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FLOCKING BEHAVIOR OF THE CUCKER-SMALE MODEL UNDER ROOTED LEADERSHIP IN A LARGE COUPLING LIMIT

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Abstract. We present an asymptotic flocking estimate for the Cucker-Smale flocking model under the rooted leadership in a large coupling limit. For this, we reformulate the Cucker-Smale model into a fast-slow dynamical system involving a small parameter which corresponds to the inverse of a coupling strength. When the coupling strength tends to infinity, the spatial configuration will be frozen instantaneously, whereas the velocity configuration shrinks to the global leader's velocity immediately. For the rigorous explanation of this phenomenon, we use Tikhonov's singular perturbation theory. We also present several numerical simulations to confirm our analytical theory.

1. Introduction. Collective coherent motions such as the aggregation of bacteria, flocking of birds, swarming of fish and herding of sheep, etc., are often observed in our biological complex system [26–28], and have been extensively studied in the engineering

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community [19, 23] due to their diverse applications, e.g., a sensor network, formation control of robots and unmanned aerial vehicles, and opinion formation of social networks. In this paper, we are interested in a generalized Cucker-Smale flocking model [10] which is similar to Newton's equations in many-body interacting particle systems. Let $x_i, v_i \in \mathbb{R}^d$ be the spatial position and velocity of the i-th agent, respectively. Then our generalized Cucker-Smale (in short C-S) model reads as follows: For $i = 0, 1, \ldots, N$,

$$\begin{aligned} \frac{dx_i}{dt} &= v_i, \quad t > 0, \\ \frac{dv_i}{dt} &= \frac{1}{\varepsilon} \sum_{j=0}^{N} \psi_{ij}(x)(v_j - v_i), \end{aligned} \tag{1.1}$$

where $\psi_{ij}(x) > 0$ is a communication weight which represents the information flow from the $j-$th agent to the $i-$th agent depending on the spatial configuration $x = (x_0, x_1, \ldots, x_N)$, and $\varepsilon > 0$ denotes the inverse of a coupling strength.

The C-S type system (1.1) has been extensively studied and generalized with extra effects: for example, the stochastic effects [2, 14], noisy environments [9], informed agents [8], inter-particle bonding forces [22], general couplings [13, 15, 16, 21], collision-avoidance [1, 7], hierarchical and rooted leadership structures [6, 20, 24], and mean-field limit system [3, 5, 11, 12, 15, 17], etc. However, so far most of the literature deals with symmetric or triangular interaction topologies, and analytical flocking analysis heavily relies on these special structures of network topologies. As far as the authors know, flocking dynamics for the continuous-time C-S model (1.1) with non-symmetric and non-triangular topology has not been studied systematically. Many biological systems exhibiting the flocking behavior in our biological systems do have such non-symmetric and non-triangular network structures (see discussion in Remark 2.1). Recently, Li and Xue [20] proposed a discrete-time C-S model with a new network structure called rooted leadership (in short RL), which partially generalizes the hierarchical leadership (HL) [24], and they established the flocking estimates for their proposed model by combining the self-boundedness argument [10] and matrix analysis technique [30]. In contrast, for the continuous-time case (1.1), the techniques employed for the discrete-time case do not work. Actually, to apply the self-boundedness argument requires a prior estimate for the decay of velocity mismatch. For the RL case in discrete-time setting, this can be done by matrix analysis techniques [20]. However, for the continuous-time case, we cannot obtain an effective estimate to serve the self-boundedness argument. Thus it is still an open problem to establish the flocking estimates for the system (1.1) under RL.

The main result of this paper is to establish the flocking estimates for the continuous-time C-S model (1.1) in a large coupling limit ($\varepsilon \to 0$). In this situation, the system (1.1) can be recast as a singular perturbation problem, so we can use Tikhonov's theory [25] to derive a qualitative description for the flocking dynamics of the C-S type systems under RL.

The rest of this paper consists of three sections. In Section 2, we review the RL topology, Hurwitz stability, and present our main result. In Section 3, we present a proof of our main result by applying Tikhnov's theorem and provide several numerical

simulations illustrating the flocking behavior in a large-coupling limit. Finally Section 4 is devoted to a brief summary of our main result.

2. A framework and statement of a main result. In this section, we discuss a framework and main result on the asymptotic behavior in a large-coupling limit ($\varepsilon \to 0$).

2.1. *Review on RL topology.* In most of the literature on C-S type models, the communication weight ψ_{ji} is assumed to be dependent on the metric distance, say, the relative position between agents:

$$\psi_{ij}(x) = \psi(\|x_j - x_i\|) > 0, \qquad i, j = 0, 1, \ldots, N, \quad i \neq j;$$

for example, the following choices for ψ were used in the literature [10, 28]:

$$\psi^{cs}(s) := \frac{1}{(1+s^2)^\beta}, \quad \beta \geq 0; \qquad \psi^v(s) := \begin{cases} 1, & \text{if } \|x_i - x_j\| \leq R, \\ 0, & \text{if } \|x_i - x_j\| > R, \end{cases}$$

where R is the range of communication. However, as can be seen in a recent experiment [4], the communication weight ψ_{ji} can be dependent on the topological distance rather than on metric distance, say, some fixed finite-number of closest neighboring agents. For example, let $\mathcal{L}(i)$ be the set of neighbors of the i-th agent, i.e., field agents communicating with the i-th agent. In this case, the topological distance ψ can be defined as follows:

$$\psi_{ij}(x) = \begin{cases} 1, & \text{if } j \in \mathcal{L}(i), \\ 0, & \text{otherwise.} \end{cases}$$

We now recall the concept of rooted leadership topology introduced in [20].

Definition 2.1 (Rooted leadership [20]). An $(N+1)$-flock $\{u_0, u_1, \ldots, u_N\}$ is said to be under *rooted leadership* if there exists a root agent, say u_0, which has no incoming directed paths from other agents, whereas every other agent has a directed path from u_0; i.e., the root agent u_0 has no information inflow and has information outflow (directly or indirectly) reaching to any other agent (see Figure 1 for a simple example).

Remark 2.1. 1. We note that the above definition of RL appears different from the original one presented in [20]: every other agent has a path "from" u_0, whereas in [20] every other agent has a path leading "to" u_0 . This is because in this paper we try to interpret the associated graph as an information flow chart. Precisely, the arc from u_j to u_i means that u_j sends some information to u_i; i.e., the i-th agent is influenced by the state of the j-th agent. Throughout this paper, we will adopt this point of view to interpret ψ_{ij}.

2. Note that the hierarchy inside the group $\{u_1, u_2, \ldots, u_N\}$ is dropped. As a natural example for an RL flock, consider a large flock of flying birds or a herd of moving cattle following a leader, where some individuals may be so close to others that they can see and influence each other. Obviously, the hierarchy structure may fail to work in this situation. However, this flock is indeed under RL. It is also easy to see that the HL flock is a special case of the RL flock. Actually, the RL structure is the most general case among flocks with a single "leadership". The adjacency matrix of an HL flock is triangular, but the adjacency matrix of an RL flock is not necessarily triangular or symmetric. Thus the previous methods based on self-bounding arguments, the Lyapunov functional approach and the ℓ^2-energy method do not work for this RL case.

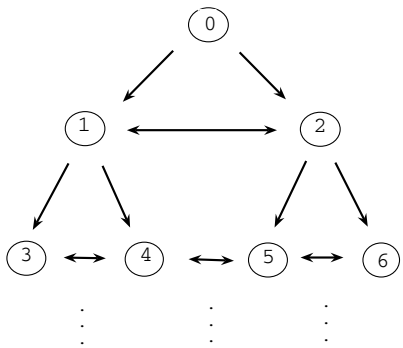


FIG. 1. An example of RL topology with seven agents.

From Definition 2.1, the $(N+1)\times(N+1)$ adjacency matrix Ψ takes the form

$$\Psi = \begin{pmatrix} \psi_{00}(x) & 0 & 0 & \cdots & 0 \\ \psi_{10}(x) & \psi_{11}(x) & \psi_{12}(x) & \cdots & \psi_{1N}(x) \\ \psi_{20}(x) & \psi_{21}(x) & \psi_{22}(x) & \cdots & \psi_{2N}(x) \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ \psi_{N0}(x) & \psi_{N1}(x) & \psi_{N2}(x) & \cdots & \psi_{NN}(x) \end{pmatrix}.$$

The first row has all entries ψ_{0j} $(j \geq 1)$ equal to zero, because the global leader u_0 cannot be influenced by any other agents. For a given solution $\{(x(t), v(t))\}$, we write

$$\psi_{ij}(t) := \psi_{ij}(x(t)), \qquad d_i(t) := \sum_{j=0, j\neq i}^{N} \psi_{ij}(t). \tag{2.1}$$

Since every agent, except u_0, is influenced by at least one agent, it is easy to see that

$$d_0 = 0, \quad d_i > 0, \ i = 1, \ldots, N.$$

Note that the velocity of the global leader u_0 is constant along the dynamics (1.1), i.e.,

$$v_0(t) = v_{00}, \quad t > 0.$$

Thus, it is natural to consider deviations (fluctuations) from the state of the global leader u_0: For a given phase configuration $\{(x_i, v_i)\}_{i=0}^N$, we set

$$\hat{x}_i := x_i - x_0, \quad \hat{v}_i := v_i - v_0, \quad i = 1, \ldots, N,$$
$$\hat{x} := (\hat{x}_1, \ldots, \hat{x}_N), \quad \hat{v} := (\hat{v}_1, \ldots, \hat{v}_N).$$

Then by (1.1) we have

$$\begin{aligned}
\frac{d\hat{x}_i}{dt} &= \frac{dx_i}{dt} - \frac{dx_0}{dt} = v_i - v_0 = \hat{v}_i, \\
\frac{d\hat{v}_i}{dt} &= \frac{dv_i}{dt} - \frac{dv_0}{dt} = \frac{1}{\varepsilon}\sum_{j=0}^{N} \psi_{ij}(v_j - v_i) \\
&= \frac{1}{\varepsilon}\sum_{j=0}^{N} \psi_{ij}(\hat{v}_j - \hat{v}_i) = \frac{1}{\varepsilon}\Big(\sum_{j=1, j\neq i}^{N} \psi_{ij}\hat{v}_j - \sum_{j=0, j\neq i}^{N} \psi_{ij}\hat{v}_i \Big).
\end{aligned}$$

We now set

$$L_{\hat{x}(t)} = \begin{pmatrix} d_1(t) & -\psi_{12}(t) & \cdots & -\psi_{1N}(t) \\ -\psi_{21}(t) & d_2(t) & \cdots & -\psi_{2N}(t) \\ \cdots & \cdots & \ddots & \cdots \\ -\psi_{N1}(t) & -\psi_{N2}(t) & \cdots & d_N(t) \end{pmatrix}; \tag{2.2}$$

then the system (1.1) can be rewritten as

$$\frac{d\hat{x}}{dt} = \hat{v}, \quad \frac{d\hat{v}}{dt} = -\frac{1}{\varepsilon} L_{\hat{x}(t)}\hat{v}, \quad t > 0. \tag{2.3}$$

The matrix $L_{\hat{x}(t)}$ is called the reduced Laplacian.

From now on, we will focus on the reduced system (2.3). In this case, to show the asymptotic flocking of the agent system is equivalent to proving that:

$$\sup_{t>0} \max_{1\le i\le N} |\hat{x}_i(t)| < \infty, \qquad \lim_{t\to\infty} \max_{1\le i\le N} |\hat{v}_i(t)| = 0.$$

To study the behavior of the system (2.3), the time-dependent matrix $L_{\hat{x}(t)}$ will play a key role as in the discrete-time system.

2.2. *Hurwitz stability.* In this part, we study the Hurwitz stability of a linear autonomous system in $\mathbb{R}^N$:

$$\frac{dy}{dt} = Ay, \quad t > 0, \qquad y(0) = y_0, \tag{2.4}$$

where A is the constant matrix. We now recall the definition of Hurwitz stability for the system (2.4) as follows.

Definition 2.2. The coefficient matrix $A \in \mathbf{M}_{N\times N}(\mathbb{R})$ in (2.4) is *Hurwitz stable* if and only if all eigenvalues of the matrix A have strictly negative real part.

Remark 2.2. It is clear that Hurwitz stability implies the global exponential stability of the trivial equilibrium $y_e = 0$.

In general, when the size of the matrix is sufficiently large, i.e., $N \gg 1$, the calculation of eigenvalues can be very demanding. Thus it is interesting to look for simpler criteria in terms of elements in the coefficient matrix A. For some class of square non-negative matrices, we provide a handy criterion for the Hurwitz stability. For this, we first introduce a set of square matrices:

$$\Sigma_0 := \{A = (a_{ij}) \in \mathbf{M}_{N\times N}(\mathbb{R}) : \ a_{ij} \ge 0 \ (i \neq j) \text{ and } \sum_{j=1}^{N} a_{ij} = a_{ii} + d_i \le 0\}.$$

Note that for $A \in \Sigma_0$, $a_{ii} < 0$. On the other hand, for a given matrix $A = (a_{ij}) \in \Sigma_0$, we can define its associated directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ without self-loops:

$$\mathcal{V} := \{u_1, u_2, \ldots, u_N\} \quad \text{and} \quad (u_j, u_i) \in \mathcal{E},\ (i \neq j) \iff a_{ij} \neq 0.$$

We next partition the vertex set $\mathcal{V}$ into two subsets depending on the sign of row sums.

DEFINITION 2.3. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph associated with the matrix $A \in \Sigma_0$.

(1) the vertex u_i is a negative-sum vertex $\iff \displaystyle\sum_{j=1}^{N} a_{ij} < 0$,

(2) the vertex u_i is a zero-sum vertex $\iff \displaystyle\sum_{j=1}^{N} a_{ij} = 0$.

In [29] the authors studied non-negative matrices; i.e., all elements are non-negative and characterize a class of matrices for which its associated discrete-time linear system is asymptotically stable by describing the distribution of non-zero elements. Later, an alternative characterization in the terminologies of graph theory, which seems to be much simpler, was proposed in [30]. This idea plays a key role in the study of Cucker-Smale flocking under RL in a discrete-time setting [20]. For the reader's convenience, we briefly review the definition of (sp) matrix, coined from "**s***im***p***le* matrix" in [29, 30]. We set

$$\mathcal{S} := \Big\{A = (a_{ij}) \in \mathbf{M}_{N\times N}(\mathbb{R}) :\ a_{ij} \geq 0,\ \sum_{j=1}^{N} a_{ij} \leq 1\Big\}.$$

We associate a matrix $A \in \mathcal{S}$ with a weighted digraph with vertices $u_1, u_2, \ldots, u_N$ such that $(u_j, u_i) \in \mathcal{E}$ is a directed arc with the weight a_{ij}.

DEFINITION 2.4 ([30]). Let $A = (a_{ij}) \in \mathcal{S}$ and $(\mathcal{V}, \mathcal{E})$ be the associated directed graph with the matrix A.

(1) The vertex u_i is called non-saturated (saturated) if and only if the i-row sum is strictly less than 1 (exactly 1).

(2) The matrix A is an (sp) matrix if and only if every saturated vertex has a directed path from a non-saturated vertex.

We may expect some analogue for the continuous-time systems (2.3) to work for our analysis. For the continuous-time case (2.4), we have a criterion (Proposition 2.1) to determine whether a matrix $A \in \Sigma_0$ is Hurwitz stable or not. We would like to acknowledge that the matrix of this kind has been shown to be non-singular, which implies asymptotic stability, in different terminologies [18]. However, we prefer to give a simple and different proof in terms of our jargons to make this paper self-contained and more readable. We first give a lemma which characterizes the relationship between the graph representation of A and its discretization.

LEMMA 2.1. Let $A = (a_{ij}) \in \Sigma_0$. Then there exists an $h > 0$ such that

(1) $\mathrm{Id} + hA \in \mathcal{S}$,

(2) the non-saturated vertices of $\mathrm{Id} + hA$ are the negative-sum vertices of A and vice versa.

Proof. Suppose that

$$A = (a_{ij}) \in \Sigma_0, \quad \text{i.e.,} \quad a_{ij} \geq 0,\ i \neq j, \quad \sum_{j=1}^{N} a_{ij} \leq 0.$$

Choose a small $h > 0$ such that

$$h < \frac{1}{\max_i\{|a_{ii}| + 1\}};$$

then the matrix $\mathrm{Id} + hA$ is a non-negative matrix and its row sums are dominated by 1, i.e.,

$$\mathrm{Id} + hA \in \mathcal{S}.$$

Note that

$$\sum_{j=1}^{N} a_{ij} < 0 \Longleftrightarrow 1 + h\sum_{j=1}^{N} a_{ij} < 1,$$

which immediately implies that the non-saturated vertices of $\mathrm{Id} + hA$ are exactly the negative-sum vertices of A. □

PROPOSITION 2.1. Let $A = (a_{ij}) \in \Sigma_0$. Then A is Hurwitz stable if and only if each zero-sum vertex has a directed path from a negative-sum vertex.

Proof. • ($\Longleftarrow$) Suppose that each zero-sum vertex has a directed path from a negative-sum vertex. By Lemma 2.1, there exists $h > 0$ such that the matrix $\mathrm{Id} + hA$ is a non-negative matrix. Observe the following implication:

$$\begin{aligned}
&\mathrm{Id} + hA \text{ is an } (sp) \text{ matrix},\\
&\Longleftrightarrow |1 + h\lambda_i(A)| < 1, \quad i = 1, 2, \ldots N,\\
&\Longrightarrow \Re(\lambda_i(A)) < 0, \quad i = 1, 2, \ldots N,\\
&\Longleftrightarrow A \text{ is Hurwitz stable},
\end{aligned}$$

where $\lambda_i(A)$ denotes an eigenvalue of A. Therefore, to prove the Hurwitz stability of A, it suffices to show that $\mathrm{Id} + hA$ is an (sp) matrix. On the other hand, it follows from Lemma 2.1 that the non-saturated vertices of $\mathrm{Id} + hA$ are exactly the negative-sum vertices of A. By Definition 2.4, we see that $\mathrm{Id} + hA$ is an (sp) matrix.
• ($\Longrightarrow$) Suppose $A \in \Sigma_0$ has a zero-sum vertex which cannot be linked to any negative-sum vertex. We use $\mathcal{V}_\infty$ to denote the set of such vertices. Let us relabel the nodes set $\{1, 2, \ldots, N\}$ such that $\mathcal{V}_\infty = \{\underbrace{N - (m-1), \ldots, N-1, N}_{m}\}$. Then the matrix A can be expressed in partitioned form as

$$A = \begin{bmatrix} D_{N-m} & * \\ \mathbf{0} & D_m \end{bmatrix},$$

where D_k represents a square matrix of order k, $\mathbf{0}$ is an $m \times (N-m)$ matrix with all elements being 0. Since the row sums of D_m are all 0, $(0, \ldots, 0, \underbrace{1, \ldots, 1}_{m})^\top$ is an eigenvector associated to the eigenvalue 0. This means that A is not Hurwitz stable. □

2.3. *Description of main results.* Recall that $(\hat{x}, \hat{v})$ satisfies

$$\frac{d\hat{x}}{dt} = \hat{v}, \quad \frac{d\hat{v}}{dt} = -\frac{1}{\varepsilon} L_{\hat{x}(t)} \hat{v}, \tag{2.5}$$

subject to initial data:

$$(\hat{x}_i, \hat{v}_i)(0) = (\hat{x}_{i0}, \hat{v}_{i0}) \in \mathbb{R}^{2d}, \quad i = 1, \ldots, N. \tag{2.6}$$

Note that the matrix $L_{\hat{x}(t)}$ in (2.5) is defined by (2.2), and $L_{\hat{x}(t)}$ is applied to $\mathbb{R}^{Nd}$ (instead of $\mathbb{R}^N$) via the d-dimensions individually. The main result of this paper can be summarized as follows.

THEOREM 2.1. The limiting dynamics as $\varepsilon \to 0$ for the reduced Cucker-Smale system (2.5)-(2.6) under rooted leadership on $0 \le t \le 1$ is given by

$$\hat{x}_i = \hat{x}_{i0}, \quad \hat{v}_i = 0, \quad i = 1, \ldots, N.$$

REMARK 2.3. 1. Since $(\hat{x}, \hat{v})$ represents the relative phase of particles $1, 2, \ldots, N$ relative to the global leader 0, Theorem 2.1 implies that the limiting dynamics as $\varepsilon \to 0$ for system (2.5) is given by the constant translational motion:

$$v_i(t) = v_{00}, \quad x_i(t) = x_{i0} + v_{00} t, \qquad i = 0, 1, \ldots, N.$$

2. Theorem 2.1 asserts that in the limiting dynamics all particles attain a velocity alignment; i.e., they form a flock and move with the leader's velocity. Note that the perturbation parameter ε going to zero means that the coupling strength which is inverse of ε goes to infinity. Thus, the singular limit $\varepsilon \to 0$ corresponds to the infinite coupling strength limit.
3. In the proof of Theorem 2.1 we only require

$$\psi_{ij}(x) > 0, \ \forall\, x,$$

if the information flow from u_j to u_i exists. In Cucker-Smale and many other works, it is assumed to be

$$\psi_{ij}(x) = \frac{1}{(1 + \|x_i - x_j\|^2)^\beta}.$$

Under this particular choice, it is non-increasing about $\|x_i - x_j\|$ and it has a positive lower bound in the evolution. Therefore, in some sense, our assumption is less restrictive.

3. A large coupling limit.

In this section, we study the continuous-time Cucker-Smale type flocking model under RL by applying Tikhonov's theory.

3.1. *Review of Tikhonov's theorem.* For the reader's convenience, we recall the classic theorem on the singular perturbation limit due to Tikhonov [25]. Consider the slow-fast dynamical system:

$$\begin{aligned} \frac{dy_i}{dt} &= f_i(y, z, t), \quad i = 1, 2, \ldots, n, \\ \mu_j \frac{dz_j}{dt} &= F_j(y, z, t), \quad j = 1, 2, \ldots, m, \end{aligned} \tag{3.1}$$

where $y = (y_1, \dots, y_n)$, $z = (z_1, \dots, z_m)$, and μ_j are small positive parameters depending on a parameter μ in such a way that

$$\lim_{\mu\to 0} \mu_j(\mu) = 0, \quad \lim_{\mu\to 0} \frac{\mu_{j+1}}{\mu_j} = 0 \text{ or } 1.$$

In the language of a singular perturbation theory, y_i's are slow variables and z_i's are fast variables.

THEOREM 3.1 (Tikhonov [25]). Suppose the following conditions hold.

(1) The degenerate system obtained by setting all $\mu_j = 0$,

$$\frac{dy_i}{dt} = f_i(y, z, t), \quad F_j(y, z, t) = 0, \tag{3.2}$$

has continuous solutions.

(2) The roots $z_j = \phi_j(y, t)$ of $F_j(y, z, t) = 0$ have continuous first partial derivatives and are exponentially asymptotically stable for the fast system

$$\frac{dz_j}{dt} = F_j(y, z, t), \tag{3.3}$$

where we treat y as a constant.

Then as $\mu \to 0$, the solutions of (3.1) tend to the corresponding solutions of the degenerate system (3.2) with the initial data (y_i^0, z_j^0, t^0), and this convergence is uniform in a closed interval $[t_1, T_*]$ for any $t_1 > t^0$ and $T_* > t_1$.

3.2. *Large coupling limit.* We recall the reduced C-S system (2.5) as follows:

$$\frac{d\hat{x}}{dt} = \hat{v}, \qquad \varepsilon\frac{d\hat{v}}{dt} = -L_{\hat{x}(t)}\hat{v}, \tag{3.4}$$

and consider its fast system:

$$\frac{d\hat{v}}{dt} = -L_{\hat{x}}\hat{v}. \tag{3.5}$$

Next we use Proposition 2.1 to study the asymptotic behavior of the fast system (3.5) where $\hat{x}$ is frozen at some fixed point $\hat{x}_f$.

LEMMA 3.1. The trivial equilibrium $\hat{v}_e = 0$ to fast system (3.5) is globally exponentially asymptotically stable.

Proof. We recall that $L_{\hat{x}(t)}$ is given by (2.1)-(2.2), which implies $-L_{\hat{x}_f} \in \Sigma_0$. To prove the desired result it suffices to show that $-L_{\hat{x}_f}$ is Hurwitz stable. First, by (2.1)-(2.2) we see that u_i is a negative-sum vertex in the directed graph of $-L_{\hat{x}_f}$ if and only if the agent u_i is directly led by u_0. Actually, if u_i is directly led by u_0, then $\psi_{i0}(\hat{x}_f) > 0$ and thus

$$\sum_{j=1}^{N}(-L_{\hat{x}_f})_{ij} = -d_i(\hat{x}_f) + \sum_{j=1, j\neq i}^{N} \psi_{ij}(\hat{x}_f) = -\psi_{i0}(\hat{x}_f) < 0,$$

and vice versa. Second, we recall the definition of an RL flock, which declares that every other agent has a directed path from u_0. This certainly means that every other agent either has a path from some agent directly led by u_0 or it is directly led by u_0. Combining the two observations above, we see that in the directed graph of $-L_{\hat{x}_f}$, every

zero-sum vertex has a directed path from a negative-sum vertex. By Proposition 2.1, it immediately follows that $-L_{\hat{x}_f}$ is Hurwitz stable. □

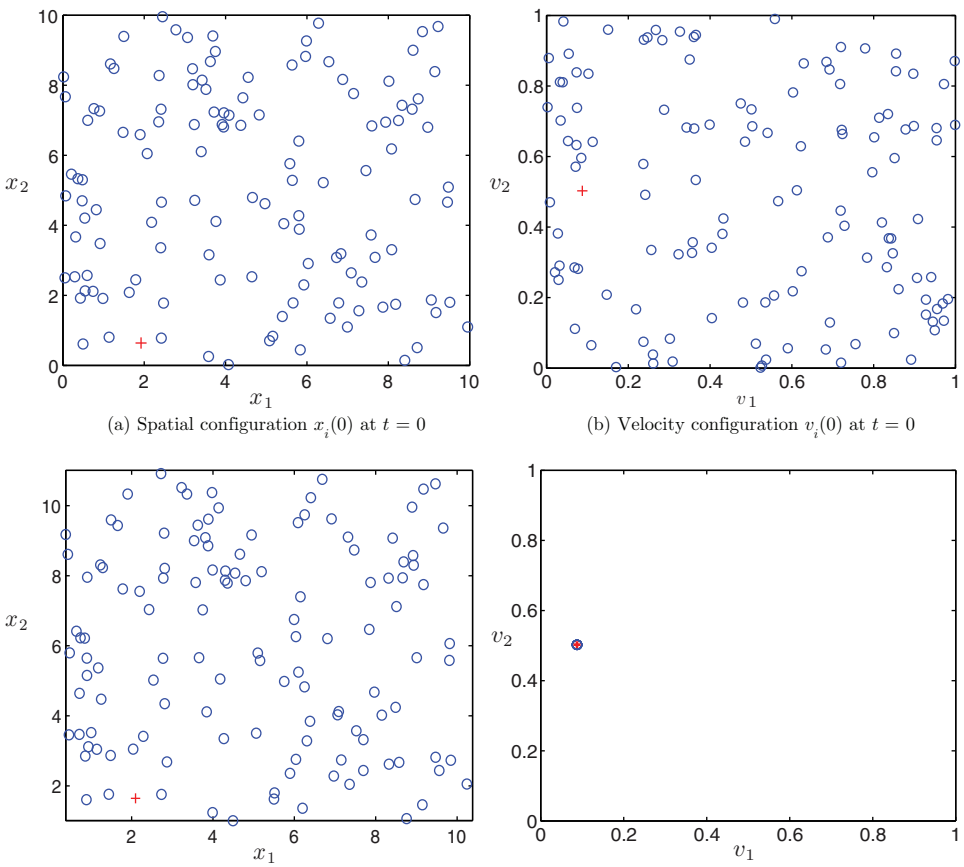
Proof of Theorem 2.1. Note that the solutions for the degenerate system

$$\frac{d\hat{x}}{dt} = \hat{v}, \qquad -L_{\hat{x}(t)}\hat{v} = 0,$$

are exactly

$$\hat{x}_i = \hat{x}_{i0}, \quad \hat{v}_i = 0, \quad i = 1, \ldots, N.$$

We observe that the conditions of Theorem 3.1 are true for (3.4). By applying Theorem 3.1 we immediately obtain the desired result in Theorem 2.1. □



(a) Spatial configuration $x_i(0)$ at $t = 0$

(b) Velocity configuration $v_i(0)$ at $t = 0$

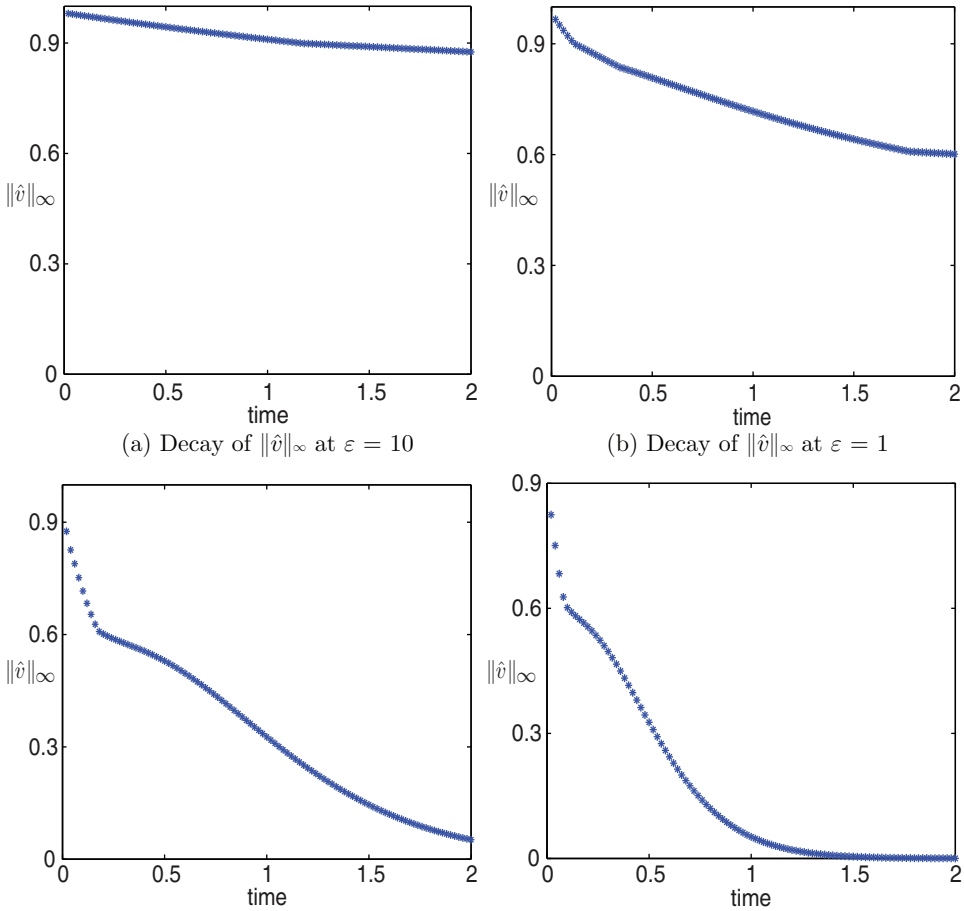
(c) Spatial configuration $x_i(2)$ at $t = 2$ for $\varepsilon = 0.05$

(d) Velocity configuration $v_i(2)$ at $t = 2$ for $\varepsilon = 0.05$

FIG. 2. Spatial and velocity configurations at $t = 0, 2$ for $\varepsilon = 0.05$ and 127 agents. The spatial position and velocity of the root agent are denoted by a red cross. *(Color available online.)*

3.3. *Numerical simulations.* In this part, we present several numerical simulations based on the 4th-order Runge Kutta method to illustrate the contents of Theorem 2.1

In Figure 2 (a)-(d), we see that for $\varepsilon = 0.05$, the initial spatial configuration at time $t = 2$ is pretty much the same as spatial configuration at $t = 0$. In contrast, the velocity configuration at time $t = 2$ almost shrinks to zero.



(a) Decay of $\|\hat{v}\|_\infty$ at $\varepsilon = 10$

(b) Decay of $\|\hat{v}\|_\infty$ at $\varepsilon = 1$

(c) Decay of $\|\hat{v}\|_\infty$ at $\varepsilon = 0.1$

(d) Decay of $\|\hat{v}\|_\infty$ at $\varepsilon = 0.05$

FIG. 3. Evolution of $\|\hat{v}\|_\infty$ at $\varepsilon = 10, 1, 0.1, 0.05$.

For numerical simulations, we take 127 agents and use the following parameters and ansatz for the communication weight:

$$d = 2, \quad \psi_{ij}(x) = \frac{\chi_{ij}}{(1 + |x_j - x_i|^2)^{\frac{1}{4}}}, \quad \chi_{ij} = \begin{cases} 1, & \text{if } j \in \mathcal{L}(i), \\ 0, & \text{otherwise}, \end{cases}$$

where the connectivity χ_{ij} follows the RL topology given by Figure 1, which can obviously be extended for 127 agents instead of 7 agents. The initial spatial and velocity configurations are randomly chosen from the uniform box $[0, 10]^2$ and $[0, 1]^2$ respectively as in Figure 2 (a) and (b). With these sets of initial data, we perform four numerical

simulations with different ε:

$$\varepsilon = 10, 1, 10^{-1}, 5 \times 10^{-2}$$

during the time-interval $[0, 2]$.

In Figure 3 (a)-(d), we see that the ℓ_∞-norms of the velocity configurations at time $t = 2$ show the relaxation to zero as $\varepsilon \to 0$. This perfectly coincides with the analytical result in Theorem 2.1.

4. Conclusion. In this paper, we presented the flocking behavior of the Cucker-Smale type model under rooted leadership in a large coupling limit. Since the symmetry and hierarchy are absent in our situation, we could not get an effective estimate for the decay of the velocity mismatch from that of the global leader in finite coupling. However, in the large coupling limit as $\varepsilon \to 0$, we can invoke Tikhonov's theory for the slow-fast dynamical systems. The advantage of introducing the fast-slow subsystems is that the fast equation itself becomes a time-invariant linear system which can be shown to be asymptotically stable. By the direct application of Tikhonov's theory, we obtain the limiting flocking behavior for the perturbed systems.

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