bounded subset \( B \) of \( X \), there exist a \( \delta > 0 \) and a sufficiently large \( s_0 > 0 \) such that when \( 0 < \lambda < \delta \),

\[
H_X(U^\lambda(t, t - s_0)x, F(s_0)\mathcal{N}(x, \varepsilon)) < \varepsilon, \quad \forall x \in B.
\]

(H4) Existence of a global attractor for the autonomous multi-valued semidynamical system: \( \{F(t)\} \) has a global attractor \( \mathcal{A} \).

Then for each \( t \in \mathbb{R} \),

\[
\lim_{\lambda \to 0} H_X^*(A^\lambda(t), \mathcal{A}) = 0.
\]

Proof. Let \( t \in \mathbb{R} \) be given arbitrarily, and let

\[
B = \bigcup_{\lambda \in (0, \delta_0)} \bigcup_{s \geq s'} A^\lambda(t - s).
\]

Noticing that \( \mathcal{A} \) is a global attractor for the MVSS \( \{F(t)\} \), there is a \( \bar{s} > s' \) such that

\[
H_X^*(F(s)\mathcal{N}(B, 1), \mathcal{A}) < \varepsilon, \quad \forall s \geq \bar{s}.
\]

On the other hand, by assumption (H3), we have for any \( 0 < \varepsilon < 1 \), there exist a \( \delta > 0 \) with \( \delta < \delta_0 \) and a \( s_0 > \bar{s} \) such that when \( 0 < \lambda < \delta \),

\[
H_X(U^\lambda(t, t - s_0)x, F(s_0)\mathcal{N}(x, \varepsilon)) < \varepsilon, \quad \forall x \in B.
\]

Then for all \( \lambda \) with \( 0 < \lambda < \delta \),

\[
H_X^*(A^\lambda(t), \mathcal{A}) \\
\leq H_X^*(U^\lambda(t, t - s_0)A^\lambda(t - s_0), F(s_0)\mathcal{N}(A^\lambda(t - s_0), \varepsilon)) \\
+ H_X^*(F(s_0)\mathcal{N}(A^\lambda(t - s_0), \varepsilon), \mathcal{A}) < 2\varepsilon.
\]

The proof of Theorem 3.1 is completed. \( \square \)

The following theorem provides a method which can be used to verify the upper semi-continuity condition (H3) in Theorem 3.1.

Theorem 3.2. Let \( \{U^\lambda(t, \tau)\} \), \( \lambda > 0 \), be a family of multi-valued processes on a Banach space \( X \), and let \( \{F(t)\} \) be an autonomous multi-valued semidynamical system on \( X \). Assume that for any \( 0 < \varepsilon < 1 \), each \( t \in \mathbb{R} \), and every bounded subset \( B \) of \( X \), there
exist a sufficiently large $s' > 0$, a $\delta'' > 0$ and a finite-dimensional subspace $X_1$ of $X$ such that

1. For all $0 < \lambda < \delta''$, $x \in B$ and all $u \in U^\lambda(t, t - s')x$,
   $$\|(I - P)u\|_X < \frac{\varepsilon}{3}.$$  

2. For all $v \in F(s')\mathcal{N}(B, 1)$,
   $$\|(I - P)v\|_X < \frac{\varepsilon}{3}.$$  

3. There exists a $\delta'$ with $0 < \delta' < \delta''$ so that for any $0 < \lambda < \delta'$, $x \in B$ and every $u \in U^\lambda(t, t - s')x$, we can find a $Pv \in PF(s')\mathcal{N}(Px, \frac{\varepsilon}{3})$ such that
   $$\|P(u - v)\|_X < \frac{\varepsilon}{3},$$

where $P : X \to X_1$ is the canonical projector. Then the upper semicontinuity condition (H3) in Theorem 3.1 holds true.

**Proof.** Let $0 < \varepsilon < 1$, $t \in \mathbb{R}$, and bounded set $B \subset X$ be given arbitrarily. Then by assumption (3), there exist $s' > 0$, $\delta' > 0$ and a finite-dimensional subspace $X_1$ of $X$, so that for any $0 < \lambda < \delta'$, $x \in B$ and every $u \in U^\lambda(t, t - s')x$, we can find a $Pv \in PF(s')\mathcal{N}(Px, \frac{\varepsilon}{3})$ such that

$$\|P(u - v)\|_X < \frac{\varepsilon}{3},$$

(3.1)

where $P : X \to X_1$ is the canonical projector.

From assumptions (1)-(2), we can deduce that for any $v \in F(s')\mathcal{N}(x, \varepsilon)$,

$$\|u - v\|_X \leq \|P(u - v)\|_X + \|(I - P)(u - v)\|_X \leq \|P(u - v)\|_X + \frac{2\varepsilon}{3}. \quad (3.2)$$

Note that $Px \in PN(x, \frac{\varepsilon}{3})$. Hence $\mathcal{N}(Px, \frac{\varepsilon}{3}) \subset PN(x, \varepsilon)$. By (3.1)-(3.2), we have

$$\inf_{v \in F(s')\mathcal{N}(x, \varepsilon)} \|u - v\|_X \leq \inf_{Pv \in PF(s')PN(x, \varepsilon)} \|P(u - v)\|_X + \frac{2\varepsilon}{3} \leq \inf_{Pv \in PF(s')\mathcal{N}(Px, \frac{\varepsilon}{3})} \|P(u - v)\|_X + \frac{2\varepsilon}{3} < \varepsilon. \quad (3.3)$$

Since $u \in U^\lambda(t, t - s')x$, $0 < \lambda < \delta'$, and $x \in B$ are arbitrary, then condition (H3) in Theorem 3.1 follows immediately from (3.3) and the definition of Hausdorff semidistance, and thus the proof is complete. \hfill \Box

**Remark 3.3.** (1) In the single-valued case, condition (H3) in Theorem 3.1 can be revised as follows: for any $\varepsilon > 0$, each $t \in \mathbb{R}$, and every bounded subset $B$ of $X$, there exist a $\delta > 0$ and a sufficiently large $s_0 > 0$ such that when $0 < \lambda < \delta$,

$$d_X(U^\lambda(t, t - s_0)x, F(s_0)x) < \varepsilon, \forall x \in B.$$  

(2) It is worth mentioning that the abstract results in this section can also be applied to pullback $\mathcal{D}$-attractors and local kernel sections defined in [8, 39] and the references cited therein.

In particular, we consider a family of autonomous multi-valued semidynamical systems $\{F^\lambda(t)\} (\lambda \in \Lambda)$, where $\Lambda$ is a metric space with metric $d_\Lambda(\cdot, \cdot)$.
Theorem 3.4. Let \( \{F^\lambda(t)\}, \lambda \in \Lambda \), be a family of autonomous multi-valued semidynamical systems on \( X \), and let \( A^\lambda \) be a global attractor of \( \{F^\lambda(t)\} \) for each \( \lambda \in \Lambda \). If there is a \( \delta_1 > 0 \) such that \( \bigcup_{d_\lambda(\lambda, \lambda_0) < \delta_1} A^\lambda \) is a bounded set, and
\[
(H5) \text{ for any } \varepsilon > 0 \text{ and every bounded subset } B \text{ of } X, \text{ there exist a } \delta_1 > 0 \text{ and a sufficiently large } s_1 > 0 \text{ such that for all } \lambda \text{ with } d_\lambda(\lambda, \lambda_0) < \delta_1,
\]
\[
H_X(F^\lambda(s_1), F^\lambda_0(s_1))N(x, \varepsilon) < \varepsilon, \forall x \in B
\]
holds true, then \( \lim_{\lambda \to \lambda_0} H_X(A^\lambda, A^\lambda_0) = 0 \).

Now it is natural to investigate the nonautonomous multi-valued dynamical systems, similar to the above. We have the following results:

Theorem 3.5. Let \( \{U^\lambda(t, \tau)\}, \lambda \in \Lambda \), be a family of multi-valued processes on \( X \), and let \( \{A^\lambda(t)\}_{t \in \mathbb{R}} \) be a pullback attractor of \( \{U^\lambda(t, \tau)\} \) for each \( \lambda \in \Lambda \). Suppose that for each fixed \( t \in \mathbb{R} \), there exist a \( \delta_2 > 0 \) and a \( \hat{s}_2 > 0 \) such that \( \bigcup_{d_\lambda(\lambda, \lambda_0) < \delta_2} \bigcup_{s \geq \hat{s}_2} A^\lambda(t-s) \) is a bounded set, and that
\[
(H6) \text{ for any } \varepsilon > 0, \text{ each } t \in \mathbb{R}, \text{ and every bounded subset } B \text{ of } X, \text{ there exist a } \delta_2 > 0 \text{ and a sufficiently large } s_2 > 0 \text{ such that for all } \lambda \text{ with } d_\lambda(\lambda, \lambda_0) < \delta_2,
\]
\[
H_X(U^\lambda(t, t-s_2)x, U^\lambda_0(t, t-s_2))N(x, \varepsilon) < \varepsilon, \forall x \in B.
\]
Then for each \( t \in \mathbb{R} \), \( A^\lambda(t) \) is upper semicontinuous at \( \lambda_0 \).

Theorem 3.6. Let \( \{U^\lambda(t, \tau)\}, \lambda \in \Lambda \), be a family of multi-valued processes on a Banach space \( X \). If for any \( 0 < \varepsilon < 1 \), each \( t \in \mathbb{R} \), and every bounded subset \( B \) of \( X \), there exist a sufficiently large \( s'_2 > 0 \), a \( \tilde{\delta}_1 > 0 \) and a finite-dimensional subspace \( X_1 \) of \( X \) such that
\[
(1) \text{ For all } \lambda \text{ with } d_\lambda(\lambda, \lambda_0) < \tilde{\delta}_1, x \in N(B, 1), \text{ and all } u \in U^\lambda(t, t-s'_2)x,
\]
\[
\|(I-P)u\|_X < \frac{\varepsilon}{3},\]
\[
(2) \text{ There exists a } \tilde{\delta}_2' \text{ with } 0 < \tilde{\delta}_2' < \tilde{\delta}_1 \text{ so that for any } \lambda \text{ with } d_\lambda(\lambda, \lambda_0) < \tilde{\delta}_2', x \in B \text{ and every } u \in U^\lambda(t, t-s'_2)x, \text{ we can find a } Pv \in PU^\lambda_0(t, t-s'_2)N(Px, \frac{\varepsilon}{3}) \text{ such that}
\]
\[
\|P(u-v)\|_X < \frac{\varepsilon}{3},\]
where \( P : X \to X_1 \) is the canonical projector. Then condition \( (H6) \) in Theorem 3.5 holds true.

Remark 3.7. Let the multi-valued process \( \{U^\lambda(t, \tau)\} \) be \( T_0 \)-periodic (i.e., \( U^\lambda(t+T_0, \tau+T_0) = U^\lambda(t, \tau), \forall t \geq \tau, \tau \in \mathbb{R} \)) for each \( \lambda \in \Lambda \), and let \( \{A^\lambda(t)\}_{t \in \mathbb{R}} \) be a \( T_0 \)-periodic uniform forward attractor for \( \{U^\lambda(t, \tau)\} \) (see [22]). By slightly modifying the proof of Theorem 4.2 in [22], we can deduce that if the condition
\[(H7) \text{ For any } \varepsilon > 0, \text{ each } T > 0 \text{ and every bounded subset } B \text{ of } X, \text{ there exists a } \delta_3 > 0 \text{ such that for all } \lambda \text{ with } d_\lambda(\lambda, \lambda_0) < \delta_3,
\]
\[
H_X(U^\lambda(t, \tau)x, U^\lambda_0(t, \tau))N(x, \varepsilon) < \varepsilon, \forall \tau \in \mathbb{R}, t \in [\tau, \tau+T], x \in B,
\]
holds, then \( A^\lambda(t) \) is uniformly upper semicontinuous at \( \lambda_0 \), i.e.,
\[
\lim_{\lambda \to \lambda_0} \sup_{t \in \mathbb{R}} H_X(A^\lambda(t), A^\lambda_0(t)) = 0.
\]
In addition, we are interested in the compactness and attracting property of the kernel in $C([-M, M]; X)$ for each $M > 0$, which will be used in Section 4.

**Theorem 3.8.** Let $\{U(t, \tau)\}$ be a multi-valued process on a Banach space $X$, and let $K$ be a semibounded kernel of $\{U(t, \tau)\}$ defined in Remark 2.9. Suppose in addition to the hypotheses in Theorem 2.8 that for any fixed $M > 0$, each $t \in \mathbb{R}$, any bounded subset $B$ of $C([-M, M]; X)$ and $\varepsilon > 0$ there exist $\tau_2 = \tau_2(t, B, \varepsilon) > 0$, a finite-dimensional subspace $X_1$ of $X$ and a $\delta_3^* > 0$ such that

(1) for all $s \geq \tau_2$, $\psi \in B$, $u(t + \theta) \in U(t + \theta, t + \theta - s)\psi(\theta)$, $\theta_1, \theta_2 \in [-M, M]$ with $|\theta_2 - \theta_1| < \delta_3^*$,

$$\|P(u(t + \theta_1) - u(t + \theta_2))\|_X < \frac{\varepsilon}{4};$$

(2) for all $s \geq \tau_2$, $\psi \in B$, $u(t + \theta) \in U(t + \theta, t + \theta - s)\psi(\theta)$,

$$\sup_{\theta \in [-M, M]} \|(I - P)u(t + \theta)\|_X < \frac{\varepsilon}{4};$$

where $P : X \rightarrow X_1$ is the canonical projector.

Then for any fixed $t \in \mathbb{R}$, $K(t + \cdot)$ is compact in $C([-M, M]; X)$, and for any bounded subset $B$ of $C([-M, M]; X)$, there exists a $\tau_2^* > 0$ such that for all $s \geq \tau_2^*$, $\psi \in B$, and all $u(t + \theta) \in U(t + \theta, t + \theta - s)\psi(\theta)$, we have

$$d_{C([-M, M]; X)}(u(t + \cdot), K(t + \cdot)) < \varepsilon.$$

**Proof.** First we prove that for any fixed $t \in \mathbb{R}$, $K(t + \theta)$ is compact in $C([-M, M]; X)$. Let a sequence $w_n \in K$ be given arbitrarily. By Remark 2.9, we see that $w_n(s) \in K(s)$ for all $n \in \mathbb{N}$ and $s \in \mathbb{R}$, and that the kernel section $K(t) = A(t)$, $\forall t \in \mathbb{R}$, where $A(t)$ is the section of the pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$ for $\{U(t, \tau)\}$ given in Theorem 2.8. Thus for each fixed $\theta \in [-M, M]$, $w_n(t + \theta)$ is precompact in $X$. On the other hand, by assumptions (1) – (2), we can deduce that for any $\varepsilon > 0$ and all $\theta_1, \theta_2 \in [-M, M]$ with $|\theta_2 - \theta_1| < \delta_3^*$,

$$\|w_n(t + \theta_1) - w_n(t + \theta_2)\|_X$$

$$\leq \|(I - P)(w_n(t + \theta_1) - w_n(t + \theta_2))\|_X + \|P(w_n(t + \theta_1) - w_n(t + \theta_2))\|_X$$

$$< \varepsilon, \forall n \in \mathbb{N}.$$

By making use of the Arzelá-Ascoli theorem, we can see that there is a subsequence (which we relabel as $w_n$) of $w_n$ such that $w_n(t + \cdot) \rightarrow w^*(t + \cdot)$ in $C([-M, M]; X)$. Notice that the kernel section $K(t) = A(t)$, $\forall t \in \mathbb{R}$, where $A(t)$ is the section of the pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$ for $\{U(t, \tau)\}$ given in Theorem 2.8. Hence for any fixed $\theta \in [-M, M]$, $K(t + \theta)$ is compact and $w^*(t + \theta) \in K(t + \theta)$ for all $\theta \in [-M, M]$. Similar to the arguments of Theorem 3.4 in [39], by the invariance of kernel section $K(t)$, we can construct a local bounded complete trajectory $w(t)$, $t \in \mathbb{R}$, of the MVP $\{U(t, \tau)\}$ such that $w \in K$ and $w(t + \theta) = w^*(t + \theta)$ for all $\theta \in [-M, M]$. Thus we have completed the proof of what we expected.

Now we show that (3.4) holds true. Suppose not. Then there exist a bounded subset $\tilde{B}$ of $C([-M, M]; X)$, $\tilde{t} \in \mathbb{R}$, $\varepsilon' > 0$, sequences $s_n \rightarrow \infty (n \rightarrow \infty)$, $\psi_n \in \tilde{B}$, and $u_n(\tilde{t} + \theta) \in U(\tilde{t} + \theta, \tilde{t} + \theta - s_n)\psi_n(\theta)$ such that

$$d_{C([-M, M]; X)}(u_n(\tilde{t} + \cdot), K(\tilde{t} + \cdot)) \geq \varepsilon', \forall n \in \mathbb{N}.$$
It follows from Remark 2.9 that the kernel section $K(t) = A(t)$, $\forall t \in \mathbb{R}$, where $A(t)$ is the section of the pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$ for $\{U(t, \tau)\}$ given in Theorem 2.8. Then by the compactness and the pullback attracting property of the pullback attractors, we can deduce that the existence of the semibounded kernel $K$ for $\{U(t, \tau)\}$ implies that for each fixed $\theta \in [-M, M]$, $u_n(\tilde{t} + \theta)$ is precompact in $X$. Using assumptions (1) – (2) again, we can conclude that there exists a $\delta'_3 > 0$ such that for all $\theta_1, \theta_2 \in [-M, M]$ with $|\theta_2 - \theta_1| < \delta'_3$, we have
\[
\|u_n(\tilde{t} + \theta_1) - u_n(\tilde{t} + \theta_2)\|_X 
\leq \|(I - P)(u_n(\tilde{t} + \theta_1) - u_n(\tilde{t} + \theta_2))\|_X + \|P(u_n(\tilde{t} + \theta_1) - u_n(\tilde{t} + \theta_2))\|_X < \varepsilon
\]
for $n$ sufficiently large. Hence, by the Arzelá-Ascoli theorem passing to a subsequence (which we still denote by $u_n(\tilde{t} + \cdot)$), we may assume that $u_n(\tilde{t} + \cdot)$ converges to a function $u^*(\tilde{t} + \cdot)$ in $C([-M, M]; X)$. Since the kernel section $K(t) = A(t)$, $\forall t \in \mathbb{R}$, where $A(t)$ is the section of the pullback attractor $\{A(t)\}_{t \in \mathbb{R}}$ for $\{U(t, \tau)\}$ given in Theorem 2.8, by the compactness and the pullback attraction of the pullback attractors, we see that $u^*(\tilde{t} + \theta) \in K(\tilde{t} + \theta)$ for all $\theta \in [-M, M]$. Similar to the above, we can construct a local bounded complete trajectory $u(t), t \in \mathbb{R}$, of the MVP $\{U(t, \tau)\}$ such that $u(\tilde{t} + \theta) = u^*(\tilde{t} + \theta)$ for all $\theta \in [-M, M]$. Therefore, $u \in K$ and $u_n(\tilde{t} + \cdot)$ converges to $u(\tilde{t} + \cdot)$ in $C([-M, M]; X)$; this contradicts (3.5). Theorem 3.8 is proved.

**Remark 3.9.** Note that all norms on finite-dimensional spaces are equivalent, analogous to the arguments in [10]; then assumptions (1)–(2) follow.

For the autonomous case, similar to Theorem 3.8, we have the following result:

**Theorem 3.10.** Let $K$ be a kernel of the multi-valued semidynamical system $\{F(t)\}$ on a Banach space $X$. If for any fixed $M > 0$, each bounded subset $B$ of $C([-M, M]; X)$ and $\varepsilon > 0$, there exist $\tau_3 = \tau_3(B, \varepsilon) > 0$, a finite-dimensional subspace $X_1$ of $X$ and a $\delta_4 > 0$ such that

1. for all $t \geq \tau_3$, $\psi \in B$, $u(t + \theta) \in F(t)\psi(\theta)$, $\theta_1, \theta_2 \in [-M, M]$ with $|\theta_2 - \theta_1| < \delta_4$, $\|P(u(t + \theta_1) - u(t + \theta_2))\|_X < \varepsilon/4$;

2. for all $t \geq \tau_3$, $\psi \in B$, $u(t + \theta) \in F(t)\psi(\theta)$, $\sup_{\theta \in [-M, M]} \|\|(I - P)u(t + \theta)\|_X < \varepsilon/4$,

where $P : X \to X_1$ is the canonical projector.

Then $K(\cdot)$ is compact in $C([-M, M]; X)$, and for any bounded set $B \subset C([-M, M]; X)$, there exists a $\tau'_3 > 0$ such that for all $t \geq \tau'_3$, $\psi \in B$, and all $u(t + \theta) \in F(t)\psi(\theta)$, we have
\[
d_{C([-M, M]; X)}(u(t + \cdot), K(\cdot)) < \varepsilon.
\]
4. Parabolic differential equations with small delays. In this section, we consider a family of nonautonomous functional parabolic equations perturbed by $0 < \varepsilon \leq \varepsilon_0$,

\[
\begin{aligned}
\frac{\partial u}{\partial t} + Au + bu &= f(u_{\varepsilon t}) + g(x), \quad \text{in } (\tau, +\infty) \times \Omega, \\
u &= 0, \quad \text{on } (\tau, +\infty) \times \partial \Omega, \\
u(t, x) &= \phi(t - \tau, x), \quad t \in [\tau - \varepsilon_0, \tau], \quad x \in \Omega,
\end{aligned}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary, $b \geq 0$, $\varepsilon_0 > 0$, $g \in L^2(\Omega)$, and the nonlinear term

\[
f(u_{\varepsilon t}(t, x)) = F(u(t - \rho_{\varepsilon}(t), x)) + \int_{-\varepsilon}^{0} G(s, u(t + s, x))ds.
\]

We assume that

(H1) $A$ is a densely defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and with compact resolvent;

(H2) there exist positive constants $k_1$, $k_2$, and positive scalar functions $m_0, m_1 \in L^1([-\varepsilon_0, 0])$ such that the functions $F \in C(\mathbb{R}; \mathbb{R})$, $\rho_{\varepsilon} \in C^1(\mathbb{R}; [0, \varepsilon])$ and $G \in C([-\varepsilon_0, 0] \times \mathbb{R}; \mathbb{R})$ satisfy

\[
|F(v)|^2 \leq k_1^2 + k_2^2|v|^2, \quad \forall v \in \mathbb{R},
\]

\[
|\rho'_{\varepsilon}(t)| \leq \rho_* < 1, \quad \forall t \in \mathbb{R}, \quad \varepsilon \in [0, \varepsilon_0],
\]

and

\[
|G(s, v)| \leq m_0(s) + m_1(s)|v|, \quad \forall v \in \mathbb{R}.
\]

We now introduce several notations. Denote

\[
m_0 = \int_{-\varepsilon_0}^{0} m_0(s)ds, \quad m_1 = \int_{-\varepsilon_0}^{0} m_1(s)ds,
\]

where $m_0, m_1 \in L^1([-\varepsilon_0, 0])$ given in (4.4). Let $H = L^2(\Omega)$ with norm $| \cdot |$ and inner product $(\cdot, \cdot)$, and let $V = D(A^\frac{1}{2})$ with norm $\| \cdot \|$ and associated scalar product $(\cdot, \cdot)$, where for $u, v \in V$, $(u, v) = (A^\frac{1}{2}u, A^\frac{1}{2}v)$. Let $C$ be the arbitrary positive constants, which may be different from line to line and even on the same line.

Given $T > \tau$ and $u : [\tau - \varepsilon_0, T] \to H$, for each $t \in [\tau, T)$ we denote by $u_t$ the function defined on $[-\varepsilon_0, 0]$ by the relation $u_t(s) = u(t + s)$, $s \in [-\varepsilon_0, 0]$. We also denote $C_H = C([-\varepsilon_0, 0]; H)$, $C_V = C([-\varepsilon_0, 0]; V)$.

It is natural to compare the dynamics of a system of parabolic equations perturbed by small delays with that of the same system without delays and to show that their attractors are “close” in the sense of Hausdorff semidistance. Thus, we have to investigate the autonomous parabolic problem:

\[
\begin{aligned}
\frac{\partial v}{\partial t} + Av + bv &= F(v) + g(x), \quad \text{in } (0, +\infty) \times \Omega, \\
v &= 0, \quad \text{on } (0, +\infty) \times \partial \Omega, \\
v(0, x) &= v_0(x), \quad x \in \Omega.
\end{aligned}
\]

By the classical Galerkin method, analogous to the arguments in [11], we have

**Theorem 4.1.** Let $g \in H$, (H1) and (4.2) hold true. Then
(1) For any \( v_0 \in H \), there exists a solution \( v(t) \) to problem (4.5), and \( v(t) \) satisfies
\[
v \in C([0, T]; H) \cap L^\infty(0, T; H) \cap L^2(0, T; V), \forall T > 0.
\]

(2) For any \( v_0 \in V \), problem (4.5) admits a strong solution,
\[
v \in C([0, T]; V) \cap L^\infty(0, T; V) \cap L^2(0, T; D(A)), \forall T > 0.
\]

**Theorem 4.2.** Suppose that \( g \in H \), (H1), (4.2), and \( 2\lambda_1 + b - \frac{k_2^2}{b} > 0 \) hold true. Then

(1) There exists a unique global attractor \( \mathcal{A} \) for the multi-valued semidynamical system \( \{F(t)\} \) on \( V \) generated by problem (4.5), and \( \mathcal{A} \) coincides with the kernel section at time \( \tau \), i.e., \( \mathcal{A} = \mathcal{K}(\tau) \) for any \( \tau \in \mathbb{R} \).

(2) Let \( \{\mathcal{F}(t)\} \) be an autonomous multi-valued semidynamical system on \( C_V \) given by
\[
[\mathcal{F}(t)\psi](s) = F(t)\psi(s), \forall \psi \in C_V,
\]
and let
\[
A_{C_V} = \{\psi \in C_V : \text{there is a bounded complete trajectory } v' \in \mathcal{K} \text{ such that } v'(t) = \psi(t), \forall t \in [-\varepsilon_0, 0]\}.
\]

Then \( A_{C_V} \) is a global attractor for the autonomous multi-valued semidynamical system \( \{\mathcal{F}(t)\} \).

**Proof.** (1) Since \( A^{-1} \) is a continuous compact operator in \( H \), by the classical spectral theory, there exist a sequence \( \{\lambda_j\}_{j=1}^\infty \),
\[
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to +\infty, \quad \text{as } j \to +\infty,
\]
and a family of elements \( \{\omega_j\}_{j=1}^\infty \) of \( D(A) \), which are orthonormal in \( H \) such that
\[
A\omega_j = \lambda_j\omega_j \quad \text{for all } j \in \mathbb{N}.
\]

Let \( V_m = \text{span}\{\omega_1, \cdots, \omega_m\} \) in \( V \) and \( P_m : V \to V_m \) be the orthogonal projector.

Let \( v = v_1 + v_2 \), where \( v_1 = P_m v \) and \( v_2 = (I - P_m)v \). We decompose Eq. (4.5) as follows:
\[
\frac{\partial v_2(t)}{\partial t} + Av_2(t) + bv_2(t) = F(v(t)) - P_m F(v_1(t)) + (I - P_m)g(x) \tag{4.6}
\]
and
\[
\frac{\partial v_1(t)}{\partial t} + Av_1(t) + bv_1(t) = P_m F(v_1(t)) + P_m g(x). \tag{4.7}
\]

Taking the inner product in \( H \) of (4.6) with \( Av_2 \), in view of (4.2) and Young’s inequality, we get
\[
\frac{d}{dt} \|v_2(t)\|^2 + 2|Av_2(t)|^2 + 2b\|v_2(t)\|^2
\]
\[
= 2(F(v(t)) - P_m F(v_1(t)), Av_2(t)) + 2((I - P_m)g(x), Av_2(t))
\]
\[
\leq \frac{3}{4}|Av_2(t)|^2 + 8k_1^2(\text{meas } \Omega) + 8k_2^2\|v(t)\|^2 + 4\|g(x)\|^2.
\]

Note that \( |Av_2|^2 \geq \lambda_{m+1}\|v_2\|^2 \). Applying Gronwall’s inequality in the interval \([0, t]\), it yields
\[
\|v_2(t)\|^2 \leq \|v_2(0)\|^2 e^{-\lambda_{m+1}t} + \int_0^t e^{-\lambda_{m+1}(t-s)}(8k_1^2(\text{meas } \Omega) + 8k_2^2\|v(s)\|^2 + 4\|g(x)\|^2)ds. \tag{4.8}
\]
Similar to the above, taking the scalar product of (4.5) and \( v(t) \), in view of \( 2\lambda_1 + b - \frac{k_2^2}{b} > 0 \), we can choose \( \varepsilon^* \) small enough such that \( 2\lambda_1 + b - \frac{k_2^2}{b} - \varepsilon^* > 0 \), then we have

\[
\frac{d}{dt}|v(t)|^2 + \left( 2\lambda_1 + b - \frac{k_2^2}{b} - \varepsilon^* \right) |v(t)|^2 \leq \frac{k_2^2|\Omega|}{b} + \frac{1}{\varepsilon^*}|g|^2;
\]

and thus Gronwall’s Lemma implies that

\[
|v(t)|^2 \leq |v(0)|^2 + C + C|g(x)|^2. \tag{4.9}
\]

Combining (4.8) and (4.9) together, we can deduce that

\[
\|v_2(t)\|^2 \leq \|v(0)\|^2 e^{-\lambda_{m+1}t} + \frac{C|v(0)|^2}{\lambda_{m+1}} + \frac{C}{\lambda_{m+1}} + \frac{C|g(x)|^2}{\lambda_{m+1}}. \tag{4.10}
\]

Similar to the proof of the existence of solutions (see [11] for details), in view of the standard estimation for the existence of a bounded absorbing set in \( V \) and Theorem 2.7 in [38], we can show that \( F(t) \) \( v_0 \) with \( v_0 \in V \) is norm-to-weak upper-semicontinuous in \( v_0 \) for any fixed \( t \geq 0 \), i.e., if \( v_{n0} \to v_0 \) in \( V \), then for any \( v_n(t) \in F(t)v_{n0} \), there exist \( v(t) \in F(t)v_0 \) and a subsequence \( v_{n_k}(t) \) such that \( v_{n_k}(t) \to v(t) \) in \( V \). Now thanks to Theorem 2.11, conclusion (1) follows from (4.10) and the standard estimation for the existence of a bounded absorbing set in \( V \), and thus the details are omitted here.

(2) It follows from the invariance of \( A \) for the MVSS \( \{F(t)\} \) on \( V \) and \( A = K(\tau) \) for any \( \tau \in \mathbb{R} \) that \( F(t)A_{C_V} = A_{C_V} \) for all \( t \in \mathbb{R}^+ \).

Invoking Theorem 3.10, now we only need to check conditions (1) – (2) in Theorem 3.10.

Replacing \( t \) in (4.10) by \( t + \theta \), we get

\[
\sup_{\theta \in [-\varepsilon_0, 0]} \|v_2(t + \theta)\|^2 \leq \|v(0)\|^2 e^{-\lambda_{m+1}(t+\theta)} + \frac{C|v(0)|^2}{\lambda_{m+1}} + \frac{C}{\lambda_{m+1}} + \frac{C|g(x)|^2}{\lambda_{m+1}}.
\]

Then for every bounded set \( B \subset C_V \) and any \( \eta > 0 \), there exist a \( \tau_3 > 0 \) and a finite-dimensional subspace \( V_m \) of \( V \) such that for all \( t \geq \tau_3 \), \( \psi \in B \), \( \theta \in [-\varepsilon_0, 0] \), and all \( v(t + \theta) \in F(t)\psi(\theta) \),

\[
\sup_{\theta \in [-\varepsilon_0, 0]} \|(I - P)v(t + \theta)\| < \frac{\eta}{4}, \tag{4.12}
\]

where \( P : V \to V_m \) is the canonical projector. On the other hand, analogous to the arguments in Theorem 4.5, noticing that all norms on finite-dimensional spaces are equivalent, we can conclude that there exists a \( \delta_4 > 0 \) such that for all \( t \geq \tau_3 \), \( \psi \in B \), \( v(t + \theta) \in F(t)\psi(\theta) \), and all \( \theta_1, \theta_2 \in [-\varepsilon, 0] \) with \( |\theta_2 - \theta_1| < \delta_4 \), we have

\[
\|P(v(t + \theta_1)) - v(t + \theta_2)\| < \frac{\eta}{4}. \tag{4.13}
\]

Thus we have completed the proof of Theorem 4.2.

**Remark 4.3.** Instead of (4.2), we can assume that there exist \( p \geq 2 \), \( C_i > 0 \), \( i = 1, \cdots, 4 \), such that

\[-C_1|v|^p - C_2 \leq F(v)v \leq -C_3|v|^p + C_4,
\]
for all \( v \in \mathbb{R} \). Then by slightly modifying the arguments in [31], we can see that there is a unique global attractor \( A \) in \( V \) for the multi-valued semidynamical system \( \{F(t)\} \) corresponding to problem (4.5). Moreover, analogous to the proof in Theorem 4.1, we can show that the autonomous multi-valued semidynamical system \( \{F(t)\} \) given in Theorem 4.2 has a global attractor in \( C_V \).

The existence of attractors for a delayed integro-differential equation without uniqueness of solutions has been studied; see, e.g., [10]. For the parabolic equations with small delays, by the arguments in [40], we have the following result:

**Theorem 4.4** ([40]). Assume that

\[
2\lambda_1 + b - \frac{k^2}{b(1 - \rho_*)} - \frac{m_1}{2} > 0, \quad 2m_1\varepsilon_0 < 1,
\]

and that \( g \in H, (H1) - (H2) \) hold. Then

1. For each \( 0 < \varepsilon \leq \varepsilon_0 \), there exists a pullback attractor \( \{A^\varepsilon(t)\}_{t \in \mathbb{R}} \) for the multi-valued process \( \{U^\varepsilon(t, \tau)\} \) on \( C_V \) generated by problem (4.1).
2. The family of multi-valued processes \( \{U^\varepsilon(t, \tau)\}, 0 < \varepsilon \leq \varepsilon_0 \), corresponding to (4.1) is uniformly equi-dissipative in \( C_V \).
3. For any \( 0 < \eta < 1 \), each \( t \in \mathbb{R} \), and every bounded subset \( B \) of \( C_V \), there exist a sufficiently large \( s > \varepsilon_0 \), a \( \delta > 0 \) and a finite-dimensional subspace \( V_m \) of \( V \) such that for all \( 0 < \varepsilon < \delta \), \( \psi \in B \) and all \( u^\varepsilon_t \in U^\varepsilon(t, t - s)\psi \),

\[
\sup_{\theta \in [-\varepsilon_0, 0]} \| (I - P)u^\varepsilon(t + \theta) \| < \frac{\eta}{3},
\]

where \( P : V \to V_m \) is the canonical projector.

Now we state and prove the main result in this section.

**Theorem 4.5.** Assume that the hypotheses in Theorem 4.4 hold. Let \( \{A^\varepsilon(t)\}_{t \in \mathbb{R}} \) be the pullback attractor for the multi-valued process \( \{U^\varepsilon(t, \tau)\} \) on \( C_V \) generated by (4.1), and let \( A_{C_V} \) given in Theorem 4.2 be the attractor for the multi-valued semidynamical system \( F(t) \) on \( C_V \). Then for each fixed \( t \in \mathbb{R} \),

\[
\lim_{\varepsilon \to 0} H^*_{C_V}(A^\varepsilon_{C_V}(t), A_{C_V}) = 0.
\]

**Proof.** Thanks to Theorems 3.1-3.2, 4.2 and 4.4, now we only need to prove that condition (3) in Theorem 3.2 holds true.

Assume on the contrary that this is not the case. Then there exist \( 0 < \eta_0 < 1, t \in \mathbb{R} \), and a bounded subset \( B \) of \( C_V \), such that for any fixed \( s' > 0 \) and finite-dimensional subspace \( V_m \) of \( V \), we can find \( \varepsilon_n \to 0 \), \( x_n \in B \), and \( \psi^n \in U^\varepsilon_n(t, t - s')x_n \), so that for all \( P\psi \in P\mathcal{F}(s')\mathcal{N}(Px_n, \frac{\eta_0}{3}) \), we have

\[
\sup_{\theta \in [-\varepsilon_0, 0]} \| P(\psi^n(\theta) - \psi(\theta)) \| \geq \frac{\eta_0}{3}, \quad \forall n \in \mathbb{N},
\]

where \( P : V \to V_m \) is the canonical projector.
Let $u = u_1 + u_2$, where $u_1 = Pu$. We consider the ordinary functional differential system

\[
\begin{aligned}
& \frac{du_1^n(T)}{dT} + Au_1^n(T) + bu_1^n(T) = PF(u_1^n(T - \rho_{\varepsilon_n}(T))) + \int_{-\varepsilon_n}^0 PG(s, u_1^n(T + s))ds \\
& + Pg(x), \ T \in [t - s', t], \\
& u_1^n(T) = Px_n, \ T \in [t - s' - \varepsilon_0, t - s'].
\end{aligned}
\]  

(4.15)

By the standard a priori estimates in [6, 7, 40], we can show that for all $n \in \mathbb{N}$ and $T \geq t - s'$,

\[
\sup_{\theta \in [-\varepsilon_0, 0]} \|u_1^n(T + \theta)\| < C.
\]  

(4.16)

On the other hand, note that all norms on finite-dimensional spaces are equivalent, so for all $n \in \mathbb{N}$ and $T \geq t - s'$,

\[
\begin{aligned}
& \|u_1^n(T + \theta_1) - u_1^n(T + \theta_2)\| \\
& \leq C|u_1^n(T + \theta_1) - u_1^n(T + \theta_2)| \\
& \leq C \int_{T + \theta_2}^{T + \theta_1} \left| \frac{du_1^n(r)}{dr} \right| dr \\
& \leq C \int_{T + \theta_2}^{T + \theta_1} (|Au_1^n(r)| + b|u_1^n(r)| + |F(u_1^n(r - \rho_{\varepsilon_n}(r)))|) \\
& + \int_{-\varepsilon_n}^0 PG(s, u_1^n(r + s))ds + |Pg(x)|dr.
\end{aligned}
\]

Then by (4.16) and similar arguments in [6, 40], we can deduce that

\[
\|u_1^n(T + \theta_1) - u_1^n(T + \theta_2)\| \to 0 \text{ as } \theta_2 \to \theta_1
\]

uniformly with respect to $n \in \mathbb{N}$ and $T \geq t - s'$. Hence, by the Arzelà-Ascoli theorem, there is a subsequence of $u_1^n$ (which we still denote by $u_1^n$) where we can assume that for any $T \geq t - s'$, $u_1^n(T + \cdot)$ converges to $v_1(T + \cdot)$ in $C_V$. We also assume that $Px_n \to Px$ in $C_V$ as $n \to \infty$. Notice that for any $T \in [t - s', t]$, 

\[
\|u_1^n(T - \rho_{\varepsilon_n}(T)) - v_1(T)\| \leq \sup_{\theta \in [-\varepsilon_0, 0]} \|u_1^n(T + \theta) - v_1(T + \theta)\| + \|v_1(T - \rho_{\varepsilon_n}(T)) - v_1(T)\|.
\]

Clearly for any $T \in [t - s', t]$,

\[
\lim_{n \to \infty} \|u_1^n(T - \rho_{\varepsilon_n}(T)) - v_1(T)\| = 0.
\]  

(4.17)

Note that the functions $u_1^n$ satisfy equality (4.15). Since the function $F$ is continuous with respect to $u$ and the function $G$ is continuous with respect to the variables $s$ and $u$, passing to the limit as $n \to \infty$ we obtain the following equality:

\[
\begin{aligned}
& \frac{dv_1(T)}{dT} + Av_1(T) + bv_1(T) = PF(v_1(T)) + Pg(x), \ T \in [t - s', t], \\
v_1(T) = Px, \ T \in [t - s' - \varepsilon_0, t - s'],
\end{aligned}
\]
5.1. Priori estimates. For the classical reaction-diffusion equations without uniqueness of solutions, we refer the reader to \[1,17,18,28\] and the references cited therein. In this section, we are mainly concerned with the dynamical behavior of the following nonautonomous nonclassical diffusion equation:

\[
\begin{cases}
\frac{\partial u}{\partial t} - \varepsilon \Delta u_t - \Delta u = f(u) + g(x) + \varepsilon h(t), \quad \text{in } (\tau, +\infty) \times \Omega, \\
u(t, x) = 0, \quad \text{on } (\tau, +\infty) \times \partial \Omega, \\
u(\tau, x) = u_0(x), \quad x \in \Omega, \quad \tau \in \mathbb{R},
\end{cases}
\]

(5.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^n$ with sufficiently regular boundary $\partial \Omega$, $\varepsilon \in [0, 1]$, and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the following condition:

\[-C_1|s|^p - C_0 \leq f(s)s \leq -C_2|s|^p + C_0, \quad p \geq 2,\]

(5.2)

for all $s \in \mathbb{R}$, $g \in L^2(\Omega)$ and $h \in L^p_{\text{loc}}(\mathbb{R}; L^2(\Omega))$ is such that

\[
\lim_{t \to -\infty} \int_{-\infty}^t e^{-\gamma(t-r)} |h(r)|_2^2 dr < \infty \quad \text{for all } \gamma > 0.
\]

(5.3)

For convenience, let $V = H^1_0(\Omega)$ and $H = L^2(\Omega)$, and the corresponding norms in $V$ and $L^p(\Omega) \ (1 \leq p < \infty)$ are denoted by $\|u\|^2 = \int_{\Omega} |\nabla u|^2$ and $|u|^p_p = \int_{\Omega} |u|^p$, respectively. Denote $\Omega(u \geq M) = \{x \in \Omega : u(x) \geq M\}$ and $\Omega(u \leq -M) = \{x \in \Omega : u(x) \leq -M\}$, and let $C$ the arbitrary positive constants, which may be different from line to line and even on the same line.

First of all, we need the following result about the existence of solutions which can be obtained by the standard Faedo-Galerkin methods; here we only give the result:

THEOREM 5.1. Let $f$ satisfy condition (5.2), $g \in H$ and $h \in L^p_{\text{loc}}(\mathbb{R}; H)$. Then for any fixed $\varepsilon \in (0, 1]$, each $\tau \in \mathbb{R}$, any $u_0 \in V$ and every $T > \tau$, there exists a solution $u$ of (5.1) such that

\[
u \in C([\tau, T]; V) \cap L^\infty(\tau, T; V), \quad u_t \in L^2(\tau, T; V), \quad \forall T > \tau.
\]

By Theorem 5.1, for any fixed $\varepsilon \in (0, 1]$, we can define a family of multi-valued mappings $U^\varepsilon(t, \tau) : V \to 2^V$, $t \geq \tau$, $\tau \in \mathbb{R}$, by setting

\[
U^\varepsilon(t, \tau)u_0 = \{u(t) \mid u(\cdot) \text{ is a solution of system (5.1) with } \varepsilon \in (0, 1]\},
\]
and we can verify that properties (1) and (2) in Definition 2.5 hold true. Hence, we can see that the family of multi-valued mappings \( \{ U^\varepsilon(t, \tau) \} \) forms a multi-valued process on \( V \). Moreover, for any fixed \( \varepsilon \in (0, 1] \), we can show that \( U^\varepsilon(t, \tau)u_0 \) is norm-to-weak upper-semicontinuous in \( u_0 \) for any fixed \( t \geq \tau, \tau \in \mathbb{R} \). Indeed, let \( u_{n_0} \to u_0 \) in \( V \), similar to the proof of the existence of solutions (see [11] for details). In view of Lemma 5.3 and Theorem 2.7 in [38], we can prove that for any fixed \( t \geq \tau, \tau \in \mathbb{R} \), and any \( u_n(t) \in U^\varepsilon(t, \tau)u_{n_0} \), there exist a \( u(t) \in U^\varepsilon(t, \tau)u_0 \) and a subsequence \( u_{n_k}(t) \) such that \( u_{n_k}(t) \to u(t) \) in \( V \cap L^p(\Omega) \), and thus the detailed arguments are omitted here.

In a similar way, we can define a norm-to-weak upper-semicontinuous multi-valued semidynamical system \( \{ F(t) \} \) on \( V \) generated by problem (5.1) with \( \varepsilon = 0 \).

**Lemma 5.2** (The Uniform Gronwall Lemma [33]). Let \( g, h, y, \) be three positive locally integrable functions on \([t_0, +\infty[\) such that \( y' \) is locally integrable on \([t_0, +\infty[\), and which satisfy

\[
\frac{dy}{dt} \leq gy + h, \quad \text{for } t \geq t_0,
\]

\[
\int_t^{t+r} g(s)ds \leq a_1, \quad \int_t^{t+r} h(s)ds \leq a_2, \quad \int_t^{t+r} y(s)ds \leq a_3, \quad \text{for } t \geq t_0,
\]

where \( r, a_1, a_2, a_3 \) are positive constants. Then

\[
y(t + r) \leq \left( \frac{a_3}{r} + a_2 \right) e^{a_1}, \quad \forall t \geq t_0.
\]

**Lemma 5.3.** The family of multi-valued processes \( \{ U^\varepsilon(t, \tau) \}, \varepsilon \in (0, 1] \), is uniformly (w.r.t. \( \varepsilon \in (0, 1] \)) pullback strongly bounded dissipative in \( V \cap L^p(\Omega) \).

**Proof.** Taking the scalar product in \( H \) of (5.1) and \( u \) we obtain

\[
\frac{d}{dt}(|u|^2 + \varepsilon \| u \|^2) + 2\| u \|^2 = 2(f(u), u) + 2(g, u) + 2\varepsilon(h(t), u).
\]

By (5.2) and Young’s inequality, we have

\[
\frac{d}{dt}(|u|^2 + \varepsilon \| u \|^2) + 2\| u \|^2 + C|u|^p \leq C + \lambda_1|u|^2 + \frac{2g^2}{\lambda_1} + \frac{2(h(t))^2}{\lambda_1},
\]

where \( \lambda_1 \) is such that \( \| u \|^2 \geq \lambda_1|u|^2 \). Using this inequality, we can choose \( \alpha > 0 \) and \( \alpha_1 > 0 \) sufficiently small such that

\[
2\| u \|^2 \geq \alpha(|u|^2 + \varepsilon \| u \|^2) + \lambda_1|u|^2 + \alpha_1\| u \|^2.
\]

Hence,

\[
\frac{d}{dt}(|u|^2 + \varepsilon \| u \|^2) + \alpha(|u|^2 + \varepsilon \| u \|^2) + C(||u||^2 + |u|^p) \leq C(1 + |g|^2 + |h(t)|^2)
\]

(5.4)

and

\[
\frac{d}{dt}(|u|^2 + \varepsilon \| u \|^2) + \alpha(|u|^2 + \varepsilon \| u \|^2) \leq C(1 + |g|^2 + |h(t)|^2).
\]

By (5.3) and Gronwall’s Lemma, we can conclude that there exists a sufficiently large \( T_0' > 0 \) such that

\[
\int_{-\infty}^{-T_0'} e^{-\alpha T_0'} e^{\alpha s}|h(s)|^2ds < C,
\]
and thus
\[
|u(t)|_2^2 + \varepsilon \|u(t)\|^2 \leq (|u_0|_2^2 + \varepsilon \|u_0\|^2) e^{-\alpha(t-s)} - C \int_t^{t_1} e^{-\alpha(t-s)}(1 + |g|^2_2 + |h(s)|_2^2) ds
\]
(5.5)
\[
\leq (|u_0|_2^2 + \varepsilon \|u_0\|^2) e^{-\alpha(t-s)} + C \int_{t_1}^{t} |h(s)|_2^2 ds + C + C|g|^2_2.
\]

Multiplying (5.1) by \( u \) and integrating over \( \Omega \), we have
\[
|u_t|_2^2 + \varepsilon \|u_t\|^2 + \frac{1}{2} \frac{d}{dt} \|u\|^2 - \frac{d}{dt} \int_{\Omega} F(u) dx = (g, u_t) + \varepsilon(h(t), u_t)
\]
(5.6)
\[
\leq \frac{1}{2} |u_t|_2^2 + |g|^2_2 + |h(t)|_2^2.
\]
(5.2) implies that
\[
-C - \tilde{C}_1 |s|^p \leq F(s) \leq -\tilde{C}_2 |s|^p + C, \quad \forall s \in \mathbb{R}.
\]
(5.7)
Combining (5.6)-(5.7) together, we get
\[
\frac{d}{dt} \left( \|u\|^2 - 2 \int_{\Omega} F(u) dx \right) \leq \|u\|^2 + 2\tilde{C}_2 |u_p|^p + 2|g|^2_2 + 2|h(t)|_2^2
\]
(5.8)
\[
\leq \|u\|^2 - 2 \int_{\Omega} F(u) dx + C + 2|g|^2_2 + 2|h(t)|_2^2.
\]

On the other hand, integrating \( t \) in (5.4) from \( t \) to \( t + 1 \), in view of (5.5), we can deduce that
\[
C \left( \int_{t}^{t+1} \|u\|^2 dt + \int_{t}^{t+1} |u|^p dt \right)
\]
\[
\leq C \left( 1 + |g|^2_2 + \int_{t}^{t+1} |h(t)|_2^2 dt \right) + |u(t)|_2^2 + \varepsilon \|u(t)\|^2
\]
(5.9)
\[
\leq (|u_0|_2^2 + \varepsilon \|u_0\|^2) e^{-\alpha(t-t')} + C \int_{-T_0}^{t+1} |h(s)|_2^2 ds + C + C|g|^2_2.
\]

It follows from (5.7) and (5.9) that
\[
\int_{t}^{t+1} \left( \|u\|^2 - 2 \int_{\Omega} F(u) dx \right) dt
\]
(5.10)
\[
\leq \int_{t}^{t+1} \|u\|^2 dt + 2\tilde{C}_1 \int_{t}^{t+1} |u|^p dt + C
\]
\[
\leq C(|u_0|_2^2 + \varepsilon \|u_0\|^2) e^{-\alpha(t-t')} + C \int_{-T_0}^{t+1} |h(s)|_2^2 ds + C + C|g|^2_2.
\]

Then by Lemma 5.2, we obtain
\[
\|u(t + 1)\|^2 - 2 \int_{\Omega} F(u(t + 1)) dx
\]
\[
\leq C(|u_0|_2^2 + \varepsilon \|u_0\|^2) e^{-\alpha(t-t')} + C \int_{-T_0}^{t+1} |h(s)|_2^2 ds + C + C|g|^2_2, \quad \forall t \geq \tau,
\]
and thus by (5.7), we have
\[\|u(t+1)\|^2 + |u(t+1)|^p_p \leq C(|u_0|^2 + |u_0|^2)e^{-\alpha(t-\tau)} + C \int_{-T_0}^{t+1} |h(s)|^2 ds + C + C|g|^2_{2,\Omega}, \forall t \geq \tau.\]  
\hspace{1cm} (5.11)

5.2. Pullback attractors in \( V \) and \( L^p(\Omega) \).

Lemma 5.4 ([11]). For any \( \varepsilon > 0 \), the bounded subset \( B \) of \( L^p(\Omega) \) (\( p > 0 \)) has a finite \( \varepsilon \)-net in \( L^p(\Omega) \) if there exists a positive constant \( M = M(\varepsilon) \) which depends on \( \varepsilon \) such that
\[ (1) \ B \ has \ a \ finite \ (3M)^{q-p}/q(\frac{q}{2})^{p/q}\-net \ in \ L^q(\Omega) \ for \ some \ q, \ 0 < q \leq p; \]
\[ (2) \ \left( \int_{\Omega(\|u\| \geq M)} |u|^p dx \right)^{1/p} < 2^{-\frac{2p+2}{p}} \varepsilon \ for \ any \ u \in B, \ where \ \Omega(\|u\| \geq M) = \{ x \in \Omega \mid |u(x)| \geq M \}. \]

Lemma 5.5 ([11]). Let \( B \) be a bounded subset of \( L^p(\Omega) \) (\( p \geq 1 \)). If \( B \) has a finite \( \varepsilon \)-net in \( L^p(\Omega) \), then there exists \( M = M(B, \varepsilon) \) such that for any \( u \in B \), the following estimate is valid:
\[ \int_{\Omega(\|u\| \geq M)} |u|^p dx \leq 2^{p+1} \varepsilon. \]

Theorem 5.6. Assume that \( f \) satisfies (5.2), \( h \in L^2_{loc}(\mathbb{R}; H) \) satisfies (5.3) and \( g \in H \). Then for any fixed \( \varepsilon \in (0,1] \), the multi-valued process \( \{U^\varepsilon(t, \tau)\} \) generated by problem (5.1) has a minimal pullback attractor \( A^\varepsilon_p = \{A^\varepsilon_p(t)\}_{t \in \mathbb{R}} \) in \( L^p(\Omega) \), i.e., \( A^\varepsilon_p \) is compact, invariant in \( L^p(\Omega) \), and pullback attracts every bounded subset of \( V \) with respect to the \( L^p \)-norm, and \( \bigcup_{\varepsilon \in (0,1]} \bigcup_{s \leq t} A^\varepsilon(s) \) is bounded in \( L^p(\Omega) \) for each \( t \in \mathbb{R} \).

Proof. In the statement of Theorem 2.8, the space of the initial data coincides with the phase space in which we study the convergence to the attractor, which now is not the case. However, from the proof of Theorem 2.8, we can see that in this case now, we still can get a minimal pullback attractor in the phase space, i.e., the attractor is compact, invariant in the topology of the phase space, and pullback attracts every bounded subset of the space of the initial data in the topology of the phase space.

By Lemma 5.3, denote by \( Q = \{Q(t)\}_{t \in \mathbb{R}} \) a uniformly (w.r.t. \( \varepsilon \in (0,1] \)) pullback strongly bounded absorbing set in \( V \cap L^p(\Omega) \). Thanks to Lemmas 5.3-5.4, Theorem 2.8 and the compact embedding \( H^1_0(\Omega) \hookrightarrow L^2(\Omega) \), now it only remains to verify that for each \( t \in \mathbb{R} \) and any \( \eta > 0 \), there are positive constants \( M = M(\eta, Q(t), t) \) and \( T_6 = T_6(\eta, Q(t), t) \) such that
\[ \int_{\Omega(\|u(t)\| \geq 2M)} |u(t)|^p dx < C\eta, \]  
\hspace{1cm} (5.12)

for all \( s \geq T_6, u_0 \in Q(t), \varepsilon \in (0,1] \) and all \( u(t) \in U^\varepsilon(t, t-s)u_0 \), where the constant \( C \) is independent of \( M, T_6, \eta, \) and \( \varepsilon \in (0,1] \).
We observe that for any $\varepsilon \in (0, 1]$ and $T \geq t - s$ with $s \geq 0$,
\[
U_\varepsilon(T, t - s)u_0 = \{u(T)|u(\cdot)| is a solution of system (5.1) with $\varepsilon \in (0, 1]$
and $u(t - s) = u_0 \in Q(t)$ for all $s \geq 0$.
\[
By (5.5), we can deduce that for all $s \geq 0$, $t - s \leq T \leq t$, $\varepsilon \in (0, 1]$, and all $u(T) \in U_\varepsilon(T, t - s)Q(t)$, we have
\[
|u(T)|^2 \leq (|u_0|^2 + \|u_0\|^2) + C \int_{-T_\varepsilon^t}^t |h(s)|^2 ds + C + C|g|^2.
\] (5.13)
Hence, by the similar arguments of Lemma 5.2 in [32], we can deduce that there exists $M > 0$ such that
\[
m(\Omega(|u(T)| \geq M)) \leq \eta \text{ and } \int_{\Omega(|u(T)| \geq M)} |g|^2 dx < \eta,
\] (5.14)
for all $s \geq 0$, $t - s \leq T \leq t$, $\varepsilon \in (0, 1]$ and all $u(T) \in U_\varepsilon(T, t - s)Q(t)$, where $m(\varepsilon)$ denotes the Lebesgue measure of $\varepsilon \subset \Omega$. Let
\[
(s - M)_+ = \begin{cases} s - M, & s \geq M, \\ 0, & s \leq M \end{cases}
\]
and
\[
(s + M)_- = \begin{cases} s + M, & s \leq -M, \\ 0, & s \geq -M. \end{cases}
\]
Thanks to the proof in [32], we have the following estimates:
\[
\int_\Omega f(u)(u - M)_+ dx \leq -C_3 \int_{\Omega(u \geq 2M)} |u|^p dx + C_4 m(\Omega(u \geq 2M))
\] (5.15)
and
\[
\int_\Omega f(u)(u + M)_- dx \leq -C_3 \int_{\Omega(u \leq -2M)} |u|^p dx + C_4 m(\Omega(u \leq -2M),
\] (5.16)
provided that $M$ is large enough, where the positive constants $C_3, C_4$ are independent of $M$.

Taking $(u - M)_+$ as a test function, in view of (5.15), we get
\[
\frac{1}{2} \frac{d}{dT} \left( \int_{\Omega(u \geq M)} (u - M)^2 dx + \varepsilon \int_{\Omega(u \geq M)} |\nabla u|^2 dx \right) + \int_{\Omega(u \geq M)} |\nabla u|^2 dx
\leq \int_{\Omega(u \geq M)} f(u)(u - M) dx + \int_{\Omega(u \geq M)} |g||u - M| dx + \int_{\Omega(u \geq M)} |h(T)||u - M| dx
\leq -C_3 \int_{\Omega(u \geq 2M)} |u|^p dx + C_4 m(\Omega(u \geq 2M))
\]
\[
+ \left( \int_{\Omega(u \geq M)} |g|^2 dx \right)^{1/2} \left( \int_{\Omega(u \geq M)} |u - M|^2 dx \right)^{1/2}
\]
\[
+ \left( \int_{\Omega(u \geq M)} |h(T)|^2 dx \right)^{1/2} \left( \int_{\Omega(u \geq M)} |u - M|^2 dx \right)^{1/2}.
\]
Similar to the above, we can choose positive constants $\alpha_2$ and $\alpha_3 < C_3$ sufficiently small such that

$$2 \int_{\Omega(u \geq M)} |\nabla u|^2 \, dx \geq \alpha_2 \left( \int_{\Omega(u \geq M)} (u - M)^2 \, dx + \varepsilon \int_{\Omega(u \geq M)} |\nabla u|^2 \, dx \right)$$

$$+ \alpha_3 \left( \int_{\Omega(u \geq M)} (u - M)^2 \, dx + \int_{\Omega(u \geq M)} |\nabla u|^2 \, dx \right).$$

By (5.14) and Young’s inequality,

$$\frac{d}{dT} \left( \int_{\Omega(u \geq M)} (u(T) - M)^2 \, dx + \varepsilon \int_{\Omega(u \geq M)} |\nabla u(T)|^2 \, dx \right)$$

$$\leq (\|u_0\|^2 + \|u_0\|^2) e^{-\alpha_2 (T-t+s)} + C\eta + C \int_{T-t}^{T} e^{-\alpha_2 (T-r)} \int_{\Omega(u(r) \geq M)} |h(r)|^2 \, dx \, dr. \quad (5.18)$$

Integrating (5.17) in $T$ from $T$ to $T+1$, in view of (5.18), we obtain

$$\int_{T}^{T+1} \int_{\Omega(u(T) \geq M)} |\nabla u(T)|^2 \, dx \, dT + \int_{T}^{T+1} \int_{\Omega(u(T) \geq 2M)} |u(T)|^p \, dx \, dT$$

$$\leq C\eta + C \int_{T}^{T+1} \int_{\Omega(u(T) \geq M)} |h(T)|^2 \, dx \, dT$$

$$+ C \int_{\Omega(u(T) \geq M)} (u(T) - M)^2 \, dx + C\varepsilon \int_{\Omega(u(T) \geq M)} |\nabla u(T)|^2 \, dx$$

$$\leq C\eta + C (\|u_0\|^2 + \|u_0\|^2) e^{-\alpha_2 (T-t+s)} + C e^{-\alpha_2 T} \int_{-\infty}^{T+1} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} |h(r)|^2 \, dx \, dr,$$

provided that $M$ is large enough, where the constant $C$ is independent of $M$, $T$, $\eta$, and $\varepsilon \in (0, 1]$. From this, (5.7) and (5.14), we can deduce that

$$\int_{T}^{T+1} \int_{\Omega(u(T) \geq 2M)} (|\nabla u(T)|^2 - 2F(u(T))) \, dx \, dT$$

$$\leq \int_{T}^{T+1} \int_{\Omega(u(T) \geq 2M)} |\nabla u(T)|^2 \, dx \, dT + 2\tilde{C}_1 \int_{T}^{T+1} \int_{\Omega(u(T) \geq 2M)} |u(T)|^p \, dx \, dT + C\eta$$

$$\leq C\eta + C (\|u_0\|^2 + \|u_0\|^2) e^{-\alpha_2 (T-t+s)} + C e^{-\alpha_2 T} \int_{-\infty}^{T+1} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} |h(r)|^2 \, dx \, dr,$$

provided that $M$ is large enough.
Taking \( \frac{\partial(u-2M)}{dT} \) as the test function, by Young’s inequality, we have
\[
\int_{\Omega(u \geq 2M)} |u_T|^2 dx + \varepsilon \int_{\Omega(u \geq 2M)} |\nabla u_T|^2 dx + \frac{1}{2} \frac{d}{dT} \int_{\Omega(u \geq 2M)} |\nabla u|^2 dx
\]
\[
= \int_{\Omega(u \geq 2M)} f(u)u_T dx + \int_{\Omega(u \geq 2M)} gu_T dx + \varepsilon \int_{\Omega(u \geq 2M)} h(T)u_T dx
\]
\[
\leq \frac{d}{dT} \int_{\Omega(u \geq 2M)} F(u) dx + \frac{1}{2} \int_{\Omega(u \geq 2M)} |u_T|^2 dx + \int_{\Omega(u \geq 2M)} |g|^2 dx
\]
\[
+ \int_{\Omega(u \geq 2M)} |h(T)|^2 dx.
\] (5.20)

By (5.7), (5.14) and (5.20), we get
\[
\frac{d}{dT} \int_{\Omega(u \geq 2M)} (|\nabla u|^2 - 2F(u)) dx
\]
\[
\leq \int_{\Omega(u \geq 2M)} |\nabla u|^2 dx + 2\bar{C}_2 \int_{\Omega(u \geq 2M)} |u|^p dx + \frac{1}{2} \int_{\Omega(u \geq 2M)} |u_T|^2 dx + \int_{\Omega(u \geq 2M)} |g|^2 dx
\]
\[
+ 2 \int_{\Omega(u \geq 2M)} |h(T)|^2 dx
\] (5.21)

Noticing that
\[
2 \int_{T}^{T+1} \int_{\Omega(u(r) \geq 2M)} |h(r)|^2 dx dr
\]
\[
\leq 2e^{-\alpha_2 T} \int_{-\infty}^{T+1} e^{\alpha_2 r} \int_{\Omega(u(r) \geq 2M)} |h(r)|^2 dx dr.
\] (5.22)

Thanks to Lemma 5.2, (5.19) and (5.21)-(5.22), we can conclude that for all \( T \geq t-s \) and all \( s \geq 0 \),
\[
\int_{\Omega(u(T+1) \geq 2M)} (|\nabla u(T+1)|^2 - 2F(u(T+1))) dx \leq C\eta
\]
\[
+ C(|u_0|^2 + \|u_0\|^2)e^{-\alpha_2 (T-t+s)} + Ce^{-\alpha_2 T} \int_{-\infty}^{T+1} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} |h(r)|^2 dx dr,
\] (5.23)

provided that \( M \) is large enough, where the constant \( C \) is independent of \( M, T, \eta, \) and \( \varepsilon \in (0, 1] \).

Let \( T+1 = t \). Then
\[
\int_{\Omega(u(t) \geq 2M)} (|\nabla u(t)|^2 - 2F(u(t))) dx
\]
\[
\leq C\eta + C(|u_0|^2 + \|u_0\|^2)e^{-\alpha_2 s} + Ce^{-\alpha_2 t} \int_{-\infty}^{t} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} |h(r)|^2 dx dr.
\] (5.24)
Now we deal with the last term of (5.24). By (5.3), we can choose a $t_7 > 0$ sufficiently large such that
\[
e^{-\alpha_2 t} \int_{-\infty}^{-t_7} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} |h(r)|^2 dx dr \leq e^{-\alpha_2 t} e^{-\alpha_2 t_7} \int_{-\infty}^{-t_7} e^{\alpha_2 t_7} e^{\alpha_2 r} |h(r)|^2 dr < \frac{\eta}{2}.
\] (5.25)

On the other hand, we can find a nonnegative simple functions sequence $\{\phi_k(r, x)\}$ such that
\[
\phi_k(r, x) \leq \phi_{k+1}(r, x) \leq |h(r, x)|^2, \phi_k(r, x) \leq k,
\]
and
\[
\lim_{k \to \infty} \phi_k(r, x) = |h(r, x)|^2 \text{ a.e. } (r, x) \in [-t_7, t] \times \Omega.
\]
By monotone convergence theorem, when $k$ is sufficiently large, we have
\[
e^{-\alpha_2 t} \int_{-t_7}^{t} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} (|h(r)|^2 - \phi_k(r)) dx dr \leq \int_{-t_7}^{t} \int_{\Omega} (|h(r)|^2 - \phi_k(r)) dx dr < \frac{\eta}{4}.
\] (5.26)

Note that when $M$ is sufficiently large,
\[
e^{-\alpha_2 t} \int_{-t_7}^{t} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} \phi_k(r) dx dr < \frac{\eta}{4}.
\] (5.27)

Combining (5.25)-(5.27) together, we can conclude that
\[
e^{-\alpha_2 t} \int_{-\infty}^{t} e^{\alpha_2 r} \int_{\Omega(u(r) \geq M)} |h(r)|^2 dx dr < \eta,
\] (5.28)

provided that $M$ is sufficiently large. Using (5.24) and (5.28), we have
\[
\int_{\Omega(u(t) \geq 2M)} (|\nabla u(t)|^2 - 2F(u(t))) dx \leq C \eta,
\] (5.29)

provided that $M$ and $s$ are large enough. By (5.7) and (5.14),
\[
\int_{\Omega(u(t) \geq 2M)} |\nabla u(t)|^2 dx + \int_{\Omega(u(t) \geq 2M)} |u(t)|^p dx \leq C \eta,
\] (5.30)

provided that $M$ and $s$ are large enough.

Similarly, taking $(u + M)_-$ and $\frac{\partial (u + 2M)_-}{\partial t}$ as the test functions, we can deduce that
\[
\int_{\Omega(u(t) \leq -2M)} |\nabla u(t)|^2 dx + \int_{\Omega(u(t) \leq -2M)} |u(t)|^p dx \leq C \eta,
\] (5.31)

provided that $M$ and $s$ are large enough. (5.12) is proved, and thus the proof of this theorem is finished.

For the multi-valued semidynamical system $\{F(t)\}$ on $V$ generated by problem (5.1) with $\varepsilon = 0$, since the proofs of Lemma 5.3 and Theorem 5.6 still work for $\varepsilon = 0$, we have the following result:

**Theorem 5.7.** Let $f$ satisfy (5.2) and $g \in H$. Then the multi-valued semidynamical system $\{F(t)\}$ on $V$ generated by problem (5.1) with $\varepsilon = 0$ has a unique global attractor $\mathcal{A}_p$ in $L^p(\Omega)$, i.e., $\mathcal{A}_p$ is compact, invariant in $L^p(\Omega)$, and attracts every bounded subset of $V$ with respect to the $L^p$-norm.
Theorem 5.8. Under the hypotheses of Theorem 5.6, for any fixed $\varepsilon \in (0, 1]$, the multi-valued process $\{U^\varepsilon(t, \tau)\}$ generated by problem (5.1) has a minimal pullback attractor $A^\varepsilon = \{A^\varepsilon(t)\}_{t \in \mathbb{R}}$ in $V$, and $\bigcup_{\varepsilon \in (0,1]} \bigcup_{s \leq t} A^\varepsilon(s)$ is bounded in $V$ for each $t \in \mathbb{R}$.

Proof. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ and $\omega_1, \omega_2, \ldots$ be the corresponding eigenvectors such that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to \infty \quad \text{as} \quad j \to \infty.$$ 

Then $\{\omega_1, \omega_2, \ldots\}$ form an orthogonal basis in $L^2(\Omega)$ and $H_0^1(\Omega)$. Let

$$H_m = \text{span}\{\omega_1, \omega_2, \ldots, \omega_m\},$$

$P_m$ be the canonical projector on $H_m$ and $I$ be the identity. Then for any $u \in L^2(\Omega)$ or $u \in H_0^1(\Omega)$, $u$ has a unique decomposition: $u = u_1 + u_2$, where $u_1 = P_m u \in H_m$ and $u_2 = (I - P_m) u \in H_m^\perp$.

Invoking Theorem 2.11 and Lemma 5.3, now it suffices to show that for any fixed $\varepsilon \in (0, 1]$, the MVP $\{U^\varepsilon(t, \tau)\}$ is pullback flattening in $V$.

Multiplying (5.1) by $u_2$ and integrating over $\Omega$, we have

$$\frac{d}{dt} \left( |u_2|^2 + \varepsilon |\nabla u_2|^2 \right) + 2|\nabla u_2|^2 \leq 2 \left( \int |f(u)| \| \mathbb{T} \| dx \right) \frac{p-1}{p} \left( \int \| u_2 \|^p dx \right)^{\frac{1}{p}} + 2|g_2|^2 |u_2|^2 + 2|h(T)| |u_2|^2.$$ 

By $|\nabla u_2|^2 \geq \lambda_{m+1} |u_2|^2$, we can choose $\beta > \max\{\alpha, \alpha_2\}$ such that for $m$ sufficiently large,

$$2|\nabla u_2|^2 - 2|u_2|^2 \geq 2\beta(|u_2|^2 + \varepsilon |\nabla u_2|^2).$$

Then

$$\frac{d}{dt} \left( |u_2|^2 + \varepsilon |\nabla u_2|^2 \right) + 2\beta(|u_2|^2 + \varepsilon |\nabla u_2|^2) \leq 2 \left( \int |f(u)| \| \mathbb{T} \| dx \right) \frac{p-1}{p} \left( \int \| u_2 \|^p dx \right)^{\frac{1}{p}} + 2|g_2|^2 |u_2|^2 + 2|h(T)| |u_2|^2,$$

and by (5.2), (5.11) and Young’s inequality, we get

$$\frac{d}{dt} \left( |u_2|^2 + \varepsilon |\nabla u_2|^2 \right) + \beta(|u_2|^2 + \varepsilon |\nabla u_2|^2) \leq C|u(T)|^p + C + C\eta + C|h(T)|^2$$

$$\leq C(|u_0|^2 + |u_0|^2)e^{-\alpha(T-1-t+s)} + C \int_{-T_0}^{T} |h(s)|^2 ds + C + C|g|^2 + C\eta + C|h(T)|^2.$$ 

Using Gronwall’s Lemma, we can deduce that

$$|u_2(T)|^2 + \varepsilon |\nabla u_2(T)|^2 \leq (|u_0|^2 + |u_0|^2)e^{-\beta(T-t+s)} + C(|u_0|^2 + |u_0|^2)e^{-\alpha(T-t+s)} + C \int_{-T_0}^{T} |h(r)|^2 dr + C + C|g|^2 + C\eta,$$
where \( u_0 \in Q(t) \), given in Theorem 5.6. Hence,

\[
\varepsilon \lambda_{m+1} |u_2(T)|^2 \leq \varepsilon |\nabla u_2(T)|^2
\]

\[
\leq (|u_0|^2 + \|u_0\|^2)e^{-\beta(T-t+s)} + C(|u_0|^2 + \|u_0\|^2)e^{-\alpha(T-t+s)} + C \int_{-T_0}^T |h(r)|^2 dr
\]

\[+ C + C|g|^2 + C\eta. \quad (5.35)\]

By (5.7), (5.14) and (5.23),

\[
\int_{\Omega(|u(T)| \geq 2M)} |u(T)|^p dx
\]

\[\leq C\eta + C(|u_0|^2 + \|u_0\|^2)e^{-\alpha_2(T-t+s)} + Ce^{-\alpha_2T} \int_{-\infty}^T e^{\alpha_2r} \int_{\Omega(|u(r)| \geq M)} |h(r)|^2 dx dr. \]

On the other hand, (5.35) implies that

\[
\int_{\Omega(|u(T)| \leq 2M)} |u_2(T)|^p dx \leq (2M)^{p-2} \int_{\Omega(|u(T)| \leq 2M)} |u_2(T)|^2 dx \leq (2M)^{p-2} |u_2(T)|^2
\]

\[\leq \frac{(2M)^{p-2}}{\varepsilon \lambda_{m+1}} (|u_0|^2 + \|u_0\|^2)e^{-\beta(T-t+s)} + \frac{(2M)^{p-2}C}{\varepsilon \lambda_{m+1}} (|u_0|^2 + \|u_0\|^2)e^{-\alpha(T-t+s)}
\]

\[+ \frac{(2M)^{p-2}C}{\varepsilon \lambda_{m+1}} \int_{-T_0}^T |h(r)|^2 dr + \frac{(2M)^{p-2}C}{\varepsilon \lambda_{m+1}} |g|^2 + \frac{(2M)^{p-2}C\eta}{\varepsilon \lambda_{m+1}}.
\]

Then

\[
\int_{\Omega} |u_2(T)|^p dx \leq \int_{\Omega(|u(T)| \geq 2M)} |u_2(T)|^p dx + \int_{\Omega(|u(T)| \leq 2M)} |u_2(T)|^p dx
\]

\[\leq C\eta + C(|u_0|^2 + \|u_0\|^2)e^{-\alpha_2(T-t+s)} + Ce^{-\alpha_2T} \int_{-\infty}^T e^{\alpha_2r} \int_{\Omega(|u(r)| \geq M)} |h(r)|^2 dx dr
\]

\[+ \frac{(2M)^{p-2}}{\varepsilon \lambda_{m+1}} (|u_0|^2 + \|u_0\|^2)e^{-\beta(T-t+s)} + \frac{(2M)^{p-2}C}{\varepsilon \lambda_{m+1}} (|u_0|^2 + \|u_0\|^2)e^{-\alpha(T-t+s)}
\]

\[+ \frac{(2M)^{p-2}C}{\varepsilon \lambda_{m+1}} \int_{-T_0}^T |h(r)|^2 dr + \frac{(2M)^{p-2}C}{\varepsilon \lambda_{m+1}} |g|^2 + \frac{(2M)^{p-2}C\eta}{\varepsilon \lambda_{m+1}}.
\]

By (5.2), (5.3), (5.11), (5.28), (5.35) and (5.36), using Hölder’s inequality and Gronwall’s Lemma, we can conclude that we fixed a sufficiently large \( M \),

\[
|u_2(t)|^2 + \varepsilon |\nabla u_2(t)|^2
\]

\[\leq (|u_0|^2 + \|u_0\|^2)e^{-\beta s}
\]

\[+ 2 \left( \int_{t-s}^t e^{-\beta(t-r)} \int_{\Omega} |f(u)|^\frac{p}{p-1} dx dr \right) \left( \int_{t-s}^t e^{-\beta(t-r)} \int_{\Omega} |u_2|^p dx dr \right)^\frac{1}{p}
\]

\[+ 2\eta |u_2|^2 + 2 \left( \int_{-T_0}^T |h(r)|^2 dx dr + C \right) \left( e^{-\beta t} \int_{t-s}^t e^{\beta r} |u_2(r)|^2 dx dr \right)^\frac{1}{2}
\]

\[\leq C\eta,
\]
provided that $m$ and $s$ are large enough, where the constant $C$ is independent of $M$, $m$, $s$ and $\eta$. Thus for any fixed $\varepsilon \in (0, 1]$ and each $t \in \mathbb{R}$, when $M$, $m$ and $s$ are sufficiently large,

$$\varepsilon |\nabla u_2(t)|_2^2 \leq C \eta, \text{ for all } u_0 \in Q(t) \text{ and all } u(t) \in U^\varepsilon(t,t-s)u_0.$$ 

The proof of Theorem 5.8 is complete. \hfill \Box

5.3. Upper semicontinuity of pullback attractors. In this subsection, we discuss the robustness of the pullback attractors for the reaction-diffusion equations with singular and nonautonomous perturbations.

**Theorem 5.9.** Suppose that the hypotheses in Theorem 5.6 hold. Let $\mathcal{A}_p^\varepsilon = \{A_p^\varepsilon(t)\}_{t \in \mathbb{R}}$ and $\mathcal{A}_p$ be the pullback attractor and global attractor for the problem (5.1) with $\varepsilon \in (0, 1]$ and $\varepsilon = 0$, respectively. Then for each $t \in \mathbb{R}$,

$$H^\varepsilon_{1,p}(A_p^\varepsilon(t), \mathcal{A}_p) \to 0, \text{ as } \varepsilon \to 0.$$ 

**Proof.** By the arguments of Theorems 5.6-5.7, we can see that conditions (H1)–(H2) and (H4) in Theorem 3.1 hold true. Thanks to Theorem 3.5 in [37], Lemma 5.4 and Theorems 5.6-5.7, it only remains to check (3) in Theorem 3.2. Noticing that all norms on finite-dimensional spaces are equivalent, we need to verify that condition (3) in Theorem 3.2 holds in the topology of $L^2(\Omega)$.

Suppose not. Then there exist $0 < \eta < 1$, $t \in \mathbb{R}$, and a bounded subset $B$ of $V$, such that for any fixed $s^* > 0$ and finite-dimensional subspace $H_m$ in $L^2(\Omega)$ and $H_0^1(\Omega)$, we can find $\varepsilon_n \to 0$, $x_n \in B$, and $u^n \in U^\varepsilon_n(t,t-s^*)x_n$, so that for all $Pv \in PF(s^*)\mathcal{N}(Px_n, \eta/3)$, we have

$$|P(u^n - v)|_2 \geq \frac{\eta}{3}, \quad \forall n \in \mathbb{N}, \quad (5.37)$$

where $P$ is the canonical projector on $H_m$. We can assume that $Px_n \to Px_0$ as $n \to \infty$. Let $u = u_1 + u_2$, where $u_1 = Pu$. We compare the ordinary differential system

$$\begin{align*}
\frac{du_1^n(T)}{dT} - \varepsilon_n \Delta \frac{du_1^n(T)}{dT} - \Delta u_1^n(T) &= Pf(u_1^n(T)) + Pg(x) + \varepsilon_n Ph(T), \quad T \in [t-s^*, t], \\
u_1^n(t-s^*) &= Px_n,
\end{align*} \quad (5.38)$$

with the following system:

$$\begin{align*}
\frac{dv_1(T)}{dT} - \Delta v_1(T) &= Pf(v_1(T)) + Pg(x), \quad T \in [t-s^*, t], \\
v_1(t-s^*) &= Px_0.
\end{align*} \quad (5.39)$$

By (5.5), we can see that for all $n \in \mathbb{N},$

$$\sup_{T \in [t-s^*, t]} |u_1^n(T)|_2 < C. \quad (5.40)$$
Without generality, we assume that \( \theta_1, \theta_2 \in [t - s^*, t] \) with \( 0 < \theta_1 - \theta_2 < 1 \). Hence for all \( n \in \mathbb{N} \),
\[
|u^n_1(\theta_1) - u^n_1(\theta_2)|^2 \leq \int_{\theta_2}^{\theta_1} \left| \frac{du^n_1(T)}{dT} \right|^2 dT
\leq \int_{\theta_2}^{\theta_1} \left( \varepsilon_n |\Delta u^n_1(T)|^2 + |\Delta u^n_1(T)|^2 + |f(u^n_1(T))|^2 + |g|^2 + |h(T)|^2 \right) dT.
\]
Since all norms on finite-dimensional spaces are equivalent, by (5.40), we can deduce that for all \( n \in \mathbb{N} \),
\[
|\Delta u^n_1(T)|^2 + |f(u^n_1(T))|^2 \leq C,
\]
and there exists a \( \alpha_4 > 0 \) such that
\[
|\Delta u^n_1(T)|^2 \leq \alpha_4 \left| \frac{du^n_1(T)}{dT} \right|^2.
\]
Notice that \( \varepsilon_n \to 0 \) as \( n \to \infty \). So when \( n \) is sufficiently large, we have \( 1 - \alpha_4 \varepsilon_n > 0 \), and by Young’s inequality,
\[
|u^n_1(\theta_1) - u^n_1(\theta_2)|^2 \leq \int_{\theta_2}^{\theta_1} \left| \frac{du^n_1(T)}{dT} \right|^2 dT
\leq \int_{\theta_2}^{\theta_1} (C + C|g|^2 + C|h(T)|^2) dT
\leq C(\theta_1 - \theta_2) + C|g|^2(\theta_1 - \theta_2)
+C(\theta_1 - \theta_2)^{\frac{3}{2}} \int_{t-s^*}^t |h(T)|^2 dT + C(\theta_1 - \theta_2)^{\frac{3}{2}} \to 0 \text{ as } \theta_2 \to \theta_1
\]
uniformly with respect to all \( n \) sufficiently large. By the Arzelá-Ascoli theorem, there exists a subsequence (which we still denote by \( u^n_1(T) \)) such that \( u^n_1(T) \) converges to a function \( v_1(T) \) in \( C([t - s^*, t]; H) \). Since \( f \) is continuous with respect to \( u \), and all norms on finite-dimensional spaces are equivalent, passing to the limit in equation (5.38), we get equality (5.39). Clearly, we can find a \( P^\ast \in PF(s^*)P^\ast x_0 \subset PF(s^*)N(P^\ast x_0, \eta/3) \) such that \( \lim_{n \to \infty} |P(u^n(t) - v(t))|^2 = 0 \), which contradicts (5.37). Theorem 5.9 is proved. \( \square \)

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**References**


