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ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO A BVP FROM FLUID MECHANICS

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Abstract. In this paper we investigate a boundary value problem (BVP) derived from a model of boundary layer flow past a suddenly heated vertical surface in a saturated porous medium. The surface is heated at a rate proportional to x^k where x measures distance along the wall and $k > -1$ is constant. Previous results have established the existence of a continuum of solutions for $-1 < k < -1/2$. Here we further analyze this continuum and determine that precisely one solution of this continuum approaches the boundary condition at infinity exponentially while all others approach algebraically. Previous results also showed that the solution to the BVP is unique for $-1/2 \leq k < 0$. Here we extend the range of uniqueness to $0 \leq k \leq 1$. Finally, the physical implications of the mathematical results are discussed and a comparison is made to the solutions for the related case of prescribed surface temperature on the surface.

1. Introduction. Convective flows in porous media appear in a wide variety of industrial processes as well as in many natural circumstances (see [16], pp. 377ff.). Models of such flows have been investigated for a number of physical configurations, including that involving a vertical flat surface. Cheng and Minkowycz [2] and Ingham and Brown [10] considered the free convection boundary layer flow next to a vertical surface in a saturated porous medium where the temperature of the surface was suddenly raised above ambient. Merkin and Zhang [13] studied a similar problem in which the convective flow is generated by a prescribed heat flux through the surface being impulsively switched on. Merkin and Zhang note the need to discuss both the prescribed wall temperature case and the prescribed wall heat flux case since the nature of the solutions can be different in

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the two cases. Indeed, one purpose of this paper is to compare the nature of the solutions for the two scenarios.

Merkin and Zhang [13] use a similarity transformation to study the large time steady state behavior of the flow. They also consider the transition from the initial configuration to the steady state in the time dependent partial differential equation and find that their numerical scheme breaks down for certain values of the parameter governing the surface heat flux. Another goal of this paper is to explain further this numerical instability.

Interest in the problem continues with the more recent paper of Merkin and Pop [12], who consider the case where there is a relationship between the wall heat flux and the wall temperature in the form of a power law variation in the temperature. Through a change of variables they reduce this new case to that considered in [13] and in the present paper. The problem also appears in [17] (Chapter 5).

The plan of the paper is as follows. Section 2 lists the governing partial differential equations as well as the ODE boundary value problem derived by Merkin and Zhang [13] through a similarity transformation. This section also discusses a second physical situation in fluid dynamics that leads to the same BVP as that derived by Merkin and Zhang. Previous mathematical results on the BVP, including the existence of a continuum of solutions for certain parameter values, are also reviewed. Section 3 contains the asymptotic analysis of the continuum of solutions. Section 4 expands the range of parameter values for which uniqueness of the solution is valid, and Section 5 discusses the implications of the results for the physical model as well as open questions.

2. Mathematical model. The non-dimensionalized model considered by Merkin and Zhang [13] reads as follows:

$$\frac{\partial \psi}{\partial y} = \theta, \tag{2.1}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{\partial^2 \theta}{\partial y^2}, \tag{2.2}$$

subject to

$$\psi = 0, \quad \theta = 0, \quad \text{for} \quad t = 0, \quad x \geq 0, \quad y \geq 0, \tag{2.3}$$

$$\psi = 0, \quad \frac{\partial \theta}{\partial y} = -x^k, \quad \text{for} \quad y = 0, \quad x \geq 0, \quad t \geq 0, \tag{2.4}$$

$$\frac{\partial \psi}{\partial y} \to 0, \quad \theta \to 0, \quad \text{as} \quad y \to \infty, \quad x \geq 0, \quad t \geq 0. \tag{2.5}$$

Here a semi-infinite vertical flat surface which is embedded in a saturated porous medium is considered. Coordinates (x, y) measure position along the surface and normal to it, respectively, with the origin at the leading edge, the x-axis directed upward and the y-axis directed to the right. The velocities in the x- and y- directions are, respectively, $u(x, y, t)$ and $v(x, y, t)$ with the stream function, $\psi(x, y, t)$, defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$. The fluid temperature is given by $\theta(x, y, t)$, and the wall heating rate, x^k, is switched on at $t = 0$.

Merkin and Zhang [13] reduce the number of independent variables by one through the similarity transformation:

$$\psi = x^{(k+2)/3} f(\eta, \tau), \quad \eta = y x^{(k-1)/3}, \quad \tau = t x^{2(k-1)/3}. \tag{2.6}$$

Equations (2.1) and (2.2) thus become

$$\theta = x^{(2k+1)/3} \frac{\partial f}{\partial \eta}, \tag{2.7}$$

$$\left(1 - \frac{2(1-k)}{3} \tau \frac{\partial f}{\partial \eta}\right) \frac{\partial^2 f}{\partial \eta \partial \tau} - \left[\left(\frac{k+2}{3}\right) f - \frac{2(1-k)}{3} \tau \frac{\partial f}{\partial \tau}\right] \frac{\partial^2 f}{\partial \eta^2} + \left(\frac{2k+1}{3}\right) \left(\frac{\partial f}{\partial \eta}\right)^2 = \frac{\partial^3 f}{\partial \eta^3}, \tag{2.8}$$

with boundary conditions

$$f = 0, \quad \text{for all} \quad \eta, \quad \tau < 0, \tag{2.9}$$

$$f = 0, \quad \frac{\partial^2 f}{\partial \eta^2} = -1, \quad \text{on} \quad \eta = 0, \quad \tau \geq 0, \tag{2.10}$$

$$\frac{\partial f}{\partial \eta} \to 0, \quad \text{as} \quad \eta \to \infty, \quad \tau \geq 0. \tag{2.11}$$

Merkin and Zhang [13] note that η and τ are the appropriate variables to use for studying the final decay to the steady state solution, and by setting $\partial/\partial\tau \equiv 0$ in (2.8) they obtain the steady state boundary layer equation

$$f''' = \left(\frac{2k+1}{3}\right) f'^2 - \left(\frac{k+2}{3}\right) f f'', \tag{2.12}$$

subject to

$$f(0) = 0, \quad f''(0) = -1, \quad f'(\infty) = 0. \tag{2.13}$$

We now list some known results regarding the BVP (2.12)-(2.13):

Theorem A ([13]). For $k \leq -1$, no solution to the BVP (2.12)-(2.13) exists.

In [14] we found it convenient to consider a related family of initial value problems, namely equation (2.12) subject to

$$f(0) = 0, \quad f'(0) = \alpha, \quad f''(0) = -1, \tag{2.14}$$

where α is a free parameter.

Theorem B ([14]). If $-1 < k < -1/2$, then the BVP (2.12)-(2.13) has uncountably many solutions given by the solutions $f(\eta; \alpha)$ to the IVP (2.12, 2.14) where

$$\alpha \geq \frac{\sqrt{3k^2 + 16k + 17}}{\sqrt{3}(k+1)}. \tag{2.15}$$

Further, these solutions satisfy $f' > 0$ and $f'' < 0$ for all $\eta > 0$.

Theorem C ([14]). If $-1/2 \leq k < 0$, then the BVP (2.12)-(2.13) has exactly one solution. If $k \geq 0$, then the BVP (2.12)-(2.13) has at least one solution.

Theorem B raises the question of which solution(s), if any, are physically relevant. In the next section we show that for $-1 < k < -1/2$, the BVP has precisely one solution whose derivative decays to zero exponentially, while the derivatives of the other solutions decay to zero algebraically. In fact, this algebraic decay of $f'(\eta;\alpha)$ is so slow that $f(\eta;\alpha)$ tends to infinity.

Finally we note that the exact same BVP (2.12)-(2.13) was derived and numerically studied in a series of papers investigating Marangoni convection [3], [4], [5], [19], [20].

3. Asymptotic behavior of the solutions for $-1 < k < -1/2$. The BVP of interest reads:

$$f''' = \left(\frac{2k+1}{3}\right) f'^2 - \left(\frac{k+2}{3}\right) f f'', \tag{3.1}$$

$$f(0) = 0, \quad f''(0) = -1, \quad f'(\infty) = 0, \tag{3.2}$$

for $k \in (-1, -1/2)$. As mentioned before, it has been proved that for each $-1 < k < -1/2$, the BVP (3.1)-(3.2) admits a continuum of solutions satisfying $f' > 0$ with $f'' < 0$. In fact it follows from the proof in [14] that the solution of the differential equation (3.1) with initial conditions

$$f(0) = 0, \quad f'(0) = \alpha, \quad f''(0) = -1 \tag{3.3}$$

satisfies (3.2) if and only if $\alpha \geq \alpha(k) > 0$, where $\alpha(k)$ is defined by

$$\alpha(k) = \inf\{\alpha : (3.1)\text{-}(3.3) \text{ has a solution } f \text{ with } f' > 0 \text{ satisfying } f'(\infty) = 0\}. \tag{3.4}$$

The goal is to find the asymptotic behavior of these solutions.

THEOREM 1. Fix $k \in (-1, -1/2)$. Then the BVP (3.1)-(3.2) admits

(i) a unique solution called the "principal solution", denoted by $\hat{f}$, corresponding to $\hat{f}'(0) = \alpha(k)$, satisfying

$$\hat{f}'(\eta) \sim c_1 \hat{f}(\eta)^{-\frac{3(k+1)}{k+2}} \exp\left(-\int_{\tilde{\eta}}^{\eta} \hat{f}(s)\, ds\right)$$

as $\eta \to \infty$ for some $\tilde{\eta} > 0$ sufficiently large, and a constant $c > 0$ that depends only on k and $\tilde{\eta}$, and

(ii) a continuum of solutions $\{f\}$, satisfying $f'(\eta) \sim c f(\eta)^{\frac{2k+1}{k+2}}$ as $\eta \to \infty$ for some constant $c > 0$ that depends only on k.

REMARK 3.1. (i) The fact that $\hat{f}'(\eta) \sim c\hat{f}(\eta)^{-\frac{3(k+1)}{k+2}} \exp(-\int_{\tilde{\eta}}^{\eta} \hat{f})$ implies that $\hat{f} \to f^\star$ as $\eta \to \infty$, for some $0 < f^\star < \infty$.

(ii) The second class of solutions of (3.1)-(3.2) grow algebraically to infinity, namely $f(\eta) \sim c\eta^{\frac{k+2}{1-k}}$ as $\eta \to \infty$.

To prove Theorem 1, we will need the following result from [8], p. 382:

LEMMA 3.1. Let $q(t) > 0$ be a positive function on $t \in [0, \infty)$ possessing a continuous second order derivative and satisfying

$$\int^{\infty} q^{1/2}(t)\, dt = \infty \quad \text{and} \quad \int^{\infty} \left| \frac{5q'^2}{16q^3} - \frac{q''}{4q^2} \right| q^{1/2}\, dt < \infty.$$

Then $u'' - q(t)u = 0$ has a pair of solutions satisfying

$$u \sim q^{-1/4} \exp\left(\pm \int^t q^{1/2}(s)\ ds\right)$$

as $t \to \infty$.

REMARK 3.2. Here $\int^\infty q(s)\ ds$ and $\int^t q(s)\ ds$ are interpreted as $\int_{t_0}^\infty q(s)\ ds$ and $\int_{t_0}^t q(s)\ ds$ for some $t_0 > 0$ respectively.

The rest of this section is devoted to the proof of Theorem 1. The proof is motivated by the paper [7], which deals with the asymptotic behavior of solutions of the Falkner-Skan equation. We will use similar arguments in this proof.

Proof of Theorem 1. Let $g = (k+2)f/3$. Then the BVP (3.1)-(3.2) transforms to

$$g''' + gg'' - \frac{2k+1}{k+2} g'^2 = 0, \tag{3.5}$$

$$g(0) = 0, \quad g''(0) = -\frac{k+2}{3}, \quad g'(\infty) = 0, \tag{3.6}$$

and we define

$$g'(0) = \gamma \geq \gamma(k), \text{ where } \gamma = (k+2)\alpha/3 \text{ and } \gamma(k) = (k+2)\alpha(k)/3. \tag{3.7}$$

It follows from [14] that (3.5)-(3.6) has a continuum of solutions. We will prove that (3.5)-(3.6) has a unique solution $\hat{g}$ with $\hat{g}'(0) = \gamma(k)$ satisfying

$$\hat{g}'(\eta) \sim c\hat{g}(\eta)^{-\frac{3(k+1)}{k+2}} \exp(-\int_{\eta_0}^\eta \hat{g})$$

as $\eta \to \infty$ for some $\eta_0 > 0$ sufficiently large and for some constant $c > 0$ that depends only on k and η_0. As a by-product of the proof we will also obtain that the other solutions $\{g\}$ of (3.5)-(3.6) satisfying $g'(0) = \gamma$, where $\gamma > \gamma(k)$, obey $g'(\eta) \sim cg(\eta)^{\frac{2k+1}{k+2}}$ as $\eta \to \infty$ for some constant $c > 0$ that depends only on k.

Let $h = g'$. Then h satisfies the equation

$$h'' + gh' - \left(\frac{2k+1}{k+2}\right) g'h = 0. \tag{3.8}$$

We also set

$$h = xe^{-\frac{1}{2}\int_{\eta_0}^\eta g\ ds}, \tag{3.9}$$

where $\eta_0 > 0$ will be chosen later. With this transformation, (3.8) can be rewritten as

$$x'' - q(\eta)x = 0, \tag{3.10}$$

where

$$q(\eta) = \frac{1}{4}g^2(\eta) + \left(\frac{5k+4}{2(k+2)}\right) g'(\eta). \tag{3.11}$$

Since $g(0) = 0$ and $g'(\eta) > 0$ for all $\eta \geq 0$, there exists a constant $A > 0$ such that $g^2(\eta) > A$ for $\eta > 0$ sufficiently large. Using this and the fact that $g'(\eta) \to 0$ as $\eta \to \infty$, it follows from (3.11) that there exists some $\eta^\star(k) > 0$ large enough (to be chosen later) such that $q(\eta) > 0$ for all $\eta \geq \eta^\star$.

We define a new variable t by $t = \eta - \eta^\star$ for $\eta \geq \eta^\star$ and consider $g(t)$. Then clearly $g(t) > 0$, $g'(t) > 0$ and $q(t) > 0$ for $t \geq 0$. Differentiating (3.11) with respect to t, we obtain that

$$q'(t) = \frac{1}{2}g(t)g'(t) + \left(\frac{5k+4}{2(k+2)}\right)g''(t) \tag{3.12}$$

and

$$q''(t) = \frac{(11k^2+17k+8)}{2(k+2)^2}g'^2(t) - \frac{2k+1}{k+2}g(t)g''(t). \tag{3.13}$$

Since $g' \to 0$ as $t \to \infty$ we have from (3.11) that

$$q(t) > \frac{1}{8}g^2(t) \tag{3.14}$$

for $t > 0$ sufficiently large. Further, since $g'' < 0$, we have from (3.12), (3.13) and (3.14) that for $t > 0$ sufficiently large, there exists a constant $C > 0$ such that

$$\begin{aligned} \frac{q'^2}{q^{5/2}} &\leq \frac{C}{g^5}(g^2g'^2 + g''^2 + gg'g'') \\ &\leq C\left(\frac{g'^2}{g^3} + \frac{g''^2}{g^5}\right) \text{ if } \quad -\frac{4}{5} \leq k < -\frac{1}{2}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \frac{q'^2}{q^{5/2}} &\leq \frac{C}{g^5}(g^2g'^2 + g''^2 - gg'g'') \\ &= C\left(\frac{g'^2}{g^3} + \frac{g''^2}{g^5} - \frac{g'g''}{g^4}\right) \text{ if } \quad -1 < k < -\frac{4}{5}, \end{aligned} \tag{3.16}$$

and

$$\frac{|q''|}{q^{3/2}} \leq C\left(\frac{g'^2}{g^3} - \frac{gg''}{g^3}\right). \tag{3.17}$$

Let $T_1 > 0$ be large enough such that (3.14), (3.15), (3.16) and (3.17) hold. Then on integrating (3.15) and (3.16) over $[T_1, \infty)$ we obtain that

$$\int_{T_1}^{\infty} \frac{q'^2}{q^{5/2}} \leq C\left(\int_{T_1}^{\infty} \frac{g'^2}{g^3} + \int_{T_1}^{\infty} \frac{g''^2}{g^5}\right) \text{ if } \quad -\frac{4}{5} \leq k < -\frac{1}{2} \tag{3.18}$$

and

$$\int_{T_1}^{\infty} \frac{q'^2}{q^{5/2}} \leq C\left(\int_{T_1}^{\infty} \frac{g'^2}{g^3} + \int_{T_1}^{\infty} \frac{g''^2}{g^5} - \int_{T_1}^{\infty} \frac{g'g''}{g^4}\right) \text{ if } \quad -1 < k < -\frac{4}{5}. \tag{3.19}$$

We will now consider each of the above integrals separately. Note that since g' and g'' are bounded with $g'' < 0$ and g increasing, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$\int_{T_1}^{\infty} \frac{g'^2}{g^3} \leq C_1 \int_{T_1}^{\infty} \frac{g'}{g^3} < \infty \tag{3.20}$$

and

$$-\int_{T_1}^{\infty} \frac{g'g''}{g^4} \leq C_2 \int_{T_1}^{\infty} \frac{g'}{g^4} < \infty. \tag{3.21}$$

Similarly an integration by parts to the integral $\int_{T_1}^{\infty} g''/g^5$ with $u = 1/g^5$ and $dv = -g''$ and an argument similar to (3.20) yield

$$\int_{T_1}^{\infty} \frac{g''^2}{g^5} \leq C_3 \int_{T_1}^{\infty} \frac{-g''}{g^5} < \infty, \tag{3.22}$$

for some constant $C_3 > 0$. Hence on combining (3.19), (3.20), (3.21) and (3.22) we obtain that

$$\int_{T_1}^{\infty} \frac{q'^2}{q^{5/2}} < \infty. \tag{3.23}$$

Similarly on integrating (3.17) we can prove that

$$\int_{T_1}^{\infty} \frac{|q''|}{q^{3/2}} < \infty. \tag{3.24}$$

Since g is increasing, we have from (3.14) that for large $t > 0$,

$$\int_{T_1}^{\infty} q^{1/2}(s)\ ds = \infty. \tag{3.25}$$

Hence on combining (3.23), (3.24) and (3.25), we conclude that $q(t)$ satisfies all the conditions in Lemma 3.1. Rewriting (3.11) yields

$$q = \frac{1}{4} g^2 \left(1 + \frac{2(5k+4)}{(k+2)} \frac{g'}{g^2} \right), \tag{3.26}$$

so that

$$\begin{aligned} q^{1/2} &= \frac{g}{2} \left(1 + \frac{(5k+4)}{(k+2)} \frac{g'}{g^2} + O\left(\frac{g'^2}{g^4} \right) \right) \\ &= \frac{g}{2} + \frac{(5k+4)}{2(k+2)} \frac{g'}{g} + O\left(\frac{g'^2}{g^3} \right) \end{aligned}$$

as $t \to \infty$. Hence for all $t > T_1$ sufficiently large, we have

$$\int_{T_1}^{t} q^{1/2}(s)\ ds = \frac{1}{2} \int_{T_1}^{t} g\ ds + \frac{(5k+4)}{2(k+2)} \ln g + c_0, \tag{3.27}$$

for some constant c_0 that depends only on k and T_1. Moreover since $g' \to 0$, it follows from (3.26) that

$$q(t) \sim \frac{1}{4} g^2(t),$$

so that

$$q^{-1/4}(t) \sim \left(\frac{1}{2} g(t) \right)^{-1/2} \tag{3.28}$$

as $t \to \infty$. From (3.27), (3.28) and applying Lemma 3.1, we obtain that (3.10) has a pair of solutions satisfying

$$x(t) \sim \hat{c} \left(\frac{1}{2} g(t) \right)^{-1/2} \exp\left(\pm \left(\frac{1}{2} \int_{T_1}^{t} g(s)\ ds + \frac{(5k+4)}{2(k+2)} \ln g(t) \right) \right)$$

as $t \to \infty$, for some constant $\hat{c} > 0$ that depends only on k and T_1.

Let $\tilde{s} = s - \eta^\star$. Define η_0 by $\eta_0 = T_1 + \eta^\star$. Then from (3.9) it follows that $h = xe^{-\frac{1}{2}\int_{T_1}^t g(s)\ d\tilde{s}}$. Hence (3.8) has a pair of solutions satisfying

$$h \sim cg^{-\frac{3(k+1)}{k+2}} \exp\left(-\int_{T_1}^t g(s)\ ds\right)$$

and

$$h \sim cg^{\frac{2k+1}{k+2}}$$

as $t \to \infty$ for some $c > 0$ that depends only on k and T_1. Since $g' = h$, (3.5)-(3.6) has solutions satisfying either

$$g' \sim cg^{-\frac{3(k+1)}{k+2}} \exp\left(-\int_{T_1}^t g\ ds\right) \tag{3.29}$$

or

$$g' \sim cg^{\frac{2k+1}{k+2}} \tag{3.30}$$

as $t \to \infty$. (3.29) implies that $g \to g^\star$ for some $0 < g^\star < \infty$, while (3.30) implies that

$$g(t) \sim ct^{\frac{k+2}{1-k}} \tag{3.31}$$

as $t \to \infty$.

To complete the proof of Theorem 1 we will now prove that (3.5)-(3.6) admits a unique solution that satisfies (3.29). For this we will need a couple of lemmas stated below.

LEMMA 3.2. The BVP (3.5)-(3.6) has a solution $\hat{g}$ that satisfies (3.29).

Proof. To prove the lemma, we will show that (3.5)-(3.6) has a solution $\hat{g}$ that is bounded, i.e. $\hat{g}(t) \to g^\star$ for some $g^\star < \infty$ as $t \to \infty$. Note that a monotonic solution of (3.5)-(3.6) satisfies (3.29) if and only if it is bounded.

Suppose that for a contradiction, all the monotonic solutions of (3.5)-(3.6) are unbounded. Consider a solution g of the differential equation (3.5). Let $-z = g'^2$ and let $\cdot$ denote d/dg. Then clearly $z < 0$ and

$$g' = \sqrt{-z}, \quad g'' = -\frac{\dot{z}}{2}, \quad g''' = -\frac{\ddot{z}}{2}\sqrt{-z}. \tag{3.32}$$

With this transformation, the problem (3.5) can be rewritten as

$$\ddot{z}\sqrt{-z} + g\dot{z} + \frac{2(2k+1)}{k+2}(-z) = 0, \tag{3.33}$$

and the boundary conditions (3.6) transform to

$$\dot{z}(0) = \frac{2}{3}(k+2), \quad z(\infty) = 0, \tag{3.34}$$

since we are assuming that $g(\infty) = \infty$. Note that if $z(g)$ is a solution of (3.33) determined by initial conditions

$$z(0) = -z_0, \quad \dot{z}(0) = \beta, \tag{3.35}$$

where $z_0, \beta > 0$, then it can be easily shown that (a) $z(g) < 0$ for small $g > 0$ and $\dot{z}(g) > 0$ as long as $-z_0 \le z < 0$ holds and (b) if $z(g)$ exists for all $g \ge 0$ and satisfies $-z_0 \le z < 0$, then $z(g) \to 0$ as $g \to \infty$.

In particular, if $z_0 = \gamma^2$, where γ is defined as in (3.7), $\beta = 2(k+2)/3$, and z satisfies (3.33) and (3.35), then $z(g)$ exists for all $g \geq 0$ with $\dot{z}(g) > 0$ and $z \to 0$ as $g \to \infty$.

For a given solution $z(g)$ of (3.33), we define

$$w = -z(g), \quad r = \dot{w}/w. \tag{3.36}$$

Then (3.33) transforms to

$$\sqrt{w}\ddot{w} + g\dot{w} - \frac{2(2k+1)}{k+2}w = 0,$$

and r satisfies

$$\dot{r} = -r^2 + \left(\frac{2(2k+1)}{k+2} - gr\right)/\sqrt{w}.$$

Consider the Weber differential equation

$$\ddot{v} + g\dot{v} - \frac{2(2k+1)}{k+2}v = 0, \quad \text{where } \dot{v} = dv/dg. \tag{3.37}$$

Let $s = \dot{v}/v$. Then s satisfies

$$\dot{s} = -s^2 + \left(\frac{2(2k+1)}{k+2} - gs\right). \tag{3.38}$$

By [8] (page 320, Exercise 17.6), it follows that (3.37) has a solution $v(g)$ satisfying $s = \dot{v}/v \sim -g$ as $g \to \infty$.

Let $g_0 > 0$ be so large that

$$0 < v(g) < 1 \quad \text{and} \quad 2(2k+1)/(k+2) - gs > 0 \quad \text{for } g \geq g_0 > 0. \tag{3.39}$$

Define $0 < v_0 < 1$ by $v_0 = v(g_0)$ and choose $\beta_0 > 0$ such that $\beta_0 < -\dot{v}(g_0)$. Let $z(g)$ be a solution of (3.33) with $z(g_0) = -v_0$ and $\dot{z}(g_0) = \beta_0$. Then

$$w(g_0) = v_0 = v(g_0), \qquad \dot{w}(g_0) = -\dot{z}(g_0) = -\beta_0 > \dot{v}(g_0).$$

In particular, we have $r(g_0) > s(g_0)$. We will show that

$$r(g) > s(g) \tag{3.40}$$

for all $g \geq g_0$ for which $r(g)$ exists. On any interval $g_o \leq g \leq g_1$, where $r(g) \geq s(g)$ holds, then an integration and the fact that $w(g_0) = v(g_0)$ yield that $w(g) \geq v(g) > 0$. Since $z(g) = -w(g)$, we conclude that $-1 < z(g) < 0$ for $g_0 \leq g \leq g_1$.

Suppose, if possible, that there exists a first $g = g_1 > g_0$, where (3.40) fails to hold; then $\dot{s}(g_1) \geq \dot{r}(g_1)$. However, (3.38) and the last part of (3.39) imply that

$$\begin{aligned} \dot{s}(g_1) &= -s^2(g_1) + \left(\frac{2(2k+1)}{k+2} - g_1 s(g_1)\right) \\ &< -r^2(g_1) + \left(\frac{2(2k+1)}{k+2} - g_1 r(g_1)\right)/\sqrt{w(g_1)} = \dot{r}(g_1). \end{aligned}$$

This contradiction proves that (3.40) holds. Consequently, if $z(g)$ is such that $-1 < z(g_0) < 0$ and (3.40) holds at $g = g_0$, then $z(g)$ exists for all $g \geq g_0$, $-1 < z(g) < 0$ for $g > g_0$, and by assertion (b), $z(g) \to 0$ as $g \to \infty$.

We will now show that the solution of (3.5) determined by

$$g(0) = 0, \quad g'(0) = \gamma(k), \quad g''(0) = -\frac{k+2}{3},$$

which also satisfies (3.6) satisfies (3.29). Let us denote this solution by $\hat{g}(t)$. Let $\hat{z}(g)$ denote the corresponding solution of (3.33), $\hat{w} = -\hat{z}(g)$ and $\hat{r}(g) = \dot{\hat{w}}/\hat{w}$. If $\gamma(k) > 1$, we define $\bar{g} \geq 0$ to be the point such that $\hat{z}(\bar{g}) = -1$, and $-\gamma^2(k) \leq \hat{z}(g) < -1$ for all $0 \leq g < \bar{g}$. Hence $\hat{w}(g) < 1$ for $g > \bar{g}$. We will prove that if $v = v(g)$ is a fixed solution of (3.37) satisfying $s \sim -g$ as $g \to \infty$, (3.39), and $v(g_0) = \hat{w}(g_0)$, where $g_0 > \bar{g}$ is chosen such that (3.39) holds, then

$$\hat{r}(g) \leq s(g) \quad \text{for } g \geq g_0. \tag{3.41}$$

Suppose, if possible, that $\hat{r}(\tilde{g}) > s(\tilde{g})$ for some $\tilde{g} \geq g_0$. Then by continuity it follows that if $z(g)$ is the solution of (3.33) with $z(0) = -\gamma^2$, $\dot{z}(0) = 2(k+2)/3$, with γ close to $\gamma(k)$ and $\gamma < \gamma(k)$, then $z(g)$ exists on the interval $[0, \tilde{g}]$, with $-1 < z(\tilde{g}) < 0$ and that the function $r(g)$ belonging to $z(g)$ satisfies (3.40) at $g = \tilde{g}$. Then by the above arguments, $z(g)$ exists for $g \geq 0$ and the corresponding solution $g(t)$ satisfies (3.5)-(3.6). But this contradicts the definition of $\gamma(k)$ (see (3.4) and (3.7)). Hence (3.41) holds.

Since $v(g_0) = \hat{w}(g_0)$, integrating (3.41), where $\hat{r} = \dot{\hat{w}}/\hat{w}$ and $s = \dot{v}/v$ yields $\hat{w}(g) \leq v(g)$ for $g \geq g_0$. This implies that $-\hat{z}(g) \leq v(g)$. Since $\dot{v}/v \sim -g$ as $g \to \infty$, $v(g) = O(\exp -(g^2/2))$ for large g. Since $\hat{z}(g) = -(\hat{g}'(t))^2$, at the t-value where $\hat{g}(t) = g$, it follows that $(\hat{g}'(t))^2 = O(\exp(-(\hat{g}(t)))^2/2)$ as $t \to \infty$. By our assumption, $\hat{g}(t) \to \infty$ as $t \to \infty$. This implies that $\hat{g}'$ cannot satisfy (3.30) and hence must satisfy (3.29). However, if $\hat{g}$ satisfies (3.29), then $\hat{g}$ must be bounded, which contradicts our assumption on unboundedness of $\hat{g}$.

Thus, we have proved that (3.5)-(3.6) has at least one solution that is bounded and hence satisfies (3.29). □

LEMMA 3.3. The BVP (3.5)-(3.6) has a unique solution that satisfies (3.29).

Proof. If possible, let $g_1(t)$ and $g_2(t)$ be two solutions of (3.5)-(3.6) satisfying (3.29) with $g_1'(0) = \gamma_1$ and $g_2'(0) = \gamma_2$. Let $g_1(t) \to g_1^\star$ and $g_2(t) \to g_2^\star$ as $t \to \infty$. Without loss of generality, assume that $\gamma_1 > \gamma_2$. Note that an integration of (3.5) yields that for $i = 1, 2$, we have that

$$g_i'' = -\frac{k+2}{3} - g_i g_i' + \frac{3(k+1)}{k+2} \int_0^t g_i'^2(s)\, ds.$$

Since g_i is bounded, letting $t \to \infty$, we obtain that the two solutions must satisfy

$$\int_0^\infty g_1'(s)^2\, ds = \int_0^\infty g_2'(s)^2\, ds = \frac{(k+2)^2}{9(k+1)}. \tag{3.42}$$

Equation (3.42) implies that g_1' and g_2' must intersect at least once, a fact that will be used later in the proof.

As in the proof of Lemma 3.2, we define $-z_i = g_i'^2$ with $z_i < 0$, $i = 1, 2$. Then z_1 and z_2 satisfy (3.32) and (3.33) with $z_1(0) = -\gamma_1^2$, $z_2(0) = -\gamma_2^2$, $\dot{z}_1(0) = \dot{z}_2(0) = 2(k+2)/3$.

Let $G_i(z)$ be the function inverse to $z = z_i(g)$ and $H_i(z) = \dot{z}_i(G_i(z))$ for $i = 1, 2$. Then

$$\frac{dG_i}{dz} = \frac{1}{H_i}, \tag{3.43}$$

while $dH_i/dz = \ddot{z}_i dG_i/dz = \ddot{z}_i/H_i$, so that (3.33) yields

$$\begin{aligned} \frac{dH_i}{dz} &= -\frac{G_i \dot{z}_i}{H_i\sqrt{-z}} - \frac{2(2k+1)}{k+2}\frac{\sqrt{-z}}{H_i} \\ &= -\frac{G_i}{\sqrt{-z}} - \frac{2(2k+1)}{k+2}\frac{\sqrt{-z}}{H_i}. \end{aligned} \tag{3.44}$$

Note that

$$H_2(-\gamma_2^2) = H_1(-\gamma_1^2) = \frac{2}{3}(k+2).$$

Also since $g'' < 0$, it follows from (3.32) that $H_1, H_2 > 0$ for all $z < 0$. Consider all z such that $-\gamma_2^2 \le z < 0$. There could be two possibilities, namely, (i) $H_2(-\gamma_2^2) \ge H_1(-\gamma_2^2)$ or (ii) $H_2(-\gamma_2^2) < H_1(-\gamma_2^2)$. We will deal with both cases separately.

(i) We will first consider the case when $H_2(-\gamma_2^2) \ge H_1(-\gamma_2^2)$. Note that since z_i increases with g, its inverse G_i also increases with z. This implies that

$$G_2(-\gamma_2^2) = 0 = G_1(-\gamma_1^2) < G_1(-\gamma_2^2); \quad \text{i.e.} \quad (G_1 - G_2)(-\gamma_2^2) > 0.$$

If $H_2(-\gamma_2^2) = H_1(-\gamma_2^2)$, then it follows from (3.44) that $(\dot{H}_2 - \dot{H}_1)(-\gamma_2^2) > 0$. Hence $H_2 > H_1$ for z sufficiently close to $-\gamma_2^2$ with $z > -\gamma_2^2$. Moreover, note that as long as $-\gamma_2^2 \le z < 0$, we have from (3.43) that

$$\frac{d}{dz}(G_1 - G_2) = \frac{H_2 - H_1}{H_1 H_2}.$$

Since $H_1, H_2 > 0$, as long as $H_2 - H_1 > 0$, we have that $d(G_1 - G_2)/dz > 0$ and that $(G_1 - G_2) > 0$. If possible, let $z_0 < 0$ be such that $H_2(z_0) - H_1(z_0) = 0$ and $H_2(z) - H_1(z) > 0$ for all $-\gamma_2^2 < z < z_0$. Then we must have $\dot{H}_2(z_0) - \dot{H}_1(z_0) \le 0$. However (3.44) implies that

$$\left.\frac{d}{dz}(H_2 - H_1)\right|_{z=z_0} = \frac{G_1(z_0) - G_2(z_0)}{\sqrt{-z_0}} > 0,$$

a contradiction. Thus, we obtain that for all $-\gamma_2^2 < z < 0$,

$$H_2(z) - H_1(z) > 0; \tag{3.45}$$

hence

$$G_1(z) - G_2(z) > 0 \quad \text{is increasing in } z. \tag{3.46}$$

Note that (3.46) implies that $z_1 < z_2$ as long as they exist, i.e. $g_1' > g_2'$ for all $t > 0$, contradicting the fact that g_1' and g_2' must intersect at least once (see (3.42)).

(ii) Now we consider the case when $H_2(-\gamma_2^2) < H_1(-\gamma_2^2)$. We will first prove that $H_2(z) < H_1(z)$ for all $-\gamma_2^2 \le z < 0$. If possible, let $-\gamma_2^2 < z_0 < 0$ be such that $H_2(z_0) = H_1(z_0)$ with $H_2(z) < H_1(z)$ for $-\gamma_2^2 \le z < z_0$. Then we must have $\dot{H}_1(z_0) - \dot{H}_2(z_0) \le 0$. Note that we cannot have $G_1(z_0) = G_2(z_0)$ unless $G_1 = G_2$ and $H_1 = H_2$. Hence, either $G_1(z_0) < G_2(z_0)$ or $G_1(z_0) > G_2(z_0)$.

Let us first assume that $G_1(z_0) < G_2(z_0)$. Then (3.44) implies that

$$\frac{d}{dz}(H_1 - H_2)\bigg|_{z=z_0} = \frac{G_2(z_0) - G_1(z_0)}{\sqrt{-z_0}} > 0,$$

a contradiction.

Now suppose that $G_1(z_0) > G_2(z_0)$. Recalling the definition of H_i and using the fact that $H_2(z) < H_1(z)$ for $-\gamma_2^2 \le z < z_0$ and $G_2(-\gamma_2^2) < G_1(-\gamma_2^2)$, we note that G_1 and G_2 do not intersect in the interval $-\gamma_2^2 \le z \le z_0$. Thus, we must have that $G_2(z) < G_1(z)$ for $-\gamma_2^2 \le z < z_0$. It follows from (3.44) that $\dot{H}_2(z_0) - \dot{H}_1(z_0) > 0$. By continuity, for some $z_1 > z_0$ sufficiently close to z_0, $H_2(z_1) - H_1(z_1) > 0$ and $G_1(z_1) - G_2(z_1) > 0$. Applying a similar analysis as in case (i) (replacing $-\gamma_2^2$ by z_1 in case (i)), we obtain a contradiction.

Hence we conclude that $H_1 - H_2$ does not change sign, i.e. $H_1 - H_2 > 0$. We will now show that $(G_1 - G_2)(z) > 0$ for all $-\gamma_2^2 \le z < 0$. If possible let $-\gamma_2^2 < \tilde{z} < 0$ be such that $(G_1 - G_2)(\tilde{z}) = 0$ with $(G_1 - G_2)(z) > 0$ for $-\gamma_2^2 \le z < \tilde{z}$. Since $H_1 - H_2 > 0$, we have from (3.43) that $(\dot{G}_1 - \dot{G}_2)(z) < 0$ for all $z \ge \tilde{z}$. Hence by continuity, there exist $\delta_1 > 0$ and $\varepsilon > 0$ such that $(G_1 - G_2)(z) < -\delta_1$ for all $0 > z > \tilde{z} + \varepsilon$. Consequently we have from (3.44) that

$$\begin{aligned} \frac{d}{dz}(H_2 - H_1) &= \frac{G_1 - G_2}{\sqrt{-z}} + O\left(G_1^{\frac{k-1}{k+2}}\right) + O\left(G_2^{\frac{k-1}{k+2}}\right) \\ &< -\frac{\delta_1}{\sqrt{-z}} + O\left(g_1^{\star\,\frac{k-1}{k+2}}\right) + O\left(g_2^{\star\,\frac{k-1}{k+2}}\right) \to -\infty \end{aligned}$$

as $z \to 0$. This in turn implies that $H_2 - H_1 \to -\infty$ as $z \to 0$, contradicting that $H_i \to 0$ as $z \to 0$ for $i = 1, 2$.

Thus, we must have $(G_1 - G_2)(z) > 0$ for all $-\gamma_2^2 \le z < 0$, which in turn implies that $g_1' > g_2'$, a contradiction.

□

Thus we have proved that there is exactly one solution of (3.5)-(3.6) that satisfies (3.29). Hence the other solutions must satisfy (3.31).

4. Uniqueness for the case $0 \le k \le 1$.

THEOREM 2. The BVP (3.1)-(3.2) has exactly one solution if $0 \le k \le 1$.

Proof. The proof of this theorem is a straightforward adaptation of the argument given in [18] which deals with uniqueness for a similar third order nonlinear differential equation. As the argument is short, we present it here in its entirety. Following Theorem 3 in [14], we know that for each $0 \le k \le 1$ the BVP admits exactly one solution f that satisfies $f' > 0$ with $f'' < 0$. Hence if the problem admits a second solution f, then f' becomes negative and achieves a minimum. Since f' cannot have a maximum, f' must then approach zero from below. Note that since $f(0) = 0$ and $f'(0) > 0$, we have $f > 0$ initially. In fact we must have $f > 0$ for all $\eta > 0$. To see this, suppose that f were to decrease through zero (necessarily after f' has already become negative); then f must remain negative thereafter (since f' is negative thereafter). Also, since f' approaches

zero from below and cannot have a maximum, we have that $f'' > 0$ for large η. This then implies from (3.1) that

$$f''' = \left(\frac{2k+1}{3}\right) f'^2 - \left(\frac{k+2}{3}\right) f f'' > 0.$$

Thus $f'' > 0$ and $f''' > 0$ for η large implies that $f' \to \infty$ as $\eta \to \infty$, contradicting $f' \to 0$.

Let $\tilde{\eta}$ be the point where f' decreases through zero. We will show that f' cannot remain negative for all $\eta > \tilde{\eta}$ and thus cannot be a second solution to the BVP. First of all note that $0 < f(\eta) \leq f(\tilde{\eta})$ for all $\eta > 0$. Next let $\hat{\eta}$ be the point where f' has its minimum, i.e. where $f''(\hat{\eta}) = 0$. For $\eta \geq \hat{\eta}$ let $r(\eta) = f''(\eta)/f'(\eta)$. Then $r(\eta)$ satisfies

$$r' + r^2 + \left(\frac{k+2}{3}\right) fr - \left(\frac{2k+1}{3}\right) f' = 0 \tag{4.1}$$

with

$$r(\hat{\eta}) = f''(\hat{\eta})/f'(\hat{\eta}) = 0, \quad r'(\hat{\eta}) = \left(\frac{2k+1}{3}\right) f'(\hat{\eta}) < 0. \tag{4.2}$$

Differentiating (4.1) we obtain that

$$r'' + \left(2r + \frac{k+2}{3} f\right) r' = \frac{k-1}{3} f'' \leq 0, \tag{4.3}$$

since $0 \leq k \leq 1$. Integrating (4.1) we obtain

$$r'(\eta) \leq r'(\hat{\eta}) \exp\left(-\int_{\hat{\eta}}^{\eta} (2r(s) + (k+2)f(s)/3)\ ds\right) < 0. \tag{4.4}$$

Let $A = r(\infty)$ and $f_\infty = f(\infty)$. Note that $A < 0$ and may be $-\infty$. First suppose that $A < 0$ is finite. Since $r' < 0$ and $r \to A$ as $\eta \to \infty$ we conclude that $r'(\infty) = 0$. Thus from (4.1) it follows that $A + (k+2)f_\infty/3 = 0$. From this and (4.4) it follows that $r'(\eta) \to -\infty$, giving a contradiction. Thus, since $r(\eta)$ cannot tend to a finite limit we conclude that $r(\eta) \to -\infty$. Since $0 < f(\eta) < f(\hat{\eta})$ for all $\eta > \hat{\eta}$, it follows from (4.1) that there exists an $\eta_1 > \hat{\eta}$ such that

$$r' < -r^2/2, \tag{4.5}$$

for all $\eta > \eta_1$. Integrating (4.5) from $\eta > \eta_1$ to $\eta_2 = \eta_1 - 2/r(\eta_1)$ we obtain that

$$r(\eta) < \frac{2}{\eta - \eta_2}. \tag{4.6}$$

However as $\eta \to \eta_2^-$, (4.6) implies that $r \to -\infty$ at finite η contradicting the fact that $r = f''/f'$ is bounded on each bounded subinterval of $[\hat{\eta}, \infty)$. Thus if $0 \leq k \leq 1$ the solution of the BVP is unique. □

In [11] a numerical study of the solutions of the boundary value problem

$$f'''(\eta) + f(\eta)f''(\eta) - \left(\frac{2n}{n+1}\right) f'(\eta)^2 = 0, \tag{4.7}$$

subject to

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \tag{4.8}$$

is conducted. For $n \geq 1$ two solutions of (4.7)-(4.8) are found in [11], but the results of [18] can be used to show that for $n = 1$ the solution is unique. In fact, $n = 1$ reduces (4.7)-(4.8) to the original problem studied in [18]. The argument of [18] does not generalize to $n > 1$, but does imply uniqueness for $0 < n \leq 1$.

5. Discussion. In this section we discuss the implications of the mathematical results derived earlier. First, Merkin and Zhang [13] emphasize the need to investigate both the prescribed surface temperature case and the prescribed surface heat flux case since the nature of the solutions can be different in the two cases. The prescribed surface temperature case was first investigated by Cheng and Minkowycz [2], and their similarity transformation led to the following BVP:

$$f''' + \left(\frac{\lambda+1}{2}\right) f f'' - \lambda f'^2 = 0, \tag{5.1}$$

subject to

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0, \tag{5.2}$$

where the wall temperature varies according to x^λ. The work of Cheng and Minkowycz [2] included a numerical study of this BVP for $-1/3 < \lambda < 1$ for which they found a unique solution. The analysis was then expanded to the range $-1/2 < \lambda < \infty$ by Ingham and Brown [10]. They showed that no solution to the BVP (5.1)-(5.2) exists for $\lambda < -1/2$. Numerically, they found a unique solution for $-1/2 < \lambda \leq 1$ and dual solutions for $\lambda > 1$. They also listed closed form solutions for $\lambda = -1/3$ ($f(\eta) = \sqrt{6}\tanh(\eta/\sqrt{6})$) and $\lambda = 0$ ($f(\eta) = 1 - \exp(-\eta)$).

We note here that for $-1/3 < \lambda < 0$ the solution of (5.1)-(5.2) is not unique. The techniques of [1], [14] and [15] can be used to show that for this range of λ an infinite continuum of solutions exists. Thus the results here bring to light a new similarity between the prescribed wall temperature case and the prescribed wall heat flux case, namely the existence of parameter range for which an infinite continuum of solutions exists. The techniques of the present paper can then be used to show similar asymptotic behavior. The range $-1/2 < \lambda \leq -1/3$ requires more detailed analysis, but here too we conjecture the existence of a continuum of solutions. As noted in [10], solutions in this range of λ necessarily require $f''(0) \geq 0$, and thus $f'(\eta)$ may not be monotonic, which may complicate the analysis. We wish to explore this problem more in future work.

Next, consider the prescribed wall heat flux case [13] and the case where the wall temperature and the wall heat flux are related [12]. This latter case can be reduced to the former by setting $N = 3k/(2k+1)$ in [12]. For $k > -1$, numerical investigations in both papers report a unique solution. But as was the case for the prescribed wall temperature case, the numerical results are incomplete. For $-1 < k < -1/2$, uniqueness does not hold and a continuum of solutions exists.

In [13], Merkin and Zhang also consider the development of the solution in the time dependent PDEs. For the large time evolution of the solution, equation (2.8) was used. In describing the transition from the initial behavior to the large time behavior, Merkin and Zhang note that a numerical instability was encountered for $k \leq -.5$. We suggest that the cause of this numerical instability may involve the continuum of solutions that

exists for k in this range. It appears that delicate numerical analysis is required for k in this range, and this also represents another potential line of future research. However, we do note that Ingham and Brown conducted a similar analysis for the prescribed wall temperature case and encountered no numerical difficulties, even in the range of λ for which a continuum of solution exists, a circumstance that raises more questions than it answers.

Next we note that the argument of Troy *et al.* [18] used in Section 4 can be used to prove that the solution of (5.1)-(5.2) is unique for $0 < \lambda \leq 1$. The techniques used for the case $k = -1/2$ in [14] can be used to show uniqueness of a solution of (5.1)-(5.2) when $\lambda = 0$. However, uniqueness or non-uniqueness remains an open question for both the prescribed surface heat flux case for $k > 1$ and for the prescribed surface temperature case for $\lambda > 1$. Based on their numerical calculations, Ingham and Brown conjectured the existence of two solutions for the prescribed surface temperature case when $\lambda > 1$.

Finally, we note the similarities between the model analyzed here and the Falkner-Skan problem. For the Falkner-Skan problem:

$$f''' + ff'' + \beta(1 - f'^2) = 0, \tag{5.3}$$

subject to

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1, \tag{5.4}$$

there exists a $\beta_c < 0$ such that for $\beta_c < \beta < 0$ a continuum of solutions exists. It was argued by Hartree [9] that the physically relevant solution would correspond to that member of the continuum that approached the boundary condition at infinity most rapidly. Davey [6] argued for a similar condition regarding the solutions of a mathematical model for flow past a wavy cylinder. Hartman [8] showed that "Hartree's condition" was satisfied by the Falkner-Skan problem, and the extension of Hartman's argument given here shows that a similar condition is satisfied by (2.12)-(2.13).

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