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EMERGENCE OF PARTIAL LOCKING STATES FROM THE ENSEMBLE OF WINFREE OSCILLATORS

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Abstract. We study the emergence of partial locking states for a subsystem whose dynamics is governed by the Winfree model. The Winfree model is the first mathematical model for synchronization. Thanks to the lack of conservation laws except for the number of oscillators, it exhibits diverse asymptotic nonlinear patterns such as partial and complete phase locking, partial and complete oscillator death, and incoherent states. In this paper, we present two sufficient frameworks for a majority sub-ensemble to evolve to the phase-locked state asymptotically. Our sufficient frameworks are characterized in terms of the mass ratio of the subsystem compared to the total system, ratio of the coupling strength to the natural frequencies, and the phase diameter of the subsystem. We also provide several numerical simulations and compare their results to the analytical results.

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1. Introduction. Biological and chemical complex systems in our nature often exhibit diverse nonlinear patterns due to the competing mechanisms between randomness, nonlinearity, diffusion, reactions, etc. Among others, we are mainly interested in the emergence of synchronization dynamics such as flashing of fireflies, collective heart beating, synchronous firing of neurons, etc. [4, 7, 8, 26, 27]. The mathematical study based on a model was first initiated by Winfree [26, 27] and Kuramoto [16, 17] five decades ago, and since then, it has become one of the hot topics in nonlinear dynamics, control theory and statistical physics. In particular, collective dynamics of complex systems has received lots of attention due to possible applications in engineering, e.g. decentralized control of sensor networks and UAVs (unmanned aerial vehicles) [18, 22, 23]. However, so far, the analytical studies were mostly restricted to the Kuramoto phase model. Although the Winfree model was the first mathematical model for synchronization, there is a rather small literature devoted to the Winfree model [2, 12, 14, 15, 19–21, 24, 25] as compared to vast literature for the Kuramoto model [1, 5, 6, 9, 10]. This is mainly due to the absence of a good mathematical structure, i.e., conservation laws. On the other hand, this absence of many conservation laws yields richer asymptotic nonlinear patterns, which makes the Winfree model more interesting from the dynamical systems viewpoint. In this paper, we are interested in the emergent asymptotic patterns and their relaxation dynamics of the Winfree phase model, in particular, the emergence of partial phase locking. We consider the complete graph consisting of N nodes and assume that at each node there is an oscillator. In this situation, the Winfree model given in [2, 12, 24] is given by the first-order system of ODEs:

$$\dot{\theta}_i = \Omega_i - \frac{K}{N}\sum_{j=1}^{N} \sin\theta_i \left(1 + \cos\theta_j\right), \quad i = 1, \ldots, N, \tag{1.1}$$

where K is a nonnegative coupling strength, and Ω_i is the natural frequency of the i-th oscillator, which is a random variable drawn from some given distribution function $g = g(\Omega)$ whose explicit form will be irrelevant in our analysis in finite-dimensional framework with $N < \infty$. The global well-posedness for (1.1) is guaranteed by the standard Cauchy-Lipschitz theory. Throughout the paper, we regard system (1.1) as a dynamical system in $\mathbb{R}^N$. As noted in [2, 12, 24], system (1.1) exhibits many asymptotic patterns such as incoherent state, partial and complete locking, partial and complete oscillator death. Thus, there are many questions from the dynamical systems viewpoint that we can think of. For example, we wish to understand how the total system $\mathcal{G}$ undergoes phase transitions between the above nonlinear patterns by varying the size of variation of natural frequency and coupling strength. In this paper, we are mainly interested in the formation of partial locking of some subsystem in the total system of Winfree oscillators. To describe the problem setting, let $\mathcal{G} := \{\theta_1, \cdots, \theta_N\}$ and $\mathcal{S} := \{\theta_{i_1}, \cdots, \theta_{i_n}\}$ be total and subsystems, respectively (see Figure 1). Without loss of generality, we can assume that

$$i_1 = 1, \cdots, i_n = n.$$

The purpose of this paper is to provide some sufficient conditions for the phase-locking of the set $\mathcal{S}$ in terms of the coupling strength K, variation of natural frequencies, and

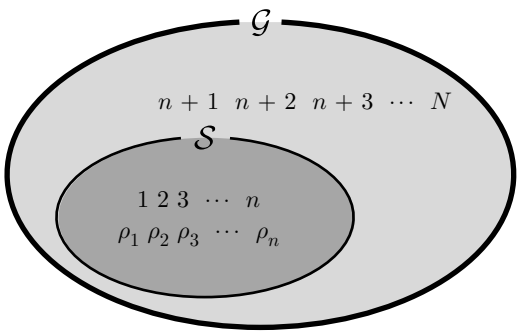


FIG. 1. Schematic diagram for subsystem $\mathcal{S}$ and total system $\mathcal{G}$

initial configurations. Here the phase-locking of the set $\mathcal{S}$ denotes the states satisfying the following relation:

$$\sup_{0\le t<\infty} \max_{1\le i,j\le n} |\theta_i(t) - \theta_j(t)| < \infty. \tag{1.2}$$

Suppose that one of the oscillators in $\mathcal{S}$, say θ_1, has a well-defined nonzero rotation number $\exists\ \rho_1 := \lim_{t\to\infty} \frac{\theta_i(t)}{t} \neq 0$. Then, the phase-locking relation (1.2) implies that the rest of the oscillators in $\mathcal{S}$ have a rotation number equal to ρ_1:

$$\exists\ \rho_i := \lim_{t\to\infty} \frac{\theta_i(t)}{t} = \rho_1, \quad 2 \le i \le n.$$

The conditions for the complete and partial oscillator death are discussed in the authors' recent works [14, 15] (see Section 2.3 for detailed discussions) in a large coupling strength regime.

The main results of this paper are two-fold: First, we consider the case where the subsystem $\mathcal{S}$ consists of oscillators with the same and nonzero natural frequency $\Omega \neq 0$. In this situation, our sufficient framework $\mathcal{F}_A$ consists of three conditions (see Section 3.1 for details). The first condition simply says that the system $\mathcal{S}$ is a majority group in the sense that its relative size $\frac{|\mathcal{S}|}{N}$ is larger than $\frac{4}{4+\pi} \approx 0.56$. The second condition says that the coupling strength K is in the moderate range compared to the common natural frequency Ω in the sense that $0 < K < C|\Omega|$ for some constant C. The third condition deals with the size of the initial phase configuration in the subsystem $\mathcal{S}$. Of course, we might not need to impose this restriction on the initial phase configuration of the subsystem $\mathcal{S}$, as is the case in the Kuramoto model [3, 11, 13]. Under the aforementioned three conditions, the relative phase differences in $\mathcal{S}$ go to zero asymptotically (see Theorem 3.1):

$$\lim_{t\to\infty} \max_{1\le i,j\le n} |\theta_i(t) - \theta_j(t)| = 0.$$

Second, we consider a set $\mathcal{S}$ consisting of oscillators with distributed natural frequencies. In this case, in addition to the first three conditions for the identical case, we need an additional fourth condition on the size of variation of natural frequencies. Under

these four conditions, the relative phase differences in $\mathcal{S}$ are uniformly bounded by some constant related to the initial phase diameter of $\mathcal{S}$ (see Theorem 4.1):

$$\sup_{0\le t<\infty}\max_{1\le i,j\le n}|\theta_i(t)-\theta_j(t)|\le\Delta_\theta^\infty<\infty.$$

The rest of this paper is organized as follows. In Section 2, we briefly review the Winfree model, discuss the relation with the Kuramoto model, and present previous results on the emergence of complete and partial oscillator deaths and the existence of periodic locked orbits for the Winfree model. In Section 3, we present a sufficient framework $\mathcal{F}_A$ for the existence of partial locking states for $\mathcal{S}$ consisting of oscillators with the same natural frequency. Our conditions $\mathcal{F}_A$ are presented in terms of the mass ratio $\frac{n}{N}$, relative ratio $\frac{K}{\Omega}$ and restriction on the initial radius of $\mathcal{S}$. In Section 4, we present another sufficient condition $\mathcal{F}_B$ for the existence of partial locking states for $\mathcal{S}$ consisting of oscillators with similar natural frequencies. Due to the heterogeneity of oscillators in the set $\mathcal{S}$, we need an extra restriction on the size of the heterogeneity of the natural frequencies. In Section 5, we present several numerical simulation results and compare them with the analytical results in Sections 3 and 4. Finally, Section 6 is devoted to the summary of our main results.

2. Preliminaries. In this section, we briefly discuss the Winfree model for synchronization and previous results on the formation of nonlinear patterns for (1.1).

2.1. *The Winfree model.* In this subsection, we briefly discuss the Winfree model with a general form of response and influence functions. Let $\theta_i=\theta_i(t)$ be a phase of the i-th oscillator. In the absence of mutual interactions between oscillators, the instantaneous phase velocity $\dot\theta_i$ is given by the natural frequency or intrinsic frequency Ω_i, i.e.,

$$\dot\theta_i=\Omega_i.$$

In 1967, Arthur Winfree [27] proposed a phase model for weakly coupled limit-cycle oscillators:

$$\dot\theta_i=\Omega_i+\frac{K}{N}\sum_{j=1}^N S(\theta_i)I(\theta_j),\quad i=1,\dots,N,$$

where S and I are response and influence functions satisfying monotonicity, regularity and parity conditions. More precisely, in [27], Winfree introduced two functions measuring the phase interactions via the influence function $I=I(\theta)$ and the sensitivity (or response) function $S=S(\theta)$ and employed two simplifying assumptions:

- The stimulus I_c impinging on the i-th oscillator is the average of the weak influences contributed by all phases of the oscillators in the ensemble (mean-field interaction):

$$I_c(\Theta):=\frac{1}{N}\sum_{j=1}^N I(\theta_j).\tag{2.1}$$

  In this case, two tacit assumptions are made. First, the influences of an individual are assumed to be propagated without attenuation in a time much shorter than the average period of the oscillators. Second, they are additive in their effect.

- The frequency perturbation $\hat{\Omega}_i$ of the i-th oscillator is proportional to the product of the sensitivity $S(\theta_i)$ and the average stimulus $I_c(\Theta)$:

$$\hat{\Omega}_i = KS(\theta_i)I_c(\Theta) = \frac{K}{N}\sum_{j=1}^{N} S(\theta_i)I(\theta_j).$$

Then, the Winfree model follows from the above assumptions.

As noted in [26], the Kuramoto model can be cast into a generalized Winfree model with two pairs of response and influence functions:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\cos\theta_i\sin\theta_j + \frac{K}{N}\sum_{j=1}^{N}\sin\theta_i \times (-\cos\theta_j). \tag{2.2}$$

Then, by setting

$$(S_1(\theta), I_1(\theta)) := (\cos\theta, \sin\theta), \qquad (S_2(\theta), I_2(\theta)) := (\sin\theta, -\cos\theta),$$

system (2.2) can be rewritten as a generalized Winfree model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} S_1(\theta_i)I_1(\theta_j) + \frac{K}{N}\sum_{j=1}^{N} S_2(\theta_i)I_2(\theta_j).$$

Note that the pairs of sensitivity and influence functions are orthogonal, in the sense that

$$\langle (S_1(\theta), I_1(\theta)), (S_2(\theta), I_2(\theta)) \rangle = 0,$$

where $\langle\ \cdot\ \rangle$ is the standard ℓ^2-inner product in $\mathbb{R}^2$.

2.2. *Remarks on the previous results.* In this subsection, we first briefly summarize the previous related results [14,15,21] on the formation of asymptotic dynamic patterns, and then discuss the comparison with our main results. We first recall the definitions on the complete and partial oscillator deaths, and complete and partial phase locking as follows.

Definition 2.1 ([14,15]). Let $\Theta := (\theta_1, \cdots, \theta_N)$ be an ensemble of Winfree oscillators whose dynamics are governed by (1.1).

(1) The configuration Θ tends to "*complete oscillator death (COD)*" if and only if the rotation numbers of all oscillators are zero, i.e.,

$$|\{i : \rho_i = 0\}| = N,$$

where $|A|$ is the size of set A.

(2) The configuration Θ tends to "*partial oscillator death (POD)*" if and only if the rotation numbers of at least two oscillators are zero, i.e.,

$$2 \le |\{i : \rho_i = 0\}| < N.$$

Note that for POD, not all rotation numbers are zero; if all oscillator rotation numbers are zero, COD is achieved.

(3) The configuration Θ tends to "*complete phase-locked state (CPLS)*" if and only if the rotation numbers of all oscillators are equal and nonzero; i.e., there exists a nonzero number ρ such that

$$|\{i : \rho_i = \rho\}| = N.$$

(4) The configuration Θ tends to "*partially phase-locked state (PPLS)*" if and only if there exist at least two oscillators whose rotation numbers are the same; i.e., there exists a nonzero constant ρ such that

$$2 \le |\{i \ : \ \rho_i = \rho\}| < N.$$

Note that for PPLS, not all rotation numbers are the same; if all oscillator rotation numbers are equal to ρ, CPLS is achieved.

2.2.1. *Complete and partial oscillator deaths.* In this part, we briefly summarize our previous results [14, 15] on the partial and complete oscillator deaths. For $\alpha \in (0, \pi)$, we set α^∞ to be the unique solution to the following trigonometric equation:

$$-\sin\theta(1+\cos\theta) = -\sin\alpha(1+\cos\alpha), \quad \theta \in \Big(0, \frac{\pi}{3}\Big].$$

Then, for such α and α^∞, we also set

$$K_e(\alpha^\infty) := \frac{\max_{1\le i\le N}|\Omega_i|}{\sin\alpha^\infty(1+\cos\alpha^\infty)}, \qquad \mathcal{R}(\alpha) := \{\Theta \in \mathbb{R}^N \ : \ \theta_i \in (-\alpha, \alpha), \ i = 1, \cdots, N\}.$$

As a direct application of Theorem 2.2 in [15] with $S(\theta) = -\sin\theta$ and $I(\theta) = 1 + \cos\theta$, we have the following two theorems on the existence of complete oscillator death.

THEOREM 2.1 ([15]). Suppose that α, coupling strength K and initial data Θ_0 satisfy

$$\alpha \in (0, \pi), \quad K > K_e(\alpha^\infty), \quad \Theta_0 \in \overline{\mathcal{R}}(\alpha).$$

Then, $\Theta(t)$ converges to a unique equilibrium state Φ; i.e., the rotation numbers of all oscillators are zero:

$$\rho_i = 0, \quad 1 \le i \le N.$$

REMARK 2.1. In [14], the following locally coupled Winfree model,

$$\dot{\theta}_i = \omega_i + K\sum_{j=1}^N c_{ji}S(\theta_i)I(\theta_j), \quad t > 0, \ i = 1, \dots, N,$$

has been studied, and it is shown that the partial oscillator death state emerges for

$$K \ge \max_{1\le i\le n} \frac{|\omega_i|}{\sum_{j=1}^n c_{ji}} \frac{1}{|SI(\alpha^\infty)|}$$

for some class of initial configurations. In particular, when $n = N$, the complete oscillator death occurs.

2.2.2. *Existence of periodic locked orbit.* In this part, we briefly discuss the result from [21]. Consider the Winfree model (1.1), and suppose that the natural frequencies ω_i are randomly chosen from the interval I:

$$\omega_i \in I := (1-\gamma, 1+\gamma) \quad \text{for } \gamma \in [0, 1), \quad i = 1, \cdots, N.$$

Let K_* be the locking bifurcation critical parameter for $\gamma = 0$ and $N = 1$ determined by

$$K_* = \sup\{K > 0 : 1 - K(1+\cos\theta)\sin\theta > 0, \forall\theta \in \mathbb{R}\} = \frac{4}{3\sqrt{3}}.$$

Define an open set U by

$$U := \{(\gamma, K) \in (0,1) \times (0, K_*) : 0 < \gamma < K\big(D(K)\big)^2\},$$

where $D(K) := L_*(1 - \frac{K}{K_*})^2$ with a constant $L_* = \frac{1}{280}\exp\big(-\frac{\pi K_*}{4(1-K_*)} - \frac{\pi K_*}{2}\big)$.

For $(\gamma, K) \in U$ and $N \geq 1$, define an open set $C^N_{\gamma,K}$ by

$$C^N_{\gamma,K} := \Big\{\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N : \max_{i,j} |\theta_i - \theta_j| < \Gamma_{\gamma,K}\Big(\frac{1}{N}\sum_{i=1}^N \theta_i\Big)\Big\},$$

where the dispersion curve $\Gamma_{\gamma,K} : \mathbb{R} \to (0,1)$ is a 2π-periodic function satisfying

$$\max_{s\in[0,2\pi]} \Gamma_{\gamma,K}(s) < D(K).$$

The main results from [21] are the existence of a positively invariant set and complete phase-locked states for each $(\gamma, K) \in U$.

THEOREM 2.2 ([21]). Let $(\gamma, K) \in U$ be an arbitrary point. Then, the following assertions hold.

(1) The set $C^N_{\gamma,K}$ is positively invariant under the Winfree flow Φ^t.
(2) There exists an initial configuration $\Theta_0 \in C^N_{\gamma,K}$ and a constant rotation vector $\rho_{\gamma,K} = (\rho_{1,\gamma,k}, \cdots, \rho_{N,\gamma,k})$ satisfying

$$\Phi^t_i(\Theta_0) = \rho_{\gamma,K} t + \Psi^N_i(t) \quad \text{for} \quad i = 1, \cdots, N, \quad t \geq 0,$$

where $\Psi^N_i : \mathbb{R}^+ \to \mathbb{R}$ is $\frac{2\pi}{\rho_{\gamma,K}}$-periodic, C^∞ and uniformly bounded with respect to N.

REMARK 2.2. As can be seen in Theorem 3.1 and Theorem 4.2, our main results mainly deal with the sufficient formation of partially phase-locked states from some admissible set of initial data and its relaxation dynamics, whereas the results from [21] are about the complete phase-locked state.

3. Partial locking for identical oscillators. In this section, we study a sufficient condition for the partial locking in an ensemble of Winfree phase oscillators. The heterogeneity of Winfree's phase oscillators is often measured by the width of natural frequencies. Throughout the paper, we say that two phase oscillators are homogeneous when they have the same natural frequency so that they are governed by the same dynamic law in the absence of coupling.

Consider an ensemble $\mathcal{G}$ of heterogeneous Winfree oscillators containing a homogeneous majority sub-ensemble $\mathcal{S}$. As shown in [14, 15], in a large coupling regime, all Winfree oscillators approach a unique equilibrium state so that all rotation numbers become zero. However, for some coupling strength in a moderate range, although the total ensemble $\mathcal{G}$ may not exhibit a synchronous behavior globally, the sub-ensemble $\mathcal{S}$ can exhibit a synchronous behavior locally, i.e., "*emergence of local synchronization*". This is one of the natural questions to consider in the absence of global synchronization. In this section, we would like to address this emergence of local synchronization by pro-

viding an analytical framework guaranteeing the synchronization of the homogeneous sub-ensemble. As the reader can see in Figure 1, since the sub-ensemble $\mathcal{S}$ is interacting with other oscillators in the whole ensemble $\mathcal{G}$, the synchronization of $\mathcal{S}$ is not clear in an *a priori* setting.

In the sequel, we are interested in the Cauchy problem for the Winfree model with all-to-all coupling:

$$\begin{cases} \dot{\theta}_i = \Omega_i - \dfrac{K}{N} \sin\theta_i \displaystyle\sum_{j=1}^{N} (1+\cos\theta_j), \quad t>0, \quad 1 \le i \le N, \\ \theta_i(0) = \theta_i^0. \end{cases} \tag{3.1}$$

3.1. *Framework A and partial locking.* In this subsection, we present a sufficient framework to guarantee the phase-locking of sub-ensemble $\mathcal{S}$ with a size $|\mathcal{S}| = n$. In the sequel, we introduce relative sizes for the sub-ensemble and coupling strength:

$$\gamma_S := \frac{n}{N}, \qquad \mu_K := \frac{K}{\Omega}.$$

Suppose that the relative size γ_S and common natural frequency of oscillators in $\mathcal{S}$ is nonzero:

$$0.56 \approx \frac{4}{4+\pi} < \gamma_S \le 1, \qquad \Omega_{i_1} = \cdots = \Omega_{i_n} = \Omega \neq 0. \tag{3.2}$$

Without loss of generality, we assume that

$$i_1 = 1, \cdots, i_n = n, \quad \Omega > 0.$$

If necessary, we replace (θ_i, Ω) by $(-\theta_i, -\Omega)$. For notational simplicity, we set the total ensemble and sub-ensemble by Θ and Θ_n, respectively:

$$\Theta := (\theta_1, \cdots, \theta_N), \qquad \Theta_n := (\theta_1, \cdots, \theta_n).$$

For a given Θ_n, we define extremal phases and functionals measuring the average and radius for Θ_n:

$$\begin{aligned} \theta_M &:= \max_{1\le i\le n} \theta_i, \qquad \theta_m := \min_{1\le i\le n} \theta_i, \\ A(\Theta_n) &:= \frac{\theta_M + \theta_m}{2}, \qquad R(\Theta_n) := \frac{\theta_M - \theta_m}{2}. \end{aligned} \tag{3.3}$$

Note that in general the extremal indices are dependent on time t. However, in our identical setting, if identical oscillators begin at different initial phases, then they cannot meet in any finite time. So the extremal indices M and m are determined by the initial configuration Θ_n^0 and are independent of t. Before we state our first main result, we introduce several constants to be used in this section. For a given K, Ω and γ, we set

$$\alpha^\infty := 1 + \frac{\pi}{4} - \frac{1}{\gamma_S}(1+2\mu_K),$$

and for $\alpha \in (0, \alpha^\infty)$, we also set positive constants $C_i^\pm$:

$$C_1^\pm(\alpha) := \frac{\mu_K}{1-2\mu_K}\left[\gamma_S\left(-2+2\alpha\pm\frac{\pi}{2}\right)+4\right], \quad C_2^\pm(\alpha) := \frac{\gamma_S\mu_K}{1+2\mu_K}\left(2-2\alpha\pm\frac{\pi}{2}\right). \tag{3.4}$$

Note that $C_i^{\pm}$ also depends on μ_K and γ_S as well, but their dependence is suppressed for notational simplicity. Note also that α^∞ satisfies the following conditions:

$$\begin{aligned}&(i)\ \alpha^\infty < \frac{\pi}{4},\\ &(ii)\ \alpha^\infty > 0 \quad\Longleftrightarrow\quad 0 < \mu_K < \frac{1}{2}\Big[\gamma_S\Big(1+\frac{\pi}{4}\Big)-1\Big],\end{aligned}$$

where the upper bound of μ_K appearing in the above relation is positive due to the relation (3.2).

We are now ready to state our Framework A for the emergence of partial locking.

- $(\mathcal{F}_A1)$: Sub-ensemble $\mathcal{S}$ is a majority group in total configuration:

$$0.56 \approx \frac{4}{4+\pi} < \gamma_S \le 1.$$

- $(\mathcal{F}_A2)$: The relative size μ_K of coupling strength compared to the common positive natural frequency Ω is moderately small:

$$0 < \mu_K < \left[\gamma_S\left(\frac{1}{2}+\frac{\pi}{8}\right)-\frac{1}{2}\right] =: \mu_K^\infty.$$

- $(\mathcal{F}_A3)$: Initial phases Θ_n^0 for sub-ensemble is confined:

$$\alpha \in (0,\alpha^\infty), \quad R(\Theta_n^0) \le \alpha e^{-C_1^-}.$$

We make a couple of observations on the above framework. We first note that the upper bound for μ_K appearing in $0 < \mu_K < \frac{1}{2}\big[\gamma_S(1+\frac{\pi}{4})-1\big]$ implies $K < \frac{\Omega}{2}$, and when the ensemble $\mathcal{S}$ is the same as the total ensemble $\mathcal{G}$, the above framework implies that

$$\alpha^\infty = \frac{\pi}{4} - 2\mu_K, \qquad \lim_{\mu_K\to\mu_K^\infty}\lim_{\alpha\to\alpha^\infty} C_1^- = \frac{\pi}{4}.$$

We are now ready to state our first result.

THEOREM 3.1. Suppose that the Framework A $(\mathcal{F}_A1)-(\mathcal{F}_A3)$ hold, and let Θ be a global solution to (3.1). Then, there exist positive constants β_i and r_i, $i=1,2$, depending only on γ_K, K and Ω such that

$$\beta_1 e^{-r_1 t} R(\Theta_n^0) \le R(\Theta_n(t)) \le \beta_2 e^{-r_2 t} R(\Theta_n^0), \quad t \ge 0,$$

for some positive β_i, where the positive constants r_i are given by the following relations:

$$\begin{aligned}r_1 &= \frac{\Omega}{2\pi}\cdot\frac{\mu_K}{1-2\mu_K}\left[4(1-\gamma_s)+\gamma_s(4\alpha+\pi)+8\mu_K\right],\\ r_2 &= \frac{\sin\alpha}{\alpha}\cdot\frac{2\gamma_s\Omega\mu_K}{(1+2\mu_K)\pi}(\alpha^\infty-\alpha).\end{aligned}$$

REMARK 3.1. Our results says that once ρ_i exists for some $i \in \{1,\cdots,n\}$, then all rotation numbers for oscillators in $\mathcal{S}$ are equal:

$$\rho_1 = \cdots = \rho_n.$$

In the following two subsections, we will provide the proof for Theorem 3.1, and as long as there is no confusion, we suppress Θ_n-dependence in $A(\Theta_n)$ and $R(\Theta_n)$:

$$A(t) := A(\Theta_n(t)), \qquad R(t) := R(\Theta_n(t)), \quad t \geq 0.$$

We first study the dynamic behavior of A and R in the following subsection.

3.2. *Dynamical system for* (A, R). It follows from (3.1) that extremal phases θ_M and θ_m satisfy

$$\dot{\theta}_M = \Omega - KI_c \sin\theta_M, \quad \dot{\theta}_m = \Omega - KI_c \sin\theta_m, \ t > 0. \tag{3.5}$$

Note that if the average stimulus $I_c := \dfrac{1}{N}\sum_{j=1}^{N}(1+\cos\theta_j)$ defined in (2.1) with $I(\theta_j) = 1+\cos\theta_j$ is constant, Adler's equations (3.5) are explicitly solvable (see Appendix in [5]), but it is clearly not a constant and satisfies the following trivial upper and lower bounds:

$$0 \leq I_c(\Theta) \leq 2.$$

LEMMA 3.1. Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to (3.1). Then, the functionals A and R defined in (3.3) satisfy the dynamics:

$$\begin{aligned} \dot{A} &= \Omega - KI_c \sin A \cos R, \\ \dot{R} &= -KI_c \cos A \sin R. \end{aligned} \tag{3.6}$$

Proof. • (Dynamics for A): Since $A = \frac{\theta_M+\theta_m}{2}$, we add two equations in (3.5) and divide by 2 to obtain

$$\begin{aligned} \dot{A} &= \Omega - \frac{KI_c}{2}(\sin\theta_M + \sin\theta_m) \\ &= \Omega - KI_c \sin\Big(\frac{\theta_M+\theta_m}{2}\Big)\cos\Big(\frac{\theta_M-\theta_m}{2}\Big) \\ &= \Omega - KI_c \sin A \cos R. \end{aligned}$$

• (Dynamics for R): Similarly, we have

$$\dot{R} = -KI_c \cos A \sin R.$$

□

3.2.1. *Rough dynamics for* (A, R). We next study the piecewise monotone dynamics of A and R (see Figure 2; the trajectory of A in current scale looks linear, but it is close to a linear motion but linear (see Figure 4)). For $K < \frac{\Omega}{2}$, we can attain $\dot{A} > 0$. Thus, there exist positive times $t_l^{\pm}$ satisfying

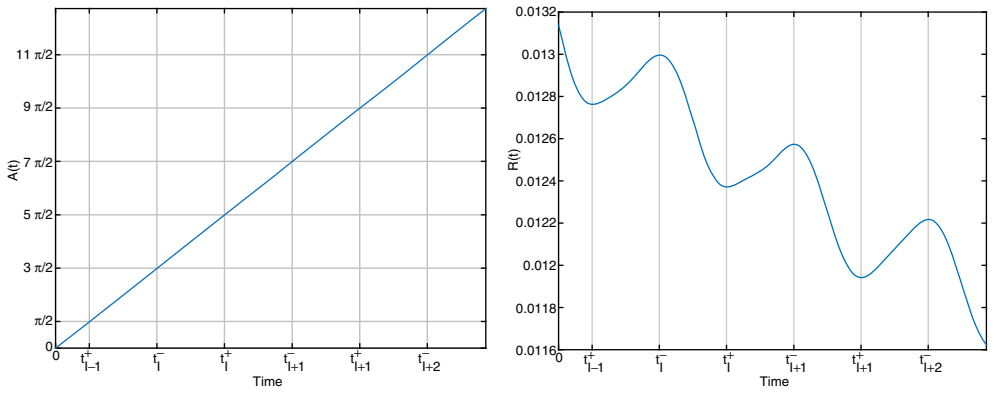
$$A(t_l^-) = 2l\pi - \frac{\pi}{2}, \qquad A(t_l^+) = 2l\pi + \frac{\pi}{2}, \tag{3.7}$$

for each integer l. Then, the sequences (t_l^+) and (t_l^-) are increasing sequences of times satisfying the relation (3.7), with

$$\cdots < t_l^- < t_l^+ < t_{l+1}^- < t_{l+1}^+ < \cdots.$$

Let N_* be the smallest integer starting from which the sequence (t_l^-) is positive:

$$0 < t_{N_*}^- < t_{N_*}^+ < t_{N_*+1}^- < t_{N_*+1}^+ < \cdots. \tag{3.8}$$



(a) Time-evolution for $A(t)$ (b) Time-evolution for $R(t)$

FIG. 2. Time-variation of A and R for identical oscillators with the following simulation setup (see Section 5 details) : $(N, n) = (20, 12)$, i.e., $\gamma_S = 0.6,\ K = 0.0178,\ \alpha = 0.0297,\ R(0) = 0.0287$

PROPOSITION 3.1. Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to (3.1) with $K < \frac{\Omega}{2}$. Then, we have the following estimates: for any integer $l \geq N_*$:

(1) The average functional A is strictly increasing in time and A and R satisfy

$$0 < \Omega - 2K \leq \dot{A} \leq \Omega + 2K, \quad t > 0,$$
$$\mathrm{sgn}\dot{R} = -\mathrm{sgn}(\cos A) \quad \text{if} \quad 0 < R < \frac{\pi}{2}.$$

(2) If $t \in (t_l^-, t_l^+)$ and $0 < R(t) < \frac{\pi}{2}$, then R is strictly decreasing.

(3) If $t \in (t_l^+, t_{l+1}^-)$ and $0 < R(t) < \frac{\pi}{2}$, then R is strictly increasing.

Proof. (1) It follows from the first equation in (3.6) that we have

$$\dot{A} = \Omega - KI_c \sin A \cos R.$$

We now use trivial estimates

$$0 \leq I_c(\Theta) \leq 2, \quad |\sin A \cos R| \leq 1$$

to see that

$$\Omega - 2K \leq \dot{A} \leq \Omega + 2K.$$

(2) Suppose that

$$t \in (t_l^-, t_l^+) \quad \text{and} \quad 0 < R(t) < \frac{\pi}{2}.$$

Then, it follows from $\dot{A} > 0$ and (3.7) that

$$2l\pi - \frac{\pi}{2} = A(t_l^-) < A(t) < A(t_l^+) = 2l\pi + \frac{\pi}{2}, \quad \text{i.e., } \cos A(t) > 0.$$

This and (3.6) yield

$$\dot{R}(t) = -KI_c(t) \cos A(t) \sin R(t) < 0.$$

Hence R is strictly decreasing as long as it stays in $(0, \frac{\pi}{2})$.

(3) Suppose that

$$t \in (t_l^+, t_{l+1}^-) \quad \text{and} \quad 0 < R(t) < \frac{\pi}{2}.$$

By the monotonicity of A, we have

$$2l\pi + \frac{\pi}{2} = A(t_l^+) < A(t) < A(t_{l+1}) = 2(l+1)\pi - \frac{\pi}{2}.$$

This yields

$$\cos A(t) < 0, \quad \dot{R}(t) = -KI_c(t)\cos A(t)\sin R(t) > 0.$$

□

REMARK 3.2. As can be seen in Figure 2 (b), although the radius of Θ_n is not monotonic, i.e., the increasing and decreasing time-interval appear periodically, it is decaying to zero in a global scale. In the next subsections, we will show that R decays to zero exponentially fast.

3.2.2. *Refined dynamics for R.* It follows from the definition of R that

$$R(t) \geq 0 \quad \text{for all times } t \geq 0.$$

Due to the uniqueness statement of Cauchy-Lipschitz theory, if $R(t_0) = 0$ for some $t_0 \geq 0$, then $R(t) = 0$ for all times $t \geq 0$, and thus our desired result is true in this case. Therefore, throughout the paper, we will assume $R > 0$ for all t. On the other hand, by Proposition 3.1, A is a strictly increasing Lipschitz function of time. This allows us to differentiate R with respect to A:

LEMMA 3.2. Suppose that positive constants l and α satisfy

$$l \geq N_* \quad \text{and} \quad \alpha \in (0, \alpha^\infty).$$

Then, we have the following assertions.

(1) If $t \in (t_l^-, t_l^+)$ and $0 < R(t) \leq \alpha$, then we have

$$L_1(A) \leq \frac{1}{\sin R}\frac{dR}{dA} \leq L_2(A).$$

(2) If $t \in (t_l^+, t_{l+1}^-)$ and $0 < R(t) \leq \alpha$, then we have

$$L_2(A) \leq \frac{1}{\sin R}\frac{dR}{dA} \leq L_1(A),$$

where the bounding functionals $L_1(A)$ and $L_2(A)$ are defined as follows:

$$\begin{aligned} L_1(A) &:= -\frac{\mu_K \cos A}{1 - 2\mu_K}\left[\gamma_S\left(-1 + \cos A + \alpha\right) + 2\right], \\ L_2(A) &:= -\frac{\mu_K \gamma_S \cos A}{1 + 2\mu_K}\left(1 + \cos A - \alpha\right). \end{aligned} \tag{3.9}$$

Proof. It follows from (3.6) that we have

$$\frac{1}{\sin R}\frac{dR}{dA} = -\frac{\mu_K I_c \cos A}{1 - \mu_K I_c \sin A \cos R}. \tag{3.10}$$

In the following estimates, we will use rough and refined estimates for I_c in the denominator and the numerator of the R.H.S. of (3.10), respectively.

(1) Suppose that $t \in (t_l^-, t_l^+)$. Then, we have

$$2l\pi - \frac{\pi}{2} < A(t) < 2l\pi + \frac{\pi}{2}, \quad \text{i.e., } \cos A > 0.$$

We use a trivial bound $I_c \le 2$ to see that the denominator in (3.10) can be estimated as follows:

$$0 < 1 - 2\mu_K \le 1 - \mu_K I_c \sin A \cos R \le 1 + 2\mu_K. \tag{3.11}$$

On the other hand, we estimate the numerator in (3.10) more carefully. We first note that the term I_c can be written as follows:

$$\begin{aligned} I_c &= \frac{1}{N}\sum_{j=1}^{N}(1+\cos\theta_j) = \frac{1}{N}\Big[\sum_{j=1}^{n}(1+\cos\theta_j) + \sum_{j=n+1}^{N}(1+\cos\theta_j)\Big] \\ &=: I_c^1 + I_c^2. \end{aligned} \tag{3.12}$$

- (Estimate of I_c^1): We use

$$\begin{aligned} \frac{1}{N}\left|\sum_{j=1}^{n}(1+\cos\theta_j) - n(1+\cos A)\right| &\le \frac{1}{N}\sum_{j=1}^{n}|\cos\theta_j - \cos A| \\ &\le \frac{1}{N}\sum_{j=1}^{n}|\theta_j - A| \le \frac{n}{N}R(t) \le \gamma_S\alpha. \end{aligned}$$

Thus, we have

$$\gamma_S(1+\cos A - \alpha) \le I_c^1 \le \gamma_S(1+\cos A + \alpha). \tag{3.13}$$

- (Estimate of I_c^2): In this case, we use a trivial estimate $0 \le 1+\cos\theta_j \le 2$ to find

$$0 \le I_c^2 \le 2(1-\gamma_S). \tag{3.14}$$

In (3.12), we combine (3.13) and (3.14) to get

$$\gamma_S(1+\cos A - \alpha) \le I_c \le 2 + \gamma_S(-1+\cos A + \alpha). \tag{3.15}$$

Finally, in (3.10), we use (3.11) and (3.15) to obtain the desired result.

(2) Suppose that $t \in (t_l^+, t_{l+1}^-)$. Then, we have

$$2l\pi + \frac{\pi}{2} < A(t) < 2(l+1)\pi - \frac{\pi}{2}, \quad \text{i.e.,} \quad \cos A < 0.$$

The rest of the arguments are the same as in (i). □

LEMMA 3.3. Suppose that positive constants l, N_* and α satisfy

$$l \ge N_* \quad \text{and} \quad \alpha \in (0, \alpha^\infty),$$

and let $L_i(A)$ be bounding functions defined in (3.9). Then, we have the following estimates:

$$\begin{aligned} &(i)\ \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_1(A)dA = -C_1^+ < 0, \qquad \int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_1(A)dA = C_1^- > 0, \\ &(ii)\ \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_2(A)dA = -C_2^+ < 0, \qquad \int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_2(A)dA = C_2^- > 0, \end{aligned}$$

$$
\begin{aligned}
&(iii)\ \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_2(A)dA + \int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_1(A)dA = -C_2^+ + C_1^- < 0,\\
&\qquad \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_1(A)dA + \int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_2(A)dA = -C_1^+ + C_2^- < 0,\\
&(iv)\ \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_1(A)dA < \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_2(A)dA,\\
&\qquad \int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_2(A)dA < \int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_1(A)dA,
\end{aligned}
$$

where the positive constants $C_i^{\pm}$ are defined in (3.4).

Proof. (i) Note that in (3.9), $L_1(A)$ can be rewritten as follows:

$$
L_1(A) = -\frac{\mu_K}{1-2\mu_K}\Big\{\frac{1}{2}\gamma_S + \Big[\gamma_S(\alpha-1)+2\Big]\cos A + \frac{\gamma_S}{2}\cos 2A\Big\}.
$$

We use the above relation to obtain

$$
\int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_1(A)dA = -\frac{\mu_K}{1-2\mu_K}\Big[\frac{\pi\gamma_S}{2} + 2\gamma_S(\alpha-1)+4\Big] = -C_1^+.
$$

Similarly, we have

$$
\int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_1(A)dA = -\frac{\mu_K}{1-2\mu_K}\Big[\frac{\pi\gamma_S}{2} - 2\gamma_S(\alpha-1)-4\Big] = C_1^-.
$$

(ii) Note that $L_2(A)$ can be rewritten as follows:

$$
L_2(A) = -\frac{\mu_K\gamma_S}{1+2\mu_K}\Big[\frac{1}{2} + (1-\alpha)\cos A + \frac{1}{2}\cos 2A\Big].
$$

Again, this yields

$$
\int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_2(A)dA = -\frac{\mu_K\gamma_S}{1+2\mu_K}\Big[\frac{\pi}{2} + 2(1-\alpha)\Big] = -C_2^+.
$$

Similarly we have

$$
\int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_2(A)dA = -\frac{\mu_K\gamma_S}{1+2\mu_K}\Big[\frac{\pi}{2} - 2(1-\alpha)\Big] = C_2^-.
$$

(iii) By direct calculation, we have

$$C_1^- - C_2^+ = -\frac{4\mu_K\gamma_S}{1-4\mu_K^2}(\alpha^\infty - \alpha) < 0$$

and

$$C_2^- - C_1^+ = -\frac{\mu_K}{1-4\mu_K^2}\Big[4(1-\gamma_S) + \gamma_S(4\alpha+\pi) + 8\mu_K\Big] < 0.$$

(iv) The inequalities in (iv) are, respectively, equivalent to the relations

$$C_1^+ > C_2^+ \quad \text{and} \quad C_1^- > C_2^-.$$

By direct calculations, we have

$$\begin{aligned} C_1^+ - C_2^+ &= \frac{\mu_K}{1-4\mu_K^2}\Big[4(1-\gamma_S) + 4\alpha\gamma_S + 2\pi\mu_K\gamma_S + 8\mu_K\Big] > 0, \\ C_1^- - C_2^- &= \frac{\mu_K}{1-4\mu_K^2}\Big[4(1-\gamma_S) + 4\alpha\gamma_S + 2\mu_K(4-\pi\gamma_S)\Big] > 0. \end{aligned}$$

□

3.3. *Decay estimate of R.* In this part, we study the exponential decay of R. We first consider a calculus type lemma as follows. Consider a function f defined by

$$f(x) := -\ln\Big(\frac{1+\cos x}{\sin x}\Big), \quad x \in \Big(0, \frac{\pi}{2}\Big). \tag{3.16}$$

Then, f is increasing on $\big(0, \frac{\pi}{2}\big)$ and has derivative

$$f'(x) = \frac{1}{\sin x}. \tag{3.17}$$

LEMMA 3.4. Let f be a function defined by the relation (3.16). Then, if $0 < x < y < \frac{\pi}{2}$, we have

$$\frac{\sin y}{y}\,(f(y) - f(x)) < \ln\frac{y}{x} < f(y) - f(x).$$

Proof. • (The second inequality): We use (3.17) and

$$\sin t < t, \quad t \in (0, \frac{\pi}{2})$$

to see that

$$\ln\frac{y}{x} = \int_x^y \frac{dt}{t} < \int_x^y \frac{dt}{\sin t} = f(y) - f(x).$$

• (The first inequality): We use (3.17) and the relation

$$\sin t \geq \frac{\sin y}{y}t, \quad x < t < y$$

to obtain

$$f(y) - f(x) = \int_x^y \frac{dt}{\sin t} < \int_x^y \frac{ydt}{t\sin y} = \frac{y}{\sin y}\ln\frac{y}{x}.$$

□

As can be seen in Figure 2, the sequences of local maxima and local minima are strictly decreasing sequences along the sequences (t_l^+) and (t_l^-), respectively. In the following lemma, we quantify the observation discussed in Proposition 3.1 for the dynamics of R.

LEMMA 3.5. Suppose that positive constants l, N_*, α and R satisfy

$$l \geq N_*, \qquad \alpha \in (0, \alpha^\infty), \qquad \sup_{0 \leq t < \infty} R(t) \leq \alpha.$$

Then, the following estimates hold:

$$\begin{aligned}
&(i)\ -C_1^+ < \ln \frac{R(t_l^+)}{R(t_l^-)} < -\frac{\sin\alpha}{\alpha} C_2^+, \qquad \frac{\sin\alpha}{\alpha} C_2^- < \ln \frac{R(t_{l+1}^-)}{R(t_l^+)} < C_1^-. \\
&(ii)\ \ln \frac{R(t_{l+1}^-)}{R(t_l^-)} < \frac{\sin\alpha}{\alpha} \underbrace{(C_1^- - C_2^+)}_{<0}, \qquad \ln \frac{R(t_{l+1}^+)}{R(t_l^+)} \geq \underbrace{-C_1^+ + C_2^-}_{<0}.
\end{aligned}$$

Proof. In this lemma, we quantify the monotonicity of R which has been discussed in Proposition 3.1.

(i) We first use Lemma 3.2 (i) and Lemma 3.3 to see that

$$-C_1^+ = \int_{2l\pi - \frac{\pi}{2}}^{2l\pi + \frac{\pi}{2}} L_1(A) dA \leq \int_{R(t_l^-)}^{R(t_l^+)} \frac{dR}{\sin R} \leq \int_{2l\pi - \frac{\pi}{2}}^{2l\pi + \frac{\pi}{2}} L_2(A) dA = -C_2^+. \tag{3.18}$$

On the other hand, it follows from Lemma 3.4 that we have

$$\int_{R(t_l^-)}^{R(t_l^+)} \frac{dR}{\sin R} = f(R(t_l^+)) - f(R(t_l^-)). \tag{3.19}$$

We now use the fact that $R(t)$ is monotonically decreasing on (t_l^-, t_l^+) and apply Lemma 3.4 with $x = R(t_l^+) < R(t_l^-) = y$ to find

$$\begin{aligned}
f(R(t_l^+)) - f(R(t_l^-)) < \ln \frac{R(t_l^+)}{R(t_l^-)} &< \frac{\sin R(t_l^-)}{R(t_l^-)} (f(R(t_l^+)) - f(R(t_l^-))) \\
&< \frac{\sin\alpha}{\alpha} (f(R(t_l^+)) - f(R(t_l^-))).
\end{aligned} \tag{3.20}$$

Thus, we use (3.18), (3.19) and (3.20) to obtain the first relation:

$$-C_1^+ < \ln \frac{R(t_l^+)}{R(t_l^-)} < -\frac{\sin\alpha}{\alpha} C_2^+.$$

For the second relation, we again use Lemma 3.2 (2) and Lemma 3.3 to find

$$C_2^- = \int_{2l\pi + \frac{\pi}{2}}^{2l\pi + \frac{3\pi}{2}} L_2(A) dA \leq \int_{R(t_l^+)}^{R(t_{l+1}^-)} \frac{dR}{\sin R} \leq \int_{2l\pi + \frac{\pi}{2}}^{2l\pi + \frac{3\pi}{2}} L_1(A) dA = C_1^-. \tag{3.21}$$

On the other hand, it follows from Lemma 3.4 that

$$\ln \frac{R(t_{l+1}^-)}{R(t_l^+)} < \int_{R(t_l^+)}^{R(t_{l+1}^-)} \frac{dR}{\sin R} < \frac{\alpha}{\sin\alpha} \ln \frac{R(t_{l+1}^-)}{R(t_l^+)}. \tag{3.22}$$

We now combine (3.21) and (3.22) to obtain

$$\frac{\sin\alpha}{\alpha} C_2^- < \ln \frac{R(t_{l+1}^-)}{R(t_l^+)} < C_1^-.$$

(ii) It follows from Lemma 3.2 and Lemma 3.3 that we have

$$\begin{aligned} & f(R(t_{l+1}^-)) - f(R(t_l^-)) \\ &\quad = f(R(t_{l+1}^-)) - f(R(t_l^+)) + f(R(t_l^+)) - f(R(t_l^-)) \\ &\quad = \int_{2l\pi+\frac{\pi}{2}}^{2(l+1)\pi-\frac{\pi}{2}} \frac{1}{\sin R}\frac{dR}{dA}dA + \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} \frac{1}{\sin R}\frac{dR}{dA}dA \\ &\quad \le \int_{2l\pi+\frac{\pi}{2}}^{2(l+1)\pi-\frac{\pi}{2}} L_1(A)dA + \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_2(A)dA \\ &\quad = C_1^- - C_2^+ < 0. \end{aligned} \tag{3.23}$$

Since f is strictly increasing on $(0, \pi/2)$, the relation (3.23) yields

$$R(t_{l+1}^-) < R(t_l^-).$$

Then, we apply Lemma 3.4 with $x = R(t_{l+1}^-)$ and $y = R(t_l^-)$ to get

$$\frac{\alpha}{\sin\alpha}\ln\frac{R(t_{l+1}^-)}{R(t_l^-)} < f(R(t_{l+1}^-)) - f(R(t_l^-)) < \ln\frac{R(t_{l+1}^-)}{R(t_l^-)}. \tag{3.24}$$

We now combine (3.23) and (3.24) to obtain the desired estimate. For the second estimate, we proceed similarly as the first relation; i.e., we obtain

$$f(R(t_{l+1}^+)) - f(R(t_l^+)) \ge \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_1(A)dA + \int_{2l\pi+\frac{\pi}{2}}^{2l\pi+\frac{3\pi}{2}} L_2(A)dA = -C_1^+ + C_2^-.$$

We again use Lemma 3.4 to derive the desired estimate. □

REMARK 3.3. The estimates on R in Lemma 3.5 show that the sub-ensemble diameter R is strictly decreasing on (t_l^-, t_l^+) and then begins to strictly increase again on (t_l^+, t_{l+1}^-). However, after one revolution of the sub-ensemble, the diameter of the sub-ensemble becomes strictly smaller than before. This illustrates that R will eventually decrease to zero.

As a corollary of Lemma 3.5, we have the following corollary.

COROLLARY 3.1. Suppose that positive constants l, N_*, α and R satisfy

$$l \ge N_*, \qquad \alpha \in (0, \alpha^\infty), \qquad \sup_{0\le t<\infty} R(t) \le \alpha.$$

Then we have the following estimates:

$$\begin{aligned} &(i)\ \ln\frac{R(t_l^-)}{R(t_{N_*}^-)} \le -\frac{\sin\alpha}{\alpha}\cdot\frac{(\Omega-2K)}{2\pi}|C_2^+ - C_1^-|\cdot|t_l^- - t_{N_*}^-|. \\ &(ii)\ \ln\frac{R(t_l^+)}{R(t_{N_*}^+)} \ge -\frac{(\Omega+2K)}{2\pi}|C_1^+ - C_2^-|\cdot|t_l^+ - t_{N_*}^+|. \end{aligned}$$

Proof. Let $l \ge N_*$ be any integer. Then, we successively apply the estimate in Lemma 3.5 to find

$$\ln\frac{R(t_l^-)}{R(t_{N_*}^-)} \le -\frac{\sin\alpha}{\alpha}|C_2^+ - C_1^-|(l - N_*). \tag{3.25}$$

On the other hand, it follows from Proposition 3.1 that we have

$$\frac{(\Omega-2K)}{2\pi}(t_l^- - t_{N_*}^-) \le (l-N_*) \le \frac{(\Omega+2K)}{2\pi}(t_l^+ - t_{N_*}^+). \tag{3.26}$$

We combine (3.25) and (3.26) to conclude that

$$\ln\frac{R(t_l^-)}{R(t_{N_*}^-)} \le -\frac{\sin\alpha}{\alpha}\cdot\frac{(\Omega-2K)}{2\pi}|C_2^+ - C_1^-|\cdot|t_l^- - t_{N_*}^-|.$$

Similarly, we have

$$\ln\frac{R(t_l^+)}{R(t_{N_*}^+)} > -|C_1^+ - C_2^-|(l-N_*) > -\frac{(\Omega+2K)}{2\pi}|C_1^+ - C_2^-|\cdot|t_l^+ - t_{N_*}^+|.$$

□

3.4. *Proof of Theorem* 3.1. In this subsection, we provide the proof of our first main result. We split the proof into two steps.

Suppose that positive constants l, N_*, α and initial datum Θ_0 satisfy

$$l \ge N_*, \quad \alpha \in (0, \alpha^\infty), \quad R_0 \le \alpha e^{-C_1^-},$$

and that $t_{N_*}^-$ is the first time that $R(t_{N*}^-)$ is the local maximum, as we defined in the equation (3.8).

Note that in the previous lemmas, the assumption $\sup_{0\le t<\infty} R(t) \le \alpha$ plays a key role in the proofs. To guarantee this condition, the first step is as follows.

• Step A (Existence of a bounding set): We define a bounding set $\mathcal{A}$:

$$\Theta_0 \in \mathcal{A} := \{\Theta \in \mathbb{R}^N \ : \ R(\Theta_n) \le \alpha\}.$$

Then, we claim that, under the Framework A, $\Theta(t) \in \mathcal{A}$ for all $t > 0$.

◇ For this, we set

$$\mathcal{T} := \{T \ge 0 : R(t) < \alpha \quad \text{for } 0 \le t \le T\},$$

$$\bar{\mathcal{T}} := \{T \ge 0 : R(t) \le \alpha \quad \text{for } 0 \le t \le T\}, \qquad \bar{T}^* := \sup\bar{\mathcal{T}}.$$

Since $0 \in \mathcal{T}$, $\bar{T}^* > 0$. Note that if $\bar{T}^* < \infty$, then

$$R(\bar{T}^*) = \alpha.$$

First, we will show that $\bar{T}^* \ge t_{N_*}^-$. Suppose $\bar{T}^* < t_{N_*}^-$. We consider two cases:

◇ Case I (R is increasing on $[0, t_{N_*}^-]$): In this case, $\cos(A_0) < 0$, $L_1(A) > 0$ for $t \in [0, t_{N_*}^-]$, hence

$$f(R(\bar{T}^*)) - f(R_0) = \int_{A_0}^{A(\bar{T}^*)} L_1(A)dA \le \int_{-\frac{3\pi}{2}}^{-\pi/2} L_1(A)dA = C_1^-.$$

Since $R_0 < \alpha e^{-C_1^-}$, $R(\bar{T}^*) < \alpha$, which is contradictory to the assumption on $\bar{T}^* < t_{N_*}^-$.

◇ Case II (R is decreasing on $[0,\delta]$ for some $\delta < t^-_{N_*}$): In this case, $\cos(A_0) > 0$. Then the time $t^+_{N_*-1}$ is well-defined with (3.8). Since R is decreasing on $[0, t^+_{N_*-1}]$ and increasing on $[t^+_{N_*-1}, t^-_{N_*}]$,

$$\begin{aligned} f(\alpha) - f(R_0) &= f(R(\bar{T}^*)) - f(R_0) \\ &\le f(R(\bar{T}^*)) - f(R(t^+_{N_*-1})) \le \int_{2N_*\pi-\frac{3\pi}{2}}^{A(\bar{T}^*)} L_1(A)dA < \int_{-\frac{3\pi}{2}}^{-\frac{\pi}{2}} L_1(A)dA. \end{aligned}$$

Then, we get the same contradiction as in Case I. In particular, we have

$$R(t^-_{N_*}) < \alpha.$$

Second, suppose that $\bar{T}^* < \infty$. From the previous two cases, there exists $l \ge N_*$ such that $t^-_l < \bar{T}^* \le t^-_{l+1}$, and $R(t^-_l) < \alpha$.

By Lemma 3.2, R is strictly decreasing on $[t^-_l, t^+_l]$, and thus it follows that $T^* > t^+_l$, hence

$$\begin{aligned} f(R(T^*)) - f(R(t^-_l)) &= \big[f(R(t^+_l)) - f(R(t^-_l))\big] + \big[f(R(T^*)) - f(R(t^+_l))\big] \\ &\le \int_{2l\pi-\frac{\pi}{2}}^{2l\pi+\frac{\pi}{2}} L_2(A)dA + \int_{2l\pi+\frac{\pi}{2}}^{A(T^*)} L_1(A)dA \\ &\le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} L_2(A)dA + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} L_1(A)dA \\ &= -C^+_2 + C^-_1 < 0, \end{aligned}$$

from Lemma 3.3.

Thus, $R(T^*) < R(t^-_l) < \alpha$, and we reach a contradiction. Therefore, we have

$$R(t) \le \alpha, \quad t \ge 0.$$

• Step B (Exponential decay estimates): In this step, we show that $R = R(t)$ in fact decays exponentially. From Corollary 3.1 and the condition $R(t) \le \alpha$ for all $t \ge 0$:

$$\begin{aligned} &(i)\ \ln\frac{R(t^-_l)}{R(t^-_{N_*})} \le -\frac{\sin\alpha}{\alpha}\cdot\frac{(\Omega-2K)}{2\pi}|C^+_2 - C^-_1|\cdot|t^-_l - t^-_{N_*}|, \\ &(ii)\ \ln\frac{R(t^+_l)}{R(t^+_{N_*})} \ge -\frac{(\Omega+2K)}{2\pi}|C^+_1 - C^-_2|\cdot|t^+_l - t^+_{N_*}|. \end{aligned}$$

◇ Step B.1 (Upper bound estimate): Note that $R(t) \le R(t^-_l)$, for $t \in [t^-_l, t^-_{l+1}]$.

Hence

$$R(t) \le R(T^-_{N_*})\exp\left[-\frac{\sin\alpha}{\alpha}\cdot\frac{(\Omega-2K)}{2\pi}|C^+_2 - C^-_1|(t - T^-_{N_*+1})\right], \text{ for } t \ge t^-_{N_*+1},$$

since the right hand side function is greater than the step function made by

$$R(t) \le R(t^-_l), \text{ for } t \in [t^-_l, t^-_{l+1}], \text{ for any } l \ge N_*.$$

Since $R(t) \le \alpha$ for $t \in [0, t^-_{N_*+1}]$, there exists $\beta_2 > 0$ such that

$$R(t) \le \beta_2 e^{-r_2 t}R(0), \quad \text{for } t \ge 0,$$

where $r_2 = -\frac{\sin\alpha}{\alpha}\cdot\frac{(\Omega-2K)}{2\pi}|C^+_2 - C^-_1|$.

$\diamond$ Step B.2 (Lower bound estimate): We again consider the fact that $R(t) \geq R(t_l^+)$, for $t \in [t_l^-, t_{l+1}^-]$. There we see that

$$R(t) \geq R(T^+_{N_*+1}) \exp\left[-\frac{(\Omega+2K)}{2\pi}|C_1^+ - C_2^-|(t - T^+_{N_*})\right], \text{ for } t \geq t^+_{N_*}.$$

This completes the proof of Theorem 3.1.

4. Partial locking for nonidentical oscillators. In this section, we show existence of partial locking for nonidentical Winfree oscillators. For a pair of nonidentical oscillators with different natural frequencies, their phase-difference may not go to zero asymptotically unlike the case of identical oscillators. Thus, what we can expect at most will be the uniform boundedness of these transversal phase-differences in time. In the sequel, we present another framework guaranteeing the uniform boundedness of phase-differences.

4.1. *Framework B and partial locking.* In this subsection, we present our second framework and main results on the emergence of partial locking on the sub-ensemble of nonidentical oscillators. Consider a sub-ensemble $\mathcal{S}$ consisting of nonidentical oscillators with slightly different natural frequencies. In the absence of coupling, the phase-differences between two nonidentical oscillators will be proportional to the product of time and differences in natural frequencies. Thus, nonidentical oscillators cannot be locked in the absence of couplings. On the other hand, when the coupling is too strong as in [15], the rotation numbers of oscillators will be zero; i.e., oscillator death appears. Hence, to get a partial locking of nonidentical oscillators, we must have a reasonable range of coupling strength. We are now ready to state our second framework, Framework B, on the emergence of partial flocking.

- ($\mathcal{F}_B1$): Sub-ensemble $\mathcal{S}$ is a majority group:

$$0.56 \approx \frac{4}{4+\pi} < \gamma_S \leq 1.$$

- ($\mathcal{F}_B2$): The relative size μ_K of coupling strength compared to the common positive natural frequency Ω is moderately small:

$$0 < \mu_K < \min\left\{\gamma_S\left(\frac{1}{2}+\frac{\pi}{8}\right) - \frac{1}{2}, \frac{1}{6}\right\}.$$

- ($\mathcal{F}_B3$): Initial phases Θ_n^0 for sub-ensemble are confined:

$$\alpha \in (0, \alpha^\infty), \quad R(\Theta_n^0) \leq \frac{1}{2}\alpha.$$

- ($\mathcal{F}_B4$): The natural frequencies Ω_i are small perturbations of the common frequency Ω:

$$\Omega > 0, \qquad 0 \leq \Delta_\Omega \leq \Delta_\Omega^\infty, \qquad \Omega_i \in \Big[\Omega - \Delta_\Omega, \Omega + \Delta_\Omega\Big], \quad i = 1, \cdots, n,$$

where the bound of frequency fluctuation Δ_Ω^∞ satisfies

$$\Delta_\Omega^\infty < \Omega - 6K \tag{4.1}$$

and

$$\frac{\Delta_\Omega^\infty}{3K}\left[\exp\left(\frac{6\pi K}{|\Omega|-\Delta_\Omega^\infty-2K}\right)-1\right]\le\min\left\{\frac{1}{2}\alpha,\ \alpha\Big(1-e^{-\frac{\sin\alpha}{\alpha}|C_2^+-C_1^-|}\Big)\right\}. \tag{4.2}$$

Note that 0 satisfies the above relations. Therefore, by the continuity of the relations, there exists a positive constant Δ_Ω^∞ satisfying the above relations.

Finally, we state our second theorem on the existence of partial synchronization. The proof will be given in Section 4.3.

THEOREM 4.1. Suppose that the Framework B $(\mathcal{F}_B1)-(\mathcal{F}_B4)$ hold, and let Θ be a global solution to (3.1). Then, the diameter of the sub-ensemble $\mathcal{S}$ is uniformly bounded:

$$\sup_{0\le t<\infty}R(\Theta_n(t))\le\frac{3}{2}\alpha.$$

4.2. *Dynamical system for* (A,R). For a given sub-ensemble Θ_n, we define functionals $A, R, t_l^-, t_l^+, \Omega_M$ and Ω_m as in Section 3. We first derive the dynamical system for (A,R) in the following lemma.

LEMMA 4.1. Let $\Theta=(\theta_1,\cdots,\theta_N)$ be a solution to (3.1). Then, the functionals A and R defined in (3.3) satisfy the dynamics:

$$\begin{aligned}\dot A&=A(\Omega)-KI_c\sin A\cos R,\\ \dot R&=R(\Omega)-KI_c\cos A\sin R,\end{aligned} \tag{4.3}$$

where $A(\Omega)$ and $R(\Omega)$ are given in terms of the natural frequencies of the extremal indices:

$$A(\Omega):=\frac{\Omega_M+\Omega_m}{2},\quad R(\Omega):=\frac{\Omega_M-\Omega_m}{2}.$$

Proof. The proof is exactly the same as that of Lemma 3.1. Hence we omit the details. □

As a corollary of Lemma 4.1, we can see that the average A of sub-ensemble $\mathcal{S}$ is strictly increasing under the Framework B.

COROLLARY 4.1. Let $\Theta=(\theta_1,\cdots,\theta_N)$ be a solution to (3.1) with $K<\dfrac{\Omega-\Delta_\Omega}{2}$. Then, the functional $A(\Theta_n)$ is strictly increasing and it satisfies

$$\Omega-\Delta_\Omega-2K\le\dot A\le\Omega+\Delta_\Omega+2K. \tag{4.4}$$

Proof. Note that

$$\Omega-\Delta_\Omega\le\frac{\Omega_M+\Omega_m}{2}\le\Omega+\Delta_\Omega,\qquad 0\le I_c\le 2.$$

Then, we use the above relations and the first equation in (4.3) to see that the sub-ensemble average A is strictly increasing and that

$$\Omega-\Delta_\Omega-2K\le\dot A\le\Omega+\Delta_\Omega+2K. \tag{4.5}$$

On the other hand, it follows from the assumption $K < \dfrac{\Omega - \Delta_\Omega}{2}$ that

$$\Omega - \Delta_\Omega - 2K > 0.$$

Hence, the L.H.S. of (4.5) is strictly positive; i.e., A is strictly increasing. □

4.3. *Uniform boundedness of R.* In this subsection, we show that R is uniformly bounded. As can be seen in the numerical simulations in Figure 3, in short-time scale, the dynamics of R looks like that of the identical case in Section 3; i.e., it decays exponentially up to some finite time, and then it begins to increase again and then relaxes to a positive constant value. To employ this numerical evidence in an analytical framework, we will introduce a virtual reference flow $\tilde{\Theta}$ which falls down to the case of Section 3, and then we will show that the fluctuation between our real Winfree flow and reference flow is uniformly bounded in time to obtain our desired uniform bound.

We now introduce a reference flow. For a given natural frequency vector $(\Omega_1, \cdots, \Omega_N)$, we introduce a modified natural frequency vector $(\tilde{\Omega}_1, \cdots, \tilde{\Omega}_N)$ as follows:

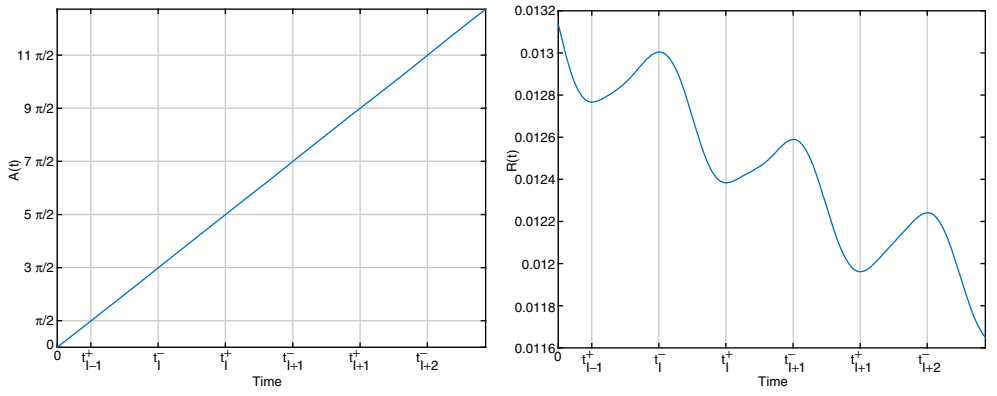
$$\tilde{\Omega}_i = \begin{cases} \Omega & \text{if } 1 \le i \le n, \\ \Omega_i & \text{if } n < i \le N. \end{cases} \tag{4.6}$$

For $l \ge N_*$, let $\tilde{\Theta}$ be a solution to (3.1) with modified natural frequencies (4.6) starting from $t = t_l^-$:

$$\begin{aligned} &\dot{\tilde{\theta}}_i = \tilde{\Omega}_i - \frac{K}{N} \sin \tilde{\theta}_i \sum_{j=1}^{N} \left(1 + \cos \tilde{\theta}_j\right), \ t > t_l^-, \\ &\tilde{\Theta}(t_l^-) = \Theta(t_l^-). \end{aligned} \tag{4.7}$$

As in Section 3, we define $\tilde{A}$, $\tilde{R}$ and the positive times $\tilde{t}_l^-$ and $\tilde{t}_l^+$ for $\tilde{\Theta}$, respectively.

LEMMA 4.2. For $l \ge N_*$, suppose that the Framework B $(\mathcal{F}_B 1) - (\mathcal{F}_B 4)$ hold, and let Θ be a global solution to (3.1). If $\tilde{R}(t_l^-) = R(t_l^-) < \alpha$, then the virtual reference flow $\tilde{\Theta}$ to



(a) Time-evolution for $A(t)$ (b) Time-evolution for $R(t)$

FIG. 3. Time-variation of A and R for nonidentical oscillators

(4.7) satisfies

$$\tilde{R}(\tilde{t}_{l+1}^-) = \max_{\tilde{t}_l^+ \le t < \infty} \tilde{R}(t) \quad \text{and} \quad \tilde{R}(t) \le \tilde{R}(t_l^-) e^{-\frac{\sin\alpha}{\alpha}|C_2^+ - C_1^-|}, \quad t \ge \tilde{t}_l^+.$$

Proof. We first note that if we set $\Delta_\Omega = 0$, then all conditions in Framework B imply the corresponding conditions in Framework A for the reference flow (4.7). Hence, we use the results of Theorem 3.1 to get the desired estimates. □

We next consider the difference of our real flow Θ to (3.1) from the virtual reference flow $\tilde{\Theta}$ to (4.7): for some $l \ge N_*$,

$$\begin{cases} \dot{\theta}_i = \Omega_i - \dfrac{K}{N} \sin\theta_i \displaystyle\sum_{k=1}^N (1 + \cos\theta_k), \quad t > t_l^-, \\ \dot{\tilde{\theta}}_i = \tilde{\Omega}_i - \dfrac{K}{N} \sin\tilde{\theta}_i \displaystyle\sum_{k=1}^N \Big(1 + \cos\tilde{\theta}_k\Big), \\ \Theta(t_l^-) = \tilde{\Theta}(t_l^-). \end{cases}$$

It follows from the above equations and mean-value theorem that we have

$$\begin{aligned} &\frac{d}{dt}(\theta_i - \tilde{\theta}_i) \\ &\quad = \Omega_i - \tilde{\Omega}_i - \frac{K}{N} \sum_{j=1}^N \Big[\sin\theta_i (1 + \cos\theta_j) - \sin\tilde{\theta}_i (1 + \cos\tilde{\theta}_j) \Big] \\ &\quad = \Omega_i - \tilde{\Omega}_i - \frac{K}{N} \sum_{j=1}^N \Big[(\cos\bar{\theta}_i^*)(\theta_i - \tilde{\theta}_i)(1 + \cos\theta_j) - (\sin\tilde{\theta}_i)(\sin\bar{\theta}_j^{**})(\theta_j - \tilde{\theta}_j) \Big], \end{aligned}$$

where $\bar{\theta}_i^*$ and $\bar{\theta}_i^{**}$ are intermediate values between θ_i and $\tilde{\theta}_i$, and θ_j and $\tilde{\theta}_j$, respectively.

This yields a Gronwall's inequality for $||\Theta - \tilde{\Theta}||_{\ell^\infty}$:

$$\frac{d}{dt}||\Theta - \tilde{\Theta}||_{\ell^\infty} \le \Delta_\Omega^\infty + 3K||\Theta - \tilde{\Theta}||_{\ell^\infty}, \quad t > t_l^-.$$

LEMMA 4.3. For $l \ge N_*$, suppose that the Framework B $(\mathcal{F}_B 1) - (\mathcal{F}_B 4)$ hold, and let Θ be a global solution to (3.1) and let $\tilde{\Theta}$ be a global solution to (4.7). Then, we have the following estimates.

(1) $t_{l+1}^- - \tilde{t}_l^+ > 0$.
(2) For positive times $t_l^\pm$ defined by the relation (3.7), we have

$$R(t_l^-) \le \alpha \quad \Longrightarrow \quad R(t_{l+1}^-) \le \alpha.$$

Proof. (1) Since $t_l^- = \tilde{t}_l^-$,

$$t_{l+1}^- - \tilde{t}_l^+ = \big(t_{l+1}^- - t_l^-\big) - \big(\tilde{t}_l^+ - \tilde{t}_l^-\big) \ge \frac{2\pi}{\Omega + \Delta_\Omega + 2K} - \frac{\pi}{\Omega - 2K} > 0.$$

(2) Fix l. Let Θ and $\tilde{\Theta}$ be solutions to the following systems:

$$\begin{cases} \dot{\theta}_i = \Omega_i - \dfrac{K}{N}\sin\theta_i \displaystyle\sum_{k=1}^{N}(1+\cos\theta_k), \quad t > t_l^-, \\ \dot{\tilde{\theta}}_i = \tilde{\Omega}_i - \dfrac{K}{N}\sin\tilde{\theta}_i \displaystyle\sum_{k=1}^{N}\Big(1+\cos\tilde{\theta}_k\Big), \\ \Theta(t_l^-) = \tilde{\Theta}(t_l^-). \end{cases}$$

It follows from the above equations and mean-value theorem that we have

$$\frac{d}{dt}||\Theta - \tilde{\Theta}||_{\ell^1} \le \Delta_\Omega^\infty + 3K||\Theta - \tilde{\Theta}||_{\ell^1}, \quad t > t_l^-.$$

Then, Gronwall's lemma yields

$$\|\Theta(t) - \tilde{\Theta}(t)\|_{\ell^1} \le \frac{\Delta_\Omega^\infty}{3K}\left[e^{3K|t-t_l^-|} - 1\right], \quad t > t_l^-. \tag{4.8}$$

We use (4.5) to see that

$$(\Omega - \Delta_\Omega^\infty - 2K)(t_{l+1}^- - t_l^-) \le 2\pi, \quad \text{i.e.,} \quad t_{l+1}^- - t_l^- \le \frac{2\pi}{\Omega - \Delta_\Omega^\infty - 2K}. \tag{4.9}$$

Finally, we combine (4.8) and (4.9) to obtain

$$\begin{aligned} R(t_{l+1}^-) &\le \tilde{R}(t_{l+1}^-) + \|\Theta(t_{l+1}^-) - \tilde{\Theta}(t_{l+1}^-)\|_{\ell^1} \\ &\le \tilde{R}(t_{l+1}^-) + \frac{\Delta_\Omega^\infty}{3K}\left[\exp\left(\frac{6\pi K}{\Omega - \Delta_\Omega^\infty - 2K}\right) - 1\right] \\ &\le \alpha e^{-\frac{\sin\alpha}{\alpha}|C_2^+ - C_1^-|} + \alpha\Big(1 - e^{-\frac{\sin\alpha}{\alpha}|C_2^+ - C_1^-|}\Big) \\ &= \alpha. \end{aligned} \tag{4.10}$$

The last inequality follows from Lemma 4.2 and condition (4.2). □

We are now ready to present our second main result.

THEOREM 4.2. Suppose that the Framework B $(\mathcal{F}_B1)-(\mathcal{F}_B4)$ hold, and let Θ be a global solution to (3.1). Then, we have

$$R(\Theta_n(t)) \le \frac{3}{2}\alpha, \quad \forall\, t \ge 0.$$

Proof. Let Θ and $\tilde{\Theta}$ be solutions to the following systems:

$$\begin{cases} \dot{\theta}_i = \Omega_i - \dfrac{K}{N}\sin\theta_i \displaystyle\sum_{k=1}^{N}(1+\cos\theta_k), \quad t > t_l^-, \\ \dot{\tilde{\theta}}_i = \tilde{\Omega}_i - \dfrac{K}{N}\sin\tilde{\theta}_i \displaystyle\sum_{k=1}^{N}\Big(1+\cos\tilde{\theta}_k\Big), \\ \Theta(0) = \tilde{\Theta}(0). \end{cases}$$

In the same way as the previous lemma, for $t \in [0, t^-_{N_*}]$,

$$\begin{aligned} R(\Theta_n(t)) &\le \tilde{R}(t) + \|\Theta(t) - \tilde{\Theta}(t)\|_{\ell^1} \\ &\le \tilde{R}(0) + \frac{\Delta^\infty_\Omega}{3K}\Big(e^{3Kt} - 1\Big) \\ &\le \frac{1}{2}\alpha + \frac{\Delta^\infty_\Omega}{3K}\Big(e^{3Kt^-_{N_*}} - 1\Big) \\ &\le \frac{1}{2}\alpha + \frac{\Delta^\infty_\Omega}{3K}\Big(\exp\big(\frac{6\pi K}{\Omega - \Delta^\infty_\Omega - 2K}\big) - 1\Big) \\ &\le \alpha. \end{aligned}$$

This implies, by Lemma 4.3 (2), that

$$R(\Theta(t^-_{N_*})) \le \alpha, \quad \text{and thus} \quad R(\Theta(t^-_l)) \le \alpha, \quad \text{for all } l \ge N_*.$$

From (4.3), we attain

$$|\dot{R}(t)| \le \Delta^\infty_\Omega + 2KR(t).$$

By Gronwall's inequality, we have

$$R(t) \le R(s) + \frac{\Delta^\infty_\Omega}{2K}(e^{2K(t-s)} - 1) \quad \text{for} \quad t > s \ge 0.$$

Thus, for $t \in [t^-_l, t^-_{l+1})$,

$$\begin{aligned} R(t) &\le R(t^-_l) + \frac{\Delta^\infty_\Omega}{2K}(e^{2K(t-t^-_l)} - 1) \\ &\le \alpha + \frac{\Delta^\infty_\Omega}{2K}\big(\exp(\frac{4\pi K}{\Omega - \Delta^\infty_\Omega - 2K}) - 1\big) \qquad (4.11) \\ &\le \alpha + \frac{1}{2}\alpha = \frac{3}{2}\alpha. \end{aligned}$$

In the last inequality of (4.11), we use (4.2) and the convexity of exponential function as

$$\begin{aligned} &\frac{\Delta^\infty_\Omega}{2K}\big(\exp(\frac{4\pi K}{\Omega - \Delta^\infty_\Omega - 2K}) - 1\big) \\ &\le \frac{1}{3}\frac{\Delta^\infty_\Omega}{2K}\big(\exp(\frac{0}{\Omega - \Delta^\infty_\Omega - 2K}) - 1\big) + \frac{2}{3}\frac{\Delta^\infty_\Omega}{2K}\big(\exp(\frac{6\pi K}{\Omega - \Delta^\infty_\Omega - 2K}) - 1\big) \\ &\le \frac{1}{2}\alpha. \end{aligned}$$

In conclusion,

$$R(\Theta_n(t)) \le \frac{3}{2}\alpha, \quad \forall\, t \ge 0.$$

□

REMARK 4.1. The result of Theorem 4.2 implies that all oscillators in Θ_n have the same rotation number: for $i, j = 1, \cdots, n$,

$$|\rho_i - \rho_j| \le \lim_{t\to\infty}\Big(\frac{|\theta_i(t) - \theta_j(t)|}{t}\Big) \le \lim_{t\to\infty}\frac{3\alpha}{2t} = 0, \quad \text{thus, we have} \quad \rho_i = \rho_j.$$

5. Numerical simulations. In this section, we provide several numerical examples and compare them with the analytical results in the previous two sections. For all simulations, we used the fourth order Runge-Kutta method with time step $\Delta t = 0.01$.

5.1. *Identical oscillators.* In this subsection, we present simulation results in the setting of Section 3; i.e., we consider a sub-ensemble of Winfree oscillators with the same nonzero natural frequency and compare simulation results to Theorem 3.1. For a numerical simulation, we choose

$$(N, n) = (20, 12), \quad \text{i.e.,} \quad \gamma_S = 0.6 > \frac{4}{4+\pi},$$

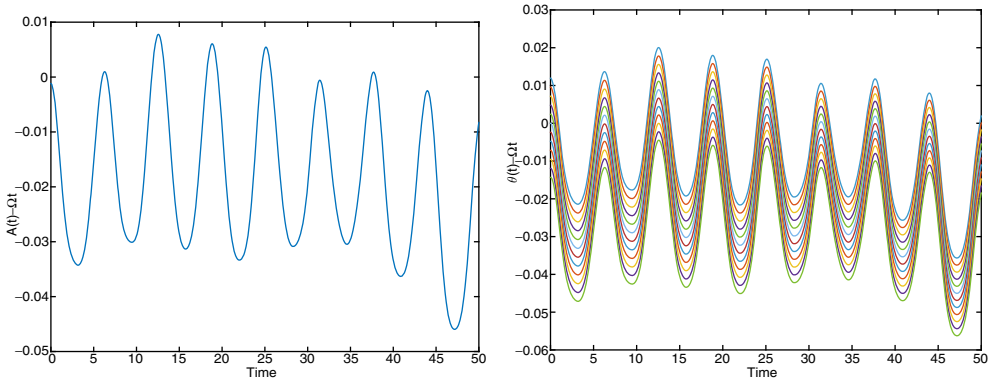
and set the following parameters to satisfy the assumptions of Theorem 3.1:

$$K = 0.0178, \quad \alpha = 0.0297, \quad R(0) = 0.0287.$$

To construct $R(0)$, we let $\{\theta_i\}_{i=1}^n$ be equally distributed in $[-R(0)/2, R(0)/2]$. Moreover, the other oscillators, $\Omega_{n+1}, \cdots, \Omega_N$ and $\theta_{n+1}, \cdots, \theta_N$ are chosen arbitrarily in $[0, 2]$ and $[0, 2\pi]$, respectively.

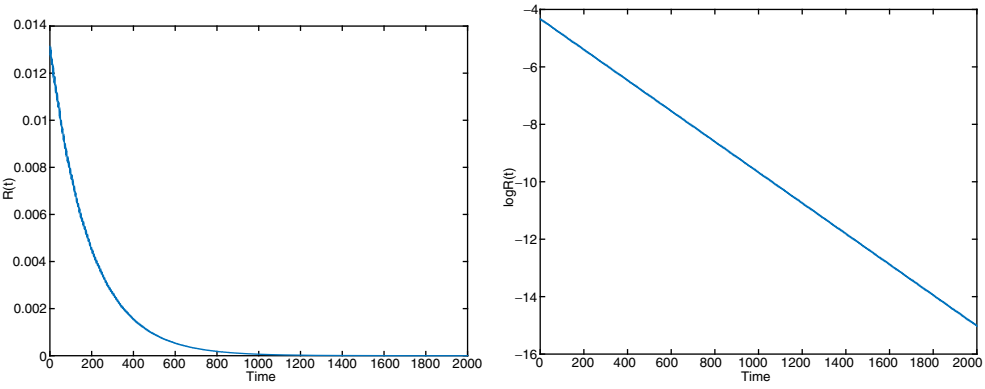
Figure 4 shows the basic behavior of the oscillators. Our principal observation is that the oscillators just rotate near their natural frequencies, according to Figure 4(b). It also shows the absence of the complete phase-locked state. Figure 2 (a) shows the middle growth of identical oscillators. They seem to follow $\Omega = 1$, but from Figure 4(a), they oscillate in a regular period, following the equation (3.6).

Figure 5 shows the partially phase-locked state. Figure 2(b) is the local behavior of $R(t)$. In each oscillation of $R(t)$, we can see the decreasing phenomena of each ridge, which is calculated in Lemma 3.3. The period of oscillation is similar to $A(t)$ in Figure 4(a) from the equations (3.6). Theorem 3.1 concludes that $R(t)$ should decay exponentially, and this result is shown in Figures 5(a) and 5(b). The straight line of Figure 5(b) is a direct evidence of the exponential decay of $R(t)$, but it is important that the graph is actually oscillating in a small scale. On the other hand, we can calculate r_1 and r_2 from



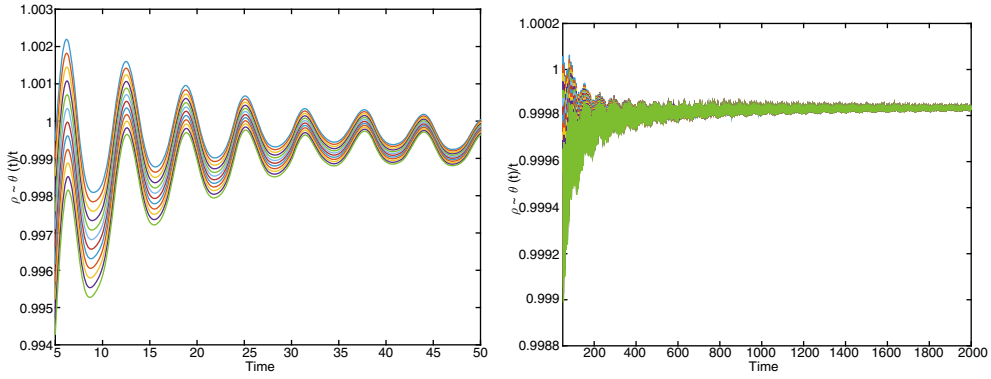
(a) Time-variation of $A(t) - \Omega t$ (b) Time-variation of $\theta_i(t) - \Omega t$ for $i = 1, \cdots, 12$

FIG. 4. Fluctuations around macroscopic motion Ωt



(a) Time-variation of $R(t)$ (b) Time-variation of $\log R(t)$

FIG. 5. Exponential decay of $R(t)$



(a) Time-variation of $\theta(t)/t$ (b) Time-variation of $\theta(t)/t$ until $t = 2000$

FIG. 6. Convergence of ρ_i

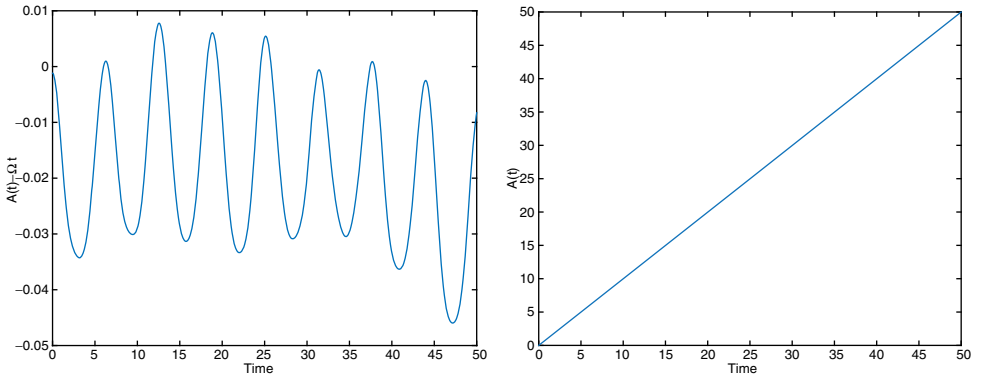
the proof of Theorem 3.1. The mean tangent value of 5(b) is -0.005354 and satisfies the conclusion of Theorem 3.1:

$$-r_1 = -0.0122 \leq -0.005354 \leq -1.950 \times 10^{-4} = -r_2.$$

Therefore, these numerical results agree with Theorem 3.1.

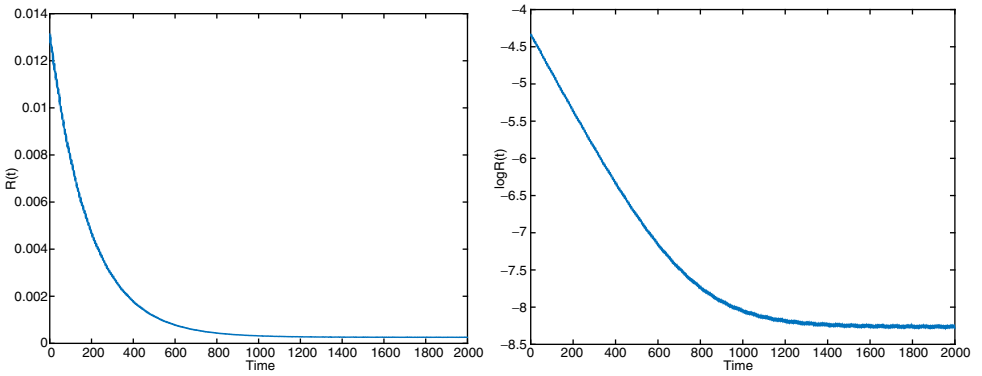
In Figure 6, we can see the existence of ρ_i and its simultaneous convergence to the same value. Figure 6(a) is the short time behavior of $\theta_i(t)/t$, as we expected in Figure 2(b). Figure 6(b) is the graph from $t = 50$ to 2000. Near $t = 2000$, the value oscillates from 0.99982 to 0.99985; hence we can see that ρ_1 is similar to Ω, but a little less than Ω.

5.2. *Nonidentical oscillators.* In this subsection, we present simulation results for Theorem 4.2. For comparison with the previous subsection, we used the same initial values and parameters, except for $\Omega_1, \cdots, \Omega_n$, which are reset to be equally distributed in $[\Omega - \Delta_\Omega^\infty, \Omega + \Delta_\Omega^\infty]$, where $\Delta_\Omega^\infty = 1.5 \times 10^{-6}$. This value satisfies the assumptions 4.2 of



(a) Time-variation of $A(t) - \Omega t$ (b) Time-variation of $A(t)$

FIG. 7. Time-variation of $A(t)$ in a nonidentical case



(a) Time-variation of $R(t)$ until $t = 2000$ (b) Time-variation of $\log R(t)$ until $t = 2000$

FIG. 8. Time-variation of $R(t)$ in a nonidentical case

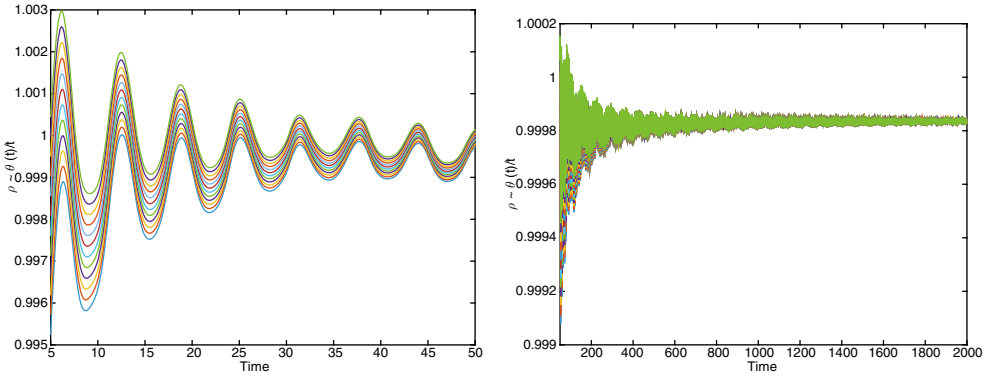
Theorem 4.2, with the following calculations:

$$\frac{1}{2}\alpha - \frac{\Delta_\Omega^\infty}{3K}\left[\exp\left(\frac{6\pi K}{|\Omega| - \Delta_\Omega^\infty - 2K}\right) - 1\right] = 0.0148 > 0,$$

$$\alpha\left(1 - e^{-\frac{\sin\alpha}{\alpha}|C_2^+ - C_1^-|}\right) - \frac{\Delta_\Omega^\infty}{3K}\left[\exp\left(\frac{6\pi K}{|\Omega| - \Delta_\Omega^\infty - 2K}\right) - 1\right] = 2.609 \times 10^{-6} > 0.$$

Figure 7 shows the same behavior as the previous subsection. Since the perturbation of Ω_i is too small to affect the graph, Figure 7(a) looks exactly the same as 4(a).

The difference comes out in the long time behavior of $R(t)$. Figure 8 shows that the partially phase-locked state also exists in the nonidentical situation. Theorem 4.2 concludes that $R(t)$ is bounded, and this result is seen in Figure 8(a). Since we fixed α with the initial value, the increasing of the ridge value of $R(t)$ does not violate the analysis in Lemma 4.3. In contrast to Figure 5(a), the value of $R(t)$ is not stabilized at 0, but decreases and stays in a nonzero value. Figure 8(b) also shows us that the value of



(a) Time-variation of $\theta(t)/t$ (b) Time-variation of $\theta(t)/t$ until $t = 2000$

FIG. 9. Convergence of ρ_i in a nonidentical case

$R(t)$ converges to some nonzero value. Therefore, for nonidentical oscillators, the result of Theorem 3.1 is too strong, and the boundedness result of Theorem 4.2 is what we can expect.

6. Conclusion. In this paper, we presented sufficient frameworks for emergent partial locking from identical and nonidentical Winfree oscillators. Our frameworks were formulated in terms of the mass ratio, the relative coupling strength and the size of admissible initial configurations. For an ensemble of Winfree oscillators with all-to-all couplings, several nonlinear patterns such as incoherence, locking, partial locking, partial death and complete death were shown to appear in their asymptotic dynamics. From the dynamical systems viewpoint, there are many interesting questions in relation to asymptotic behavior, e.g. classification of basins of attraction and phase-transitions between the aforementioned asymptotic states. Among them, in this paper, we focused on the basins of attraction of the partial locking states where part of the oscillators were confined in a bounded domain. In a large coupling regime, a given initial configuration will evolve toward complete oscillator death where all oscillators stop to rotate; i.e., the rotation vector becomes zero. Thus, in order to get a partial locked state, our coupling strength compared to the reference frequency of the subsystem should not be large. Of course, we have not discussed other important issues, such as the existence of a critical coupling strength for partial phase-locking and the stability of several nonlinear patterns. We leave these interesting problems for future investigation.

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