

ASYMPTOTICS OF THE EIGENVALUES  
OF THE DIRICHLET-LAPLACE PROBLEM  
IN A DOMAIN WITH THIN TUBE EXCLUDED

BY

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**Abstract.** We consider a Laplace problem with Dirichlet boundary condition in a three dimensional domain containing an inclusion taking the form of a thin tube with small thickness  $\delta$ . We prove convergence in operator norm of the resolvent of this problem as  $\delta \rightarrow 0$ , establishing that the perturbation induced by the inclusion on the resolvent is not greater than  $O(|\ln \delta|^{-\gamma})$  for some  $\gamma > 0$ . We deduce convergence of the eigenvalues of the perturbed operator toward the limit operator.

**1. Introduction.** Analysis of singular perturbations of elliptic boundary value problems by inclusions of small volume has received a lot of attention in the past decades due to its numerous applications to modeling and numerical simulation of multi-scale problems and its possible use in efficient inverse problem [1, 2] or shape optimization [20] strategies.

For a Laplace equation, many works of the existing literature have been dedicated to asymptotic analysis involving the presence of an inclusion that has small size in all directions of space (typically small balls) in the case of either impenetrable (homogeneous Dirichlet, Neumann, or Robin) or penetrable (transmission problem) boundary conditions. Such asymptotics have been provided for both 2-D and 3-D problems in the case of either a single inclusion or many; see [10, 16–19]. The books [9, 14] are landmarks on this type of problem.

Regarding a Laplace equation perturbed by a small inclusion that takes the form of a thin tube, much less work is available in the literature. Some works can be found in the case of a tube centered around a smooth closed contour without self-intersecting point. The analysis of the Laplace equation in 3-D with Neumann and Dirichlet boundary condition was provided in [5, 6], where the author considers either a tube centered at

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a non-self-intersecting smooth closed contour or a tube centered at a straight segment. This analysis is refined and generalized to arbitrary dimensions in [15] and [14, Chap. 12]. Similar results were established for acoustics and the Helmholtz equation in [3, 7, 22].

It should be pointed out that in the case of a 3-D Laplace equation perturbed by a thin elongated inclusion, the cases of the homogeneous Dirichlet or Robin boundary condition lead to much more challenging analysis than the case of Neumann or transmission conditions: while Neumann/transmission conditions lead to rather standard asymptotic constructions, matching of asymptotics in the Dirichlet/Robin case leads to an ill-posed 1-D integral equation.

On the other hand, asymptotic analysis of the Laplace problem with Dirichlet boundary condition has many important applications, in particular in electrical engineering and antenna-based electric/electronic devices, as it is commonly taken as a rough model for electromagnetic diffraction by perfectly conducting thin wires; see for example [11, Chap. 6]. In this context, the study of eigenvalues is of particular importance as it is related to resonance phenomena. Yet, in the case of thin elongated inclusions with Dirichlet boundary condition, only Planida [21] has addressed this issue so far. In this reference, the author establishes convergence of eigenvalues in the case where the inclusion is centered around a smooth closed curve that does not self-intersect. However no explicit rate of convergence is provided.

In the present article, we study the Laplace equation with homogeneous Dirichlet condition and the associated eigenvalue problem, perturbed by the presence of a small elongated inclusion that takes the form of a thin tube. We establish convergence in the operator norm of the resolvent of the perturbed problem toward the resolvent of the limit problem. A first novel ingredient here compared to previously established results concerns the geometrical configurations under consideration that require only that the inclusion be concentrated around a parametrized curve. Contrary to pre-existing works, the present analysis does not require the inclusion to be connected or admit any symmetry of revolution. The limit curve considered here may admit self-crossing points, which, to our knowledge, has never been investigated in the existing literature (at least in the case of a homogeneous Dirichlet boundary condition). In addition, we establish a convergence result (with error estimates) for eigenvalues with explicit error estimate showing that the presence of an elongated inclusion induces a shift of order  $O(|\ln \delta|^{-\gamma})$  of the eigenvalues, for some  $\gamma > 0$ .

The outline of the present article is as follows. In Section 2 we describe in detail the geometry under consideration. In Section 3 we introduce weighted  $\delta$ -dependent norms adapted to our analysis and establish a stability property for the Laplace operator in terms of these norms. The next section is dedicated to proving Hardy type inequalities that will be necessary for the subsequent error estimates. In the last section we derive an upper bound for the difference, in operator norm, between the resolvents of the perturbed problem and the limit problem.

**2. Geometry under consideration.** We start by describing in detail the geometrical setting under consideration. Let  $\Omega \subset \mathbb{R}^3$  refer to a Lipschitz domain. Consider a  $\mathcal{C}^1$ -function  $\gamma : \mathbb{R} \mapsto \Omega$  that is  $L$ -periodic for some  $L > 0$  and such that

$0 < \alpha_- \leq \left| \frac{d\gamma}{dz}(z) \right| \leq \alpha_+, \forall z \in \mathbb{R}$  for fixed constants  $\alpha_{\pm} > 0$ . Then we set  $\Gamma := \gamma(\mathbb{R})$ , which is a Lipschitz curve. We shall need to refer to the distance function

$$d(\mathbf{x}) := \inf_{\mathbf{y} \in \Gamma} |\mathbf{x} - \mathbf{y}|. \tag{1}$$

Then we consider  $(\Xi_{\delta})_{\delta > 0}$  as any family of Lipschitz domains such that there exists a fixed constant  $C > 0$  independent of  $\delta$  for which

$$\sup_{\mathbf{x} \in \Xi_{\delta}} d(\mathbf{x}, \Gamma) \leq C \delta. \tag{2}$$

This condition imposes that, as  $\delta \rightarrow 0$ , the sets  $\Xi_{\delta}$  are more and more concentrated around  $\Gamma$ . Changing the index parameter  $\delta$  through multiplication by some factor if necessary, we may assume that  $C < 1$  in (2) without restricting generality. Then we denote

$$\Omega_{\delta} := \Omega \setminus \Xi_{\delta}.$$

The geometrical configurations described above may be used to consider resonance problems in cavities containing inclusions whose geometry is similar to wires. Note that we do not require  $\gamma$  to be one-to-one over a period so that, under the assumptions above, the curve  $\Gamma$  may self-intersect. Note also that  $\Xi_{\delta}$  do not need to be connected and may degenerate, as  $\delta \rightarrow 0$ , to only a part of  $\Gamma$  with tips. As a consequence, the present setting covers such geometrical configurations as the ones presented in Figure 1.

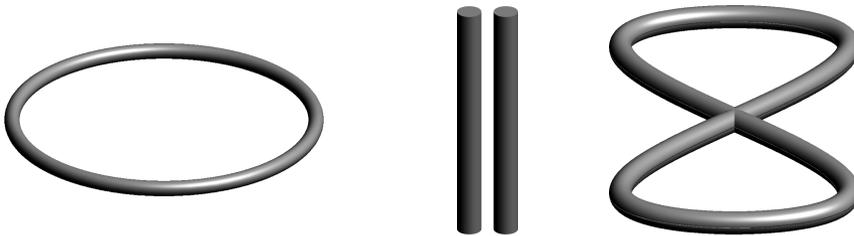


FIG. 1. Three examples of geometrical configurations covered by the analysis presented here.

**3. Weighted Sobolev spaces.** Given any bounded open Lipschitz domain  $\omega \subset \mathbb{R}^3$ , the set  $L^2(\omega)$  shall refer to the space of square integrable functions over  $\omega$  equipped with the pairing  $u, v \mapsto (u, v)_{\omega} := \int_{\omega} u v d\mathbf{x}$  and the norm  $\|u\|_{L^2(\omega)} = \sqrt{(u, \bar{u})_{\omega}}$ . As usual,  $H_0^1(\omega)$  shall refer to the completion of  $\mathcal{C}_0^{\infty}(\omega) := \{\varphi \in \mathcal{C}^{\infty}(\mathbb{R}^3), \varphi = 0 \text{ on } \mathbb{R}^3 \setminus \bar{\omega}, \text{supp}(\varphi) \text{ bounded}\}$  for the norm  $\|\varphi\|_{H_0^1(\omega)} := \|\varphi\|_{L^2(\omega)} + \|\nabla \varphi\|_{L^2(\omega)}$ . Finally  $H^{-1}(\omega)$  will refer to the topological dual to  $H_0^1(\omega)$ .

For a given parameter  $\delta > 0$  small enough, we are interested in studying the eigenvalues of the operator  $A_{\delta} : H_0^1(\Omega_{\delta}) \rightarrow H^{-1}(\Omega_{\delta})$  defined by

$$\langle A_{\delta}(u), v \rangle := (\nabla u, \nabla v)_{\Omega} \quad \forall u, v \in H_0^1(\Omega_{\delta}). \tag{3}$$

We shall compare its eigenvalues with those of the limit operator  $A_0 : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$  that is defined similarly but in a domain without any perturbation, i.e.  $\langle A_0(u), v \rangle := (\nabla u, \nabla v)_\Omega \forall u, v \in H_0^1(\Omega)$ .

To study these two operators, we need to introduce adapted  $\delta$ -dependent weighted Sobolev norms. These are not classical weighted norms such as those characterizing Kondratiev’s spaces; see e.g. [13]. The norms we consider here involve  $\delta$ -dependent logarithmic weights: given a  $\beta \in \mathbb{R}$  and a Lipschitz open set  $\omega \subset \mathbb{R}^2$ , we define

$$\begin{aligned} \|v\|_{V_{\beta,\delta}^1(\omega)}^2 &:= |v|_{V_{\beta,\delta}^1(\omega)}^2 + |v|_{V_{\beta,\delta}^0(\omega)}^2 \\ \text{where } |v|_{V_{\beta,\delta}^0(\omega)} &:= \int_\omega |\ln d_\delta(\mathbf{x})|^{-2\beta} \frac{|v(\mathbf{x})|^2}{|d_\delta(\mathbf{x}) \ln d_\delta(\mathbf{x})|^2} d\mathbf{x}, \\ |v|_{V_{\beta,\delta}^1(\omega)} &:= \int_\omega |\ln d_\delta(\mathbf{x})|^{-2\beta} |\nabla v|^2 d\mathbf{x} \end{aligned} \tag{4}$$

with  $d_\delta(\mathbf{x}) := d_\star(\mathbf{x}) + \delta$  where  $d_\star(\mathbf{x}) := \min(d(\mathbf{x}), e^{-1}/2)$ .

For  $\delta > 0, \beta \in \mathbb{R}$  fixed, the norm above is obviously equivalent to  $\|\cdot\|_{H^1(\omega)}$ . However these norms “behave” differently as  $\delta \rightarrow 0$ . The norms (4) are specifically tailored for our analysis.

The condition  $d_\star(\mathbf{x}) \leq e^{-1}/2 \forall \mathbf{x} \in \Omega$  guarantees that  $|d_\star(\mathbf{x}) \ln d_\star(\mathbf{x})| \leq |d_\delta(\mathbf{x}) \ln d_\delta(\mathbf{x})|$  for all  $\mathbf{x} \in \Omega$  and for  $\delta < e^{-1}/2$ . On the other hand, recall that  $d$  is a distance function to a compact set so it is Lipschitz with  $\|\nabla d\|_{L^\infty(\Omega)} = 1$ ; see [4, Prop. 1D.4] for example. Hence  $d_\star$  is a Lipschitz function, and  $d_\delta$  is non-vanishing with  $d_\delta(\mathbf{x}) \geq \delta, \forall \mathbf{x} \in \Omega$ . We deduce that  $\Upsilon_\beta^\delta(v) := v(\mathbf{x}) \ln^\beta d_\delta(\mathbf{x})$  isomorphically maps  $H_0^1(\Omega)$  onto itself, and there exist constants  $c_\pm, \delta_0 > 0$  independent of  $\delta$  such that

$$0 < c_- \leq \frac{\|\Upsilon_\beta^\delta(v)\|_{V_{\beta,\delta}^1(\Omega)}}{\|v\|_{V_{0,\delta}^1(\Omega)}} \leq c_+ \quad \forall v \in H_0^1(\Omega), \quad \forall \delta \in (0, \delta_0). \tag{5}$$

We shall also consider the corresponding dual norms, each providing a norm for  $H^{-1}(\Omega)$  and carrying at the same time some dependency with respect to  $\delta$ ,

$$\|f\|_{V_{\beta,\delta}^{-1}(\omega)} := \sup_{v \in H_0^1(\omega) \setminus \{0\}} \frac{|\langle f, v \rangle|}{\|v\|_{V_{-\beta,\delta}^1(\omega)}}.$$

Let us examine the continuity properties of the operators  $A_0, A_\delta$  with respect to these norms. First of all, both  $A_0$  and  $A_\delta$  easily appear as uniformly continuous with respect to these norms; i.e. a direct calculus shows that

$$\limsup_{\delta \rightarrow 0} \left( \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\|A_0(v)\|_{V_{\beta,\delta}^{-1}(\Omega)}}{\|v\|_{V_{\beta,\delta}^1(\Omega)}} + \sup_{v \in H_0^1(\Omega_\delta) \setminus \{0\}} \frac{\|A_\delta(v)\|_{V_{\beta,\delta}^{-1}(\Omega_\delta)}}{\|v\|_{V_{\beta,\delta}^1(\Omega_\delta)}} \right) < +\infty.$$

Uniform invertibility of these operators also holds under certain hypotheses on the weight exponent  $\beta$ .

PROPOSITION 3.1. There exists  $\beta_\star > 0$  such that for  $|\beta| < \beta_\star$ , the inverses of both  $A_0$  and  $A_\delta$  remain uniformly bounded with respect to  $\delta$  in terms of the weighted norms

$$\limsup_{\delta \rightarrow 0} \left( \sup_{f \in H^{-1}(\Omega) \setminus \{0\}} \frac{\|A_0^{-1}(f)\|_{V_{\beta,\delta}^1(\Omega)}}{\|f\|_{V_{\beta,\delta}^{-1}(\Omega)}} + \sup_{f \in H^{-1}(\Omega_\delta) \setminus \{0\}} \frac{\|A_\delta^{-1}(f)\|_{V_{\beta,\delta}^1(\Omega_\delta)}}{\|f\|_{V_{\beta,\delta}^{-1}(\Omega_\delta)}} \right) < +\infty.$$

*Proof.* We need to provide an upper bound for both  $A_\delta^{-1}$  and  $A_0^{-1}$ . We prove such a bound only for  $A_\delta^{-1}$ , as the derivation for  $A_0^{-1}$  is very similar and slightly easier. First of all observe that, according to the uniform bounds (5), this is equivalent to proving that for  $\beta \in \mathbb{R}$  fixed such that  $|\beta| < \beta_\star$ , there exists a constant  $C_\beta > 0$  independent of  $\delta$  such that

$$C_\beta \leq \sup_{v \in H_0^1(\Omega_\delta) \setminus \{0\}} \frac{|\langle A_\delta \cdot \Upsilon_\beta^\delta(u), \Upsilon_{-\beta}^\delta(v) \rangle|}{\|u\|_{V_{0,\delta}^1(\Omega)} \|v\|_{V_{0,\delta}^1(\Omega)}}. \tag{6}$$

We have  $\langle A_\delta \cdot \Upsilon_\beta^\delta(u), \Upsilon_{-\beta}^\delta(v) \rangle = (\nabla \Upsilon_\beta^\delta(u), \nabla \Upsilon_{-\beta}^\delta(v))_{\Omega_\delta}$ , according to the definition of  $A_\delta$ . Expanding this expression yields

$$\begin{aligned} (\nabla \Upsilon_{+\beta}(u), \nabla \Upsilon_{-\beta}(v))_{\Omega_\delta} &= (\nabla u, \nabla v)_{\Omega_\delta} \\ &+ \beta \int_\Omega \nabla d_\star(\mathbf{x}) \cdot \frac{u \nabla v - v \nabla u}{d_\delta(\mathbf{x}) \ln d_\delta(\mathbf{x})} d\mathbf{x} - \beta^2 \int_\Omega \frac{u v}{|d_\delta(\mathbf{x}) \ln d_\delta(\mathbf{x})|^2} d\mathbf{x}. \end{aligned} \tag{7}$$

Observe that, taking  $v = \bar{u}$ , the second term in the right hand side above is purely imaginary. In addition, since  $|d_\star(\mathbf{x}) \ln d_\star(\mathbf{x})| \leq |d_\delta(\mathbf{x}) \ln d_\delta(\mathbf{x})|$  for all  $\mathbf{x} \in \Omega$ , using Proposition 4.1 below and the fact that  $v \in H_0^1(\Omega_\delta) \subset H_0^1(\Omega)$ , we obtain

$$\begin{aligned} \Re\{ (\nabla \Upsilon_{+\beta}(u), \nabla \Upsilon_{-\beta}(v))_{\Omega_\delta} \} &\geq (1 - |\beta/\beta_\star|^2) \|\nabla u\|_{L^2(\Omega_\delta)}^2 \\ &\geq \frac{1 - |\beta/\beta_\star|^2}{1 + 1/\beta_\star^2} \|u\|_{V_{0,\delta}^1(\Omega_\delta)}^2. \end{aligned} \tag{8}$$

Inequality (8) leads to the conclusion of the proof since it yields the inf-sup condition (6) with a constant independent of  $\delta$ .  $\square$

**4. Hardy type inequalities.** This section is dedicated to proving two inequalities that take a form similar to Hardy’s inequality, but adapted to our geometrical setting. The first result below may be understood as a cylindrical version of Hardy’s inequality. The proof, though, is made tricky by the rather general geometry under consideration here and, in particular, the possibility for  $\Gamma$  to admit self-crossing points.

PROPOSITION 4.1. There exists a constant  $\beta_\star > 0$  such that

$$\frac{1}{\beta_\star^2} := \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \left\{ \frac{1}{\|\nabla v\|_{L^2(\Omega)}^2} \int_\Omega \frac{|v(\mathbf{x})|^2 d\mathbf{x}}{|d_\star(\mathbf{x}) \ln d_\star(\mathbf{x})|^2} \right\} < +\infty. \tag{9}$$

*Proof.* During this proof, we shall refer to  $\mathbf{e}_3(t) := \partial_t \gamma(t)/|\partial_t \gamma(t)|$ , consider a  $\mathcal{C}^0$ -vector field  $t \mapsto \mathbf{e}_1(t) \in \mathbb{R}^3$  such that  $\mathbf{e}_1(t) \cdot \mathbf{e}_3(t) = 0, \forall t \in \mathbb{R}$ , and set  $\mathbf{e}_2(t) := \mathbf{e}_3(t) \times \mathbf{e}_1(t)$ . Denoting  $I_\epsilon := (-\epsilon, +\epsilon)$  and  $D_\epsilon \subset \mathbb{R}^2$  the disk of center 0 and radius  $\epsilon$ , for each  $t \in \mathbb{R}$ , there exists a small cylinder  $\hat{Q}_t = D_{\epsilon_t} \times I_{\epsilon_t}$  with  $\epsilon_t > 0$  such that the map  $\phi_t : \hat{Q}_t \rightarrow \mathbb{R}^3$  defined by

$$\phi_t(x, y, z) := \gamma(t + z) + x\mathbf{e}_1(t) + y\mathbf{e}_2(t)$$

is an immersion. Pick finitely many  $t_1, t_2, \dots, t_n$  such that  $[0, L] \subset \bigcup_{j=1}^n (t_j - \epsilon_{t_j}, t_j + \epsilon_{t_j})$ . Denote  $\hat{Q}_j := \hat{Q}_{t_j}$ ,  $Q_j := \phi_{t_j}(\hat{Q}_j)$  and  $\Gamma_j := \Gamma \cap Q_j$  so that, in particular,  $\Gamma \subset Q_1 \cup \dots \cup Q_n$ .

Let  $\omega_\eta = \{\mathbf{x} \in \Omega \mid d(\mathbf{x}) < \eta\}$ . Let us prove that if  $\eta$  is chosen small enough, then for any  $\mathbf{x} \in \omega_\eta$  there is one  $j$  such that  $\mathbf{x} \in Q_j$  and  $d(\mathbf{x}) = \inf_{\mathbf{y} \in \Gamma_j} |\mathbf{x} - \mathbf{y}|$ . To show this, proceed by contradiction, assuming for a moment that such is not the case.

This means that there exists a sequence  $\mathbf{x}_p \in \Omega$  with  $\lim_{p \rightarrow \infty} d(\mathbf{x}_p) = 0$  and such that, for any  $\mathbf{x}_p^* \in \Gamma$  satisfying  $|\mathbf{x}_p - \mathbf{x}_p^*| = d(\mathbf{x}_p)$ , none of the  $Q_j$ 's contain both  $\mathbf{x}_p$  and  $\mathbf{x}_p^*$ . Since  $\Gamma$  is compact and extracting a sub-sequence if necessary, we may assume that  $\lim_{p \rightarrow \infty} \mathbf{x}_p = \lim_{p \rightarrow \infty} \mathbf{x}_p^* = \mathbf{x}_\infty \in \Gamma$ . Take a  $Q_m$  containing  $\mathbf{x}_\infty$ . Then, since  $Q_m$  is an open neighborhood of  $\mathbf{x}_\infty$ , for sufficiently large  $p$  we have both  $\mathbf{x}_p \in Q_m$  and  $\mathbf{x}_p^* \in Q_m$ , which contradicts our initial assumption.

From now on, we assume that  $\eta > 0$  is chosen small enough to guarantee the property discussed in the previous paragraph. Denote  $d_j(\mathbf{x}) = \inf_{\mathbf{y} \in \Gamma_j} |\mathbf{x} - \mathbf{y}|$ , and let  $\mathbf{1}_{Q_j}$  refer to the characteristic function of  $Q_j$ . What precedes shows that, for any  $\mathbf{x} \in \omega_\eta$ , we have

$$\frac{1}{|d_\star(\mathbf{x}) \ln d_\star(\mathbf{x})|} \leq \sum_{j=1}^n \frac{\mathbf{1}_{Q_j}(\mathbf{x})}{|d_j(\mathbf{x}) \ln d_j(\mathbf{x})|}. \tag{10}$$

Choose a smooth open set  $Q_0 \subset \mathbb{R}^3$  such that  $Q_0 \cup Q_1 \cup \dots \cup Q_n$  achieves a covering of  $\bar{\Omega}$ . Using a partition of unity subordinated to this covering, one may decompose any  $v \in H_0^1(\Omega)$  in the form  $v = v_0 + \dots + v_n$  where  $v_j \in H_0^1(Q_j)$ . This remark, together with inequality (10), shows that it suffices to prove the existence of a constant  $C > 0$  such that

$$\int_{Q_j} \frac{|v(\mathbf{x})|^2 d\mathbf{x}}{|d_j(\mathbf{x}) \ln d_j(\mathbf{x})|^2} \leq C \int_{Q_j} |\nabla v|^2 d\mathbf{x} \quad \forall v \in H_0^1(Q_j) \quad \forall j = 1 \dots n. \tag{11}$$

From now on and until the end of the proof, we take a  $j$  fixed. Let us relabel for a moment  $\epsilon = \epsilon_{t_j}$ ,  $\phi = \phi_{t_j}$  and set  $\hat{Q} = \hat{Q}_j = D_\epsilon \times I_\epsilon$ ,  $\Sigma := \{0\} \times I_\epsilon$  and  $\hat{v} = v \circ \phi$ . For any  $\hat{\mathbf{x}} \in \hat{Q}$  we also set  $\hat{d}(\hat{\mathbf{x}}) = \inf_{\hat{\mathbf{y}} \in \Sigma} |\hat{\mathbf{x}} - \hat{\mathbf{y}}|$ . The diffeomorphism  $\phi$  and its inverse  $\phi^{-1}$  are both Lipschitz, so there are constants  $c_\pm > 0$  such that  $c_- |\hat{\mathbf{x}} - \hat{\mathbf{y}}| \leq |\phi(\hat{\mathbf{x}}) - \phi(\hat{\mathbf{y}})| \leq c_+ |\hat{\mathbf{x}} - \hat{\mathbf{y}}|$ . Routine verifications show that this implies existence of two constants  $c', c'' > 0$  such that

$$c' \hat{d}(\hat{\mathbf{x}}) \leq d(\phi(\hat{\mathbf{x}})) \leq c'' \hat{d}(\hat{\mathbf{x}}) \quad \forall \hat{\mathbf{x}} \in \hat{Q}. \tag{12}$$

Let  $r, \theta$  refer to the polar coordinates in  $\mathbb{R}^2$  so that the cylinder  $\hat{Q}$  is parametrized by the cylindrical coordinates  $(r, \theta, z) \in [0, +\epsilon) \times [0, 2\pi] \times I_\epsilon$ . The classical Hardy's inequality applied in  $\hat{Q}$  (see [8]) shows that

$$\int_{\hat{Q}} \frac{|\hat{v}|^2 r dr d\theta dz}{|r \ln(r)|^2} \leq C \int_{\hat{Q}} |\nabla \hat{v}| + |\hat{v}|^2 d\hat{\mathbf{x}} \quad \forall \hat{v} \in H^1(\hat{Q}). \tag{13}$$

There only remains to observe that  $\hat{d}(\hat{\mathbf{x}}) = r$ , to use (12), and to apply the change of variables  $\mathbf{x} = \phi(\hat{\mathbf{x}})$  in the integrals (13). Denoting by  $D\phi$  the differential of  $\phi$ , since  $\|D\phi\|_{L^\infty(\hat{Q})}$  and  $\|D\phi^{-1}\|_{L^\infty(Q_j)}$  are both bounded, this finally yields inequality (11) with  $\|\nabla v\|_{L^2(Q_j)}^2 + \|v\|_{L^2(Q_j)}^2$  instead of just  $\|\nabla v\|_{L^2(\Omega)}^2$ . We conclude by using Poincaré's inequality in  $Q_j$ . □

We will also need another  $\delta$ -dependent weighted inequality. This one involves a weighted  $L^2$ -norm evaluated only over a coronal cylinder of radius  $\delta$  and is not primarily based on the classical Hardy’s inequality. Instead it is derived by means of Kondratiev’s analysis.

PROPOSITION 4.2. Denoting  $\mathbb{Q}_\delta := \{\mathbf{x} \in \Omega \mid \delta < d(\mathbf{x}) < 2\delta\}$  we have

$$\limsup_{\delta \rightarrow 0} \sup_{v \in H^2(\Omega) \cap H_0^1(\Omega) \setminus \{0\}} \left\{ \frac{1}{\|\Delta v\|_{L^2(\Omega)}^2} \int_{\mathbb{Q}_\delta} \left| \frac{v(\mathbf{x})}{d_\star(\mathbf{x})} \right|^2 d\mathbf{x} \right\} < +\infty.$$

*Proof.* To establish this result, we follow a path similar to that of the proof of Proposition 4.1 and use the same notation. Take any  $\mathbf{x} \in \omega_\eta$ . According to the first part of the preceding proof, there exists  $j$  such that  $\mathbf{x} \in Q_j$  and  $d_j(\mathbf{x}) = d_\star(\mathbf{x})$ , hence  $\delta < d(\mathbf{x}) < 2\delta \Rightarrow \delta < d_j(\mathbf{x}) < 2\delta$ . As a consequence  $\mathbb{Q}_\delta \subset \bigcup_{j=1}^n \mathbb{Q}_\delta^j$  where  $\mathbb{Q}_\delta^j := \{\mathbf{x} \in Q_j, \delta < d_j(\mathbf{x}) < 2\delta\}$ , which implies  $\mathbf{1}_{\mathbb{Q}_\delta} \leq \sum_{j=1}^n \mathbf{1}_{\mathbb{Q}_\delta^j}$ . Hence it suffices to prove, for each  $j$ , the existence of constants  $C, \delta_0 > 0$  independent of  $\delta$  such that

$$\int_{\mathbb{Q}_\delta^j} \left| \frac{v(\mathbf{x})}{d_j(\mathbf{x})} \right|^2 d\mathbf{x} \leq C \int_{\Omega} |\Delta v|^2 d\mathbf{x} \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega), \quad \forall \delta \in (0, \delta_0).$$

Fixing  $j$  and using a change of variables as in the previous proof, the inequality above boils down to establishing the existence of constants  $C, \delta_0 > 0$  independent of  $\delta$  such that for any  $v \in H^2(\hat{Q}) \cap H_0^1(\hat{Q})$  and any  $\delta \in (0, \delta_0)$ , we have

$$\int_{\hat{Q}_\delta} \frac{|v|^2}{r^2} r dr d\theta dz \leq C \int_{\hat{Q}} |\Delta v|^2 + |\nabla v|^2 + |v|^2 d\hat{\mathbf{x}}, \tag{14}$$

where we recall that  $\hat{Q} = D_\epsilon \times I_\epsilon$ , the variables  $r, \theta, z$  refer to the cylindrical coordinates in  $\hat{Q}$ , and  $\hat{Q}_\delta := \{\hat{\mathbf{x}} \in \hat{Q} \mid \delta < r < 2\delta\}$ . To establish (14), set  $\Delta_\perp v = r^{-2}((r\partial_r)^2 + \partial_\theta^2)v$ . Decompose each point  $\hat{\mathbf{x}} \in \hat{Q}$  as  $\hat{\mathbf{x}} = (\hat{\mathbf{x}}_\perp, z)$ , so that  $\hat{\mathbf{x}}_\perp = (r \cos \theta, r \sin \theta)$  and  $r = |\hat{\mathbf{x}}_\perp|$ . Introduce the Fourier decomposition of  $v$  in the  $z$  variable, setting

$$v_p(\hat{\mathbf{x}}_\perp) := \frac{1}{2\epsilon} \int_{-\epsilon}^{+\epsilon} v(\hat{\mathbf{x}}_\perp, z) \exp(-i\pi z/\epsilon) dz. \tag{15}$$

Then we have  $v_p \in H^2(D_\epsilon) \cap H_0^1(D_\epsilon)$  and  $-\Delta_\perp v_p \in L^2(D_\epsilon)$  for all  $p \in \mathbb{Z}$  if  $v \in H^2(\hat{Q}) \cap H_0^1(\hat{Q})$ . Applying Kondratiev’s analysis in  $D_\epsilon \setminus \{0\}$  (see Chapter 6 of [13]), we find the existence of coefficients  $\alpha_p \in \mathbb{C}$  and a constant  $C > 0$  independent of  $p$  such that

$$|\alpha_p|^2 + \int_{D_\epsilon} \frac{|v_p(\hat{\mathbf{x}}_\perp) - \alpha_p|^2}{|\hat{\mathbf{x}}_\perp|^3} d\hat{\mathbf{x}}_\perp \leq C \int_{D_\epsilon} |\Delta_\perp v_p|^2 d\hat{\mathbf{x}}_\perp \quad \forall p \in \mathbb{Z}.$$

Parseval identity associated to decomposition (15), together with the estimate above, indicates that  $\sum_{p=-\infty}^{+\infty} |\alpha_p|^2 < +\infty$  so that there exists a function  $\alpha = \alpha(z) \in L^2(I_\epsilon)$  and constants  $C, C' > 0$  independent of  $v$  satisfying

$$\begin{aligned} & \|\alpha\|_{L^2(I_\epsilon)}^2 + \int_{\hat{Q}} \frac{|v(\hat{\mathbf{x}}_\perp, z) - \alpha(z)|^2}{|\hat{\mathbf{x}}_\perp|^3} d\hat{\mathbf{x}}_\perp dz \\ & \leq C \|f\|_{L^2(\hat{Q})}^2 = C \|\Delta_\perp v\|_{L^2(\hat{Q})}^2 \leq C' \|\Delta v\|_{L^2(\hat{Q})}^2 \quad \forall v \in H^2(\hat{Q}) \cap H_0^1(\hat{Q}). \end{aligned} \tag{16}$$

The inequality above is justified by standard elliptic a priori estimates for the Laplace operator; see [13, Chap.3] for example. Finally, let us pick an arbitrary  $v \in H^2(\hat{Q}) \cap H_0^1(\hat{Q})$ . Plugging (16) into the left hand side of (14) yields

$$\begin{aligned} \int_{\hat{Q}_\delta} \frac{|v|^2}{r^2} r dr d\theta dz &= \int_{\hat{Q}_\delta} \frac{|v(\hat{\mathbf{x}})|^2}{|\hat{\mathbf{x}}_\perp|^2} d\hat{\mathbf{x}} \leq \int_{\hat{Q}_\delta} \frac{|\alpha(z)|^2}{|\hat{\mathbf{x}}_\perp|^2} d\hat{\mathbf{x}} + \int_{\hat{Q}} \frac{|v(\hat{\mathbf{x}}) - \alpha(z)|^2}{|\hat{\mathbf{x}}_\perp|^3} d\hat{\mathbf{x}} \\ &\leq \|\alpha\|_{L^2(I_\varepsilon)}^2 \underbrace{\left( \int_\delta^{2\delta} \frac{dr}{r} \right)}_{=\ln(2)} + C \|\Delta v\|_{L^2(\hat{Q})}^2 \leq C' \|\Delta v\|_{L^2(\hat{Q})}^2. \end{aligned}$$

Since  $v$  was chosen arbitrarily in  $H^2(\hat{Q}) \cap H_0^1(\hat{Q})$ , and the constant  $C' > 0$  is independent of  $\delta$ , this concludes the proof.  $\square$

**5. Norm convergence of the resolvent.** We will now use the previous analysis to show that  $A_\delta^{-1}$  strongly converges toward  $A_0^{-1}$  in some appropriate operator norm. Before stating this result let us just point out that using extension by 0, we have  $H_0^1(\Omega_\delta) \subset H_0^1(\Omega)$  so that  $H^{-1}(\Omega) \subset H^{-1}(\Omega_\delta)$ . The expression  $A_\delta^{-1}(f)$  with  $f \in H^{-1}(\Omega)$  should be understood according to these inclusions.

**PROPOSITION 5.1.** For  $\beta_\star > 0$  as in Proposition 3.1 and for each  $\beta \in \mathbb{R}$  satisfying  $|\beta| < \beta_\star$ , there exist constants  $c_\beta, \delta_0 > 0$  independent of  $\delta$  such that

$$\sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|A_0^{-1}(f) - A_\delta^{-1}(f)\|_{V_{\beta,\delta}^1(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq \frac{c_\beta}{|\ln \delta|^\beta} \quad \forall \delta \in (0, \delta_0).$$

*Proof.* First of all consider a  $\mathcal{C}^\infty$  cut-off function  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\chi(t) = 0$  for  $t \leq 1$  and  $\chi(t) = 1$  for  $t \geq 2$ , and set  $\chi_\delta(\mathbf{x}) := \chi(d(\mathbf{x})/\delta)$  and  $\psi_\delta := 1 - \chi_\delta$ . In the remainder of this proof, we shall denote  $\mathbb{Q}_\delta := \text{supp}(\nabla \chi_\delta)$  so that, for any  $\mathbf{x} \in \mathbb{Q}_\delta$ , we have  $\delta \leq d(\mathbf{x}) \leq 2\delta$ . We introduce the operator  $R_\delta : H^{-1}(\Omega) \rightarrow H_0^1(\Omega_\delta)$  defined by

$$R_\delta(f) := \chi_\delta A_0^{-1}(f) \quad \forall f \in H^{-1}(\Omega).$$

Now take any  $f \in L^2(\Omega)$ , and set  $u_\delta := A_\delta^{-1}(f) \in H_0^1(\Omega_\delta)$ ,  $\hat{u}_\delta := R_\delta(f) \in H_0^1(\Omega_\delta)$  and  $u_0 := A_0^{-1}(f) \in H^2(\Omega) \cap H_0^1(\Omega)$ . Let us look at the problem solved by the difference  $u_\delta - \hat{u}_\delta$ . There exist constants  $c, c' > 0$  independent of  $\delta$  such that for any  $v \in H_0^1(\Omega_\delta)$ , we have

$$\begin{aligned} |\langle A_\delta(u_\delta - \hat{u}_\delta), v \rangle| &= |(\nabla(u_\delta - \hat{u}_\delta), \nabla v)_\Omega| \\ &= |(f, \psi_\delta v)_\Omega + (\nabla \chi_\delta, u_0 \nabla v - v \nabla u_0)_{\mathbb{Q}_\delta}| \\ &\leq \|f\|_{L^2(\Omega)} \|\psi_\delta v\|_{L^2(\Omega)} + |(\nabla \chi_\delta, u_0 \nabla v - v \nabla u_0)_{\mathbb{Q}_\delta}|. \end{aligned} \tag{17}$$

To derive an upper bound for the second term in the right hand side above, observe that  $\delta \leq d(\mathbf{x}) \leq 2\delta$  for  $\mathbf{x} \in \text{supp}(\nabla \chi_\delta)$ . Since  $|\nabla d| \leq 1$ , we have  $\sup_{\mathbf{x} \in \bar{\Omega}} |\nabla \chi_\delta| \leq 2d(\mathbf{x})^{-1} \sup_{\mathbf{x} \in \bar{\Omega}} |\partial_t \chi|$ . So Cauchy-Schwarz inequality, together with Proposition 4.1 applied to  $\nabla u_0$  and Proposition 4.2 applied to  $u_0$ , yields the existence of a constant  $C > 0$

independent of  $\delta$  such that

$$\begin{aligned} |(\nabla\chi_\delta, u_0 \nabla v - v \nabla u_0)_{\mathbb{Q}_\delta}| &\leq C( \|\nabla v\|_{L^2(\mathbb{Q}_\delta)} + \|v\|_{L^2(\mathbb{Q}_\delta)} )\|\Delta u_0\|_{L^2(\Omega)} \\ &\leq C' |\ln \delta|^{-\beta} \|v\|_{V_{-\beta, \delta}^1(\Omega)} \|f\|_{L^2(\Omega)}. \end{aligned} \tag{18}$$

The first term in the right hand side of (17) can be bounded by noting that  $\delta \leq d_\delta(\mathbf{x}) \leq 2\delta$  on  $\text{supp}(\psi_\delta)$ . So applying Hardy’s inequality (9) yields a constant  $C > 0$  independent of  $\delta$  such that  $\|\psi_\delta v\|_{L^2(\Omega)} \leq C\delta |\ln \delta|^{1-\beta} \|v\|_{V_{-\beta, \delta}^1(\Omega)}$  for all  $v \in H_0^1(\Omega_\delta)$ . Plugging this together with estimate (18) into inequality (17) provides a constant  $C > 0$  independent of  $\delta$  such that

$$|\langle A_\delta(u_\delta - \hat{u}_\delta), v \rangle| \leq C |\ln \delta|^{-\beta} \|v\|_{V_{-\beta, \delta}^1(\Omega)} \|f\|_{L^2(\Omega)}.$$

Observe that  $A_\delta(u_\delta - \hat{u}_\delta) = (\text{Id} - A_\delta R_\delta)f \in H^{-1}(\Omega)$ . Since  $v \in H_0^1(\Omega_\delta)$  was arbitrary in the calculus above, what precedes shows that there exists a constant  $C > 0$  independent of  $\delta$  such that

$$\|(\text{Id} - A_\delta R_\delta)f\|_{V_{\beta, \delta}^{-1}(\Omega_\delta)} \leq C \|f\|_{L^2(\Omega)} |\ln \delta|^{-\beta}. \tag{19}$$

Following the same calculus but choosing a test function  $v \in H_0^1(\Omega)$ , the same result as (19) holds with  $A_\delta, \Omega_\delta$  replaced by  $A_0, \Omega$ . To conclude the proof, there only remains to use the stability estimates of Proposition 3.1 that yield constants  $C > 0$  independent of  $\delta$  such that for any  $f \in L^2(\Omega)$ , we have

$$\begin{aligned} \|A_0^{-1}(f) - A_\delta^{-1}(f)\|_{V_{\beta, \delta}^{-1}(\Omega)} &\leq \|A_0^{-1}(f) - R_\delta(f)\|_{V_{\beta, \delta}^{-1}(\Omega)} + \|A_\delta^{-1}(f) - R_\delta(f)\|_{V_{\beta, \delta}^{-1}(\Omega_\delta)} \\ &\leq C( \|f - A_0 R_\delta(f)\|_{V_{\beta, \delta}^{-1}(\Omega)} + \|f - A_\delta R_\delta(f)\|_{V_{\beta, \delta}^{-1}(\Omega_\delta)} ) \\ &\leq C |\ln \delta|^{-\beta} \|f\|_{L^2(\Omega)}. \end{aligned}$$

□

The proposition above yields consistency estimates for the asymptotic source problem. Indeed, let  $f \in L^2(\Omega)$  refer to a fixed function not depending on  $\delta$ , let  $u_0 \in H_0^1(\Omega_\delta)$  satisfy  $-\Delta u_0 = f$  in  $\Omega$ , and let  $u_\delta \in H_0^1(\Omega_\delta)$  satisfy  $-\Delta u_\delta = f$  in  $\Omega_\delta$ . Then Proposition 5.1 implies that  $\|u_\delta - u_0\|_{H^1(\Omega \cup \overline{V})} = O(|\ln \delta|^{-\beta})$  for any neighborhood  $U$  of  $\Gamma$  and for  $|\beta| < \beta_\star$ .

Sharper results can be obtained in terms of the  $L^2$ -norm. Observe that there exists a constant  $C > 0$  independent of  $\delta$  such that  $\|v\|_{L^2(\Omega)} \leq C \|v\|_{V_{\beta, \delta}^1(\Omega)}$  for any  $\beta \in \mathbb{R}$ . Plugging this into the estimate of Proposition 5.1 yields the following result.

**COROLLARY 5.1.** For  $\beta_\star > 0$  as in Proposition 3.1 and for any  $\epsilon \in (0, 2\beta_\star)$ , there exist constants  $c_\epsilon, \delta_0 > 0$  independent of  $\delta$  such that

$$\sup_{f \in L^2(\Omega) \setminus \{0\}} \frac{\|A_0^{-1}(f) - A_\delta^{-1}(f)\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}} \leq \frac{c_\epsilon}{|\ln \delta|^{\beta_\star - \epsilon}} \quad \forall \delta \in (0, \delta_0).$$

Recall that  $L^2(\Omega) \subset H^{-1}(\Omega_\delta)$  and  $H_0^1(\Omega_\delta) \subset L^2(\Omega)$ , so that  $A_\delta^{-1}$  is a continuous operator mapping  $L^2(\Omega)$  to  $L^2(\Omega)$ . As such, it is self-adjoint and compact, and, as can be checked by routine verifications, its eigenvalues with those of  $A_\delta^{-1}$  are considered to be an operator mapping  $L^2(\Omega_\delta)$  to  $L^2(\Omega_\delta)$ . Then, since  $A_0^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$  is also self-adjoint and compact, straightforward application of Theorem 4.10 of Chapter V of [12] yields that the spectra of  $A_\delta^{-1}$  and  $A_0^{-1}$  are closed to each other.

PROPOSITION 5.2. For  $\beta_\star > 0$  as in Proposition 3.1, and for any  $\epsilon \in (0, 2\beta_\star)$ , there exist constants  $\delta_0, c_\epsilon > 0$  independent of  $\delta$  such that

$$\sup_{\mu \in \mathfrak{S}(A_0)} \inf_{\lambda \in \mathfrak{S}(A_\delta)} \left| \frac{1}{\mu} - \frac{1}{\lambda} \right| + \sup_{\mu \in \mathfrak{S}(A_\delta)} \inf_{\lambda \in \mathfrak{S}(A_0)} \left| \frac{1}{\mu} - \frac{1}{\lambda} \right| \leq \frac{c_\epsilon}{|\ln \delta|^{\beta_\star - \epsilon}} \quad \forall \delta \in (0, \delta_0).$$

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