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BOLTZMANN SYSTEMS FOR GAS MIXTURES IN 1D TORUS

By

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Abstract. We study the 1D Boltzmann equation for a mixture of two gases on a torus with the initial condition of one gas near a vacuum and the other near a Maxwellian equilibrium state. An $L^\infty_x L^\infty_{\xi,\beta}$ analysis is developed to study this mass diffusion problem, which is based on the Boltzmann equation for the single species hard sphere collision in an earlier work of the author. The decay rate of the solution is algebraic for a small time region and exponential for a large time region. Moreover, the exponential rate depends on the size of the domain.

1. Introduction.

1.1. *The model.* The Boltzmann equation for a mixture of two gases takes the form of the following system:

$$\begin{cases} \partial_t F_A + \xi_1 \partial_x F_A = Q^{AA}(F_A, F_A) + Q^{AB}(F_A, F_B)\,, \\ \partial_t F_B + \xi_1 \partial_x F_B = Q^{BB}(F_B, F_B) + Q^{BA}(F_B, F_A)\,. \end{cases} \tag{1.1}$$

The right hand side consists of the usual collision terms which for $X, Y \in \{A, B\}$ are given by:

$$Q^{XY}(F_X, F_Y) = \sigma^2_{XY} \int_U \left(F'_X F'_{Y_*} - F_X F_{Y_*}\right) |(\xi - \xi_*) \cdot \Omega| d\xi_* d\Omega$$

with

$$U = \left\{(\xi_*, \Omega) \in \mathbb{R}^3 \times \mathbb{S}^2 : (\xi - \xi_*) \cdot \Omega \geq 0\right\},$$

where $F'_{Y_*} = F_Y(t, x, \xi'_*), F_{Y_*} = F_Y(t, x, \xi_*)$ and $F'_X = F_X(t, x, \xi')$. The constant factor σ_{XY} denotes the sum of the radii of molecule X and molecule Y (for simplicity, we assume that $\sigma_{XY} = 1$ for all $X, Y \in \{A, B\}$). Let m_X and m_Y be the molecule masses

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of the two gases. The pre-collision velocities (ξ, ξ_*) and post-collision velocities (ξ', ξ'_*) satisfy the conservations of momentum and energy

$$m_X \xi' + m_Y \xi'_* = m_X \xi + m_Y \xi_* \,,$$
$$m_X |\xi'|^2 + m_Y |\xi'_*|^2 = m_X |\xi|^2 + m_Y |\xi_*|^2 \,.$$

These conservation laws result in the collision transformation: $\Omega \in S^2$,

$$\xi' = \xi + \frac{2m_Y}{m_X + m_Y}[(\xi - \xi_*) \cdot \Omega]\Omega \,, \quad \xi'_* = \xi_* - \frac{2m_X}{m_X + m_Y}[(\xi - \xi_*) \cdot \Omega]\Omega \,.$$

Let $X \neq Y \in \{A, B\}$; then we have the following invariant properties of the collision operator (see [4]):

$$\begin{cases} \displaystyle\int_{\mathbb{R}^3} \{1, \xi, |\xi|^2\} Q^{XX}(F_X, F_X) d\xi = 0 \,, \\ \displaystyle\int_{\mathbb{R}^3} Q^{XY}(F_X, F_Y) d\xi = 0 \,, \\ \displaystyle\int_{\mathbb{R}^3} \{1, \xi, |\xi|^2\} \big(m_X Q^{XY}(F_X, F_Y) + m_Y Q^{YX}(F_Y, F_X)\big) d\xi = 0 \,. \end{cases} \tag{1.2}$$

1.2. *Review of previous works and the linearized problem.* The studies of gas mixtures in terms of the Boltzmann equations were first introduced in [8] to study the condensation-vaporization problem for the mixture of vapors of different species. Many interesting physical problems such as "ghost effects" and "Knudsen layer" for gas mixture have been investigated in [1], [6]. For mathematical points of view, the explicit spectrum gap estimate and the hypocoercivity for a linearized multi-species Boltzmann system can be found in [2]. Moreover, the global existence and stability of mild solutions to the Boltzmann system was completed by Ha, Noh and Yun in [4].

This research is also interested in phenomena related to vapor-vapor mixtures. Its focus is on the basic phenomenon of mass diffusion of a finite amount of gas A dissolved into the surrounding gas B. A qualitative-quantitative mathematical analysis in the whole space case was introduced by Sotirov and Yu [7]. In this paper, we want to study the torus case and to see the relation between the decay rate and the size of the domain. For this consideration, we write

$$\begin{cases} F_A = \sqrt{M_A} f_A \,, \\ F_B = M_B + \sqrt{M_B} f_B \,, \end{cases} \tag{1.3}$$

and the initial conditions of F_A and F_B satisfy

$$\begin{cases} F_A(0, x, \xi) = \sqrt{M_A} g_{in}(x, \xi) \,, \\ F_B(0, x, \xi) = M_B + \sqrt{M_B} h_{in}(x, \xi) \,. \end{cases}$$

Here, we fix the Maxwellian state of the surrounding gas B as the standard Gaussian, i.e.,

$$M_B(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\Big(\frac{-|\xi|^2}{2}\Big) \,.$$

We want to show M_A, the Maxwellian state of gas A, satisfies

$$Q^{AB}(M_A, M_B) = Q^{BA}(M_B, M_A) = 0.$$

This implies

$$M_A(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\Big(- \frac{m_A}{m_B} \frac{|\xi|^2}{2} \Big).$$

Under this choice, the two Maxwellian states satisfy

$$\begin{cases} Q^{AA}(M_A, M_A) = 0, \quad Q^{BB}(M_B, M_B) = 0\,, \\ Q^{AB}(M_A, M_B) = 0, \quad Q^{BA}(M_B, M_A) = 0\,. \end{cases} \tag{1.4}$$

When we show both the total mass of gas A and the perturbations in gas B are sufficiently small, we can assume that the initial density distribution function $F_A(0, x, \xi)$ is near a vacuum and $F_B(0, x, \xi)$ has the same total mass, momentum and energy as the Maxwellian M_B. Then we assume that

$$\int_{\mathbb{T}^1_{1/\varepsilon}} g_{in}(x, \xi) dx = 0\,, \tag{1.5}$$

and

$$\int_{\mathbb{T}^1_{1/\varepsilon}} \int_{\mathbb{R}^3} \sqrt{M_B}(\xi) \Big\{1, \xi, |\xi|^2\Big\} h_{in}(x, \xi) d\xi dx = 0\,. \tag{1.6}$$

This means that the initial condition $g_{in}(x, \xi)$ satisfies the zero mean condition and $h_{in}(x, \xi)$ satisfies the zero moments condition.

By (1.1) and (1.3), we find that the system for (f_A, f_B) satisfies

$$\begin{cases} \partial_t f_A + \xi_1 \partial_x f_A = L_{AB} f_A + \Gamma^{AA}(f_A, f_A) + \Gamma^{AB}(f_A, f_B)\,, \\ \partial_t f_B + \xi_1 \partial_x f_B = L_{BB} f_B + \Gamma^{BB}(f_B, f_B) + L_{BA} f_A + \Gamma^{BA}(f_B, f_A)\,, \end{cases} \tag{1.7}$$

where

$$\begin{cases} L_{AB} f_A = \dfrac{1}{\sqrt{M_A}} Q^{AB}(f_A \sqrt{M_A}, M_B)\,, \\ L_{BA} f_A = \dfrac{1}{\sqrt{M_B}} Q^{BA}(M_B, f_A \sqrt{M_A})\,, \\ L_{BB} f_B = \dfrac{1}{\sqrt{M_B}} \Big(Q^{BB}(M_B, f_B \sqrt{M_B}) + Q^{BB}(f_B \sqrt{M_B}, M_B) \Big)\,, \\ \Gamma^{XY}(f_X, f_Y) = \dfrac{1}{\sqrt{M_X}} Q^{XY}(f_X \sqrt{M_X}, f_Y \sqrt{M_Y})\,. \end{cases}$$

In our consideration, i.e., the total mass of gas A and the perturbations in gas B are sufficiently small. In general, the basic time asymptotic behavior of the nonlinear solution is governed by a linear equation. It is reasonable to drop the nonlinear term. Therefore in great generality we have the following decoupled linear Boltzmann system:

$$\begin{cases} \partial_t g + \xi_1 \partial_x g = L_{AB} g\,, \\ \partial_t h + \xi_1 \partial_x h = L_{BB} h + L_{BA} g\,. \end{cases} \tag{1.8}$$

In this paper, we consider the space variable $x \in \mathbb{T}^1_{1/\varepsilon}$, the 1-dimensional torus with size $1/\varepsilon$. In order to see the generic feature of mass diffusion, the following form of the initial data $(g,h)|_{t=0} = (g_{in}, h_{in})$ will be sufficient:

$$\begin{cases} g_{in}(x,\xi) = 0 \quad \text{for} \quad |x| \geq 1\,, \\ \sup\limits_{|x|\leq 1, \xi\in\mathbb{R}^3} (1+|\xi|)^\beta |g_{in}(x,\xi)| < \infty, \quad \beta > 3/2\,, \\ \displaystyle\int_{\mathbb{T}^1_{1/\varepsilon}} g_{in}(x,\xi)dx = 0 \quad \text{for all} \quad \xi \in \mathbb{R}^3, \end{cases} \tag{1.9}$$

and

$$\begin{cases} h_{in}(x,\xi) = 0 \quad \text{for} \quad |x| \geq 1\,, \\ \sup\limits_{|x|\leq 1, \xi\in\mathbb{R}^3} (1+|\xi|)^\beta |h_{in}(x,\xi)| < \infty, \quad \beta > 3/2\,, \\ \displaystyle\int_{\mathbb{T}^1_{1/\varepsilon}} \int_{\mathbb{R}^3} M_B^{1/2}(\xi)\Big\{1, \xi, |\xi|^2\Big\} h_{in}(x,\xi) d\xi dx = 0\,. \end{cases} \tag{1.10}$$

1.3. *Main theorem.* Before the presentation of the main theorem, let us define some notation in this paper. For the microscopic variable ξ, we shall use L^2_ξ to represent the classical Hilbert space with norm

$$\|f\|_{L^2_\xi} = \Big(\int_{\mathbb{R}^3} |f|^2 d\xi\Big)^{1/2}.$$

The Sobolev space of functions with all its s-th partial derivatives in L^2_ξ will be denoted by H^s_ξ. The L^2_ξ inner product in $\mathbb{R}^3$ will be denoted by $\langle\cdot,\cdot\rangle_\xi$ and the weighted sup norm is denoted by

$$\|f\|_{L^\infty_{\xi,\beta}} = \sup_{\xi\in\mathbb{R}^3} |f(\xi)|(1+|\xi|)^\beta\,.$$

For the space variable x, we have a similar notation. In fact, L^2_x is the classical Hilbert space with norm

$$\|f\|_{L^2_x} = \Big(\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|}\int_{\mathbb{T}^1_{1/\varepsilon}} |f|^2 dx\Big)^{1/2},$$

and the Sobolev space of functions with all its s-th partial derivatives in L^2_x will be denoted by H^s_x. We define the sup norm by

$$\|f\|_{L^\infty_x} = \sup_{x\in\mathbb{T}^1_{1/\varepsilon}} |f(x)|\,.$$

For simplicity of notation, hereafter, we abbreviate "$\leq C$" to "$\lesssim$", where C is a positive constant depending only on fixed number.

In the following, we describe our main result.

THEOREM 1.1. Assuming that $0 < \varepsilon \ll 1$, $\beta > 3/2$ and the initial conditions (g_{in}, h_{in}) satisfy the assumptions (1.9) and (1.10). Then the solution of (1.8) has the following estimates:

$$\|g\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim e^{-\overline{a}\varepsilon^2 t}(1+t)^{-1/2}\|g_{in}\|_{L^\infty_x L^\infty_{\xi,\beta}}\,,$$

and

$$\|h\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim e^{-\overline{a}\varepsilon^2 t}(1+t)^{-1/2}\|(g_{in}, h_{in})\|_{L^\infty_x L^\infty_{\xi,\beta}}\,.$$

Now, let us compare the results in the torus case (Theorem 1.1) and in the whole space case (constructed by Sotirov-Yu [7]).

The result in [7] basically gave full wave structures of the solutions in the whole space:

(a) Outside the wave cone ($|x| > Mt$ for some $M > 0$): g decays exponentially in both t and x and h propagates along the Euler characteristic, i.e.,

$$\|g\|_{L^2_\xi} \leq e^{-(|x|+t)}$$

and

$$\|h\|_{L^2_\xi} \leq \sum_{j=1}^{3} (1+t)^{-1/2} e^{-\frac{(x-a_j t)^2}{C(t+1)}} + e^{-c(|x|+t)},$$

where a_j, $j = 1, 2, 3$ are the wave speed of h.

(b) Inside the wave cone ($|x| \leq Mt$ for some $M > 0$): Both g and h propagate along the Euler characteristic, i.e.,

$$\|g\|_{L^2_\xi} \leq (1+t)^{-1/2} e^{-\frac{|x|^2}{C(t+1)}} + e^{-c(|x|+t)}$$

and

$$\|h\|_{L^2_\xi} \leq \sum_{j=1}^{3} (1+t)^{-1/2} e^{-\frac{(x-a_j t)^2}{C(t+1)}} + e^{-t^{1/2}}.$$

Here we remark that the wave speed of g is 0.

However, in the torus case, we only focus on large time behavior. This is because the domain is bounded and the "outside wave cone" seems meaningless. Moreover, our estimate in the velocity variable ξ is $L^\infty_{\xi,\beta}$ norm, which is better than the L^2_ξ estimate.

On the other hand, in the whole space case [5], [7], they got dissipative behavior of the type of the Navier-Stokes equation by using the complex analysis method, but in the torus case, the Fourier transform in space x is replaced by the Fourier series (it is discrete). It may be difficult to introduce the complex analysis method in the discrete version.

If we only consider the large time behavior of g and h in the whole space, then the time decay rate is $(1+t)^{-1/2}$. This coincides with the large time behavior of the torus case. However, we have an extra observation for very large time behavior of g, h in the torus case (exponential decay depends on the size of the domain).

1.4. *Method of proof and plan of the paper.* The basic principle used to develop the theory is the "separation of scales", which is a concept created in [5], [9]. In [5], [9], a long wave-short wave decomposition and a kinetic-remainder decomposition were introduced. Under those decompositions, one can decompose the solution of the linearized Boltzmann equation as the fluid like wave, kinetic like wave and remainder, which separate the scales, and hence get different regularity and decay rate in each part.

In this paper, we need to modify the idea in [9]. Let us consider the long wave-short wave decomposition of the solution first, then apply the kinetic-remainder decomposition in the short wave part only (in [9], the kinetic-remainder decomposition was applied to the whole solution). Under this consideration, one can separate the decay rate of the long wave part and the short wave part.

Let us explain why we need this modification. If we apply the kinetic-remainder decomposition to the whole solution g as in [9], then g can be decomposed as the fluid part and the nonfluid part, where the fluid part is part of the long wave expansion, and the nonfluid part is the sum of the kinetic part and the remainder part. However, when we estimate h, the fluid and nonfluid will intersect in the long wave expansion, but the nonfluid part of g does not separate as the long wave nonfluid part and short wave nonfluid part. This is why we need to modify our method. Under the new decomposition in g, we will get the fluid part, the long wave nonfluid part and the short wave nonfluid part. Then it will be no problem to estimate the solution of h.

Another interesting part of this paper is the fluid-fluid interaction of the long wave expansion of h. The long wave-short wave decomposition allows one to work in Fourier space. Then we need some cancellation properties to deal with the fluid-fluid interaction.

When the linear estimate is established, it is natural to consider the nonlinear problem (1.7) as our future project. Assuming that we have a smallness assumption of the initial conditions, we expect that (f_A, f_B) of (1.7) have a time decay estimate $e^{-a\varepsilon^2 t}(1+t)^{-1/2}\ln(1+t)$. The extra $\ln(1+t)$ is due to the 1D spatial domain. However, there are some difficulties for this project. First, the collision invariants of Γ^{AB} and L_{BB} are different. This means that one cannot estimate f_B directly. Second, this problem is in 1D space. The nonlinear effect is much stronger than the 3D case, hence we need more careful analysis for nonlinear interaction (see [5] for nonlinear analysis in the one species 1D whole space case).

The paper is organized as follows: we list some properties of the linearized collision operator in Section 2, then prove the main theorem in Section 3.

2. Preliminaries.

Lemma 2.1 (The null spaces of L_{BB} and L_{AB} [7]). The null space of L_{BB} is a five-dimensional vector space with orthogonal basis $\{\chi_i\}_{i=0}^4$, where

$$\{\chi_0, \chi_j, \chi_4\} = \left\{\sqrt{M_B}, \xi_j\sqrt{M_B}, \frac{1}{\sqrt{6}}(|\xi|^2-3)\sqrt{M_B}\right\}, \quad j=1,2,3\,.$$

On the other hand, the null space of L_{AB} is a one-dimensional vector space $\{E_D\}$, where $E_D = \sqrt{M_A}$.

Let $\mathrm{Inv}(L)$ be the collision invariants of the given operator L. The following lemma can be obtained directly from (1.2):

Lemma 2.2 (Collision invariants).

$$\begin{aligned}\mathrm{Inv}(L_{BB}) &= \left\{\sqrt{M_B}, \xi\sqrt{M_B}, |\xi|^2\sqrt{M_B}\right\},\\ \mathrm{Inv}(L_{AB}) &= \left\{\sqrt{M_A}\right\},\\ \mathrm{Inv}\Big(L_{AB} + \frac{m_B}{m_A}\frac{\sqrt{M_B}}{\sqrt{M_A}}L_{BA}\Big) &= \left\{\sqrt{M_A}, \xi\sqrt{M_A}, |\xi|^2\sqrt{M_A}\right\}.\end{aligned}$$

Lemma 2.3 (Grad's decomposition [7]). Let $(X,Y) = (B,B)$ or (A,B). Then the collision operator L_{XY} consists of a multiplicative operator $\nu^{XY}(\xi)$ and an integral operator

K^{XY}: $L_{XY}f = -\nu^{XY}(\xi)f + K^{XY}f$, where

$$K^{XY}f = \int_{\mathbb{R}^3} W^{XY}(\xi,\xi_*)f(\xi_*)d\xi_* .$$

The multiplicative operator $\nu^{XY}(\xi) \sim 1 + |\xi|$. The kernel $W^{XY}(\xi,\xi_*)$ has a singularity of type $|\xi - \xi_*|^{-1}$ and has the following regularization effects:

$$\|K^{XY}h\|_{L^\infty_\xi} \lesssim \|h\|_{L^2_\xi}, \quad \|K^{XY}h\|_{H^1_\xi} \lesssim \|h\|_{L^2_\xi}, \quad \|K^{XY}h\|_{L^\infty_{\xi,\beta+1}} \lesssim \|h\|_{L^\infty_{\xi,\beta}}, \tag{2.1}$$

for any $\beta \geq 0$. Moreover K^{XY} can be decomposed as a singular part and a regular part, i.e., $K^{XY} = K^{XY}_s + K^{XY}_r$, $K^{XY}_s \equiv K^{XY}_{s,d}$ and $K^{XY}_r \equiv K^{XY}_{r,d}$:

$$\begin{cases} K^{XY}_s f = \displaystyle\int_{\mathbb{R}^3} \chi\Big(\frac{|\xi-\xi_*|}{d}\Big) W^{XY}(\xi,\xi_*)f(\xi_*)d\xi_* , \\ K^{XY}_r f = K^{XY}f - K^{XY}_s f , \end{cases}$$

where $\chi \in C^\infty_c(\mathbb{R})$ is nonnegative and satisfies $\chi(r) = 1$ for $r \in [-1,1]$, and $\mathrm{supp}(\chi) \subset [-2,2]$. The singular part K^{XY}_s shares the same smoothing properties as K^{XY} and has strength of the order of the cut-off parameter d:

$$\|K^{XY}_s h\|_{H^1_\xi} \lesssim d\|h\|_{L^2_\xi}, \quad \|K^{XY}_s h\|_{L^\infty_{\xi,\beta}} \lesssim d\|h\|_{L^\infty_{\xi,\beta}} . \tag{2.2}$$

The regular part K^{XY}_r has better smoothing property in ξ: for all $s > 0$,

$$\|K^{XY}_r h\|_{H^s_\xi} \lesssim \|h\|_{L^2_\xi} . \tag{2.3}$$

The behavior of the operator L_{BA} is different from L_{BB} and L_{AB}.

Lemma 2.4 ([7]). The operator L_{BA} is a bounded integral operator:

$$L_{BA}f = \int_{\mathbb{R}^3} W^{BA}(\xi,\xi_*)f(\xi_*)d\xi_* ,$$

where the kernel $W^{BA}(\xi,\xi_*)$ is a smooth function in ξ and ξ_*:

$$\|L_{BA}h\|_{H^s_\xi} \lesssim \|h\|_{L^2_\xi}, \quad \|L_{BA}h\|_{L^\infty_{\xi,\beta}} \lesssim \|h\|_{L^\infty_\xi}, \tag{2.4}$$

for any $s, \beta \geq 0$.

We need to recall the spectrum $\mathrm{Spec}_{XY}(\varepsilon k)$, $k \in \mathbb{Z}$, of $L^k_{XY} = -i\pi\varepsilon\xi_1 k + L_{XY}$ in the classical Hilbert space L^2_ξ, where $(X,Y) = (B,B)$ or (A,B).

Lemma 2.5 (Spectrum of L^k_{BB} [3]). Given $\delta > 0$,
(i) there exists $\tau_1 = \tau_1(\delta) > 0$ such that if $|\varepsilon k| > \delta$,

$$\mathrm{Spec}_{BB}(\varepsilon k) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < -\tau_1\} . \tag{2.5}$$

(ii) If $|\varepsilon k| < \delta$, the spectrum within the region $\{z \in \mathbb{C} : \mathrm{Re}(z) > -\tau_1\}$ consists of exactly three eigenvalues $\{\sigma_j(\varepsilon k)\}^3_{j=1}$,

$$\mathrm{Spec}_{BB}(\varepsilon k) \cap \{z \in \mathbb{C} : \mathrm{Re}(z) > -\tau_1\} = \{\sigma_j(\varepsilon k)\}^3_{j=1} , \tag{2.6}$$

and the corresponding eigenvectors $\{e_j(\varepsilon k)\}_{j=1}^3$. They have the expansions

$$\begin{aligned} \sigma_j(\varepsilon k) &= ia_{j,1}|\varepsilon k| - a_{j,2}|\varepsilon k|^2 + O(|\varepsilon k|^3)\,, \\ e_j(\varepsilon k) &= E_j + O(|\varepsilon k|)\,. \end{aligned} \tag{2.7}$$

Here $a_{j,2} > 0$, $\langle e_j(-\varepsilon k), e_l(\varepsilon k)\rangle_\xi = \delta_{jl}$, $1 \le j, l \le 3$ and

$$\begin{cases} a_{11} = \sqrt{\frac{5}{3}}\,, \quad a_{21} = 0\,, \quad a_{31} = -\sqrt{\frac{5}{3}}\,, \\ E_1 = \sqrt{\frac{3}{10}}\chi_0 + \sqrt{\frac{1}{2}}\chi_1 + \sqrt{\frac{1}{5}}\chi_4\,, \\ E_2 = -\sqrt{\frac{2}{5}}\chi_0 + \sqrt{\frac{3}{5}}\chi_4\,, \\ E_3 = \sqrt{\frac{3}{10}}\chi_0 - \sqrt{\frac{1}{2}}\chi_1 + \sqrt{\frac{1}{5}}\chi_4\,. \end{cases} \tag{2.8}$$

More precisely, the semigroup $e^{(-i\pi\varepsilon\xi_1 k + L_{BB})t}$ can be decomposed as

$$\begin{aligned} e^{(-i\pi\varepsilon\xi_1 k + L_{BB})t} f &= e^{(-i\pi\varepsilon\xi_1 k + L_{BB})t}\Pi_k^\perp f \\ &\quad + \chi_{\{|\varepsilon k|<\delta\}} \sum_{j=1}^3 e^{\sigma_j(\varepsilon k)t}\langle e_j(-\varepsilon k), f\rangle_\xi e_j(\varepsilon k)\,, \end{aligned} \tag{2.9}$$

where $\chi_{\{\cdot\}}$ is the indicator function. Moreover, there exist $a(\tau_1) > 0$, $\overline{a}_1 > 0$ such that $\|e^{(-i\pi\varepsilon\xi_1 k + L_{BB})t}\Pi_k^\perp\|_{L^2_\xi} \lesssim e^{-a(\tau_1)t}$ and $|e^{\sigma_j(\varepsilon k)t}| \le e^{-\overline{a}_1|\varepsilon k|^2 t}$ for all $1 \le j \le 3$.

Lemma 2.6 (Spectrum of L_{AB}^k [7]). Given $\delta > 0$,
(i) there exists $\tau_2 = \tau_2(\delta) > 0$ such that if $|\varepsilon k| > \delta$,

$$\mathrm{Spec}(\varepsilon k) \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < -\tau_2\}\,. \tag{2.10}$$

(ii) If $|\varepsilon k| < \delta$, the spectrum within the region $\{z \in \mathbb{C} : \mathrm{Re}(z) > -\tau_2\}$ consists of exactly one eigenvalue $\{\lambda(\varepsilon k)\}$,

$$\mathrm{Spec}(\varepsilon k) \cap \{z \in \mathbb{C} : \mathrm{Re}(z) > -\tau_2\} = \{\lambda(\varepsilon k)\}\,, \tag{2.11}$$

and the corresponding eigenvector $\{e_D(\varepsilon k)\}$. They have the expansions

$$\begin{aligned} \lambda(\varepsilon k) &= -a_2(|\varepsilon k|)^2 + O(|\varepsilon k|^3)\,, \\ e_D(\varepsilon k) &= E_D + iE_{D,1}|\varepsilon k| + O(|\varepsilon k|^2)\,, \end{aligned} \tag{2.12}$$

where $E_D = \sqrt{M_A}$, $a_2 > 0$, $E_{D,1} = L_{AB}^{-1}\xi_1 E_D$ and $\langle e_D(-\varepsilon k), e_D(\varepsilon k)\rangle_\xi = 1$.
More precisely, the semigroup $e^{(-i\pi\varepsilon\xi_1 k + L_{AB})t}$ can be decomposed as

$$e^{(-i\pi\varepsilon\xi_1 k + L_{AB})t} f = e^{(-i\pi\varepsilon\xi_1 k + L_{AB})t}\Pi_k^{D\perp} f + \chi_{\{|\varepsilon k|<\delta\}} e^{\lambda(\varepsilon k)t}\langle e_D(-\varepsilon k), f\rangle_\xi e_D(\varepsilon k)\,. \tag{2.13}$$

Moreover, there exist $a(\tau_2) > 0$, $\overline{a}_2 > 0$ such that $\|e^{(-i\pi\varepsilon\xi_1 k + L_{AB})t}\Pi_k^{D\perp}\|_{L^2_\xi} \lesssim e^{-a(\tau_2)t}$ and $|e^{\lambda(\varepsilon k)t}| \le e^{-\overline{a}_2|\varepsilon k|^2 t}$.

LEMMA 2.7 ([10]). Assuming that $0 < \varepsilon \ll 1$, $\beta > 3/2$, h_{in} satisfies assumptions (1.10), then the solution of the linearized Boltzmann equation

$$\begin{cases} \partial_t h + \xi_1 \partial_x h = L_{BB} h\,, \\ h(0,x,\xi) = h_{in}(x,\xi)\,, \end{cases} \tag{2.14}$$

has the following $L^\infty_x L^\infty_{\xi,\beta}$ estimate:

$$\|h\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim (1+t)^{-1/2} e^{-\overline{a}_1 \varepsilon^2 t} \|h_{in}\|_{L^\infty_x L^\infty_{\xi,\beta}}\,. \tag{2.15}$$

The following lemma is useful in the estimate of the fluid part.

LEMMA 2.8. If $0 < \varepsilon \ll 1$, $k \in \mathbb{Z}$, $a > 0$, $s \geq 0$ and t is in a short time region, i.e., $\varepsilon^2 t \ll 1$, then

$$\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta, k \neq 0} |\varepsilon k|^s e^{-a|\varepsilon k|^2 t} \lesssim e^{-a\varepsilon^2 t}(1+t)^{-(1+s)/2}\,.$$

Proof. Note that

$$\begin{aligned} &\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta, k \neq 0} |\varepsilon k|^s e^{-a|\varepsilon k|^2 t} \\ &= e^{-a\varepsilon^2 t} \frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta, k \neq 0} |\varepsilon k|^s e^{-a\varepsilon^2(|k|^2-1)t} \\ &= e^{-a\varepsilon^2 t} \frac{1}{t^{(1+s)/2}} \frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta t^{1/2}, k \neq 0} |\varepsilon k|^s e^{-a\varepsilon^2(|k|^2-1)} \\ &= e^{-a\varepsilon^2 t} \frac{1}{t^{(1+s)/2}} \Big[\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{k \in \mathbb{Z}} \varepsilon^s (|k|+1)^s e^{-a\varepsilon^2[(|k|+1)^2-1]} \Big] \\ &\lesssim e^{-a\varepsilon^2 t} \frac{1}{t^{(1+s)/2}} \Big[\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{k \in \mathbb{Z}} \varepsilon^s |k|^s e^{-a\varepsilon^2|k|^2} + O(\varepsilon^{1+s}) \Big] \\ &\to e^{-a\varepsilon^2 t} \frac{1}{t^{(1+s)/2}} \int_{\mathbb{R}} |y|^s e^{-a|y|^2} dy \end{aligned} \tag{2.16}$$

as $\varepsilon \to 0$. This means for any $\alpha_0 > 0$ there exists $\delta_0 > 0$ such that if $\varepsilon < \delta_0$, we have

$$\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{k \in \mathbb{Z}} |\varepsilon k|^s e^{-a|\varepsilon k|^2} < \alpha_0 + \int_{\mathbb{R}} |y|^s e^{-a|y|^2} dy \equiv C_0\,. \tag{2.17}$$

Combining (2.16) and (2.17) together yields

$$\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \sum_{|\varepsilon k| < \delta, k \neq 0} |\varepsilon k|^s e^{-a|\varepsilon k|^2 t} \lesssim e^{-a\varepsilon^2 t}(1+t)^{-(1+s)/2}\,. \tag{2.18}$$

□

3. Proof of the main theorem.

3.1. *Analysis of g.* Consider the linear diffusive Boltzmann equation

$$\begin{cases} \partial_t g + \xi_1 \partial_x g = L_{AB} g\,, \quad (t,x,\xi) \in (\mathbb{R}^+, \mathbb{T}^1_{1/\varepsilon}, \mathbb{R}^3)\,, \\ g(0,x,\xi) = g_{in}(x,\xi)\,, \end{cases} \tag{3.1}$$

where the initial condition g_{in} satisfies the assumptions (1.9). Consider the Fourier series of the initial condition g_{in} in x

$$\begin{cases} g_{in}(x,\xi) = \sum_{k\in\mathbb{Z}} (\hat{g}_{in})_k(\xi) e^{i\pi\varepsilon k x}\,, \\ (\hat{g}_{in})_k(\xi) = \dfrac{1}{|\mathbb{T}^1_{1/\varepsilon}|} \displaystyle\int_{\mathbb{T}^1_{1/\varepsilon}} g_{in}(\cdot,\xi) e^{-i\pi\varepsilon k x} dx\,. \end{cases} \tag{3.2}$$

One has the long wave-short wave decomposition of g_{in}, i.e. $g_{in} = g^L_{in} + g^S_{in}$, where

$$\begin{cases} g^L_{in}(x,\xi) = \displaystyle\sum_{|\varepsilon k|\le\delta, k\neq 0} (\hat{g}_{in})_k(\xi) e^{i\pi\varepsilon k x}\,, \\ g^S_{in}(x,\xi) = \displaystyle\sum_{|\varepsilon k|>\delta} (\hat{g}_{in})_k(\xi) e^{i\pi\varepsilon k x}\,. \end{cases} \tag{3.3}$$

Now, the solution g can be decomposed into the long wave part and short wave part:

$$\begin{cases} \partial_t g^L + \xi_1 \partial_x g^L = L_{AB} g^L\,, \quad (t,x,\xi) \in (\mathbb{R}^+, \mathbb{T}^1_{1/\varepsilon}, \mathbb{R}^3)\,, \\ g^L(0,x,\xi) = g^L_{in}(x,\xi)\,, \end{cases} \tag{3.4}$$

and

$$\begin{cases} \partial_t g^S + \xi_1 \partial_x g^S = L_{AB} g^S\,, \quad (t,x,\xi) \in (\mathbb{R}^+, \mathbb{T}^1_{1/\varepsilon}, \mathbb{R}^3)\,, \\ g^S(0,x,\xi) = g^S_{in}(x,\xi)\,. \end{cases} \tag{3.5}$$

This construction is a modification of the previous work [9] and hence we only sketch the proof of the following two lemmas. We refer to [9] for more details.

Lemma 3.1 (Long wave g^L). We have the following estimate of the long wave part of g:

$$\|g^L\|_{L^\infty_x L^2_\xi} \lesssim e^{-\overline{a}_2 \varepsilon^2 t} (1+t)^{-1/2} \|g_{in}\|_{L^\infty_x L^2_\xi}\,. \tag{3.6}$$

Proof. By Lemma 2.6 and assumption of the initial condition $(1.9)_3$, we have the fluid-nonfluid decomposition of the long wave part,

$$\begin{aligned} g^L(t,x,\xi) &= \sum_{|\varepsilon k|<\delta, k\neq 0} e^{i\pi\varepsilon k x} e^{\lambda(\varepsilon k)t} \big\langle e_D(-\varepsilon k), (\hat{g}_{in})_k \big\rangle_\xi e_D(\varepsilon k) \\ &\quad + \sum_{|\varepsilon k|<\delta, k\neq 0} e^{i\pi\varepsilon k x} e^{(-i\pi\varepsilon\xi_1 k + L_{AB})t} \Pi^{D\perp}_k (\hat{g}_{in})_k \\ &= A_1 + A_2\,. \end{aligned} \tag{3.7}$$

For A_1, we have

$$\|A_1\|_{L^2_\xi} \lesssim \sum_{|\varepsilon k|\le\delta, k\neq 0} e^{-\overline{a}_2|\varepsilon k|^2 t}\|(\hat{g}_{in})_k\|_{L^2_\xi}$$
$$\lesssim \big(e^{-\overline{a}_2\varepsilon^2 t}\|g_{in}\|_{L^\infty_x L^2_\xi}\big)\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|}\sum_{|\varepsilon k|<\delta} e^{-\overline{a}_2|k\varepsilon|^2 t}\,.$$

By Lemma 2.8, we have

$$\|A_1\|_{L^\infty_x L^2_\xi} \lesssim e^{-\overline{a}_2\varepsilon^2 t}(1+t)^{-1/2}\|g_{in}\|_{L^\infty_x L^2_\xi}\,.$$

The estimate of A_2 is the same as Section 3 of the previous work [9], and we get

$$\|A_2\|_{H^s_x L^2_\xi} \lesssim e^{-a(\tau_2)t}\|g_{in}\|_{L^2_x L^2_\xi} \quad \text{for all} \quad s>0\,.$$

This completes the proof of the lemma. □

Lemma 3.2 (Short wave g^S). We have the following estimate of the short wave part of g:

$$\|g^S\|_{L^\infty_x L^2_\xi} \lesssim e^{-c_2 t}\|g_{in}\|_{L^\infty_x L^\infty_{\xi,\beta}}\,. \tag{3.8}$$

Proof. In order to estimate g^S, we need the kinetic-remainder decomposition as in [9]. Now, we design a Picard type iteration, which treats the regular part $K^{AB}g^S$ as a source term, but we split the first step. The -1 order approximation of the linear diffusive Boltzmann equation (3.1) is the damped transport equation

$$\begin{cases} \partial_t g^{(S,-1)} + \xi_1\partial_x g^{(S,-1)} + \nu^{AB}(\xi)g^{(S,-1)} - K^{AB}_s g^{(S,-1)} = 0\,, \\ g^{(S,-1)}(0,x,\xi) = g^S_{in}(x,\xi)\,. \end{cases} \tag{3.9}$$

The zero order approximation $g^{(S,0)}$ is

$$\begin{cases} \partial_t g^{(S,0)} + \xi_1\partial_x g^{(S,0)} + \nu^{AB}(\xi)g^{(S,0)} = K^{AB}_r g^{(S,-1)}\,, \\ g^{(S,0)}(0,x,\xi) = 0\,. \end{cases} \tag{3.10}$$

In general, we can define the j-th order approximation $g^{(S,j)}$, $j\ge 1$, as

$$\begin{cases} \partial_t g^{(S,j)} + \xi_1\partial_x g^{(S,j)} + \nu^{AB}(\xi)g^{(S,j)} = K^{AB} g^{(S,j-1)}\,, \\ g^{(S,j)}(0,x,\xi) = 0\,. \end{cases} \tag{3.11}$$

Therefore the solution g^S of the linear diffusive Boltzmann equation (3.1) can be represented as a series

$$g^S = \sum_{j=-1}^{4} g^{(S,j)} + \mathcal{R}^S_g\,,$$

where the tail part $\mathcal{R}^S_g$ satisfies

$$\begin{cases} \partial_t \mathcal{R}^S_g + \xi_1\partial_x \mathcal{R}^S_g = L_{AB}\mathcal{R}^S_g + K^{AB} g^{(S,4)}\,, \\ \mathcal{R}^S_g(0,x,\xi) = 0\,. \end{cases}$$

Using Lemma 2.3, we have

$$\Big\| \sum_{j=-1}^{4} g^{(S,j)} \Big\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim (1+t)^5 e^{-c_2 t} \|g_{in}\|_{L^\infty_x L^\infty_{\xi,\beta}} ,$$

and

$$\|\mathcal{R}^S_g\|_{H^2_x L^2_\xi} \lesssim e^{-c_2 t} \|g_{in}\|_{L^2_x L^2_\xi} .$$

These estimates are the same as Section 4 of [9] and hence we omit the details. Combining those two estimates, one has

$$\|g^S\|_{L^\infty_x L^2_\xi} \lesssim e^{-c_2 t} \|g_{in}\|_{L^\infty_x L^\infty_{\xi,\beta}} .$$

This completes the proof of the lemma. □

One can apply Lemma 2.3 and the bootstrap process to get the $L^\infty_x L^\infty_{\xi,\beta}$ estimate of g^L and g^S:

Proposition 3.3. Assuming that $0 < \varepsilon \ll 1$, $\beta > 3/2$ and the initial condition g_{in} satisfies the assumption (1.9), then the solutions of g^L and g^S have the following estimates:

$$\|g^L\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim e^{-\overline{a}_2 t}(1+t)^{-1/2} \|g_{in}\|_{L^\infty_x L^\infty_{\xi,\beta}} ,$$
$$\|g^S\|_{L^\infty_x L^\infty_{\xi,\beta}} \lesssim e^{-c_2 t} \|g_{in}\|_{L^\infty_x L^\infty_{\xi,\beta}} .$$

The proof of this lemma is similar to Subsection II.A of [10] and hence we omit the details. This completes the estimate of g.

3.2. *Analysis of h.* Consider the linearized Boltzmann equation

$$\begin{cases} \partial_t h + \xi_1 \partial_x h = L_{BB} h + L_{BA} g , \quad (t,x,\xi) \in (\mathbb{R}^+, \mathbb{T}^1_{1/\varepsilon}, \mathbb{R}^3) , \\ h(0,x,\xi) = h_{in}(x,\xi) , \end{cases} \tag{3.12}$$

where the initial condition h_{in} satisfies the assumptions (1.10). Now, we decompose the solution into the long wave part h^L and the short wave part h^S, where the long wave part satisfies

$$\begin{cases} \partial_t h^L + \xi_1 \partial_x h^L = L_{BB} h^L + L_{BA} g^L , \quad (t,x,\xi) \in (\mathbb{R}^+, \mathbb{T}^1_{1/\varepsilon}, \mathbb{R}^3) , \\ h^L(0,x,\xi) = h^L_{in}(x,\xi) , \end{cases} \tag{3.13}$$

and the short wave part satisfies

$$\begin{cases} \partial_t h^S + \xi_1 \partial_x h^S = L_{BB} h^S + L_{BA} g^S , \quad (t,x,\xi) \in (\mathbb{R}^+, \mathbb{T}^1_{1/\varepsilon}, \mathbb{R}^3) , \\ h^S(0,x,\xi) = h^S_{in}(x,\xi) . \end{cases} \tag{3.14}$$

Lemma 3.4 (Long wave h^L). We have the following estimate of the long wave part of h:

$$\|h^L\|_{L^\infty_x L^2_\xi} \lesssim e^{-\overline{a}\varepsilon^2 t}(1+t)^{-1/2} \|(g_{in}, h_{in})\|_{L^\infty_x L^2_\xi} . \tag{3.15}$$

Proof. The estimate of the homogeneous part is the same as g^L and hence we omit the details. In the following proof, we may assume $h^L_{in} = 0$. Let us rewrite the solution

$h^L(t,x,\xi)$ of (3.13) in terms of the Fourier series as

$$\begin{cases} h^L(t,x,\xi) = \sum_{k\in\mathbb{Z}} (\hat{h}^L)_k(t,\xi) e^{i\pi\varepsilon kx}\,, \\ (\hat{h}^L)_k(t,\xi) = \dfrac{1}{|\mathbb{T}^1_{1/\varepsilon}|}\displaystyle\int_{\mathbb{T}^1_{1/\varepsilon}} h^L(t,\cdot,\xi) e^{-i\pi\varepsilon kx} dx\,. \end{cases} \tag{3.16}$$

Note that the Fourier modes of (3.16) satisfy the following equations:

$$\begin{cases} \partial_t (\hat{h}^L)_k + i\pi\varepsilon\xi_1 k (\hat{h}^L)_k - L_{BB}(\hat{h}^L)_k = L_{BA}(\hat{g}^L)_k\,, \\ (\hat{h}^L)_k(0,\xi) = 0\,, \end{cases}$$

and can be solved explicitly as

$$(\hat{h}^L)_k(t,\xi) = \int_0^t e^{(-i\pi\varepsilon\xi_1 k + L_{BB})(t-s)} L_{BA}(\hat{g}^L)_k(\cdot,s) ds\,.$$

Using Lemma 2.5, the solution of (3.13) is given by

$$\begin{aligned} h^L(t,x,\xi) &= \sum_{k\in\mathbb{Z}} \int_0^t e^{i\pi\varepsilon kx} e^{(-i\pi\varepsilon\xi_1 k + L_{BB})(t-s)} L_{BA}(\hat{g}^L)_k(\cdot,s) ds \\ &= \sum_{|\varepsilon k|\le\delta, k\neq 0} \int_0^t e^{i\pi\varepsilon kx} e^{(-i\pi\varepsilon\xi_1 k + L_{BB})(t-s)} L_{BA} e^{(-i\pi\varepsilon\xi_1 k + L_{AB})s} (\hat{g}_{in})_k(\cdot,s) ds \\ &= I_1 + I_2 + I_3 + I_4\,, \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sum_{|\varepsilon k|\le\delta, k\neq 0} \sum_{j=1}^3 \int_0^t e^{i\pi\varepsilon kx} e^{\sigma_j(\varepsilon k)(t-s) + \lambda(\varepsilon k)s} \big\langle e_j(\varepsilon k), L_{BA} e_D(\varepsilon k)\big\rangle_\xi \big\langle (\hat{g}_{in})_k, e_D(\varepsilon k)\big\rangle_\xi e_j(\varepsilon k) ds, \\ I_2 &= \sum_{|\varepsilon k|\le\delta, k\neq 0} \sum_{j=1}^3 \int_0^t e^{i\pi\varepsilon kx} e^{\sigma_j(\varepsilon k)(t-s)} \big\langle e_j(\varepsilon k), L_{BA} e^{(-i\pi\varepsilon\xi_1 k + L_{AB})s} \Pi_k^{D\perp} (\hat{g}_{in})_k\big\rangle_\xi e_j(\varepsilon k) ds, \\ I_3 &= \sum_{|\varepsilon k|\le\delta, k\neq 0} \int_0^t e^{i\pi\varepsilon kx} e^{\lambda(\varepsilon k)s} e^{(-i\pi\varepsilon\xi_1 k + L_{BB})(t-s)} \Pi_k^\perp L_{BA} e_D(\varepsilon k) \big\langle e_D(\varepsilon k), (\hat{g}_{in})_k\big\rangle_\xi ds, \\ I_4 &= \sum_{|\varepsilon k|\le\delta, k\neq 0} \int_0^t e^{i\pi\varepsilon kx} e^{(-i\pi\varepsilon\xi_1 k + L_{BB})(t-s)} \Pi_k^\perp L_{BA} e^{(-i\pi\varepsilon\xi_1 k + L_{AB})s} \Pi_k^{D\perp} (\hat{g}_{in})_k(\cdot,s) ds\,. \end{aligned}$$

In order to estimate I_1, we need to observe some cancellation of $\big\langle e_j(\varepsilon k), L_{BA} e_D(\varepsilon k)\big\rangle_\xi$. Note that $L_{BA} E_D = 0$ (where we recall that $Q^{BA}(M_B, M_A) = 0$), and we have the "macroscopic cancellation"

$$\big\langle e_j(\varepsilon k), L_{BA} e_D(\varepsilon k)\big\rangle_\xi = \big\langle e_j(\varepsilon k), i L_{BA} E_{D,1}(k/|k|)|\varepsilon k| + O(|\varepsilon k|^2)\big\rangle_\xi\,.$$

Let $I_1 = I_{11} + I_{12} + I_{13}$, where I_{1j} corresponds to the solution project to $e_j(\varepsilon k)$, i.e.,

$$I_{1j} = \sum_{|\varepsilon k|\le\delta, k\neq 0} \int_0^t e^{i\pi\varepsilon k x} e^{\sigma_j(\varepsilon k)(t-s)+\lambda(\varepsilon k)s} \big\langle e_j(\varepsilon k), L_{BA} e_D(\varepsilon k)\big\rangle_\xi \big\langle (\hat{g}_{in})_k, e_D(\varepsilon k)\big\rangle_\xi e_j(\varepsilon k) ds\,.$$

By the fundamental theorem of calculus, one can check that

$$\int_0^t e^{\sigma_j(\varepsilon k)(t-s)+\lambda(\varepsilon k)s} ds = \frac{1}{\lambda(\varepsilon k) - \sigma_j(\varepsilon k)} \Big[e^{\lambda(\varepsilon k)t} - e^{\sigma_j(\varepsilon k)t} \Big]$$

and

$$\big\langle e_j(\varepsilon k), L_{BA} e_D(\varepsilon k)\big\rangle_\xi = O(|\varepsilon k|)\,.$$

This implies

$$\|I_{1j}\|_{L^2_\xi} \le \sum_{|\varepsilon k|\le\delta, k\neq 0} \frac{|\varepsilon k|}{|\lambda(\varepsilon k) - \sigma_j(\varepsilon k)|} \Big| e^{\lambda(\varepsilon k)t} + e^{\sigma_j(\varepsilon k)t} \Big| \|(\hat{g}_{in})_k\|_{L^2_\xi}\,.$$

Moreover, one can calculate that

$$\frac{|\varepsilon k|}{|\lambda(\varepsilon k) - \sigma_j(\varepsilon k)|} = \frac{1}{|(-a_2 + a_{j,2})|\varepsilon k| + i a_{j,1}|} = O(1)$$

if $|\varepsilon k|$ is small and $a_{j,1} \neq 0$. In other words, the above calculation only works for $j = 1$ and $j = 3$. Then by Lemma 2.8, one gets

$$\|I_{11} + I_{13}\|_{L^2_\xi} \le (1+t)^{-1/2} e^{-\overline{a}\varepsilon^2 t} \|g_{in}\|_{L^\infty_x L^2_\xi}\,.$$

In order to estimate I_{12}, we need more cancellation property. Let Φ denote any linear combination of $\{1, \xi, |\xi|^2\}$. We have that (Lemma 2.2) the operator

$$L_{AB} J + \frac{m_B}{m_A} \frac{\sqrt{M_B}}{\sqrt{M_A}} L_{BA} J$$

is orthogonal to all collision invariants $\{\sqrt{M_A}, \xi\sqrt{M_A}, |\xi|^2\sqrt{M_A}\}$. With this applied to $J = E_{D,1} = L_{AB}^{-1}\xi_1 E_D$ (where we recall that $E_D = \sqrt{M_A}$) it follows

$$\int_{\mathbb{R}^3} \Big(L_{AB}\big[L_{AB}^{-1}\xi_1 E_D\big] + \frac{m_B}{m_A}\frac{\sqrt{M_B}}{\sqrt{M_A}} L_{BA}\big[L_{AB}^{-1}\xi_1 E_D\big] \Big) \Phi\sqrt{M_A} d\xi = 0$$

so that

$$\frac{m_B}{m_A} \int_{\mathbb{R}^3} \Phi\sqrt{M_B} L_{BA}\big[L_{AB}^{-1}\xi_1 E_D\big] d\xi = -\int_{\mathbb{R}^3} \Phi\xi_1 M_A d\xi\,.$$

For I_{12}, choose $\Phi = \sqrt{\frac{1}{10}}(-1 + |\xi|^2)$; then we have

$$\int_{\mathbb{R}^3} E_2 L_{BA}\big[L_{AB}^{-1}\xi_1 E_D\big] d\xi = -\frac{m_A}{m_B} \int_{\mathbb{R}^3} \sqrt{\frac{1}{10}}(-1 + |\xi|^2)\xi_1 M_A d\xi = 0$$

since the integrand is odd.

The above calculation shows that $\big\langle e_2(\varepsilon k), L_{BA}e_D(\varepsilon k)\big\rangle_\xi = O(|\varepsilon k|^2)$. Hence

$$\begin{aligned}\|I_{12}\|_{L^2_\xi} &\le \sum_{|\varepsilon k|\le\delta, k\neq 0}\int_0^t e^{\sigma_2(\varepsilon k)(t-s)+\lambda(\varepsilon k)s}|\varepsilon k|^2\|(\hat g_{in})_k\|_{L^2_\xi}ds\\ &\le \int_0^t \sum_{|\varepsilon k|\le\delta, k\neq 0} e^{-\overline{a}|\varepsilon k|^2 t}|\varepsilon k|^2\|(\hat g_{in})_k\|_{L^2_\xi}ds\,,\end{aligned}$$

where $\overline{a}=\min\{\overline{a}_1,\overline{a}_2\}$. One can calculate that

$$\begin{aligned}&\sum_{|\varepsilon k|\le\delta, k\neq 0} e^{-\overline{a}|\varepsilon k|^2 t}|\varepsilon k|^2\|(\hat g_{in})_k\|_{L^2_\xi}\\ &\lesssim \Big(\int_{\mathbb{T}^1_{1/\varepsilon}}\|g_{in}\|_{L^2_\xi}dx\Big)\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|}\sum_{|\varepsilon k|<\delta;|k|\neq 0}|\varepsilon k|^2 e^{-\overline{a}|k\varepsilon|^2 t}\\ &\lesssim \big(e^{-\overline{a}\varepsilon^2 t}\|g_{in}\|_{L^\infty_x L^2_\xi}\big)\frac{1}{|\mathbb{T}^1_{1/\varepsilon}|}\sum_{|\varepsilon k|<\delta}|\varepsilon k|^2 e^{-\overline{a}|k\varepsilon|^2 t}\,.\end{aligned}$$

By Lemma 2.8, we get

$$\sum_{|\varepsilon k|\le\delta, k\neq 0} e^{-\overline{a}|\varepsilon k|^2 t}|\varepsilon k|^2\|(\hat g_{in})_k\|_{L^2_\xi} \lesssim \big(e^{-\overline{a}\varepsilon^2 t}\|g_{in}\|_{L^\infty_x L^2_\xi}\big)(1+t)^{-3/2}\,,$$

and hence

$$\|I_{12}\|_{L^2_\xi}\le (1+t)^{-1/2}e^{-\overline{a}\varepsilon^2 t}\|g_{in}\|_{L^\infty_x L^2_\xi}\,.$$

This completes the estimate of I_1. For I_2, by Lemma 2.8, we have

$$\begin{aligned}\|I_2\|_{L^2_\xi} &\le \sum_{|\varepsilon k|\le\delta, k\neq 0}\sum_{j=1}^3\int_0^t e^{-a(\tau_2)s}e^{\sigma_j(\varepsilon k)(t-s)}\|(\hat g_{in})_k\|_{L^2_\xi}ds\\ &\le \int_0^t e^{-a(\tau_2)s}e^{-\overline{a}_1\varepsilon^2(t-s)}(1+t-s)^{-1/2}ds\|g_{in}\|_{L^\infty_x L^2_\xi}\\ &= e^{-\overline{a}_1\varepsilon^2 t}\Big(\int_0^{t/2}+\int_{t/2}^t e^{-(a(\tau_2)-\overline{a}_1\varepsilon^2)s}(1+t-s)^{-1/2}ds\Big)\|g_{in}\|_{L^\infty_x L^2_\xi}\,.\end{aligned}$$

One can check that

$$\begin{aligned}\int_0^{t/2} e^{-(a(\tau_2)-\overline{a}_1\varepsilon^2)s}(1+t-s)^{-1/2}ds &\le (1+t/2)^{-1/2}\int_0^{t/2} e^{-(a(\tau_2)-\overline{a}_1\varepsilon^2)s}ds\\ &\lesssim (1+t/2)^{-1/2}\,,\end{aligned}$$

and

$$\begin{aligned}\int_{t/2}^t e^{-(a(\tau_2)-\overline{a}_1\varepsilon^2)s}(1+t-s)^{-1/2}ds &\le \int_{t/2}^t e^{-(a(\tau_2)-\overline{a}_1\varepsilon^2)s}ds\\ &\lesssim te^{-(a(\tau_2)-\overline{a}_1\varepsilon^2)t/2}\,.\end{aligned}$$

This implies

$$\|I_2\|_{L^2_\xi}\le (1+t)^{-1/2}e^{-\overline{a}_1\varepsilon^2 t}\|g_{in}\|_{L^\infty_x L^2_\xi}\,.$$

For I_3, by Lemma 2.8, we have

$$\begin{aligned}\|I_3\|_{L^2_\xi} &\le \sum_{|\varepsilon k|\le\delta, k\neq 0}\int_0^t e^{-a(\tau_1)(t-s)}e^{\lambda(\varepsilon k)s}\|(\hat{g}_{in})_k\|_{L^2_\xi}ds\\ &\le \int_0^t e^{-a(\tau_1)(t-s)}e^{-\overline{a}_2\varepsilon^2 s}(1+s)^{-1/2}ds\|g_{in}\|_{L^\infty_x L^2_\xi}\\ &= e^{-a(\tau_1)t}\Big(\int_0^t e^{(a(\tau_1)-\overline{a}_2\varepsilon^2)s}(1+s)^{-1/2}ds\Big)\|g_{in}\|_{L^\infty_x L^2_\xi}\,.\end{aligned}$$

The integration by parts gives

$$\begin{aligned}&\int_0^t e^{(a(\tau_1)-\overline{a}_2\varepsilon^2)s}(1+s)^{-1/2}ds\\ &= \frac{e^{(a(\tau_1)-\overline{a}_2\varepsilon^2)s}}{(a(\tau_1)-\overline{a}_2\varepsilon^2)}(1+s)^{-1/2}\Big|_{s=0}^t + \frac{2}{(a(\tau_1)-\overline{a}_1\varepsilon^2)}\int_0^t e^{(a(\tau_1)-\overline{a}_2\varepsilon^2)s}(1+s)^{-3/2}ds\,.\end{aligned} \tag{3.17}$$

Note that if t is large enough, the maximum of $e^{(a(\tau_1)-\overline{a}_2\varepsilon^2)s}(1+s)^{-3/2}$ for $s\in[0,t]$ occurs at $s=t$. This implies

$$\int_0^t e^{(a(\tau_1)-\overline{a}_2\varepsilon^2)s}(1+s)^{-3/2}ds \lesssim e^{(a(\tau_1)-\overline{a}_2\varepsilon^2)t}t(1+t)^{-3/2}\,, \tag{3.18}$$

and hence we get

$$\|I_3\|_{L^2_\xi} \le (1+t)^{-1/2}e^{-\overline{a}_1\varepsilon^2 t}\|g_{in}\|_{L^\infty_x L^2_\xi}\,.$$

Finally,

$$\|I_4\|_{L^2_\xi} \le \int_0^t e^{-a(\tau_1)(t-s)-a(\tau_2)s}ds\|g_{in}\|_{L^\infty_x L^2_\xi} \le te^{-a(\tau)t}\|g_{in}\|_{L^\infty_x L^2_\xi}\,,$$

where $a(\tau)=\min\{a(\tau_1),a(\tau_2)\}$. In conclusion, we have

$$\|h^L\|_{L^\infty_x L^2_\xi} \lesssim e^{-\overline{a}\varepsilon^2 t}(1+t)^{-1/2}\|h_{in}\|_{L^\infty_x L^2_\xi}\,. \tag{3.19}$$

This completes the proof of the lemma. □

Lemma 3.5 (Short wave h^S). We have the following estimate of the short wave part of h:

$$\|h^S\|_{L^\infty_x L^2_\xi} \lesssim e^{-c_1 t}\|(g_{in},h_{in})\|_{L^\infty_x L^\infty_{\xi,\beta}}\,. \tag{3.20}$$

The estimate of h^S is similar to g^S (the extra term is $L_{BA}g$, but L_{BA} has super nice properties in Lemma 2.4) and hence we omit the details. Finally, we can apply the bootstrap process to get $L^\infty_x L^\infty_{\xi,\beta}$ estimates of h^S and h^L.

Proposition 3.6. Assuming that $0<\varepsilon\ll 1$, $\beta>3/2$ and the initial condition h_{in} satisfies the assumption (1.10). Then the solutions of h^L and h^S have the following estimates:

$$\begin{aligned}\|h^L\|_{L^\infty_x L^\infty_{\xi,\beta}} &\lesssim e^{-\overline{a}_1 t}(1+t)^{-1/2}\|(g_{in},h_{in})\|_{L^\infty_x L^\infty_{\xi,\beta}}\,,\\ \|h^S\|_{L^\infty_x L^\infty_{\xi,\beta}} &\lesssim e^{-c_1 t}\|(g_{in},h_{in})\|_{L^\infty_x L^\infty_{\xi,\beta}}\,.\end{aligned}$$

With Propositions 3.3 and 3.6, we have our main theorem.

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