QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXIV, NUMBER 2
JUNE 2016, PAGES 275–296
http://dx.doi.org/10.1090/qam/1418
Article electronically published on March 21, 2016

BIFURCATION ANALYSIS OF A SINGLE-GROUP ASSET FLOW MODEL

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Abstract. We study the stability and Hopf bifurcation analysis of an asset pricing model that is based on the model introduced by Caginalp and Balenovich, under the assumption of a fixed amount of cash and stock in the system. First, we analyze stability of equilibrium points. Choosing the momentum coefficient as a bifurcation parameter, we also show that Hopf bifurcation occurs when the bifurcation parameter passes through a critical value. Analytical results are supported by numerical simulations. A key conclusion for economics and finance is the existence of periodic solutions in the absence of exogenous factors for an interval of the bifurcation parameter, which is the trend-based (or momentum) coefficient.

1. Introduction. A central theme in classical finance is that market participants all have access to the same information, and all seek to optimize their returns so that a unique equilibrium price is established (see, for example, [3], [18], [20]). The approach to equilibrium is often assumed to be a process involving some randomness or noise, but otherwise smooth and rapid. Aside from noise, one can expect that prices will not

Received August 15, 2014.
2010 *Mathematics Subject Classification.* Primary 91B25, 91B26, 91B50, 34D20, 37G15; Secondary 91G80, 91G99, 34C60, 37N40.
Key words and phrases. Asset price dynamics, stability of price dynamics, Hopf bifurcation, price trend, momentum, market dynamics, liquidity, periodic solutions.
The first author was supported by TUBITAK (The Scientific and Technological Research Council of Turkey).
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overshoot the equilibrium price since the equation governing the change in price, P, is a first order in time, i.e., $P' = F(D/S)$ where S and D are supply and demand that depend on price but not on the recent price derivative history. As such, there is no mechanism for oscillations or cyclic behavior within this setting.

A well known example of cyclic behavior in economics is called the "cobweb theorem", whereby prices oscillate periodically due to the time lag between supply and demand decisions. Agricultural commodities provide a simple example with a delay between planting and harvesting (see [21] (pages 133-134 give two agricultural examples: rubber and corn) and [36]). In financial markets, however, the prevailing theory (at least during the latter part of the 20th century), called the efficient market hypothesis (EMH), maintains the existence of infinite arbitrage capital that would quickly exploit any deviations between the trading price and the intrinsic or fundamental value of the asset, which are necessarily unique since the participants have the same information and calculation of future returns. The absence of any delay in information or trading excludes, mathematically, the possibilities of overshooting the equilibrium price or oscillating about it.

While policymakers often discuss instabilities in asset prices, classical finance tends to treat these as rare occurrences within a stochastic setting. In particular, much of classical finance is based on the concept that an asset's price, $P(t)$, is governed by the equation

$$d\log P(t) = \mu dt + \sigma dW(t), \tag{1.1}$$

where $d\log P = P^{-1}dP$ is the relative change in price, μ is the expected return, W is Brownian motion, and σ is the standard deviation [37]. This means that any abrupt changes in asset prices, such as the 1000 point (9%) drop and rapid recovery in the Dow Jones Industrial Average on May 6, 2010, can be attributed to rare probabilistic events. Consequently, from this perspective little insight can be gained into causes of such market crashes that occur in the absence of changes in valuation.

An alternative perspective into asset price dynamics that has been studied since 1990 is the asset flow approach in which infinite arbitrage is not assumed (see [4], [5], [6], [7], [9], [11], [14], [27], [28] and references therein). These models, described below, stipulate that relative price change depends on supply and demand, but these quantities can depend on other factors such as the price trend. Trading motivations beyond valuation are a main issue for behavioral finance (see, for example, [3], [13], [22], [31], [32], [33]). Furthermore, the cash and stock are finite, so that one cannot rely on near perfect arbitrage to enforce an equation such as (1.1) above. The dependence on trend and the finiteness of assets are factors that market practitioners generally assume. There are also statistical studies that confirm the role of momentum (or trend-based) investing in asset markets [30].

In this paper we pursue two related issues. First, we study the onset of instability as parameters such as the trend (or momentum) coefficient are varied. In particular, we can use this as the bifurcation parameter in the system of ODE's. Second, we demonstrate numerically that there is an interval in the trend parameter for which one has cyclic behavior in price and other variables. We prove the existence of a Hopf bifurcation and explore it further with numerical computation. The cyclic behavior within a particular interval of the bifurcation parameter (the trend coefficient) is significant in terms of economics and finance since the model assumes a single valuation (as does classical

finance) but yields results that show that equilibrium in the classical context does not exist for a range of parameters. Indeed the introduction of trend as a motivation for decisions and the finiteness of assets are the only prerequisites for a radical change in the type of equilibrium that is attained. In financial economics, the equilibrium price has generally been assumed to be a single value. However, our work suggests that for a range of parameters, the equilibrium must be regarded as a limit cycle, i.e., a periodic orbit to which the solution trajectories converge.

The solution trajectory starts at some initial value and oscillates until it reaches a cyclic orbit and remains cyclic indefinitely. Thus the notion of equilibrium is that the variables (including price) oscillate periodically, even though there are no new assets, strategies or information introduced into the equations after the initial conditions. The periodicity in the asset flow equations is not a feature of an exogenous factor such as time lag due to planting/harvesting, etc., as it is for the cobweb theorem.

Studies on Hopf bifurcation and its properties has attracted substantial interest in mathematical biology, medicine, ecology, economics, and so on. Many theoreticians and experimentalists have paid great attention to differential equations having periodic solutions which have significant biological and physical meaning. More specifically, they have concentrated on the stability of solutions and Hopf bifurcation occurrence (see, for example, the references [2], [15], [17], [24], [25], [29], [35] and the references therein).

Stability and bifurcation analysis of the asset flow models have recently been studied by several researchers. DeSantis and Caginalp [9] have studied stability of a multi-group (two-group) asset flow model of a financial instrument with one group focused on price trend, the other one on value. They determined a curve of equilibrium points of the model given the basic parameters governing the investor group motivations and assets, and showed that a strong motivation based on price trend is associated with instability. DeSantis, Swigon and Caginalp [14] have studied stability and bifurcation properties of that model and have shown that as key parameters were varied, one moved from an unstable point to a nearby stable point along this curve of equilibria. In many cases, however, the numerical computations showed that the trajectory underwent a large "excursion" prior to converging to the stable point. In this paper, the stability and bifurcation properties were established for the curve of equilibria. Duran [16] has also studied the numerical stability of a single group asset flow model. In the current work, we use a related model to determine conditions on parameters for which an equilibrium point is stable. Moreover, we show the existence of a Hopf bifurcation by taking the trend-based (or momentum) coefficient as a bifurcation parameter, which is an important conclusion in economics and finance points of view. We also support the analytical result numerically by using MATLAB.

This paper is organized as follows. Section 2 presents the mathematical model. In Section 3 we study theoretical results such as conditions for equilibrium and stability. Section 4 gives a detailed Hopf bifurcation analysis of the model. Section 5 presents numerical results including a family of periodic solutions at the critical bifurcation value. While the vast majority of analysis in dynamical systems focuses on the initial onset of stability, existence of a periodic solution is of great importance in many applications.

Finally, in the conclusion (Section 6) we discuss the implications of our results in terms of finance and directions for future research.

2. The model. Caginalp and his collaborators have modeled a closed market containing N shares of a risky asset (e.g., a stock) and a total of M dollars distributed arbitrarily among a single group of investors at the outset. The dynamics of the market is determined by the following system of differential equations (see [5] and references therein):

$$\delta^{-1}\frac{1}{P}\frac{dP}{dt} = F\left(\frac{k}{1-k}\frac{1-B}{B}\right), \tag{2.1}$$

$$\frac{dB}{dt} = k(1-B) + (k-1)B + B(1-B)\frac{1}{P}\frac{dP}{dt}, \tag{2.2}$$

$$\frac{d\zeta_1}{dt} = c_1 q_1 \frac{1}{P}\frac{dP}{dt} - c_1\zeta_1, \tag{2.3}$$

$$\frac{d\zeta_2}{dt} = c_2 q_2 \frac{P_a - P}{P_a} - c_2\zeta_2, \tag{2.4}$$

where

1. $P(t)$ is the trading price of the asset at time t, $P_a(t)$ is the fundamental value;
2. δ^{-1} is a time scale;
3. F is an increasing function satisfying $F(1) = 0$, such as $x-1$ or $\log x$, and $F'(1) > 0$, so prices remain unchanged if supply and demand are equal, while prices increase if demand exceeds supply;
4. $B(t)$ is the fraction of total funds in the risky asset, i.e. $B(t) = \frac{NP(t)}{NP(t)+M}$, so the fraction in cash is $1 - B(t) = \frac{M}{NP(t)+M}$;
5. k is the transition rate function defined in terms of the sentiment functions, $\zeta_1(t)$ and $\zeta_2(t)$, as $k(t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\zeta_1(t) + \zeta_2(t)\right)$. Here, $\zeta_1(t)$ is the trend-based component of the investor preference that quantifies the investor group's sentiment toward the price trend, and $\zeta_2(t)$ is the value-based component that quantifies that sentiment toward the deviation from fundamental value. They are defined mathematically as follows:

$$\zeta_1(t) := q_1 c_1 \int_{-\infty}^{t} \frac{1}{P(\tau)}\frac{dP(\tau)}{d\tau} e^{-c_1(t-\tau)} d\tau, \tag{2.5}$$

$$\zeta_2(t) := q_2 c_2 \int_{-\infty}^{t} \frac{P_a(\tau) - P(\tau)}{P_a(\tau)} e^{-c_2(t-\tau)} d\tau. \tag{2.6}$$

Equation (2.5) is the mathematical expression of a sum of the impact of all former relative price changes before time t. Equation (2.6) represents the fundamentalist sentiment, i.e., the tendency to buy, with some finite reaction time, when the stock is undervalued. In both equations, the parameters q_i and c_i, $i = 1, 2$, are the amplitude constants and the inverses of the time scales for the two motivations, respectively. For example, one may consider that c_1^{-1} is a measure of the memory length. A linear sum of the functions ζ_1 and ζ_2 gives the sentiment function that quantifies the investor sentiment toward the price trend and deviation from fundamental value. The function $k(t)$ may include other motivations and behavioral effects besides the trend and discount, such as high/low liquidity, price and trading history, and inherent behavioral biases (fear/hope). Furthermore, the

structure of the sentiment function could be nonlinear rather than simply linear additive (see [34], [23], [31] for more discussions). Differentiating these equations using the Leibnitz rule, one can obtain the differential equations (2.3) and (2.4).

As in [5], let the liquidity be defined as the ratio of cash to asset quantity, $L := M/N$. It was observed there that liquidity is a third variable (in addition to trading price and fundamental value) that has the units of dollars per share, providing a natural unit to measure price (see scaling below). With this definition one also has a simple relation for the ratio between the fraction of total funds in the risky asset, B, and that in cash, $1 - B$, namely,

$$\frac{B}{1-B} = \frac{N}{M}P = \frac{P}{L} \quad \text{and} \quad B = \frac{P}{L+P}. \tag{2.7}$$

If M and N are fixed in the system, then L is a constant so that the rate of change of B can be obtained from that of P by using the equation in (2.7). Therefore, we will consider a system of the equations involving only the equations (2.1), (2.3) and (2.4) instead of the equations (2.1)-(2.4) above as long as M and N are fixed. Next, we scale these equations as follows.

Let

$$\widetilde{P} := \frac{P}{L} \quad \text{and} \quad \widetilde{P}_a := \frac{P_a}{L}, \tag{2.8}$$

where L is constant. Then we can write (2.1) as

$$\frac{d\widetilde{P}}{dt} = \delta \widetilde{P} F\left(\frac{k}{1-k}\frac{1}{\widetilde{P}}\right). \tag{2.9}$$

Similarly, (2.3) and (2.4) can be written as

$$\frac{d\zeta_1}{dt} = c_1 q_1 \frac{1}{\widetilde{P}}\frac{d\widetilde{P}}{dt} - c_1\zeta_1, \tag{2.10}$$

$$\frac{d\zeta_2}{dt} = c_2 q_2 \frac{\widetilde{P}_a - \widetilde{P}}{\widetilde{P}_a} - c_2\zeta_2, \tag{2.11}$$

where all parameters are the same as in the system of equations (2.1)-(2.4).

3. Local stability analysis of equilibrium points. We study the stability analysis of the rescaled model defined by the equations (2.9), (2.10) and (2.11) under the following constraints:

i. $F(x) = x - 1$,
ii. $k(t) \approx \frac{1}{2} + \frac{1}{2}(\zeta_1(t) + \zeta_2(t))$ for small values of $\zeta_1(t) + \zeta_2(t)$,
iii. $|\zeta_1(t) + \zeta_2(t)| < \varepsilon$, where ε is a small positive number,
iv. $\widetilde{P}_a(t) = \widetilde{P}_a > 0$, $\widetilde{P}_a$ is constant,
v. c_1, c_2, q_1 and q_2 are all positive parameters.

If we rewrite the model under the constraints (i)-(v), then we have the following equations:

$$\frac{d\widetilde{P}}{dt} = \frac{\delta(1+\zeta_1+\zeta_2)}{1-\zeta_1-\zeta_2} - \delta\widetilde{P}, \tag{3.1}$$

$$\frac{d\zeta_1}{dt} = \frac{\delta c_1 q_1(1+\zeta_1+\zeta_2)}{\widetilde{P}(1-\zeta_1-\zeta_2)} - \delta c_1 q_1 - c_1\zeta_1, \tag{3.2}$$

$$\frac{d\zeta_2}{dt} = c_2 q_2 \left(1 - \frac{\widetilde{P}}{\widetilde{P}_a}\right) - c_2\zeta_2. \tag{3.3}$$

As a vector equation form one can write them as

$$\mathbf{X}' = \mathbf{F}(\mathbf{X}), \tag{3.4}$$

where $\mathbf{X} = (\widetilde{P}, \zeta_1, \zeta_2)^t$, $\mathbf{F} = (f_1, f_2, f_3)^t$ and

$$f_1(\widetilde{P}, \zeta_1, \zeta_2; q_1, q_2, \widetilde{P}_a, c_1, c_2) \quad : \quad = \frac{\delta(1+\zeta_1+\zeta_2)}{1-\zeta_1-\zeta_2} - \delta\widetilde{P}, \tag{3.5}$$

$$f_2(\widetilde{P}, \zeta_1, \zeta_2; q_1, q_2, \widetilde{P}_a, c_1, c_2) \quad : \quad = \frac{\delta c_1 q_1(1+\zeta_1+\zeta_2)}{\widetilde{P}(1-\zeta_1-\zeta_2)} - \delta c_1 q_1 - c_1\zeta_1, \tag{3.6}$$

$$f_3(\widetilde{P}, \zeta_1, \zeta_2; q_1, q_2, \widetilde{P}_a, c_1, c_2) \quad : \quad = c_2 q_2 \left(1 - \frac{\widetilde{P}}{\widetilde{P}_a}\right) - c_2\zeta_2. \tag{3.7}$$

The equilibrium points of system (3.4) can be obtained by solving the following equation for $\widetilde{P}$, ζ_1 and ζ_2:

$$\mathbf{F}(\mathbf{X}) = \mathbf{0}.$$

Also, $f_1 = 0$ yields that

$$\widetilde{P} = \frac{(1+\zeta_1+\zeta_2)}{1-\zeta_1-\zeta_2}. \tag{3.8}$$

From $f_2 = 0$ together with (3.8) we obtain $\zeta_1^{eq} = 0$ yielding

$$\widetilde{P}_{eq} = \frac{1+\zeta_2^{eq}}{1-\zeta_2^{eq}}, \tag{3.9}$$

i.e.,

$$\zeta_2^{eq} = \frac{\widetilde{P}_{eq} - 1}{\widetilde{P}_{eq} + 1}. \tag{3.10}$$

Finally, from $f_3 = 0$ we have

$$\zeta_2^{eq} = q_2 \frac{\widetilde{P}_a - \widetilde{P}_{eq}}{\widetilde{P}_a}. \tag{3.11}$$

Now combining (3.9) and (3.11) yields the following quadratic equation in $\widetilde{P}_{eq}$:

$$\widetilde{P}_{eq}^2 + \left(1 - \widetilde{P}_a + \frac{\widetilde{P}_a}{q_2}\right)\widetilde{P}_{eq} - (1+q_2)\frac{\widetilde{P}_a}{q_2} = 0, \tag{3.12}$$

which has the positive root

$$\widetilde{P}_{eq} = \frac{\left(\widetilde{P}_a - \frac{\widetilde{P}_a}{q_2} - 1\right) + \sqrt{\left(1 - \widetilde{P}_a + \frac{\widetilde{P}_a}{q_2}\right)^2 + 4(1+q_2)\frac{\widetilde{P}_a}{q_2}}}{2}. \tag{3.13}$$

From a market point of view, only the positive equilibrium points are relevant, so we ignore $\widetilde{P}_{eq} = \frac{(.)-\sqrt{.}}{2}$, which is negative. Under the constraints (i)-(v) the equilibrium

points have the following forms:

$$\left(\widetilde{P}_{eq}, \zeta_1^{eq}, \zeta_2^{eq}\right) = \left(\frac{1+\zeta_2^{eq}}{1-\zeta_2^{eq}}, 0, \zeta_2^{eq}\right) \tag{3.14}$$

$$= \left(\frac{\widetilde{P}_a + q_2(\widetilde{P}_a - \widetilde{P}_{eq})}{\widetilde{P}_a - q_2(\widetilde{P}_a - \widetilde{P}_{eq})}, 0, q_2\frac{\widetilde{P}_a - \widetilde{P}_{eq}}{\widetilde{P}_a}\right) \tag{3.15}$$

$$= \left(\widetilde{P}_{eq}, 0, q_2\frac{\widetilde{P}_a - \widetilde{P}_{eq}}{\widetilde{P}_a}\right) = \left(\widetilde{P}_{eq}, 0, \frac{\widetilde{P}_{eq}-1}{\widetilde{P}_{eq}+1}\right), \tag{3.16}$$

where $\widetilde{P}_{eq}$ denotes the scaled equilibrium price determined by (3.13). Notice that the system has infinitely many equilibrium points, and the form of the equilibrium in (3.16) shows that each equilibrium point depends only on the parameters $\widetilde{P}_a$ and q_2. Notice also that (3.15) indicates that if $\widetilde{P}_a = \widetilde{P}_{eq}$, then the equilibrium point will be $(1, 0, 0)$ or, equivalently, $(P_{eq}, \zeta_1^{eq}, \zeta_2^{eq}) = (L, 0, 0)$ so that $P_{eq} = P_a = L$ at this equilibrium.

Next we shift the equilibrium point $\left(\widetilde{P}_{eq}, \zeta_1^{eq}, \zeta_2^{eq}\right)$ to the origin. Let $y_1(t) := \widetilde{P}(t) - \widetilde{P}_{eq}$, $y_2(t) := \zeta_1(t) - \zeta_1^{eq}$ and $y_3(t) := \zeta_2(t) - \zeta_2^{eq}$ be new variables. Then, system (3.4) can be written as

$$\mathbf{Y}' = \mathbf{J}\mathbf{Y} + \mathbf{h.o.t.}, \tag{3.17}$$

where $\mathbf{Y} = (y_1, y_2, y_3)^t$, **h.o.t.** represents higher order terms in (y_1, y_2, y_3) and $\mathbf{J}$ is the Jacobian matrix of the system at these equilibria with the form

$$\mathbf{J} = \begin{bmatrix} -\delta & \frac{\delta(1+\widetilde{P}_{eq})^2}{2} & \frac{\delta(1+\widetilde{P}_{eq})^2}{2} \\ -\frac{c_1 q_1 \delta}{\widetilde{P}_{eq}} & \frac{c_1 q_1 \delta(1+\widetilde{P}_{eq})^2}{2\widetilde{P}_{eq}} - c_1 & \frac{c_1 q_1 \delta(1+\widetilde{P}_{eq})^2}{2\widetilde{P}_{eq}} \\ -\frac{c_2 q_2}{\widetilde{P}_a} & 0 & -c_2 \end{bmatrix}, \tag{3.18}$$

in which $\widetilde{P}_{eq}$ is determined by (3.13). The nonlinear system (3.17) is linearly equivalent to the following linear system locally:

$$\mathbf{Y}' = \mathbf{J}\mathbf{Y}. \tag{3.19}$$

Its characteristic polynomial is

$$Q(\lambda) := \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \tag{3.20}$$

where

$$a_2 \quad : \quad = -trace(J) = \delta + c_1 + c_2 - \frac{\delta c_1 q_1 (1+\widetilde{P}_{eq})^2}{2\widetilde{P}_{eq}}, \tag{3.21}$$

$$a_1 \quad : \quad = \sum_{k=1}^{3} J_{kk} = \text{Sum of the principle minors of } J$$

$$= \delta(c_1 + c_2) + c_1 c_2 + \frac{\delta c_2 (1+\widetilde{P}_{eq})^2}{2}\left(\frac{q_2}{\widetilde{P}_a} - \frac{c_1 q_1}{\widetilde{P}_{eq}}\right), \tag{3.22}$$

$$a_0 \quad : \quad = -\det(J) = \delta c_1 c_2 \left(1 + \frac{q_2}{2\widetilde{P}_a}(1+\widetilde{P}_{eq})^2\right). \tag{3.23}$$

Note that a_0 is always positive under the constraints (iv)-(v).

THEOREM 1. Equilibria of the system of equations (3.1)-(3.3) are asymptotically stable if the following condition holds:

(C1): either $q_1 \in (0, K_1) \cap (0, K_2)$ or $q_1 \in (0, K_1) \cap (K_3, \infty)$, where

$$K_1 := \min \left\{ \begin{array}{c} \frac{2(\delta+c_1+c_2)\widetilde{P}_{eq}}{\delta c_1(1+\widetilde{P}_{eq})^2}, \frac{2(c_1+c_2)\widetilde{P}_{eq}}{c_1c_2(1+\widetilde{P}_{eq})^2} \\ +\frac{2\widetilde{P}_{eq}}{\delta(1+\widetilde{P}_{eq})^2} + \frac{q_2\widetilde{P}_{eq}}{c_1\widetilde{P}_a} \end{array} \right\}, \tag{3.24}$$

$$K_2 := \frac{B - \sqrt{B^2 - 4AC}}{2A}, \tag{3.25}$$

$$K_3 := \frac{B + \sqrt{B^2 - 4AC}}{2A}, \tag{3.26}$$

in which

$$A := \frac{c_2c_1^2\delta^2(1+\widetilde{P}_{eq})^4}{4\widetilde{P}_{eq}^2}, \tag{3.27}$$

$$\begin{aligned} B &:= \frac{\delta c_1c_2(\delta+c_1)(1+\widetilde{P}_{eq})^2}{\widetilde{P}_{eq}} + \frac{\delta c_1(\delta c_1 + c_2^2)(1+\widetilde{P}_{eq})^2}{2\widetilde{P}_{eq}} \\ &\quad + \frac{c_1c_2q_2\delta^2(1+\widetilde{P}_{eq})^4}{4\widetilde{P}_a\widetilde{P}_{eq}}, \end{aligned} \tag{3.28}$$

$$\begin{aligned} C &:= \delta^2(c_1+c_2) + \delta(c_1+c_2)^2 + c_1c_2(c_1+c_2) \\ &\quad + \frac{\delta c_2q_2(\delta+c_2)(1+\widetilde{P}_{eq})^2}{2\widetilde{P}_a}. \end{aligned} \tag{3.29}$$

Proof. The Routh-Hurwitz criteria give necessary and sufficient conditions for all of the roots of a polynomial (with real coefficients) to lie in the left half of the complex plane (see [1], page 150). Thus, all of the roots of the characteristic polynomial $Q(\lambda)$ (see (3.20)) are negative or have negative real part if and only if

(1) all coefficients a_0, a_1 and a_2 are strictly positive,
(2) $a_1a_2 - a_0 > 0$.

From (3.23) together with the constraints (iv)-(v) one can see that a_0 is always positive. From (3.22), the coefficient a_1 is positive if

$$q_1 < \frac{2(c_1+c_2)\widetilde{P}_{eq}}{c_1c_2(1+\widetilde{P}_{eq})^2} + \frac{2\widetilde{P}_{eq}}{\delta(1+\widetilde{P}_{eq})^2} + \frac{q_2\widetilde{P}_{eq}}{c_1\widetilde{P}_a}.$$

Similarly, from (3.21), a_2 is positive if

$$q_1 < \frac{2(\delta+c_1+c_2)\widetilde{P}_{eq}}{\delta c_1(1+\widetilde{P}_{eq})^2}.$$

Let K_1 denote the minimum of the right hand sides of the inequalities above, i.e.,

$$K_1 := \min \left\{ \frac{2(\delta+c_1+c_2)\widetilde{P}_{eq}}{\delta c_1(1+\widetilde{P}_{eq})^2}, \frac{2(c_1+c_2)\widetilde{P}_{eq}}{c_1c_2(1+\widetilde{P}_{eq})^2} + \frac{2\widetilde{P}_{eq}}{\delta(1+\widetilde{P}_{eq})^2} + \frac{q_2\widetilde{P}_{eq}}{c_1\widetilde{P}_a} \right\}.$$

Thus, both coefficients a_1 and a_2 are positive if

$$q_1 < K_1.$$

Next we determine the conditions on the parameters such that $a_1a_2 - a_0 > 0$. Using (3.21)-(3.23) one obtains that

$$a_1a_2 - a_0 > 0 \quad \text{iff} \quad Aq_1^2 - Bq_1 + C > 0,$$

where the coefficients A, B and C of the quadratic polynomial in q_1 are all positive and defined by (3.27)-(3.29). The roots of the polynomial $Aq_1^2 - Bq_1 + C$ are

$$(q_1)_{1,2} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Note that q_1 is a (real) positive parameter. Then, the polynomial has real roots if $B^2 - 4AC \geq 0$. Thus, the inequality $Aq_1^2 - Bq_1 + C > 0$ is satisfied iff

$$\text{either} \quad q_1 < K_2 \quad \text{or} \quad q_1 > K_3,$$

where

$$K_2 \quad : \quad = \frac{B - \sqrt{B^2 - 4AC}}{2A},$$
$$K_3 \quad : \quad = \frac{B + \sqrt{B^2 - 4AC}}{2A},$$

which are both positive. Finally, both conditions (1) and (2) of the Routh-Hurwitz criteria are satisfied together if either $q_1 \in (0, K_1) \cap (0, K_2)$ or $q_1 \in (0, K_1) \cap (K_3, \infty)$. This completes the proof. □

4. Bifurcation analysis of the model.

LEMMA 1. The (characteristic) polynomial $Q(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ has a pair of pure imaginary roots if and only if $a_1a_2 = a_0$ and $a_1 > 0$.

Proof. Let $\lambda = iw$ be a root of $Q(\lambda)$ where $w \in R$ and $w > 0$. Substituting $\lambda = iw$ into the equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$ one obtains

$$(a_0 - a_2w^2) + iw(a_1 - w^2) = 0.$$

The left hand side of the equation above is zero if and only if both $a_0 - a_2w^2 = 0$ and $a_1 - w^2 = 0$, which leads to the following equality:

$$\frac{a_0}{a_2} = w^2 = a_1 \quad \Rightarrow \quad a_1a_2 = a_0. \tag{4.1}$$

Now using the first identity, one can determine $w = \sqrt{a_1}$ where $a_1 > 0$.

Let us now assume that $a_1a_2 = a_0$ and $a_1 > 0$. Then, the polynomial $Q(\lambda)$ can be factored and the equation $\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$ can be written as

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_1a_2 = \lambda(\lambda^2 + a_1) + a_2(\lambda^2 + a_1) = (\lambda + a_2)(\lambda^2 + a_1) = 0,$$

which yields the following roots:

$$\lambda = -a_2 \text{ and } \lambda = \pm i\sqrt{a_1}. \tag{4.2}$$

This completes the proof. □

REMARK 1. In Lemma 1, if the two conditions $a_1a_2 = a_0$ and $a_1 > 0$ are satisfied, then both a_2 and a_0 have the same sign. In other words, if $Q(\lambda)$ has a pair of pure imaginary roots, then the coefficients a_2 and a_0 must have the same sign. Furthermore, since a_0 is always positive, a_2 must be positive in order to have pure imaginary roots.

THEOREM 2. The characteristic polynomial $Q(\lambda)$ has pure imaginary roots, namely $\lambda_{1,2} = \pm i\sqrt{a_1}$, if the following condition holds:

(C2): either $q_1 = K_2 < K_1$ or $q_1 = K_3 < K_1$, where K_1, K_2 and K_3 are defined by (3.24)-(3.29).

Proof. If q_1 is equal to either K_2 or K_3, then $a_1a_2 = a_0$. On the other hand, both a_1 and a_2 are positive if $q_1 < K_1$. Then the proof follows from Lemma 1. □

Next we determine the conditions on parameters such that $Q(\lambda)$ has a pair of complex conjugate roots of the form

$$\lambda_{1,2}(\alpha) = m(\alpha) \pm in(\alpha), \tag{4.3}$$

where $m(\alpha)$ and $n(\alpha)$ are real numbers, and $n(\alpha)$ is nonzero. Here, α represents the bifurcation parameter. In order to show that the system undergoes a Hopf bifurcation at the equilibrium point we first need to find a critical value, α_c, of α at which

$$m(\alpha_c) = 0 \quad \text{and} \quad n(\alpha_c) = n_0 > 0. \tag{4.4}$$

Notice that all coefficients of $Q(\lambda)$ depend on the parameters q_1, q_2, c_1, c_2 and $\widetilde{P}_a$. Therefore, α can be chosen as one of them to study Hopf bifurcation analysis. In this work, we will choose q_1 to be a bifurcation parameter and fix the others. As a result, the equilibrium point will be unique and also isolated for these fixed parameters since it is independent of the parameter q_1. In other words, varying q_1 does not change the equilibrium point. In this case, we can apply the existence theory, i.e., the general Hopf bifurcation theorem (see the references [19] and [26]), in literature to show the existence of Hopf bifurcation.

Let us now assume that $Q(\lambda)$ has roots of the form of (4.3). Then, substituting it into $Q(\lambda)$ yields the following two equations:

$$n^2 - m^2 - 2a_2m - a_1 = 0, \tag{4.5}$$

$$2a_2m^2 + 2a_2^2m + a_1a_2 - a_0 = 0. \tag{4.6}$$

One can see that if a_2 was zero, then a_0 had to be zero due to (4.6). However, we know that a_0 is always positive (see (3.23)) so that a_2 cannot be zero. Furthermore, from Remark 1, a_2 cannot be negative either in order to have pure imaginary roots. Then, the roots of the quadratic equation (4.6) have the following forms:

$$m_1 := m_1(\alpha) = \frac{-a_2^2 + \sqrt{(a_2^2)^2 - 2a_2(a_1a_2 - a_0)}}{2a_2}, \tag{4.7}$$

$$m_2 := m_2(\alpha) = \frac{-a_2^2 - \sqrt{(a_2^2)^2 - 2a_2(a_1a_2 - a_0)}}{2a_2}, \tag{4.8}$$

where a_0, a_1 and a_2 are all functions of α. Notice first that the quantity $a_1a_2 - a_0$ in the radicand determines the sign of the roots $m_{1,2}$ so that it also determines the critical value of the bifurcation parameter, namely α_c. Notice also that the radicand

$(a_2^2)^2 - 2a_2(a_1a_2 - a_0)$ cannot be negative since we assume that m is a real number! (see (4.3)). A complete analysis of the roots m_1 and m_2 together with that of the characteristic polynomial $Q(\lambda)$ is given below:

CASE 1. If $a_2 > 0$ and $a_1a_2 - a_0 = 0$ in (4.7), then

$$m_1(\alpha_c) = \frac{-a_2^2 + \sqrt{(a_2^2)^2}}{2a_2} = 0$$

so that $Q(\lambda)$ has a pair of pure imaginary roots, namely $\lambda_{1,2} = \pm i\sqrt{a_1}$, where $a_1 = \frac{a_0}{a_2}$ and a_0 is always positive (see (3.23)) (see also (4.3) and (4.5)). The third root of $Q(\lambda)$, which is denoted by λ_3 in the rest of the work, will be $\lambda_3 = -a_2$, which is negative (see Lemma 1).

CASE 2. If $a_2 > 0$ and $a_1a_2 - a_0 > 0$ in (4.7), then $m_1 < 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with negative real parts. On the other hand, since we assume that $a_2 > 0$ and $a_1a_2 - a_0 > 0$ and also know that a_0 is always positive (see (3.23)), a_1 must be positive. Thus, by Theorem 1, the third root of $Q(\lambda)$ is either a negative real number or a complex number with negative real part, i.e., $\mathrm{Re}(\lambda_3) < 0$. In this case, the equilibrium point is asymptotically stable.

CASE 3. If $a_2 > 0$ and $a_1a_2 - a_0 < 0$ in (4.7), then $m_1 > 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with positive real parts. Furthermore, since we assume that $a_1a_2 - a_0 < 0$, $\mathrm{Re}(\lambda_3) > 0$ by Theorem 1. Therefore, the equilibrium point is unstable for this case.

CASE 4. If $a_2 > 0$ and $a_1a_2 - a_0 = 0$ in (4.8), then

$$m_2(\alpha_c) = \frac{-a_2^2 - \sqrt{(a_2^2)^2}}{2a_2} = -a_2$$

so that $Q(\lambda)$ has a pair of complex conjugate roots whose real parts are $-a_2$ which is negative. However, since we assume that $a_1a_2 - a_0 = 0$, Theorem 1 leads to the result that $\mathrm{Re}(\lambda_3)$ cannot be negative so that the equilibrium point is unstable in this case.

CASE 5. If $a_2 > 0$ and $a_1a_2 - a_0 > 0$ in (4.8), then $m_2 < 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with negative real parts. On the other hand, since we assume that $a_2 > 0$ and $a_1a_2 - a_0 > 0$ and also know that a_0 is always positive (see (3.23)), a_1 must be positive. Thus, $\mathrm{Re}(\lambda_3) < 0$ by Theorem 1. In this case, the equilibrium point is asymptotically stable.

CASE 6. If $a_2 > 0$ and $a_1a_2 - a_0 < 0$ in (4.8), then $m_2 < 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with negative real parts. However, since we assume that $a_1a_2 - a_0 < 0$, $\mathrm{Re}(\lambda_3)$ cannot be negative by Theorem 1. Thus, the equilibrium point is unstable for this case.

CASE 7. If $a_2 < 0$ and $a_1a_2 - a_0 = 0$ in (4.7), then $m_1(\alpha_c) = 0$. However, since $a_1 = \frac{a_0}{a_2} < 0$, $n = \sqrt{a_1} \notin R$ (see also (4.3) and (4.5)) . Thus, there is no solution for the parameters satisfying the conditions in this case.

CASE 8. If $a_2 < 0$ and $a_1a_2 - a_0 > 0$ in (4.7), then $m_1 < 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with negative real parts. But, since $a_2 < 0$, $\mathrm{Re}(\lambda_3) > 0$ by Theorem 1. In this case, the equilibrium point is unstable.

CASE 9. If $a_2 < 0$ and $a_1a_2 - a_0 < 0$ in (4.7), then $m_1 > 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with positive real parts, so the equilibrium point is unstable.

CASE 10. If $a_2 < 0$ and $a_1a_2 - a_0 = 0$ in (4.8), then $m_2(\alpha_c) = -a_2$ so that $Q(\lambda)$ has a pair of complex conjugate roots whose real parts are $-a_2$ that is positive. In this case, the equilibrium point is unstable.

CASE 11. If $a_2 < 0$ and $a_1a_2 - a_0 > 0$ in (4.8), then $m_2 > 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with positive real parts. In this case, the equilibrium point is unstable.

CASE 12. If $a_2 < 0$ and $a_1a_2 - a_0 < 0$ in (4.8), then $m_2 > 0$ so that $Q(\lambda)$ has a pair of complex conjugate roots with positive real parts. In this case, the equilibrium point is again unstable.

We conclude from the analysis above that Cases 1, 2 and 3 all lead to a Hopf bifurcation so long as one has a particular transversality condition (specified below) that is satisfied at the critical bifurcation value. Next we start by determining α_c and then checking the transversality condition. Once again, q_1 will be taken as the Hopf bifurcation parameter; i.e., we set $\alpha = q_1$ in this work. Notice that q_2 or $\widetilde{P}_a$ can also be chosen as a bifurcation parameter to study the bifurcation analysis of the system as well; however, the equilibrium point is no longer isolated for these parameters. As we mentioned earlier, the quantity $a_1a_2 - a_0$ determines the critical value of the bifurcation parameter. Therefore, solving the equation $a_1a_2 - a_0 = 0$ for q_1 we determine the critical value of the bifurcation parameter as

$$\alpha_c = q_1^* = K_2,\ K_3 \tag{4.9}$$

where K_2 and K_3 are defined by (3.25)-(3.29). Thus, from the analysis above (see Case 1), we have

$$m(q_1^*) = 0 \quad \text{and} \quad n(q_1^*) = \sqrt{a_1} \quad \Rightarrow \quad \lambda_{1,2} = \pm i\sqrt{a_1},\ \lambda_3 = -a_2 \tag{4.10}$$

when the condition (C2) holds. In other words, $Q(\lambda)$ has a pair of pure imaginary roots under the condition (C2) (see also Lemma 1). Furthermore, when either $q_1 < K_2 \leq K_1$ or $K_3 < q_1 < K_1$ (i.e., the condition (C1) holds), the equilibrium point is asymptotically stable (see Case 2). However, when $K_2 < q_1 < K_3$ the equilibrium point is unstable (see Case 3).

LEMMA 2. If the following condition holds:

$$\left.\frac{d(a_1a_2 - a_0)}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} \neq 0,$$

then

$$\left.\frac{dm}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} \neq 0,$$

where a_0, a_1, a_2 and m are all functions of α.

Proof. Since $m_2(\alpha)$ in (4.8) does not give a pair of pure imaginary roots for any choice of the parameters, we will take

$$m(\alpha) := m_1(\alpha) = -\frac{1}{2}a_2 + \frac{1}{2a_2}\sqrt{(a_2^2)^2 - 2a_2(a_1a_2 - a_0)}$$

as in (4.7) for the bifurcation analysis. Therefore, it is enough to show that

$$\left.\frac{dm_1}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} \neq 0$$

for the proof.

Now differentiating $m_1(\alpha)$ with respect to the bifurcation parameter α one obtains

$$\begin{aligned}\frac{dm_1}{d\alpha} &= -\frac{1}{2}\frac{d(a_2)}{d\alpha} - \frac{1}{2a_2^2}\frac{d(a_2)}{d\alpha}\sqrt{(a_2^2)^2 - 2a_2(a_1a_2 - a_0)} \\ &\quad + \frac{1}{2a_2}\frac{\left(4a_2^3\frac{d(a_2)}{d\alpha} - \left(2\frac{d(a_2)}{d\alpha}\right)(a_1a_2 - a_0) - 2a_2\frac{d(a_1a_2-a_0)}{d\alpha}\right)}{2\sqrt{(a_2^2)^2 - 2a_2(a_1a_2 - a_0)}}.\end{aligned}$$

Recall that when $\alpha = \alpha_c$, $a_1a_2 - a_0 = 0$. Then, evaluating the derivative of m_1 at $\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)$ yields that

$$\begin{aligned}\left.\frac{dm_1}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} &= -\frac{d(a_2)}{d\alpha} + \frac{1}{4a_2^3}\left(4a_2^3\frac{d(a_2)}{d\alpha} - 2a_2\frac{d(a_1a_2 - a_0)}{d\alpha}\right) \\ &= -\frac{1}{2a_2^2}\left(\left.\frac{d(a_1a_2 - a_0)}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)}\right).\end{aligned}$$

Since a_2 cannot be zero,

$$\left.\frac{dm_1}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} \neq 0$$

as long as

$$\left.\frac{d(a_1a_2 - a_0)}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} \neq 0.$$

□

COROLLARY 1. If $h(\alpha) := a_1(\alpha)a_2(\alpha) - a_0(\alpha)$ does not have a double root, then the transversality condition is satisfied.

Proof. One can show that

$$\mathrm{Re}\left(\frac{d\lambda(\alpha)}{d\alpha}\right)\Bigg|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} = \left.\frac{dm_1}{d\alpha}\right|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)}.$$

The proof follows from Lemma 2. □

The next lemma proves that the roots of $m(\alpha)$ are indeed simple.

LEMMA 3. The two roots of $m(\alpha)$ are simple at $\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)$.

Proof. Suppose that $m(\alpha)$ had double roots. Then, from (4.7) and (4.8) one can see that $(a_2^2)^2 - 2a_2(a_1a_2 - a_0) = 0$. However, when $\alpha = \alpha_c$, $a_1a_2 - a_0 = 0$ so that $(a_2^2)^2 = 0$, that is, $a_2(\alpha_c) = 0$. But, this forces $a_0(\alpha_c)$ to be zero due to (4.6). This contradicts the fact that $a_0(\alpha)$ is always nonzero (see (3.23)). This completes the proof. □

LEMMA 4. The following transversality condition holds:

$$\mathrm{Re}\left(\frac{d\lambda(\alpha)}{d\alpha}\right)\bigg|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} \neq 0.$$

Proof. One can show that

$$\mathrm{Re}\left(\frac{d\lambda(\alpha)}{d\alpha}\right)\bigg|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)} = \frac{dm}{d\alpha}\bigg|_{\left(\alpha_c;\widetilde{P}_{eq},\zeta_1^{eq},\zeta_2^{eq}\right)}.$$

Then the proof follows from Lemma 3. □

From the analysis above and the general Hopf bifurcation theorem (see the references [19] and [26] for hypotheses of the theorem), we can deduce the following result.

THEOREM 3. For the system of the equations (3.1)-(3.3) the following hold:

(1) If the condition (C1) is satisfied, then the equilibria of the system are asymptotically stable.
(2) If $K_2 < q_1 < K_3$, then the equilibria of the system are unstable.
(3) The system undergoes a Hopf bifurcation at these positive equilibria when the condition (C2) is satisfied.

5. Numerical simulations. In this section, we perform some numerical simulations in order to support and extend the analytical results that we have obtained in the former sections. As an example, we consider a closed market involving 500 dollars cash and 100 units of a particular stock distributed arbitrarily among a single group of investors at the outset. The system is conserved; that is, there is no cash or stock flow in and out of the system. We assume that the investor group values the stock as $P_a(t) = 4$, so the scaled fundamental value is $\widetilde{P}_a(t) = P_a(t)/L = 0.8$, which is fixed for all time t. For simplicity, the time scale δ is taken as 1. Also, we fix magnitude for the valuation motivation as $q_2 = 45$ and also take large time scales ($c_1 = 0.001$, $c_2 = 0.001$) for both trend and valuation motivations, respectively. Here, the liquidity value is $L = 5$. Our aim is to study numerically the dynamics of the market by using the nonlinear system (3.4) with the following parameters:

$$\delta = 1,\ c_1 = c_2 = 0.001,\ q_2 = 45 \quad \text{and} \tag{5.1}$$

$$M = 500,\ N = 100,\ L = \frac{M}{N} = 5,\ P_a = 4,\ \widetilde{P}_a = \frac{P_a}{L} = 0.8. \tag{5.2}$$

We vary the trend-based (or momentum) coefficient, q_1, which is the bifurcation parameter. With respect to the parameters above, the system has only one positive equilibrium point, namely $E^* := \left(\widetilde{P}_{eq}, \zeta_1^{eq}, \zeta_2^{eq}\right) = (0.8020, 0, -0.1099)$. K_1, K_2 and K_3 are calculated as follows:

$$K_1 = 494.9482,\ K_2 = 494.4543 \text{ and } K_3 = 46098.8148.$$

Note that $(0, K_1) \cap (0, K_2) = (0, 494.4543)$ and $(0, K_1) \cap (K_3, \infty) = \emptyset$. Therefore, by Theorem 1, the equilibrium point E^* is asymptotically stable when $q_1 \epsilon (0, K_2) = (0, 494.4543)$ and unstable when $q_1 > 494.4543$. Moreover, the Hopf bifurcation occurs at $q_1^* = K_2 =$

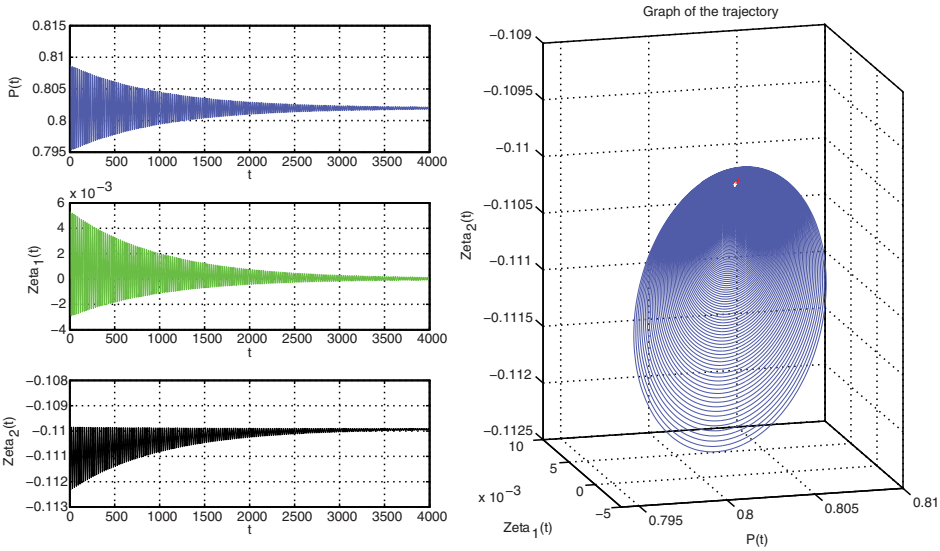

FIG. 1. Graphs of $P(t)$, $\zeta_1(t)$ and $\zeta_2(t)$ for $q_1 = 493.5 < q_c$ are on the left. Graph of the trajectory is on the right. The star denotes the equilibrium point.

$494.4543 < K_1$ (see the condition (C2) in Theorem 2), as illustrated by computer simulations below. For each simulation, we use the ODE package (ode45) in MATLAB (7.6.0). All simulations have been done in a time interval $[0, 4000]$ with increment 0.01.

In Figure 1 we take $\widetilde{P}(0) = 0.8000$, $\zeta_1(0) = 0$ and $\zeta_2(0) = -0.1099$ as an initial condition and plot the graph of each component of the solution for $q_1 = 493.5 < q_1^*$. Following this, we plot the graph of the trajectory $(P(t), \zeta_1(t), \zeta_2(t))$ as follows. These graphs illustrate that the equilibrium point is stable for the q_1 values that are smaller than the critical bifurcation value q_1^*.

In Figure 2 we use the same initial conditions as in Figure 1 and plot again the graph of each component of the solution for $q_1 = 495.12 > q_1^*$. Following this, we also plot the graph of the trajectory $(P(t), \zeta_1(t), \zeta_2(t))$. The graphs in this figure shows only the segment $t \in [3500, 4000]$. These graphs illustrate that the equilibrium point is unstable for the q_1 values that are larger than the critical bifurcation value q_1^*.

In Figure 3 we use the same initial conditions as in Figure 1 and plot again the graph of each component of the solution for $q_1 = 494.4543 = q_1^*$. Following this, we plot the graph of the trajectory $(P(t), \zeta_1(t), \zeta_2(t))$. Figure 3 shows only the segment $t \in [3500, 4000]$.

Figure 4 illustrates the graphs of the trajectories for the different bifurcation values, namely $q_1 = 494.46$ (inner), 494.48 (middle) and 494.50 (outer), respectively, which lie just above the critical bifurcation value. This figure underlines that there exists a small interval $[q_1^*, q^*)$ on which periodic solutions occur. Note that the amplitude of each periodic solution gets larger as the critical value q_1 gets larger from q_1^* to q^*. These graphs support Theorem 3.

In addition to numerical simulations above, Figure 5 shows the graphs of the trajectories for the different initial values, namely $\widetilde{P}(0) = 0.8015$, $\zeta_1(0) = 0$ and $\zeta_2(0) = -0.1099$

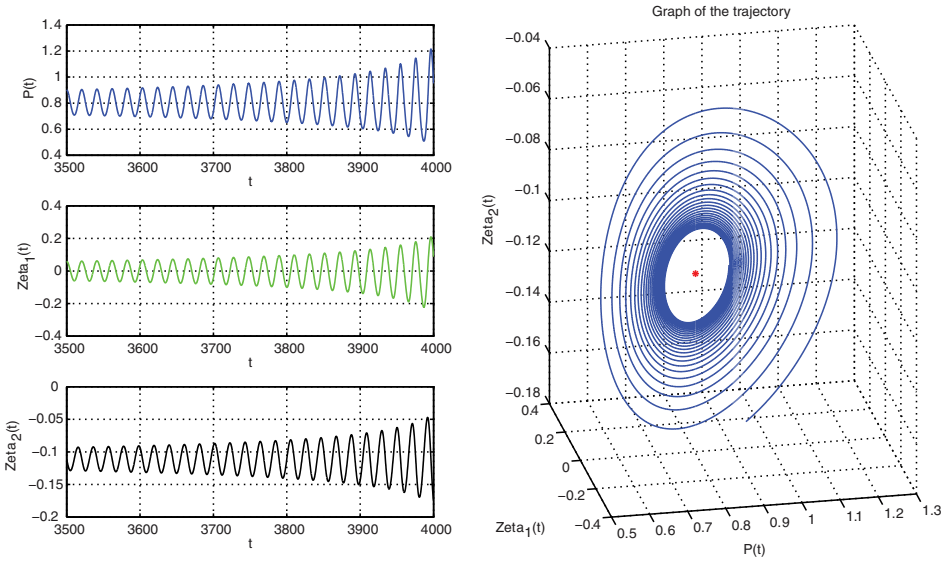

FIG. 2. Graphs of $P(t)$, $\zeta_1(t)$ and $\zeta_2(t)$ for $q_1 = 493.5 < q_c$ are on the left. Graph of the trajectory is on the right. The star denotes the equilibrium point.

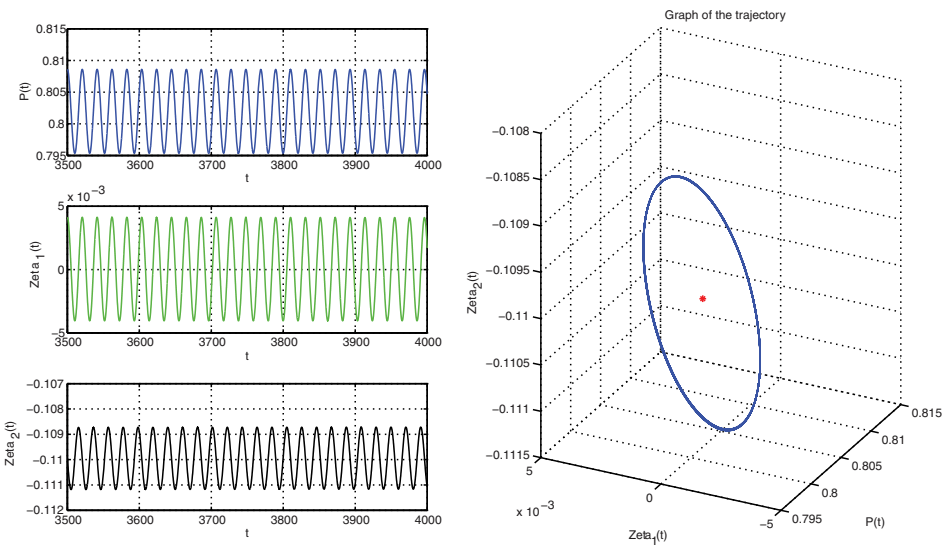

FIG. 3. Graphs of $P(t)$, $\zeta_1(t)$ and $\zeta_2(t)$ for $q_1 = 494.4543 = q_c$ are on the left. Graph of the trajectory is on the right. The star denotes the equilibrium point.

(inner), $\widetilde{P}(0) = 0.8010$, $\zeta_1(0) = 0$ and $\zeta_2(0) = -0.1099$ (middle) and $\widetilde{P}(0) = 0.8000$, $\zeta_1(0) = 0$ and $\zeta_2(0) = -0.1099$ (outer), respectively, for $q_1 = 494.4543 = q_1^*$. We also observe that we have a similar picture when we vary $\zeta_1(0)$ (or $\zeta_2(0)$) fixing other

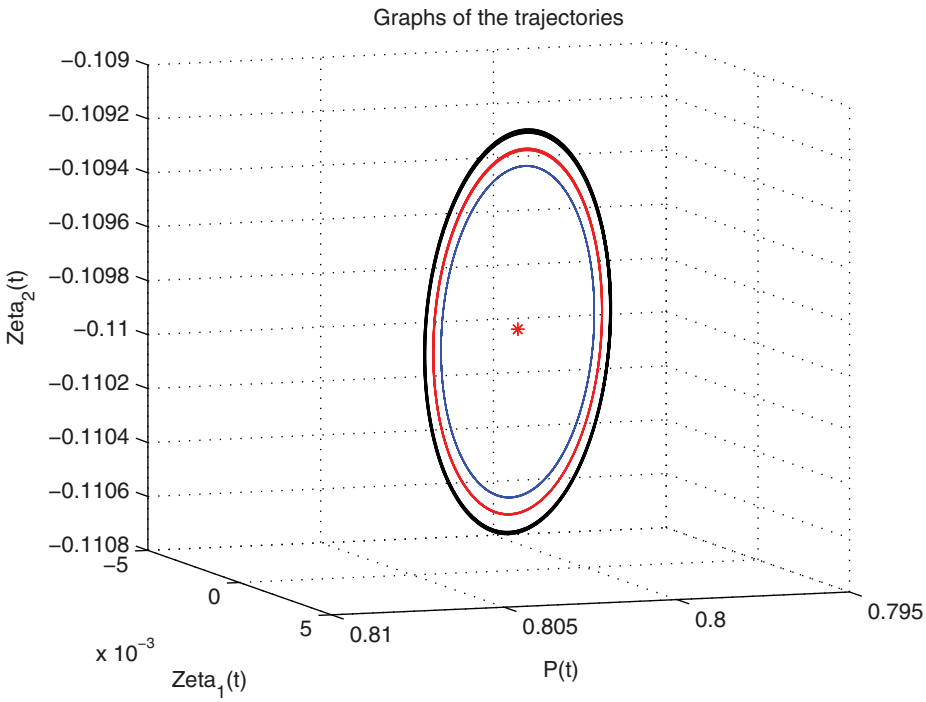

FIG. 4. Graphs of the trajectories for $q_1 = 494.46$ (inner), 494.48 (middle) and 494.50 (outer), respectively. Initial conditions are $\widetilde{P}(0) = 0.8010$, $\zeta_1(0) = 0$ and $\zeta_2(0) = -0.1099$ for each trajectory.

components. Figure 5 underlines that there exists a family of periodic solutions when $q_1 = 494.4542 = q_1^*$. In other words, the nonlinear system may have a pair of pure imaginary eigenvalues.

6. Discussion and conclusion. Classical finance models of asset price dynamics are largely based on equation (1.1). An immediate consequence of this model is the exclusion of any periodic behavior. Furthermore, any large, rapid deviation in price is attributed to a rare probabilistic event that is possible within Brownian motion. When such a large rapid change (sometimes called a "flash crash") occurs, there is usually a quest for a cause, and few believe the event is a purely random one. Unfortunately, using (1.1) as a starting point offers no hope of resolving this issue. In other words, the model starts with the assumption that prices behave according to Brownian motion (or in more generalized models, a Levy process in which the price path can include discontinuities). Thus, there is no possibility of analyzing market microstructure or participant strategies. In other words, any sharp drop in prices would be attributed to a rapid change in the fundamental value of the asset. If it is clear from an assessment of value (e.g., a calculation of potential stream of earnings) that there has been no fundamental change, then the "flash crash" can only be attributed to a very low probability random event.

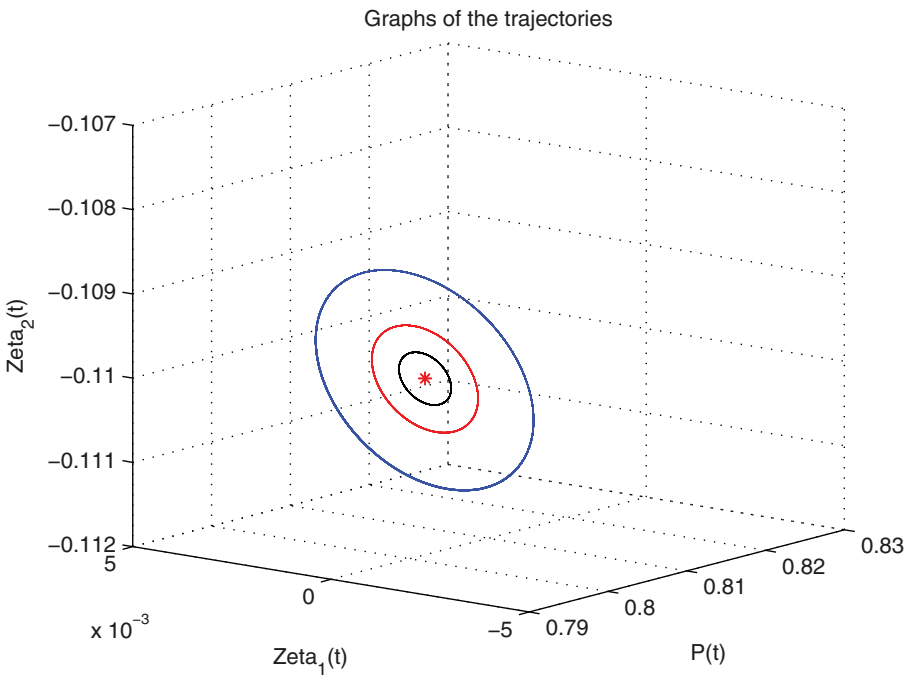

FIG. 5. Graphs of the trajectories for the initial values $\widetilde{P}(0) = 0.8015$ (inner), 0.8010 (middle) and 0.8000 (outer) for $q_1 = 494.4543 = q_c$. Other components of initial values are fixed as $\zeta_1(0) = 0$ and $\zeta_2(0) = -0.1099$ in each plot.

We use a dynamical systems approach to study the asset flow model developed by Caginalp and collaborators since 1990 (see [4]-[12], [14], [27], [28] and the references therein) to examine the extent to which the phenomenon of instability arises endogenously. Earlier studies have explored this possibility in the context of two groups of investors with different motivations, namely, trend based and value based (see [14] and references therein). In this study we have examined the issue in a more basic context in which there is a single group that is motivated by both trend and value. The mathematical explanations are consistent with the way many practitioners have viewed major crashes. If one examines the news analyses of market professionals after the broad market crashes of October 29, 1929, and October 19, 1987, in which the major US stock indices fell by 20% in two days and one day, respectively, the mechanisms for the crashes can be attributed to a combination of the finiteness of assets and strategies that depend on the trend. In particular, a major factor of the 1929 crash is believed to be the use of large amount of "margin" related selling. This means that an investor who has bought shares largely with borrowed money is forced to sell some of it (or provide more capital) when the ratio of the investor's capital to the borrowed money falls below a set fraction and the broker issues a "margin call". In this way, trend-based selling is a key factor as prices are falling. Classical finance would counter this by saying that there is a nearly infinite amount of cash that is available for arbitrage, and this arbitrage capital would be used to

profit by buying the shares that others are compelled to sell, thereby implicitly remedying the situation. While the arbitrage capital may be adequate under normal circumstances, during major crashes much of this arbitrage capital is usually invested utilizing similar strategies, so that the capital that is supposed to come to the rescue is actually a major part of the problem. In fact, if one examines the buildup to these crashes, including more recent ones, the problem often arises as an asset class makes steady gains that are far in excess of the prevailing interest rates. Large investors then observe that borrowing at low rates and buying these assets (sometimes called "carry trade") appears very profitable. Within the competitive environment in finance, it would be difficult for an asset manager to forgo such profits with the idea of waiting until prices crash. While more stringent margin requirements were imposed after the 1929 crash, other mechanisms were created over the ensuing decades, including derivatives and "portfolio insurance", which were supposed to protect a portfolio with the idea of selling options as the market fell. In essence this amounted to selling when there was a downtrend. The central idea here was that there would be a buyer for every seller at prevailing prices (as one would expect from the classical theory). Large amounts of capital that are supposedly ready to buy when prices drop and thereby rectify the situation were absent. This allowed prices to fall about 20% for the major indices and more for the smaller capitalized stocks. Ultimately, in each of these crashes, prices found some support at much lower prices than previously existed. From our perspective, these would be the value-based investors who had been awaiting a bargain without much regard for price trend.

Using the asset flow model introduced by Caginalp and Balenovich in 1999 [5] we have examined these phenomena from a dynamical systems approach. In this paper we have studied the stability and Hopf bifurcation properties of one of the most basic models. We have determined a curve of equilibria of the model and determined the required conditions for the stability of the equilibrium point by utilizing the Routh-Hurwitz criteria. Following this, we have given a detailed Hopf bifurcation analysis of the model by choosing the trend-based (or momentum) coefficient as a bifurcation parameter. Theorem 3 shows that the Hopf bifurcation occurs as the bifurcation parameter, q_1, passes through a critical value, q_1^*.

The stability analysis and bifurcation properties of other asset flow models (see [9], [14], [16]) have also been studied.

Another interesting aspect of markets involves the issue of periodicity. As discussed in the introduction, there is no mechanism for cyclic behavior in the absence of some exogenous periodicity, such as the seasons in agricultural products. In the context of the asset flow model, we show numerically that there exists an interval of the bifurcation parameter on which the system has periodic solutions.

The existence of endogenous periodic solutions is highly significant in terms of altering the notion of equilibrium in economics and finance. In classical economics and finance, an equilibrium is a fixed point to which prices converge. Using assumptions that are compatible with those of market practitioners, we show the existence of limit cycles in which prices approach periodic behavior, rendering cyclic solutions the new equilibrium concept. Once again, if investors had infinite arbitrage together with complete insight into the market behavior, they would seek to capitalize on the periodicity by buying low

and selling high, thereby gradually eliminating the cyclic behavior in favor of a single point equilibrium.

The issue of endogenous periodic behavior has not garnered as much attention as instability in market studies. There are studies indicating that some months feature worse performance of the markets, for example, late September and early October, though these are sometimes attributed to exogenous events during several years with very negative returns or some regular cyclic agricultural events related to agriculture in the early part of the 20th century.

The results above show that as investors focus their strategies more on trend (thereby increasing the aggregate q_1 value), the equilibrium varies from a stable point to a periodic orbit to an unstable trajectory. One can also choose the valuation coefficient, q_2, as a bifurcation parameter to study the bifurcation properties of the same model. Comparing these two results may give a better understanding of the financial markets by providing a similar understanding in terms of the effect of increased attention to valuation. This will be the topic of a future study.

7. Appendix. Positivity of the trading price, $P(t)$, and the equilibrium price, P_{eq}.

A. Consider the basic model (without approximation and normalization by L)

$$\delta^{-1}\frac{dP}{dt} = \frac{k}{1-k}L - P. \tag{7.1}$$

We assume of course that $L > 0$ (and, at this stage, a fixed constant). As P becomes very small, the first term on the RHS of (7.1) will dominate. In particular, since $0 < k < 1$, when t is such that $P(t) = 0$, we have that $k(1-k)^{-1}L > 0$, so that the RHS is strictly positive, and therefore, $dP(t)/dt > 0$. Hence there is no possibility of negative $P(t)$ in this basic model.

B. Next, suppose that we approximate $k = \frac{1}{2} + \frac{1}{2}\tanh\zeta$ for small ζ by $k \tilde{=} \frac{1}{2} + \frac{1}{2}\zeta$. This means that

$$\frac{k}{1-k} \tilde{=} \frac{\frac{1}{2}+\frac{1}{2}\zeta}{\frac{1}{2}-\frac{1}{2}\zeta} = \frac{1+\zeta}{1-\zeta} \tilde{=} 1 - 2\zeta.$$

Whether or not we use the second approximate equality above, we see that for small ζ,

$$\frac{1+\zeta}{1-\zeta} > 0.$$

Thus substituting this in (7.1) above, we have

$$\delta^{-1}\frac{dP}{dt} = \frac{1+\zeta}{1-\zeta}L - P, \tag{7.2}$$

and the same conclusion follows in exactly the same way.

C. Now suppose we are using the model (7.2) above, without any restriction that ζ is small. In this case we know that with $\zeta := \zeta_1 + \zeta_2$, the value sentiment, ζ_2, must be positive when $P = 0$ since $P_a > 0$ and $P_a - P > 0$. However, depending on the trend up to time t, we may have $\zeta_1 < 0$, and this may be larger in magnitude than ζ_2.

Thus, $(1+\zeta)/(1-\zeta)$ could be negative, and for $P(t) = 0$, we could have $dP(t)/dt < 0$, so that prices can become negative. Of course, when $\zeta > 1$, the approximation

$k = \frac{1}{2} + \frac{1}{2}\tanh\zeta \tilde{=} \frac{1}{2} + \frac{1}{2}\zeta$ is not valid at the outset, so it is not surprising that the model can yield negative prices.

D. The question of whether $\tilde{P}_{eq}$ is necessarily positive. From (3.13) we have with $a := \tilde{P}_a - \tilde{P}_a/q_2 - 1$,

$$2\tilde{P}_{eq} = a + \sqrt{a^2 + 4\left(1 + q_2\right)\tilde{P}_a/q_2}.$$

Since the second term in the radical, $4\left(1 + q_2\right)\tilde{P}_a/q_2$, is strictly positive, we know that

$$\tilde{P}_{eq} > 0.$$

Hence, we know even in the approximated model that this is true. For the purpose of understanding stability, we have $P(t)$ that is near $\tilde{P}_{eq}$, so this allows us to assume that $P > 0$.

References

[1] L. J. S. Allen, *An Introduction to Mathematical Biology*, Pearson/Prentice Hall, 2007.

[2] J. Bélair and S. A. Campbell, *Stability and bifurcations of equilibria in a multiple-delayed differential equation*, SIAM J. Appl. Math. **54** (1994), no. 5, 1402–1424, DOI 10.1137/S0036139993248853. MR1293106 (95j:34100)

[3] Z. Bodie, A. Kane and A. J. Marcus, *Investments*, 7th edition, McGraw-Hill Education, Boston, 2008.

[4] G. Caginalp and D. Balenovich, *Market oscillations induced by the competition between value-based and trend-based investment strategies*, Appl. Math. Finance **1** (1994), 29–164.

[5] G. Caginalp and D. Balenovich, *Asset flow and momentum: deterministic and stochastic equations*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci. **357** (1999), no. 1758, 2119–2133, DOI 10.1098/rsta.1999.0421. MR1712407 (2000e:91067)

[6] G. Caginalp and G. B. Ermentrout, *A kinetic thermodynamics approach to the psychology of fluctuations in financial markets*, Appl. Math. Lett. **3** (1990), 17-19.

[7] G. Caginalp and G. B. Ermentrout, *Numerical studies of differential equations related to theoretical financial markets*, Appl. Math. Lett. **4** (1991), no. 1, 35–38, DOI 10.1016/0893-9659(91)90118-F. MR1088032

[8] G. Caginalp and M. DeSantis, *Nonlinearity in the dynamics of financial markets*, Nonlinear Anal. Real World Appl. **12** (2011), no. 2, 1140–1151, DOI 10.1016/j.nonrwa.2010.09.008. MR2736200 (2011i:91155)

[9] G. Caginalp and M. DeSantis, *Multi-group asset flow equations and stability*, Discrete Contin. Dyn. Syst. Ser. B **16** (2011), no. 1, 109–150, DOI 10.3934/dcdsb.2011.16.109. MR2799544 (2012c:91257)

[10] G. Caginalp and M. Desantis, *Stock price dynamics: nonlinear trend, volume, volatility, resistance and money supply*, Quant. Finance **11** (2011), no. 6, 849–861, DOI 10.1080/14697680903220356. MR2806968 (2012e:91315)

[11] G. Caginalp and H. Merdan, *Asset price dynamics with heterogeneous groups*, Phys. D **225** (2007), no. 1, 43–54, DOI 10.1016/j.physd.2006.09.036. MR2304913 (2007k:91130)

[12] G. Caginalp, D. Porter, and V. Smith, *Initial cash/asset ratio and asset prices: an experimental study*, Proc. Natl. Acad. Sci. USA **95** (1998), no. 2, 756–761 (electronic), DOI 10.1073/pnas.95.2.756. MR1613147

[13] K. D. Daniel, D. Hirshleifer, and A. Subrahmanyam, *Investor psychology and security market under and overreaction*, Journal of Finance **53** (1998), 1839-1885.

[14] M. DeSantis, D. Swigon, and G. Caginalp, *Nonlinear dynamics and stability in a multigroup asset flow model*, SIAM J. Appl. Dyn. Syst. **11** (2012), no. 3, 1114–1148, DOI 10.1137/120862211. MR3022062

[15] R. C. Dorf and R. H. Bishop, *Modern Control Systems*, 11th edition, Pearson Prentice-Hall, Upper Saddle River, NJ, 2008.

[16] A. Duran, *Stability analysis of asset flow differential equations*, Appl. Math. Lett. **24** (2011), no. 4, 471–477, DOI 10.1016/j.aml.2010.10.044. MR2749729

[17] R. Frisch and H. Holme, *The characteristic solutions of a mixed difference and differential equation occurring in economic dynamics*, Econometrica **3** (1935), 225–239.

[18] D. Fudenberg and J. Tirole, *Game theory*, MIT Press, Cambridge, MA, 1991. MR1124618 (92m:90001)

[19] J. K. Hale and H. Koçak, *Dynamics and bifurcations*, Texts in Applied Mathematics, vol. 3, Springer-Verlag, New York, 1991. MR1138981 (93e:58047)

[20] J. M. Henderson and R. E. Quant, *Microeconomic Theory, A Mathematical Approach*, 3rd edition, McGraw-Hill, 1980.

[21] N. Kaldor, *A classificatory note on the determinateness of equilibrium*, The Review of Economic Studies **1** (1934), 122-136.

[22] D. Kahneman and A. Tversky, *Prospect theory: an analysis of decision making under risk*, Econometrica **47** (1979), 263–291.

[23] L. Lopes, *Between hope and fear: the psychology of risk*, Adv. Exp. Soc. Psychol. **20** (1987), 255–295.

[24] X. Li, S. Ruan, and J. Wei, *Stability and bifurcation in delay-differential equations with two delays*, J. Math. Anal. Appl. **236** (1999), no. 2, 254–280, DOI 10.1006/jmaa.1999.6418. MR1704582 (2001a:34131)

[25] M. Mackey and L. Glass, *Oscillations and chaos in physiological control systems*, Science **197** (1977), 287–289.

[26] R. May, *Stability and Complexity in Model Ecosystem*, Princeton University Press, Princeton, NJ, 1973.

[27] H. Merdan and M. Alisen, *A mathematical model for asset pricing*, Appl. Math. Comput. **218** (2011), no. 4, 1449–1456, DOI 10.1016/j.amc.2011.06.028. MR2831653

[28] H. Merdan and H. Cakmak, *Liquidity effect on the asset price forecasting*, Journal of Nonlinear Systems and Applications (2012), 82-87.

[29] H. Akkocaoğlu, H. Merdan, and C. Çelik, *Hopf bifurcation analysis of a general non-linear differential equation with delay*, J. Comput. Appl. Math. **237** (2013), no. 1, 565–575, DOI 10.1016/j.cam.2012.06.029. MR2966929

[30] J. M. Poterba and L. H. Summers, *Mean reversion in stock prices: Evidence and implications*, Journal of Financial Economics **22** (1988), 27-59.

[31] H. Shefrin, *A Behavioral Approach to Asset Pricing*, Elsevier, London, 2005.

[32] H. Shefrin and M. Statman, *The disposition to sell winners too early and ride losers too long: Theory and evidence*, Journal of Finance **40** (1985), 777-790.

[33] V. L. Smith, G. L. Suchanek and A. W. Williams, *Bubbles, crashes and endogenous expectations in experimental spot asset markets*, Econometrica **56** (1988), 1119-1151.

[34] A. Tversky and D. Kahneman, *Judgment under uncertainty: Heuristics and biases* Science **185** (1974), 1109-1186.

[35] R. Yafia, *Hopf bifurcation in differential equations with delay for tumor-immune system competition model*, SIAM J. Appl. Math. **67** (2007), no. 6, 1693–1703 (electronic), DOI 10.1137/060657947. MR2350003

[36] D. S. Watson and M. Getz, *Price Theory and Its Uses*, 5th edition, University Press of America, Lanham, MD, 1993.

[37] P. Wilmott, *Paul Wilmott Introduces Quantitative Finance*, John Wiley & Sons, 2007.