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ON THE HIGHER-ORDER BOUNDARY CONDITIONS FOR INCOMPRESSIBLE NONLINEAR BIPOLAR FLUID FLOW

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Abstract. The higher-order boundary conditions associated with the flow of an incompressible, nonlinear, bipolar viscous fluid in a bounded domain are derived; these boundary conditions differ from the various ad hoc sets of higher-order boundary conditions that have been used in work involving fluid dynamics models employing higher-order gradients of the velocity field. The derivation presented is based on a principle of virtual work and some deep results of Heron on higher-order traces of divergence-free vector fields.

1. Introduction. The theory of multipolar materials is due to Green and Rivlin [1], [2], who considered the constitutive equations for an elastic, nonviscous material; a model for a bipolar fluid may be found in the paper of Bleustein and Green [3]. Nečas and Šilhavý [4] developed a thermodynamic theory of constitutive equations for multipolar viscous fluids within the framework of the theory of Green and Rivlin, [1], [2]; the general constitutive theory developed in [4] is consistent with the principle of material-frame indifference and the second law of thermodynamics as expressed by the Clausius-Duhem inequality. In [5], Bellout, Bloom, and Nečas expanded upon some of the consequences of the multipolar fluid model with particular emphasis on the nonlinear, isothermal, incompressible bipolar case.

The Navier-Stokes model of fluid flow is based upon the Stokes hypothesis, which restricts the relation between the stress tensor and the velocity. By relaxing the constraints of the Stokes hypothesis, the mathematical theory of multipolar viscous fluids generalizes the usual Stokes model in three important respects: it allows for nonlinear constitutive relations between the viscous part of the stress tensor and velocity gradients, it allows

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for a dependence of the viscous stress on velocity gradients of order two or higher, and it introduces constitutive relations for higher-order stress tensors (moments of stress), which must be present in the balance of energy equations as soon as higher-order velocity gradients are admitted into the theory.

The simplest expression for the viscous stress $\boldsymbol{\tau}^v$, which is consistent with the primitive conceptual idea of a viscous fluid, i.e., that $\boldsymbol{\tau}^v$ be zero when there is no relative motion between neighboring portions of the fluid, is one of the form (see, e.g., Shinbrot [6])

$$\begin{cases} \boldsymbol{\tau}^v = \boldsymbol{f}(\nabla \boldsymbol{v}, \nabla\nabla \boldsymbol{v}, \ldots), \\ \boldsymbol{f}(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0}, \end{cases} \tag{1.1}$$

where $\boldsymbol{v}$ is the velocity vector. The Stokes hypothesis consists of two simplifying assumptions: namely, in (1.1), $\boldsymbol{f}$ depends linearly on the first velocity gradient $\nabla \boldsymbol{v}$ and is independent of all higher-order velocity gradients. The possible utility of considering more general relations, which allow for a nonlinear velocity-dependent viscosity, has been clearly indicated in Ruelle [7], and various theories of viscous fluid response which allow for nonlinearity in the constitutive theory, as well the presence of higher-order velocity gradients, have been considered, e.g., in Ladyzhenskaya [8], [9], Kaniel [10], and Du and Gunzburger [11]. Perturbations of the Navier-Stokes equations which incorporate higher-order velocity gradients may be found, e.g., in the papers of Lions [12], Ou and Sritharan [13], [14], and the references cited therein, as well as in the book [15] by Temam.

The constitutive relations for isothermal, nonlinear, incompressible bipolar viscous fluids which were introduced in [5] have the form

$$\tau_{ij} = -p\delta_{ij} + 2\mu_0(\epsilon + |\boldsymbol{e}|^2)^{-\alpha/2} e_{ij} - 2\mu_1 \Delta e_{ij}, \tag{1.2}$$

$$\tau_{ijk} = 2\mu_1 \frac{\partial e_{ij}}{\partial x_k}, \tag{1.3}$$

with the τ_{ij} being the (components of the) stress tensor, τ_{ijk} the first multipolar stress tensor, and p the pressure. In (1.2), (1.3) the e_{ij} are the components of the rate of deformation tensor, i.e.

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.4}$$

while $\epsilon, \mu_0, \mu_1 > 0$ and α, $0 \le \alpha < 1$, are constitutive parameters. Besides the presence of the multipolar stress tensor τ_{ijk} (which affects only higher-order boundary conditions), and the higher-order velocity gradients, the theory also involves a nonlinear viscosity

$$\mu = \mu_0(\epsilon + |\boldsymbol{e}|^2)^{-\alpha/2} \tag{1.5}$$

in which μ_0 has the form $\mu_0 = \bar{\mu}_0 \bar{\epsilon}^{\alpha/2}$, with $\bar{\mu}_0$ the usual Stokes viscosity, while $\bar{\epsilon}$ has the same physical dimensions as ϵ, i.e., t^{-2}.

With μ_0 having the form indicated viz., $\mu_0 = \bar{\mu}_0 \bar{\epsilon}^{\alpha/2}$, the theory represented by (1.2), (1.3) reduces, in fact, precisely to that given by the Stokes model when $\alpha = \mu_1 = 0$.

The constitutive relations (1.2), (1.3) and the condition of incompressibility produce for the nonlinear bipolar model of a viscous fluid the following system of nonlinear partial differential equations:

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \nabla \cdot (2\mu \boldsymbol{e}) - 2\mu_1 \nabla \cdot (\Delta \boldsymbol{e}) + \rho \boldsymbol{f} \tag{1.6}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{1.7}$$

where ρ is the constant density, $\boldsymbol{f}$ is the external body force vector, and $\mu(|\boldsymbol{e}|)$ is given by (1.5). The system of equations (1.6), (1.7) holds in some domain $\Omega \times [0, T)$, $\Omega \subseteq R^n$, $n = 1, 2, 3$, $T > 0$, and is subject to initial conditions of the form

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \tag{1.8}$$

When $\Omega \subseteq R^3$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, (1.6)–(1.8) is supplemented by the boundary conditions

$$\boldsymbol{v} = \boldsymbol{0},\ \tau_{ijk}\nu_j\nu_k\tau_i = 0,\ i = 1, 2, 3,\ \text{on } \partial\Omega \times [0, T) \tag{1.9}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$, while $\boldsymbol{\tau}$ is any vector lying in the tangent space to $\partial\Omega$ at $\boldsymbol{x}$. The first set of conditions in (1.9) represent the usual no-slip condition associated with a viscous fluid, while the second set expresses the fact that the first moments of the traction vanish on $\partial\Omega$. It has often been cited in the literature that with the first moments of the tractions equal to zero on $\partial\Omega$, the conditions $\tau_{ijk}\nu_j\nu_k = 0$, $i = 1, 2, 3$, hold on $\partial\Omega$ as a consequence of the principle of virtual work (e.g., [5]); however, this latter set of conditions is valid only in the absence of the incompressibility constraint. The situation for an elastic material in the absence of the incompressibility constraint has been studied by Toupin [37]. In either case, neither the conditions $\tau_{ijk}\nu_j\nu_k = 0$, $i = 1, 2, 3$, nor the appropriate form of the higher-order boundary conditions as stated in (1.9) for an incompressible fluid have been established, rigorously, in the literature; that is the goal of the present research note. We also note that the second set of boundary conditions in (1.9) differs from the rather ad hoc sets of higher-order boundary conditions which have been used in some of the other papers cited in this introduction which also involve fluid dynamics models employing higher-order spatial gradients of the velocity. For example, the analysis in [12], [13], and [14] imposes the artificial assumption that the normal derivative of the velocity field vanishes on $\partial\Omega$. It will be seen in §3 that the form of the higher-order boundary condition delineated in (1.9) depends, in a crucial manner, on a deep result of Heron in [16], which was written under the direction of Roger Temam at Orsay.

Various studies of the incompressible bipolar fluid flow model and its specialization to the non-Newtonian case, in which $\mu_1 = 0$, have appeared in the recent literature: special types of flows have been analyzed in [17]–[21], exterior flow and flow over non-smooth boundaries have been studied, respectively, in [22] and [23], general existence and uniqueness results have appeared in [24]–[27], and theorems related to asymptotic stability and the existence of global attractors and inertial manifolds may be found in [28]–[34]. A complete overview of the entire body of work cited above will appear in the forthcoming monograph [35].

2. The structure of the higher-order boundary conditions 1: Virtual work without the incompressibility constraint. We now turn to the issue of the structure of the higher-order boundary conditions associated with the constitutive equations defining an incompressible nonlinear bipolar fluid under isothermal conditions. Following a similar analysis in [2] for elastic materials, we obtain an appropriate set of boundary conditions by applying the principle of virtual work. Our analysis in this subsection will be carried out without imposing the constraint of incompressibility on the virtual velocities and velocity gradients; this constraint is imposed in §3, and the resulting modification of the computations presented below will then yield the final form of the higher-order boundary conditions associated with the multipolar stress tensor τ_{ijk}.

We begin by rewriting the constitutive equations in the form

$$t_{ij} = -p\delta_{ij} + \tau_{ij}^{(0)} - \frac{\partial}{\partial x_k}\tau_{ijk} \tag{2.1}$$

where τ_{ijk} is given by (1.3) and (with $\alpha = 2 - p$)

$$\tau_{ij}^{(0)} = 2\mu_0(\epsilon + e_{ij}e_{ij})^{\frac{p-2}{2}} e_{ij} \tag{2.2}$$

Next, we define

$$\Gamma(e_{ij}) = \mu_0 \int_0^{e_{ij}e_{ij}} (\epsilon + s)^{\frac{p-2}{2}}\, ds \tag{2.3a}$$

$$\Phi\left(\frac{\partial e_{ij}}{\partial x_k}\right) = \mu_1 \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k} \tag{2.3b}$$

and

$$W\left(e_{ij}, \frac{\partial e_{ij}}{\partial x_k}\right) = \Gamma(e_{ij}) + \Phi\left(\frac{\partial e_{ij}}{\partial x_k}\right) \tag{2.4}$$

Then, clearly, if we set $\xi_{ijk} = \frac{\partial e_{ij}}{\partial x_k}$,

$$\tau_{ij}^{(0)} = \frac{\partial \Gamma}{\partial e_{ij}} = \frac{\partial W}{\partial e_{ij}} \tag{2.5a}$$

$$\tau_{ijk} = \frac{\partial \Phi}{\partial \xi_{ijk}} = \frac{\partial W}{\partial \xi_{ijk}} \tag{2.5b}$$

so that the complete residual stress tensor $\tau_{ij} = \tau_{ij}^{(0)} - \frac{\partial}{\partial x_k}\tau_{ijk}$ is given by

$$\tau_{ij} = \frac{\partial W}{\partial e_{ij}} - \frac{\partial}{\partial x_k}\left(\frac{\partial W}{\partial \xi_{ijk}}\right) \tag{2.6}$$

The potential energy of the fluid in a fixed, bounded domain $\Omega \subseteq R^n$, $n = 2, 3$, with smooth boundary $\partial\Omega$ and exterior unit normal $\boldsymbol{\nu}$ at $\boldsymbol{x} \in \partial\Omega$ is now defined by

$$E(\Omega) = \int_\Omega W(e_{ij}, \xi_{ijk})\, d\boldsymbol{x} \tag{2.7}$$

The principle of virtual work, then, assumes the form

$$\delta E(\Omega) = \int_\Omega f_i \delta v_i\, d\boldsymbol{x} + \oint_{\partial\Omega} (T_i \delta v_i + M_i D\delta v_i)\, dS_{\boldsymbol{x}} \tag{2.8}$$

for arbitrary values of the virtual velocity variations δv_i, subject only to the constraint that the $\delta v_i = 0$ on $\partial\Omega$, and arbitrary variations of the virtual velocity gradients $\delta v_{i,j}$, where $D\delta v_i = \delta v_{i,j}\nu_j$, are the normal derivatives of the virtual velocity components on

$\partial\Omega$, $f_i = \rho F_i - \frac{\partial p}{\partial x_i}$, T_i are the tractions/area on $\partial\Omega$, and M_i are the hypertractions (or moments/area) on $\partial\Omega$. From (2.7) we have

$$\delta E(\Omega) = \int_\Omega \left[\frac{\partial W}{\partial e_{ij}} \delta e_{ij} + \frac{\partial W}{\partial \xi_{ijk}} \delta(\xi_{ijk}) \right] d\boldsymbol{x} \tag{2.9}$$

Now,

$$\delta e_{ij} = \frac{1}{2}(\delta v_{i,j} + \delta v_{j,i})$$

so

$$\frac{\partial W}{\partial e_{ij}} \delta e_{ij} = \frac{1}{2} \frac{\partial W}{\partial e_{ij}} (\delta v_{i,j} + \delta v_{j,i}) = \frac{\partial W}{\partial e_{ij}} \delta v_{i,j} \tag{2.10}$$

by the symmetry of e_{ij}. In a similar manner,

$$\delta\left(\xi_{ijk}\right) = \frac{1}{2}\left(\delta v_{i,jk} + \delta v_{j,ik}\right)$$

and

$$\frac{\partial W}{\partial \xi_{ijk}} \delta\left(\xi_{ijk}\right) = \frac{\partial W}{\partial \xi_{ijk}} \delta v_{i,jk} \tag{2.11}$$

Employing (2.10), (2.11) in (2.9) we obtain

$$\delta E(\Omega) = \int_\Omega \left[\frac{\partial W}{\partial e_{ij}} \delta v_{i,j} + \frac{\partial W}{\partial \xi_{ijk}} \delta v_{i,jk} \right] d\boldsymbol{x} \tag{2.12}$$

However, by integration by parts and the divergence theorem, we obtain for the first integral in (2.12)

$$\begin{aligned} \int_\Omega \frac{\partial W}{\partial e_{ij}} \delta v_{i,j}\, d\boldsymbol{x} &= \int_\Omega \frac{\partial}{\partial x_j} \left(\frac{\partial W}{\partial e_{ij}} \delta v_i \right) d\boldsymbol{x} - \int_\Omega \frac{\partial}{\partial x_j} \left(\frac{\partial W}{\partial e_{ij}} \right) \delta v_i \, d\boldsymbol{x} \\ &= -\int_\Omega \frac{\partial}{\partial x_j} \left(\frac{\partial W}{\partial e_{ij}} \right) \delta v_i \, d\boldsymbol{x} \end{aligned} \tag{2.13}$$

as $\delta v_i = 0$ on $\partial\Omega$. In an analogous fashion, two consecutive integrations by parts, applied to the second integral in (2.9), yields

$$\begin{aligned} \int_\Omega \frac{\partial W}{\partial \xi_{ijk}} \delta v_{i,jk}\, d\boldsymbol{x} &= \int_\Omega \frac{\partial}{\partial x_k} \left(\frac{\partial W}{\partial \xi_{ijk}} \delta v_{i,j} \right) d\boldsymbol{x} \\ &\quad - \int_\Omega \frac{\partial}{\partial x_k} \left(\frac{\partial W}{\partial \xi_{ijk}} \right) \delta v_{i,j}\, d\boldsymbol{x} \\ &= \oint_{\partial\Omega} \frac{\partial W}{\partial \xi_{ijk}} \delta v_{i,j} \nu_k \, dS_{\boldsymbol{x}} \\ &\quad - \int_\Omega \frac{\partial}{\partial x_j} \left[\frac{\partial}{\partial x_k} \left(\frac{\partial W}{\partial \xi_{ijk}} \right) \delta v_i \right] d\boldsymbol{x} \\ &\quad + \int_\Omega \frac{\partial^2}{\partial x_j \partial x_k} \left(\frac{\partial W}{\partial \xi_{ijk}} \right) \delta v_i \, d\boldsymbol{x} \end{aligned}$$

or

$$\begin{aligned} \int_\Omega \frac{\partial W}{\partial \xi_{ijk}} \delta v_{i,jk}\, d\boldsymbol{x} &= \oint_{\partial\Omega} \frac{\partial W}{\partial \xi_{ijk}} \delta v_{i,j} \nu_k \, dS_{\boldsymbol{x}} \\ &\quad + \int_\Omega \frac{\partial^2}{\partial x_j \partial x_k} \left(\frac{\partial W}{\partial \xi_{ijk}} \right) \delta v_i \, d\boldsymbol{x} \end{aligned} \tag{2.14}$$

where we have again used the fact that $\delta v_i = 0$ on $\partial\Omega$. Employing (2.13) and (2.14) in (2.12) and taking note of (2.5a),(2.5b), we find that

$$\delta E(\Omega) = -\int_\Omega \frac{\partial}{\partial x_j}\tau_{ij}^{(0)}\delta v_i\, d\boldsymbol{x} + \oint_{\partial\Omega} \tau_{ijk}\delta v_{i,j}\nu_k\, dS_{\boldsymbol{x}} + \int_\Omega \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}\tau_{ijk}\right)\delta v_i\, d\boldsymbol{x}$$

or, by combining like integrals,

$$\delta E(\Omega) = \int_\Omega \frac{\partial}{\partial x_j}\left\{-\tau_{ij}^0 + \frac{\partial}{\partial x_k}\tau_{ijk}\right\}\delta v_i\, d\boldsymbol{x} + \oint_{\partial\Omega} \tau_{ijk}\delta v_{i,j}\nu_k\, dS_{\boldsymbol{x}} \tag{2.15}$$

If we now combine (2.8) and (2.15) and recall that $\tau_{ij} = \tau_{ij}^{(0)} - \frac{\partial}{\partial x_k}\tau_{ijk}$, we obtain the equation

$$\int_\Omega \left(f_i + \frac{\partial}{\partial x_j}\tau_{ij}\right)\delta v_i\, d\boldsymbol{x} + \oint_{\partial\Omega}(M_i\nu_j - \tau_{ijk}\nu_k)\delta v_{i,j}\, dS_{\boldsymbol{x}} = 0 \tag{2.16}$$

By virtue of the independence of the variations of the virtual velocity components (in Ω) and their gradients (on $\partial\Omega$), and in the absence, to this point, of the incompressibility constraint, we obtain from (2.16)

$$\frac{\partial}{\partial x_j}\tau_{ij} = -f_i = -\rho F_i + \frac{\partial p}{\partial x_i},\ \text{in } \Omega \tag{2.17}$$

or

$$-\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\tau_{ij}^{(0)} - \frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}\tau_{ijk}\right) + \rho F_i = 0,\ \text{in } \Omega \tag{2.18}$$

as well as

$$\tau_{ijk}\nu_k = M_i\nu_j,\ \text{on } \partial\Omega \tag{2.19}$$

If we multiply (2.19) through by ν_j, sum on j, and use the fact that $\nu_j\nu_j = 1$, it follows that

$$\tau_{ijk}\nu_j\nu_k = M_i,\ \text{on } \partial\Omega \tag{2.20}$$

for $i = 1, 2, 3$ (in R^3), which is the form the higher-order boundary conditions assume at each fixed time t if we do not impose the incompressibility constraint on the velocity variations; in the next subsection we will modify the calculations presented here so as to take into account this constraint.

3. The structure of the higher-order boundary conditions 2: Virtual work subject to the incompressibility constraint. If we restrict ourselves in (2.16) to smooth, divergence-free virtual velocity fields with compact support in Ω, then (2.16) reduces to

$$\int_\Omega \left(f_i + \frac{\partial}{\partial x_j}\tau_{ij}\right)\delta v_i\, d\boldsymbol{x} = 0 \tag{3.1}$$

for all δv_i that are smooth and divergence-free. It then follows (see, e.g., [36], Theorem 2.3) that (2.17) holds. Using (2.17), it is now a direct consequence of (2.16) that the boundary integrals

$$\oint_{\partial\Omega}(M_i\nu_j - \tau_{ijk}\nu_k)\,\delta v_{i,j}\, ds = 0 \tag{3.2}$$

for all δv_i that are divergence-free and zero on $\partial\Omega$. From (3.2) it follows that for any tensor $\boldsymbol{w}$ that is the restriction of the gradient of a divergence-free, smooth, vector function $\delta\boldsymbol{v}$ that vanishes on the boundary, we have

$$\oint_{\partial\Omega} (M_i \nu_j - \tau_{ijk}\nu_k)\, w_{ij}\, ds = 0 \tag{3.3}$$

As δv_i is zero on $\partial\Omega$, it follows that all tangential derivatives of δv_i are zero on the boundary. Thus, in (3.2), the only non-zero part of $\delta v_{i,j}$ is that component which corresponds to the normal derivative of δv_i. By the normal derivative of the vector $\delta\boldsymbol{v}$ with components δv_i, we mean, of course, the vector whose ith component is $\frac{\partial \delta v_i}{\partial x_j}\nu_j$ and, by Theorem 3.1 of [16], all such vectors are tangential to $\partial\Omega$. Furthermore, for any smooth vector $\boldsymbol{g}$ tangential to $\partial\Omega$, there exists a function δv_i such that

$$\begin{aligned} \frac{\partial \delta v_i}{\partial \boldsymbol{\nu}} &= g_i, \text{ on } \partial\Omega \\ \delta v_i &= 0, \text{ on } \partial\Omega \\ \operatorname{div} \delta v_i &= 0, \text{ in } \Omega \end{aligned} \tag{3.4}$$

Remarks. Heron, in [16], studied higher-order traces of divergence-free fields. We use here only a particular form of his Theorem 3.1; for the convenience of the reader, we state below this simplified version of Heron's theorem.

Theorem (Heron [16]). Let Ω be a connected, bounded subset of R^N with C^3 boundary $\partial\Omega = \Gamma$. Given a vector $\boldsymbol{g}_1 \in \boldsymbol{H}^{1/2}(\Gamma)$, there exists $\boldsymbol{u} \in \boldsymbol{W}^{2,2}(\Omega)$ with $\operatorname{div}\boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} = \boldsymbol{0}$ on Γ, and $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = \boldsymbol{g}_1$ on Γ if and only if $\boldsymbol{g}_1$ is tangential to Γ, i.e., if and only if $\boldsymbol{g}_1 \cdot \boldsymbol{\nu} = 0$ where $\boldsymbol{\nu}$ is the exterior unit normal to Γ.

Remarks. In the notation used in [16], $\boldsymbol{u} = \boldsymbol{g}_0$ on Γ and $\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} = \boldsymbol{g}_1$ on Γ; thus, in the special case considered above, $\boldsymbol{g}_0 = \boldsymbol{0}$.

It now follows from (3.2) that the tangential component of $M_i\nu_j - \tau_{ijk}\nu_k$ is zero on $\partial\Omega$; thus, we may state the following

Lemma 3.1. For an incompressible, nonlinear, bipolar viscous fluid defined by the constitutive relations (1.2), (1.3) in an open bounded domain $\Omega \subseteq R^3$ with smooth boundary $\partial\Omega$, the higher-order boundary condition

$$\tau_{ijk}\nu_j\nu_k\tau_i = M_i\tau_i, \text{ on } \partial\Omega \tag{3.5}$$

must be satisfied, where for $n = 2$, $\boldsymbol{\nu}$ is the exterior unit normal to the smooth curve $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$ and $\boldsymbol{\tau}$ is the unit tangent vector, while for $n = 3$, $\boldsymbol{\nu}$ is the exterior unit normal to the surface $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$ and $\boldsymbol{\tau}$ is any unit vector in the tangent plane to $\partial\Omega$ at $\boldsymbol{x}$.

Remarks. If we assume that $\boldsymbol{M} = \boldsymbol{0}$ on $\partial\Omega$, then (3.5) reduces to

$$\tau_{ijk}\nu_j\nu_k\tau_i = 0, \text{ on } \partial\Omega \tag{3.6}$$

An alternative form of the higher-order boundary conditions is then given by

Lemma 3.2. Let Ω be an open bounded domain in R^3 with smooth boundary $\partial\Omega$ and let $\boldsymbol{x} \in \partial\Omega$. Let $\boldsymbol{\nu}$ be the exterior unit normal to $\partial\Omega$ at $\boldsymbol{x}$ and let $\boldsymbol{\tau}$ denote any unit

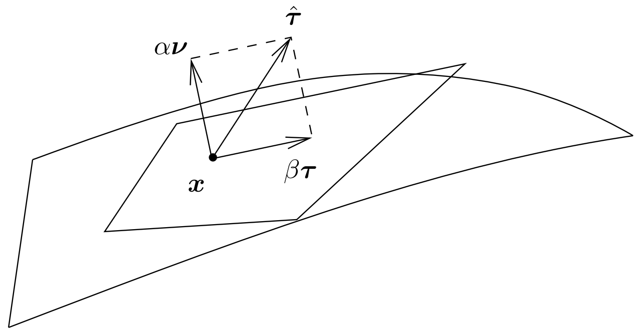

FIG. 1

vector in the tangent space to $\partial\Omega$ at $\boldsymbol{x}$. Then, at $\boldsymbol{x}$,

$$\tau_{ijk}\nu_j\nu_k\tau_i = 0 \Leftrightarrow \tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0,\ i = 1,2,3 \tag{3.7}$$

whenever $\boldsymbol{M} = \boldsymbol{0}$.

Proof. Suppose $\tau_{ijk}\nu_j\nu_k - \tau_{jkl}\nu_j\nu_k\nu_l\nu_i = 0$, $i = 1,2,3$. Multiplying through by τ_i, summing on i, and using the fact that $\nu_i\tau_i = 0$, we get

$$\tau_{ijk}\nu_j\nu_k\tau_i = 0, \quad i = 1,2,3$$

Now, let $\tau_{ijk}\nu_j\nu_k\tau_i = 0$, where the τ_i are the components of any unit (tangent) vector in the tangent space to $\partial\Omega$ at $\boldsymbol{x} \in \partial\Omega$, and let $\hat{\boldsymbol{\tau}}$ be the vector at $\boldsymbol{x}$ with i-th component $(\hat{\boldsymbol{\tau}})_i = \tau_{ijk}\nu_j\nu_k$. Also, let $\beta\boldsymbol{\tau}$ be the projection of $\hat{\boldsymbol{\tau}}$ onto the tangent plane to $\partial\Omega$ at $\boldsymbol{x}$, where $\boldsymbol{\tau}$ is a unit vector in the tangent space, so that (see Fig. 1)

$$\hat{\boldsymbol{\tau}} = \alpha\boldsymbol{\nu} + \beta\boldsymbol{\tau} \tag{3.8}$$

where $\alpha\boldsymbol{\nu}$ is the projection of $\hat{\boldsymbol{\tau}}$ onto the normal direction to $\partial\Omega$ at $\boldsymbol{x}$. Therefore,

$$\hat{\tau}_i = \tau_{ijk}\nu_j\nu_k = \alpha\nu_i + \beta\tau_i \tag{3.9}$$

which, as $\nu_i\tau_i = 0$, $\nu_i\nu_i = 1$, implies that

$$\alpha = \tau_{ijk}\nu_j\nu_k\nu_i \equiv \tau_{jkl}\nu_j\nu_k\nu_l \tag{3.10}$$

As $\tau_i\tau_i = 1$, by virtue of our hypothesis and the definition of $\hat{\boldsymbol{\tau}}$,

$$\tau_{ijk}\nu_j\nu_k\tau_i = \beta = 0 \tag{3.11}$$

in which case

$$\tau_{ijk}\nu_j\nu_k = (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i, \quad i = 1,2,3 \tag{3.12}$$

so that $\tau_{ijk}\nu_j\nu_k - (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i = 0$, $i = 1,2,3$ whenever $\tau_{ijk}\nu_j\nu_k\tau_i = 0$. $\square$

REMARKS. The proofs of Lemmas 3.1 and 3.2 have been constructed for the case $n = 3$; the statements and proofs for $n = 2$ are trivial modifications of the proofs given above.

It will now be shown that the second of the conditions in (3.7) implies that

$$\tau_{ijk}e_{ij}\nu_k = 0, \text{ on } \partial\Omega \tag{3.13}$$

By virtue of Lemma 3.2 it then follows that (3.13) is also a consequence of the condition (3.6).

LEMMA 3.3. Let $S \subseteq \mathbb{R}^3$ be a smooth surface, and let $\boldsymbol{v}(\cdot)$ be a divergence-free C^2 vector field defined on a neighborhood of $\partial\Omega$, with $\boldsymbol{v} = \mathbf{0}$ on $\partial\Omega$. If $\tau_{ijk}(\boldsymbol{v})\nu_j\nu_k - \tau_{jkl}(\boldsymbol{v})\nu_j\nu_k\nu_l\nu_i|_{\partial\Omega} = 0$, for $i = 1, 2, 3$, where $\boldsymbol{\nu}$ is the exterior unit normal on $\partial\Omega$, then $\tau_{ijk}(\boldsymbol{v})e_{ij}(\boldsymbol{v})\nu_k|_S = 0$.

Proof. As τ_{ijk} and e_{ij} are both symmetric in i and j,

$$\tau_{ijk}e_{ij}\nu_k = \tau_{ijk}\frac{\partial v_i}{\partial x_j}\nu_k \tag{3.14}$$

Let $\boldsymbol{p} \in \partial\Omega$ and let $(\boldsymbol{t}, \boldsymbol{\tau})$ be a pair of orthonormal vectors at $\boldsymbol{p}$ that lie in the tangent plane to $\partial\Omega$ at $\boldsymbol{p}$; thus, $(\boldsymbol{t}, \boldsymbol{\tau}, \boldsymbol{\nu})$ form an orthonormal triplet at $\boldsymbol{p}$. Define the vectors $\boldsymbol{\lambda}^{(i)}$, $i = 1, 2, 3$, by

$$\lambda_j^{(i)} \equiv \frac{\partial v_i}{\partial x_j} = \alpha^{(i)}\nu_j + \beta^{(i)}t_j + \gamma^{(i)}\tau_j \tag{3.15}$$

By virtue of (3.14), (3.15), it follows that

$$\begin{aligned}\tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k|_{\boldsymbol{p}} &= \tau_{ijk}(\boldsymbol{v})\lambda_j^{(i)}\nu_k|_{\boldsymbol{p}} \\ &= \alpha^{(i)}\tau_{ijk}(\boldsymbol{v})\nu_j\nu_k|_{\boldsymbol{p}} \\ &\quad + \beta^{(i)}\tau_{ijk}(\boldsymbol{v})t_j\nu_k|_{\boldsymbol{p}} + \gamma^{(i)}\tau_{ijk}(\boldsymbol{v})\tau_j\nu_k|_{\boldsymbol{p}}\end{aligned} \tag{3.16}$$

We now choose curves $\boldsymbol{\eta}_1(\xi) \subset \partial\Omega$, $\boldsymbol{\eta}_2(\xi) \subset \partial\Omega$, $|\xi| \leq \bar{\xi}$, such that $\boldsymbol{\eta}_1(0) = \boldsymbol{\eta}_2(0) = \boldsymbol{p}$ and $\boldsymbol{\eta}_1(0) = \boldsymbol{t}$, $\boldsymbol{\eta}_2(0) = \boldsymbol{\tau}$. As $\boldsymbol{v}|_{\partial\Omega} = \mathbf{0}$, $v_i(\boldsymbol{\eta}_1(\xi)) = v_i(\boldsymbol{\eta}_2(\xi)) = 0$, $|\xi| \leq \xi$, for $i = 1, 2, 3$; thus

$$\begin{cases} \dfrac{dv_i}{d\xi}|_{\boldsymbol{\eta}_i} \equiv \dfrac{\partial v_i}{\partial x_j}(\boldsymbol{\eta}_1(\xi))\dfrac{dx_j}{d\xi} = 0 \\ \dfrac{dv_i}{d\xi}|_{\boldsymbol{\eta}_2} \equiv \dfrac{\partial v_i}{\partial x_j}(\boldsymbol{\eta}_2(\xi))\dfrac{dx_j}{d\xi} = 0 \end{cases} \quad (i = 1, 2, 3) \tag{3.17}$$

Setting $\xi = 0$ in (3.17), we have

$$\left.\frac{\partial v_i}{\partial x_j}\right|_{\boldsymbol{p}} \cdot t_j = \left.\frac{\partial v_i}{\partial x_j}\right|_{\boldsymbol{p}} \cdot \tau_j = 0, \quad i = 1, 2, 3 \tag{3.18}$$

However, by virtue of (3.15), $\beta^{(i)} = \dfrac{\partial v_i}{\partial x_j}t_j$ and $\gamma^{(i)} = \dfrac{\partial v_i}{\partial x_j}\tau_j$, for $i = 1, 2, 3$. Thus, at $\boldsymbol{p}$, $\gamma^{(i)} = \beta^{(i)} = 0$, for $i = 1, 2, 3$; therefore, by (3.16) and the second set of higher-order boundary conditions in (3.7),

$$\tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k|_{\boldsymbol{p}} = \alpha^{(i)}\nu_i\tau_{ljk}(\boldsymbol{v})\nu_l\nu_j\nu_k|_{\boldsymbol{p}} \tag{3.19}$$

But, as $\beta^{(i)} = \gamma^{(i)} = 0$, $i = 1, 2, 3$, (3.15) reduces to

$$\lambda_j^{(i)} = \frac{\partial v_i}{\partial x_j} = \alpha^{(i)}\nu_j \tag{3.20}$$

Therefore, setting $i = j$ in (3.20), summing on i, and using the fact that $\boldsymbol{v}$ is a solenoidal vector field, we obtain

$$\frac{\partial v_i}{\partial x_i} \equiv \nabla \cdot \boldsymbol{v} = \alpha^{(i)}\nu_i = 0 \tag{3.21}$$

in which case (3.19) yields the required result, i.e., $\tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k|_{\boldsymbol{p}} = 0$. □

REMARKS. Lemma 3.3 has been used, repeatedly, in the work on incompressible bipolar fluid flow to effectuate the following integration by parts computation: for $\boldsymbol{v}$ sufficiently smooth and satisfying (3.7), as well as $\boldsymbol{v}|_{\partial\Omega} = \mathbf{0}$,

$$\begin{aligned}\int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij})v_i d\boldsymbol{x} &= \int_\Omega \left[\frac{\partial}{\partial x_j}(\Delta e_{ij}v_i) - \Delta e_{ij}\frac{\partial v_i}{\partial x_j}\right] d\boldsymbol{x}\\ &= \oint_{\partial\Omega} \Delta e_{ij} v_i \nu_j dS - \int_\Omega \frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_k}\right)\frac{\partial v_i}{\partial x_j} d\boldsymbol{x}\\ &= -\int_\Omega \left[\frac{\partial}{\partial x_k}\left(\frac{\partial e_{ij}}{\partial x_k}\frac{\partial v_i}{\partial x_j}\right) - \frac{\partial e_{ij}}{\partial x_k}\frac{\partial v_i}{\partial x_j \partial x_k}\right] d\boldsymbol{x} \ (\text{as } \boldsymbol{v}|_{\partial\Omega} = \mathbf{0})\\ &= -\oint_{\partial\Omega} \frac{\partial e_{ij}}{\partial x_k}\frac{\partial v_i}{\partial x_j}\nu_k dS + \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x}\end{aligned} \tag{3.22}$$

Therefore,

$$\begin{aligned}2\mu_1 \int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij})v_i d\boldsymbol{x} &= -\oint_{\partial\Omega} \tau_{ijk} e_{ij}\nu_k dS + 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x}\\ &= 2\mu_1 \int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k} d\boldsymbol{x}\end{aligned} \tag{3.23}$$

REMARKS. As shown in Lemma 3.3, $\tau_{ijk}e_{ij}\nu_k|_{\partial\Omega} = 0$ in (3.23) is a consequence of the boundary conditions (3.7) that apply whenever $M_k\tau_k = 0$ on $\partial\Omega$ so, in particular, $\tau_{ijk}e_{ij}\nu_k|_{\partial\Omega} = 0$ if $M_i = 0$, on $\partial\Omega$, $i = 1, 2, 3$. Now, suppose that $\tau_{ijk}\nu_j\nu_k\tau_i = M_k\tau_k \neq 0$ on $\partial\Omega$, where we sum on all repeated indices. Then, by virtue of (3.9) and (3.10), we have

$$\tau_{ijk}\nu_j\nu_k - (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i = \beta\tau_i \tag{3.24}$$

Multiplying (3.24) by τ_i, summing on i, and using the facts that $\nu_i\tau_i = 0$, $\tau_i\tau_i = 1$, we obtain

$$\beta = \tau_{ijk}\nu_j\nu_k\tau_i = M_k\tau_k \tag{3.25}$$

the last result being valid on $\partial\Omega$ by virtue of the higher-order boundary condition (3.5). Substituting (3.25) back into (3.24), we obtain

$$\tau_{ijk}\nu_j\nu_k - (\tau_{jkl}\nu_j\nu_k\nu_l)\nu_i = (M_k\tau_k)\tau_i, \text{ on } \partial\Omega \tag{3.26}$$

as the form of the higher-order boundary conditions on $\partial\Omega$ that are equivalent to (3.5) whenever $M_i\tau_i \neq 0$ on $\partial\Omega$. The term on the right-hand side of (3.26) is the projection of the vector $\boldsymbol{M}$ onto the tangent plane to the surface $\partial\Omega$ at a point $\boldsymbol{x} \in \partial\Omega$, if we are working in space dimension $n = 3$; if $n = 2$, it is the projection of $\boldsymbol{M}$ onto the direction of the tangent vector to the curve $\partial\Omega$ at a point $\boldsymbol{x} \in \partial\Omega$.

Assume now that $M_k\tau_k \neq 0$ on $\partial\Omega$. Then by virtue of (3.16) with $\beta^{(i)} = \nu^{(i)} = 0$, at any $\boldsymbol{p} \in \partial\Omega$,

$$\tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k|_{\boldsymbol{p}} = \alpha^{(i)}\tau_{ijk}\nu_j\nu_k|_{\boldsymbol{p}} \tag{3.27}$$

However, if $M_k\tau_k \neq 0$, then by (3.26)

$$\tau_{ijk}(\boldsymbol{v})\nu_j\nu_k - (\tau_{ljk}(\boldsymbol{v})\nu_l\nu_j\nu_k)\nu_i = (M_k\tau_k)\tau_i$$

at each $\boldsymbol{p} \in \partial\Omega$, and (3.27) becomes

$$\tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k|_p = \alpha^{(i)}\left[(\tau_{ljk}\nu_l\nu_j\nu_k)\nu_i - (M_k\tau_k)\tau_i\right]_{\boldsymbol{p}} \tag{3.28}$$

From (3.20) we obtain

$$\alpha^{(i)} = \frac{\partial v_i}{\partial x_j}\nu_j = \frac{\partial v_i}{\partial \boldsymbol{\nu}}, \text{ on } \partial\Omega \tag{3.29}$$

Also, as $\boldsymbol{v}$ is divergence-free,

$$\alpha^{(i)}\nu_i = \nabla \cdot \boldsymbol{v} = 0 \tag{3.30}$$

Combining (3.28), (3.29), and (3.30), we obtain, at each $\boldsymbol{p} \in \partial\Omega$,

$$\begin{aligned} \tau_{ijk}(\boldsymbol{v})e_{ij}\nu_k = & \quad (M_k\tau_k)\alpha^{(i)}\tau_i \\ = & \quad (M_k\tau_k)\tau_i\frac{\partial v_i}{\partial \boldsymbol{\nu}} \end{aligned} \tag{3.31}$$

in which case the last result in (3.23) becomes

$$2\mu_1\int_\Omega \frac{\partial}{\partial x_j}(\Delta e_{ij})v_i d\boldsymbol{x} = 2\mu_1\int_\Omega \frac{\partial e_{ij}}{\partial x_k}\frac{\partial e_{ij}}{\partial x_k}d\boldsymbol{x} - \oint_{\partial\Omega}(M_k\tau_k)\tau_i\frac{\partial v_i}{\partial \boldsymbol{\nu}}dS \tag{3.32}$$

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