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SINGULAR INVARIANT INTEGRALS FOR ELASTIC BODY WITH DELAMINATED THIN ELASTIC INCLUSION

By

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Abstract. We consider an equilibrium problem for a $2D$ elastic body with a thin elastic inclusion. It is assumed that the inclusion is partially delaminated, therefore providing the presence of a crack. Inequality type boundary conditions are imposed at the crack faces to prevent a mutual penetration of the faces. Differentiability properties of the energy functional with respect to the crack length are analyzed. We prove an existence of the derivative and find a formula for this derivative. It is shown that the formula for the derivative can be written in the form of a singular invariant integral.

1. Introduction. To analyze composite materials, one has to consider mathematical models of deformable bodies with elastic or rigid inclusions and cracks. It is known that inclusions can be divided into thin and volume ones. The terminology "thin" inclusion is used in the case when the inclusion dimension is less than a dimension of the body. On the other hand, among thin inclusions we can distinguish rigid and elastic ones. Thin rigid inclusions can be viewed as "unticracks". On the other hand, cracks can be viewed as thin inclusions of zero rigidity. There are different approaches to model cracks in solids. The classical approaches are characterized by linear boundary conditions at the crack faces [1, 11, 12]. Suitable linear models allow the opposite crack faces to penetrate each other, which demonstrates a shortcoming of the model from a mechanical standpoint. During the last few years, a crack theory with non-penetration conditions at the crack faces has been under active study. This theory is characterized by inequality type boundary conditions that lead to free boundary value problems. The book [2] contains results on crack models with the non-penetration conditions for a wide class of constitutive laws. The elastic behavior of bodies with cracks and inequality type boundary conditions are analyzed in [3, 6, 10, 15], etc. In particular, a differentiability of energy functionals with respect to the crack length is investigated. Finding the derivatives of the energy

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functionals is important from the standpoint of the Griffith rupture criterion. We can mention the publications [2, 3, 9, 15], among many others. The considered inclusions may be delaminated, hence the crack approach with non-penetration conditions is to be used. In the case of rigid inclusions with delaminations, a new type of boundary value problem and non-local boundary condition appears. Existence theorems and qualitative properties of solutions in equilibrium problems for elastic bodies with rigid inclusions can be found in [3–5, 13, 14]. On the other hand, in many cases, derivatives of the energy functional can be rewritten in the form of invariant integrals over curves surrounding crack tips. As for the linear crack model, we refer the reader to the paper [8]. Invariant integrals for the non-linear crack model can be found in [2, 3, 9, 15].

In this paper, we consider a model of a thin elastic inclusion inside an elastic body recently proposed in [7]. A behavior of the inclusion is modeled by the Kirchhoff-Love equations. The inclusion may be delaminated, therefore providing the presence of a crack. To exclude a mutual penetration between the crack faces, non-linear boundary conditions of inequality type are considered at the cracks. We prove an existence of the derivative of the energy functional with respect to the crack length and write the formula for this derivative in the form of an invariant integral. It turns out that the invariant integral consists of two terms, regular and singular ones. Different geometries of curves surrounding the crack tip are considered.

2. Problem formulation. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary Γ such that $\bar{\gamma} \subset \Omega$, $\gamma = (-1, 1) \times \{0\}$. Denote by $\nu = (0, 1)$ the unit normal vector to γ, $\tau = (1, 0)$, and set $\Omega_\gamma = \Omega \setminus \bar{\gamma}$, $\gamma_0 = (-1, 0) \times \{0\}$; see Fig. 1.

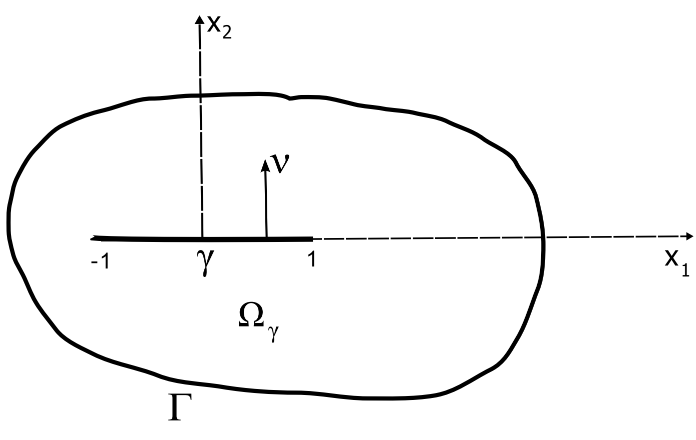

Fig. 1

In what follows the domain Ω_γ represents a region filled with an elastic material, and γ is an elastic inclusion with specified properties. In particular, we consider γ as a bending beam incorporated in the elastic body. We assume that the beam is delaminated at γ_0^+, thus providing a presence of a crack, and there is no delamination at $\gamma \setminus \gamma_0$.

Let $A = \{a_{ijkl}\}, i, j, k, l = 1, 2$, be a given elasticity tensor with the usual properties of symmetry and positive definiteness,

$$a_{ijkl} = a_{jikl} = a_{klij}, \ \ i, j, k, l = 1, 2, \ a_{ijkl} = const,$$
$$a_{ijkl}\xi_{ij}\xi_{kl} \geq c_0|\xi|^2 \quad \forall \xi_{ji} = \xi_{ij}, \ c_0 = const > 0 \ .$$

Summation convention over repeated indices is used; all functions with two lower indices are assumed to be symmetric in those indices.

The equilibrium problem for the body Ω_γ with the elastic inclusion γ is formulated as follows. For given external forces $f = (f_1, f_2) \in C^1(\bar{\Omega})^2$ acting on the body, we want to find a displacement field $u = (u_1, u_2)$, a stress tensor $\sigma = \{\sigma_{ij}\}, i, j = 1, 2$, and a (vertical) displacement w of the beam, defined in $\Omega_{\gamma\setminus\gamma_0}, \Omega_\gamma, \gamma$, respectively, such that

$$-\text{div}\,\sigma = f \quad \text{in} \quad \Omega_\gamma, \tag{1}$$
$$\sigma - A\varepsilon(u) = 0 \quad \text{in} \quad \Omega_{\gamma\setminus\gamma_0}, \tag{2}$$
$$w_{,1111} = [\sigma_\nu] \quad \text{on} \quad \gamma, \tag{3}$$
$$u = 0 \quad \text{on} \quad \Gamma, \tag{4}$$
$$w_{,11} = w_{,111} = 0 \quad \text{for} \quad x_1 = -1, 1, \tag{5}$$
$$w = u_\nu, \ [\sigma_\tau] = 0 \quad \text{on} \quad \gamma \setminus \gamma_0, \tag{6}$$
$$[u_\nu] \geq 0, \ w = u_\nu^-, \ \sigma_\nu^+ \leq 0, \ \sigma_\nu^+[u_\nu] = 0, \ \sigma_\tau^\pm = 0 \quad \text{on} \quad \gamma_0. \tag{7}$$

Here $[v] = v^+ - v^-$ denotes the jump of v on γ, and $v^\pm$ are the traces of v on the crack faces $\gamma^\pm$ (the signs $\pm$ correspond to the positive and the negative directions of ν). Then, $\varepsilon(u) = \{\varepsilon_{ij}(u)\}$ is the strain tensor, $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ i, j = 1, 2$; $\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j), \ \sigma_\nu = \sigma_{ij}\nu_j\nu_i, \ \sigma_\tau = \sigma_{ij}\nu_j\tau_i, \ u_\nu = u\nu$. In our setting $\nu = (0, 1)$ so that $\sigma\nu = (\sigma_{12}, \sigma_{22})$, $\sigma_\nu = \sigma_{22}$, $\sigma_\tau = \sigma_{12}$, and $u_\nu = u_2$.

Relations (1), (3) are the equilibrium equations for the elastic body and the inclusion, while (2) represents Hooke's law. The first equation of (6) represents the non-debonding condition since the vertical displacements of the elastic body coincide with the displacement of the beam γ. The first inequality in (7) provides a mutual non-penetration between the crack faces. The second relation of (7) shows that the inclusion displacement coincides with the vertical displacement of the elastic body at γ_0^-.

First, we provide a variational formulation of the problem (1)-(7). Introduce the set of admissible displacements

$$K_0 = \{(u, w) \in H^1_\Gamma(\Omega_{\gamma\setminus\gamma_0})^2 \times H^2(\gamma) \mid [u_\nu] \geq 0, \ w = u_\nu^- \text{ on } \gamma_0; \ w = u_\nu \text{ on } \gamma \setminus \gamma_0\}$$

and the energy functional

$$\pi(u, w) = \frac{1}{2} \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \sigma(u)\varepsilon(u)dx - \int\limits_{\Omega_{\gamma\setminus\gamma_0}} fu dx + \frac{1}{2} \int\limits_\gamma w_{,11}^2 dx_1,$$

where the Sobolev space $H^1_\Gamma(\Omega_{\gamma\setminus\gamma_0})$ is defined as

$$H^1_\Gamma(\Omega_{\gamma\setminus\gamma_0}) = \{v \in H^1(\Omega_{\gamma\setminus\gamma_0}) \mid v = 0 \text{ on } \Gamma\},$$

and $\sigma(u) = \sigma$ is defined from the relation (2). For simplicity we write $\sigma(u)\varepsilon(u) = \sigma_{ij}(u)\varepsilon_{ij}(u), fu = f_i u_i$.

The solvability of a problem like (1)-(7) was proved in [7], and we omit the details. To this end a variational approach was used. In particular, the coercivity of the functional π on the set K_0 can be established. The problem (1)-(7) is equivalent to the following minimization problem:

$$\text{find } (u,w) \in K_0 \text{ such that } \pi(u,w) = \inf_{(\bar{u},\bar{w})\in K_0} \pi(\bar{u},\bar{w}). \tag{8}$$

A unique solution of problem (8) exists, and it satisfies the variational inequality

$$(u,w) \in K_0, \tag{9}$$

$$\int\limits_{\Omega_{\gamma\setminus\gamma_0}} \sigma(u)\varepsilon(\bar{u}-u)dx - \int\limits_{\Omega_{\gamma\setminus\gamma_0}} f(\bar{u}-u)dx \tag{10}$$

$$+\int\limits_{\gamma} w_{,11}(\bar{w}_{,11}-w_{,11})dx_1 \geq 0 \quad \forall\, (\bar{u},\bar{w}) \in K_0.$$

3. Derivative of energy functional. We introduce the problem perturbed with respect to (9)-(10). Denote $\gamma_\delta = (-1,\delta)\times\{0\}$, where δ is a positive parameter. We want to find a displacement field $u^\delta = (u_1^\delta, u_2^\delta)$, a stress tensor $\sigma^\delta = \{\sigma_{ij}^\delta\}, i,j = 1,2$, and a (vertical) displacement w^δ of the beam, defined in $\Omega_{\gamma\setminus\gamma_\delta}$, Ω_γ, γ, respectively, such that

$$-\mathrm{div}\sigma^\delta = f \quad \text{in} \quad \Omega_\gamma, \tag{11}$$

$$\sigma^\delta - A\varepsilon(u^\delta) = 0 \quad \text{in} \quad \Omega_{\gamma\setminus\gamma_\delta}, \tag{12}$$

$$w_{,1111}^\delta = [\sigma_\nu^\delta] \quad \text{on} \quad \gamma, \tag{13}$$

$$u^\delta = 0 \quad \text{on} \quad \Gamma, \tag{14}$$

$$w_{,11}^\delta = w_{,111}^\delta = 0 \quad \text{for} \quad x_1 = -1, 1, \tag{15}$$

$$w^\delta = u_\nu^\delta,\ [\sigma_\tau^\delta] = 0 \quad \text{on} \quad \gamma\setminus\gamma_\delta, \tag{16}$$

$$[u_\nu^\delta] \geq 0,\ w^\delta = u_\nu^{\delta -},\ \sigma_\nu^{\delta +} \leq 0,\ \sigma_\nu^{\delta +}[u_\nu^\delta] = 0,\ \sigma_\tau^{\delta\pm} = 0 \quad \text{on} \quad \gamma_\delta. \tag{17}$$

The problem (11)-(17) also admits a variational formulation. To this end, consider the set of admissible displacements

$$K_\delta = \{(u,w) \in H_\Gamma^1(\Omega_{\gamma\setminus\gamma_\delta})^2 \times H^2(\gamma) \mid [u_\nu] \geq 0,\ w = u_\nu^- \text{ on } \gamma_\delta; w = u_\nu \text{ on } \gamma\setminus\gamma_\delta\}$$

and the energy functional

$$\pi^\delta(u,w) = \frac{1}{2}\int\limits_{\Omega_{\gamma\setminus\gamma_\delta}} \sigma(u)\varepsilon(u)dy - \int\limits_{\Omega_{\gamma\setminus\gamma_\delta}} fudy + \frac{1}{2}\int\limits_{\gamma} w_{,11}^2 dy_1.$$

In the domain $\Omega_{\gamma\setminus\gamma_\delta}$ consider a minimization problem: find $(u^\delta, w^\delta) \in K_\delta$ such that

$$\pi^\delta(u^\delta,w^\delta) = \inf_{(\bar{u},\bar{w})\in K_\delta} \pi^\delta(\bar{u},\bar{w}). \tag{18}$$

Problem (18) has a unique solution $(u^\delta, w^\delta) \in K_\delta$ satisfying the variational inequality

$$(u^\delta, w^\delta) \in K_\delta, \tag{19}$$

$$\int\limits_{\Omega_{\gamma\setminus\gamma_\delta}} \sigma(u^\delta)\varepsilon(\bar u - u^\delta)dy - \int\limits_{\Omega_{\gamma\setminus\gamma_\delta}} f(\bar u - u^\delta)dy \tag{20}$$

$$+ \int\limits_\gamma w^\delta_{,11}(\bar w_{,11} - w^\delta_{,11})dy_1 \geq 0 \quad \forall\, (\bar u, \bar w) \in K_\delta.$$

Consider a perturbation $y = \Phi_\delta(x)$ of the domain $\Omega_{\gamma\setminus\gamma_0}$ describing a change of γ_0 along the axis Ox_1:

$$y_1 = x_1 + \delta\theta(x_1, x_2), \quad y_2 = x_2, \quad x \in \Omega_{\gamma\setminus\gamma_0}, \ \ y \in \Omega_{\gamma\setminus\gamma_\delta}. \tag{21}$$

The function $\theta \in C_0^\infty(\mathbb{R}^2)$ is chosen such that $\theta = 1$ in a small neighborhood of the point $(0,0)$, $\operatorname{supp}\theta \cap \{(-1,0)\} = \emptyset$.

There exists $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the transformation (21) is one-to-one between $\Omega_{\gamma\setminus\gamma_0}$ and $\Omega_{\gamma\setminus\gamma_\delta}$. Since γ_0 is rectilinear, the mapping (21) provides a one-to-one mapping between K_δ and K_0.

Next we formulate and prove auxiliary statements to be useful below. Consider a matrix of the transformation (21),

$$\frac{\partial \Phi_\delta}{\partial x} = I + \delta\frac{\partial V}{\partial x},$$

where $V = (\theta, 0)$. The Jacobian of the transformation (21) is as follows:

$$J_\delta(x) = 1 + \delta\theta_{,1}(x), \tag{22}$$

and it is positive for all $\delta \in (0, \delta_0)$. Hence there exists an inverse matrix

$$\Psi_\delta = \Big(\frac{\partial \Phi_\delta}{\partial x}\Big)^{-1},$$

which admits a representation

$$\Psi_\delta = I - \delta\frac{\partial V}{\partial x} + r_1(\delta, x), \ \|r_1(\delta, x)\|_{[W^{1,\infty}_{loc}(\mathbb{R}^2)]^4} = o(\delta). \tag{23}$$

Also we have

$$\frac{d^2}{dx_1^2} = (1 + \delta\theta_{,1})^2\frac{d^2}{dy_1^2} + \delta\theta_{,11}\frac{d}{dy_1}.$$

Hence it follows that

$$\frac{d^2}{dy_1^2} = k_1(\delta)\frac{d^2}{dx_1^2} - k_2(\delta)\frac{d}{dx_1}, \tag{24}$$

where

$$k_1(\delta) = (1 + \delta\theta_{,1})^{-2} = 1 - 2\delta\theta_{,1} + r_2(\delta, x),$$

$$k_2(\delta) = \delta\theta_{,11}(1 + \delta\theta_{,1})^{-3} = \delta\theta_{,11} + r_3(\delta, x),$$

$$\|r_2(\delta, x)\|_{W^{1,\infty}_{loc}(\mathbb{R}^2)} = o(\delta), \quad \|r_3(\delta, x)\|_{W^{1,\infty}_{loc}(\mathbb{R}^2)} = o(\delta).$$

Next we apply the transformation (21) to functions inserted in (20):

$$\int\limits_{\Omega_{\gamma\setminus\gamma_0}} J_\delta(x) a_{ijkl} E_{kl}(\Psi_\delta; u_\delta) E_{ij}(\Psi_\delta; \bar{u} - u_\delta) dx \tag{25}$$

$$+ \int\limits_{\gamma} J_\delta(x)(k_1(\delta) w_{\delta,11} - k_2(\delta) w_{\delta,1})(k_1(\delta)(\bar{w}_{,11} - w_{\delta,11}) - k_2(\delta)(\bar{w}_{,1} - w_{\delta,1})) dx_1$$

$$\geq \int\limits_{\Omega_{\gamma\setminus\gamma_0}} J_\delta(x) f_\delta(\bar{u} - u_\delta) dx \quad \forall (\bar{u}, \bar{w}) \in K_0.$$

Here $v_\delta(x) = v^\delta(\Phi_\delta(x))$, and $E_{ij}(\Psi_\delta; w)$ is the transformed strain tensor

$$E_{ij}(\Psi_\delta; u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}\Psi_{\delta kj} + \frac{\partial u_j}{\partial x_k}\Psi_{\delta ki}\right).$$

Thus, the following statement holds.

THEOREM 1. For all $\delta \in (0, \delta_0)$ the solution $(u^\delta, w^\delta) \in K_\delta$ of problem (20), transformed to $\Omega_{\gamma\setminus\gamma_0}$ with the help of (21), is a unique solution $(u_\delta, w_\delta) \in K_0$ of the variational inequality (25).

By the above presentations (22), (23), (24), we have

$$\int\limits_{\Omega_{\gamma\setminus\gamma_0}} J_\delta(x) a_{ijkl} E_{kl}(\Psi_\delta; \bar{u}) E_{ij}(\Psi_\delta; \bar{w}) dx \tag{26}$$

$$= \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \Big(\sigma_{ij}(\bar{u})\varepsilon_{ij}(\bar{w}) + \delta B_1(V; \bar{u}, \bar{w})\Big) dx + o(\delta) r_1(\bar{u}, \bar{w}),$$

$$\int\limits_{\gamma} J_\delta(x)(k_1(\delta)\bar{w}_{,11} - k_2(\delta)\bar{w}_{,1})(k_1(\delta)\bar{v}_{,11} - k_2(\delta)\bar{v}_{,1}) dx_1 \tag{27}$$

$$= \int\limits_{\gamma} \Big(\bar{w}_{,11}\bar{v}_{,11} + \delta B_2(V; \bar{w}, \bar{v})\Big) dx_1 + o(\delta) r_2(\bar{w}, \bar{v}),$$

$$\int\limits_{\Omega_{\gamma\setminus\gamma_0}} J_\delta(x) f_\delta \bar{u} dx = \int\limits_{\Omega_{\gamma\setminus\gamma_0}} f\bar{u} dx + \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \delta \mathrm{div}(V f_i)\bar{u}_i dx + o(\delta) r_3(\bar{u}), \tag{28}$$

where

$$B_1(V; \bar{u}, \bar{w}) = \mathrm{div}V \cdot \sigma_{ij}(\bar{u})\varepsilon_{ij}(\bar{w}) - \sigma_{ij}(\bar{u}) E_{ij}\left(\frac{\partial V}{\partial x}; \bar{w}\right) - \sigma_{ij}(\bar{w}) E_{ij}\left(\frac{\partial V}{\partial x}; \bar{u}\right), \tag{29}$$

$$B_2(V; \bar{w}, \bar{v}) = -(3\theta_{,1}\bar{w}_{,11}\bar{v}_{,11} + \theta_{,11}\bar{w}_{,1}\bar{v}_{,11} + \theta_{,11}\bar{w}_{,11}\bar{v}_{,1}), \tag{30}$$

and r_1, r_2, r_3 are bounded functionals.

Now we prove a theorem characterizing a continuous dependence on δ of the solution of (19)-(20).

THEOREM 2. Let (u_δ, w_δ) and (u, w) be solutions of (25), (9)-(10), respectively. Then

$$\|u_\delta - u\|_{H^1_\Gamma(\Omega_{\gamma\setminus\gamma_0})^2} \le c\delta, \quad \|w_\delta - w\|_{H^2(\gamma)} \le c\delta, \tag{31}$$

where a constant c does not depend on δ.

Proof. First, we substitute in (25) test functions of the form $(\bar{u}, \bar{w}) = (0, 0)$ and $(\bar{u}, \bar{w}) = 2(u_\delta, w_\delta)$. By (26)-(28), we obtain uniformly in δ:

$$\|u_\delta\|_{H^1_\Gamma(\Omega_{\gamma\setminus\gamma_0})^2} \le c, \quad \|w_\delta\|_{H^2(\gamma)} \le c. \tag{32}$$

Next, we substitute in (25) a test function (u, w); simultaneously, in (10) we substitute a test function (u_δ, w_δ). By (32) and Korn's inequality we derive (31). Theorem 2 is proved. □

Now consider $\pi^\delta(\tilde{u}, \tilde{w})$ and apply the transformation (21), where $(\tilde{u}, \tilde{w}) \in H^1_\Gamma(\Omega_{\gamma\setminus\gamma_\delta})^2 \times H^2(\gamma)$. We have

$$\pi^\delta(\tilde{u}, \tilde{w}) = \pi_\delta(\bar{u}, \bar{w}), \ \tilde{u}(y) = \bar{u}(x), y = \Phi_\delta(x).$$

In view of (26)-(28), the following representation holds:

$$\pi_\delta(\bar{u}, \bar{w}) = \frac{1}{2} \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \sigma_{ij}(\bar{u})\varepsilon_{ij}(\bar{u})dx + \frac{1}{2}\int\limits_\gamma \bar{w}^2_{,11}dx_1 - \int\limits_{\Omega_{\gamma\setminus\gamma_0}} f\bar{u}dx$$

$$+\frac{1}{2}\delta \int\limits_{\Omega_{\gamma\setminus\gamma_0}} B_1(V; \bar{u}, \bar{u})dx + \frac{1}{2}\delta \int\limits_\gamma B_2(V, \bar{w}, \bar{w})dx_1$$

$$-\delta \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \operatorname{div}(Vf_i)\bar{u}_i dx + o(\delta) r_4(\bar{u}, \bar{w}),$$

where r_4 is a bounded functional. We have

$$\pi^\delta(u^\delta, w^\delta) = \pi_\delta(u_\delta, w_\delta).$$

Since K_δ is transformed one-to-one at K_0, as $\delta \to 0$, we obtain

$$\frac{\pi^\delta(u^\delta, w^\delta) - \pi(u, w)}{\delta} = \frac{\pi_\delta(u_\delta, w_\delta) - \pi(u, w)}{\delta} \le \frac{\pi_\delta(u, w) - \pi(u, w)}{\delta}$$

$$\longrightarrow \frac{1}{2} \int\limits_{\Omega_{\gamma\setminus\gamma_0}} B_1(V; u, u)dx + \frac{1}{2}\int\limits_\gamma B_2(V, w, w)dx_1 - \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \operatorname{div}(Vf_i)u_i dx.$$

On the other hand, by (31), as $\delta \to 0$,

$$\frac{\pi^\delta(u^\delta, w^\delta) - \pi(u, w)}{\delta} = \frac{\pi_\delta(u_\delta, w_\delta) - \pi(u, w)}{\delta} \ge \frac{\pi_\delta(u_\delta, w_\delta) - \pi(u_\delta, w_\delta)}{\delta}$$

$$\longrightarrow \frac{1}{2} \int\limits_{\Omega_{\gamma\setminus\gamma_0}} B_1(V; u, u)dx + \frac{1}{2}\int\limits_\gamma B_2(V, w, w)dx_1 - \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \operatorname{div}(Vf_i)u_i dx.$$

We see that the following limit exists:

$$G = \lim_{\delta\to 0} \frac{\pi^\delta(u^\delta, w^\delta) - \pi(u, w)}{\delta}, \tag{33}$$

and that the formula for this limit can be written. The limit (33) provides a derivative of the energy functional $\pi^\delta(u^\delta, w^\delta)$ with respect to δ at the point $\delta = +0$:

$$G = \frac{d\pi^\delta(u^\delta, w^\delta)}{d\delta}\Big|_{\delta=+0}.$$

Hence the following statement is proved.

THEOREM 3. The derivative of the energy functional with respect to the crack length exists, and the following formula takes place:

$$G = \frac{1}{2}\int\limits_{\Omega_{\gamma\setminus\gamma_0}} B_1(V; u, u)dx \tag{34}$$

$$+\frac{1}{2}\int\limits_{\gamma} B_2(V, w, w)dx_1 - \int\limits_{\Omega_{\gamma\setminus\gamma_0}} \operatorname{div}(Vf_i)u_i dx,$$

where (u, w) is the solution of the unperturbed problem (9)–(10), and B_1, B_2 are defined in (29), (30).

REMARKS. 1) It is interesting to compare (34) with the formula obtained in [16] where a contact problem for a $2D$ elastic body and a thin elastic beam is analyzed.

2) In deriving (34) we assumed $\delta > 0$. The case $\delta > 0$ is more interesting from the standpoint of mechanics since the healing of cracks in elastic materials is hardly expected. In fact, formula (34) can be derived in a general case, where δ is a small parameter, $\delta \to 0$.

4. Invariant integrals. In this section we write formula (34) in the form of a singular invariant integral over a curve, assuming a sufficient regularity of the solution. The invariance means an independence on the curve of integration. Assume that $f = 0$ in a neighborhood of the point $(0, 0)$. By $V = (\theta, 0)$, from (34) we have

$$G = \int\limits_{\Omega_\gamma} (\frac{1}{2}\theta_{,1}\sigma_{ij}\varepsilon_{ij} - \sigma_{ij}u_{i,1}\theta_{,j})dx - \frac{1}{2}\int\limits_{\gamma} (3\theta_{,1}w_{,11}^2 + 2\theta_{,11}w_{,1}w_{,11})dx_1. \tag{35}$$

Here we have changed the domain integration $\Omega_{\gamma\setminus\gamma_0}$ by Ω_γ. This was done because we are planning to integrate by parts in (35) and to use the equilibrium equations (1), and the equilibrium equations are fulfilled in Ω_γ. Assume that $\theta = 1$ in the bounded domain with a smooth boundary L, and $\theta = 0$ outside the bounded domain with a smooth boundary M. Hence in (35) we should integrate over the domain located between curves L and M, instead of Ω_γ; see Fig. 2. Denote by $n = (n_1, n_2)$ a unit normal vector to L as is shown in Fig. 2. Providing the integration by parts for the integral included in (35), we derive

$$-\int\limits_{\Omega_\gamma} \sigma_{ij}u_{i,1}\theta_{,j}dx = \int\limits_{\Omega_\gamma} \theta(\sigma_{ij}u_{i,1})_{,j}dx + \int\limits_{L} \sigma_{ij}u_{i,1}n_j ds + \int\limits_{N\cup Q} [\sigma_{ij}u_{i,1}\nu_j\theta]dx_1. \tag{36}$$

Here $N = (c, e) \times \{0\},\ Q = (a, b) \times \{0\}$; see Fig. 2. Also, from (35) we have

$$\int\limits_{\Omega_\gamma} \frac{1}{2}\theta_{,1}\sigma_{ij}\varepsilon_{ij}dx = -\frac{1}{2}\int\limits_{\Omega_\gamma} \theta(\sigma_{ij}\varepsilon_{ij})_{,1}dx - \frac{1}{2}\int\limits_{L} \sigma_{ij}\varepsilon_{ij}n_1 ds. \tag{37}$$

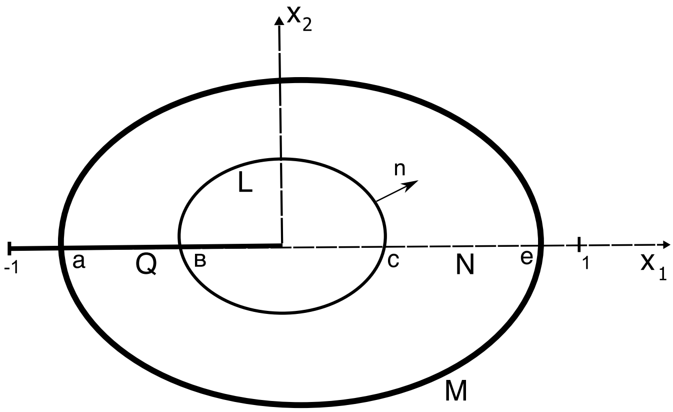

FIG. 2

Let us transform the integral in (36) over N. By $[u_{1,1}] = [u_{2,1}] = 0,\ [\sigma_{12}] = 0,\ w = u_2$ on N, we have

$$\int_N [\sigma_{ij} u_{i,1} \nu_j \theta] dx_1 = \int_N \theta[\sigma_\nu] w_{,1} dx_1. \tag{38}$$

Next we calculate the integral over Q in (36). By $\sigma_{12}^{\pm} = 0$ on Q, it gives

$$\int_Q [\sigma_{ij} u_{i,1} \nu_j \theta] dx_1 = \int_Q \theta[\sigma_{22} u_{2,1}] dx_1,$$

and hence

$$\int_Q [\sigma_{ij} u_{i,1} \nu_j \theta] dx_1 = \int_Q \theta \sigma_{22}^{+} [u_{2,1}] dx_1 + \int_Q \theta[\sigma_{22}] w_{,1} dx_1. \tag{39}$$

Therefore, by the Lemma proved below, we derive

$$\int_Q [\sigma_{ij} u_{i,1} \nu_j \theta] dx_1 = \int_Q \theta[\sigma_\nu] w_{,1} dx_1. \tag{40}$$

The formulas (38), (40) imply

$$\int_{N \cup Q} [\sigma_{ij} u_{i,1} \nu_j \theta] dx_1 = \int_{N \cup Q} \theta[\sigma_\nu] w_{,1} dx_1. \tag{41}$$

Now we transform the integral over γ in (35) by integrating by parts. Observe that, in fact, we should integrate over $N \cup Q$ since, outside of $N \cup Q$, derivatives of θ are equal to zero. Denote

$$F(h) = \frac{1}{2} w_{,11}^2(h) - w_{,1}(h) w_{,111}(h). \tag{42}$$

Then, by (5), direct calculations show that

$$-\frac{1}{2}\int_\gamma (3\theta_{,1}w_{,11}^2 + 2\theta_{,11}w_{,1}w_{,11})dx_1 = -\int_{N\cup Q} \theta w_{,1}w_{,1111}dx_1 + F(c) - F(b). \tag{43}$$

Collecting the formulas (35), (36), (37), (41), (43) we find

$$G = \int_L (-\frac{1}{2}\sigma_{ij}\varepsilon_{ij}n_1 + \sigma_{ij}u_{i,1}n_j)ds + \int_{N\cup Q} \theta(-w_{,1}w_{,1111} + [\sigma_\nu]w_{,1})dx_1 + F(c) - F(b). \tag{44}$$

In so doing we take into account that the volume integral (after integrating by parts over Ω_γ in (35)) is equal to zero. By (3), from (44) there follows the singular invariant integral

$$G = \int_L (-\frac{1}{2}\sigma_{ij}\varepsilon_{ij}n_1 + \sigma_{ij}u_{i,1}n_j)ds + F(c) - F(b). \tag{45}$$

Note that there is no dependence on L (and hence on b, c) in (45). Denote by μ_c the Dirac measure on γ with support at the point $x_1 = c$. Then (45) can be rewritten as

$$G = \int_L (-\frac{1}{2}\sigma_{ij}\varepsilon_{ij}n_1 + \sigma_{ij}u_{i,1}n_j)ds + \int_\gamma F d\mu_c - \int_\gamma F d\mu_b.$$

Now consider a case corresponding to a different choice of the curve L as it is shown in

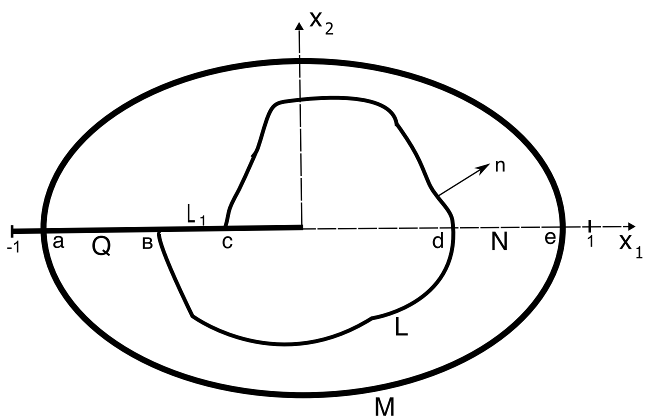

FIG. 3

Fig. 3. Let the curve $L \cup L_1$ surround the domain where θ is equal to one, $L_1 = (b, c) \times \{0\}$. Let $Q = (a, b) \times \{0\}$ $N = (d, e) \times \{0\}$. The formula for the derivative of the energy functional has the same form as (35) in this case. But now the situation is different. We have

$$G = \int_{\Omega_\gamma} (\frac{1}{2}\theta_{,1}\sigma_{ij}\varepsilon_{ij} - \sigma_{ij}u_{i,1}\theta_{,j})dx - \frac{1}{2}\int_\gamma (3\theta_{,1}w_{,11}^2 + 2\theta_{,11}w_{,1}w_{,11})dx_1. \tag{46}$$

Integrating by parts in (46) over Ω_γ provides

$$-\int\limits_{\Omega_\gamma} \sigma_{ij} u_{i,1} \theta_{,j} dx = \int\limits_{\Omega_\gamma} \theta(\sigma_{ij} u_{i,1})_{,j} dx + \int\limits_{L \cup L_1^+} \sigma_{ij} u_{i,1} n_j ds + \int\limits_{N \cup Q} [\sigma_{ij} u_{i,1} \nu_j \theta] dx_1.$$

Hence the singular invariant integral in this case is as follows:

$$G = \int\limits_{L \cup L_1^+} (-\frac{1}{2}\sigma_{ij}\varepsilon_{ij} n_1 + \sigma_{ij} u_{i,1} n_j) ds + F(d) - F(b), \tag{47}$$

where $F(h)$ is defined by formula (42). In formula (47), there is no dependence on $L \cup L_1$, and consequently, on b, d.

Now we have to prove the following Lemma used in deriving invariant integrals.

LEMMA. For smooth solutions of the problem (1)-(7), the following boundary condition holds:

$$\sigma_\nu^+ [u_{2,1}] = 0 \;\text{ on }\; Q. \tag{48}$$

Proof. By (7), the following condition takes place:

$$\sigma_\nu^+ [u_2] = 0 \;\text{ on }\; Q.$$

Assume that at a given point y we have $\sigma_\nu^+(y) > 0$. Then this inequality holds in a neighborhood of the point y. In this neighborhood, we have $[u_2] = 0$, hence $[u_{2,1}] = 0$, and (48) follows. If at the point y, the equality $\sigma_\nu^+(y) = 0$ holds, then (48) clearly takes place. The lemma is proved. □

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