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ON THE INTEGRO-DIFFERENTIAL GENERAL SOLUTION FOR THE UNSTEADY MICROPOLAR STOKES FLOW OF A CONDUCTING FERROFLUID

By

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Abstract. The three–dimensional (3–D) unsteady creeping motion, corresponding to Stokes flow, of a non–conductive colloidal suspension of ferromagnetic particles, which are embedded within an otherwise electrically conducting, viscous and incompressible, carrier liquid, is considered in this contribution. This group of micropolar conducting ferrofluids comprises a novel class of engineering materials that respond in the presence of a general externally applied magnetic field, which is arbitrarily orientated in the three–dimensional domain of practical interest. Therein, an induced magnetic field of minor importance is created, while the effective viscosity of the fluid is increasing and an additional magnetic pressure appears. In order to be compatible with the principles of both ferrohydrodynamics and magnetohydrodynamics, we readily include the magnetization and the electrical conductivity of the magnetic fluid, respectively into the governing partial differential equations of the particular physical system. Employing the potential representation theory, we fabricate a new integro–differential general solution for the situation under investigation, which provides the time–dependent velocity and total pressure fields in a 3–D spaced closed form and in terms of easy–to–find potentials, via a semi–analytical shape. This generalized representation is proved to be complete, whilst it is valid for any non–axisymmetric geometry. We demonstrate the applicability of our analytical approach, by introducing a basic degenerate case of the aforementioned method to simulate the time–dependent creeping flow of a micropolar fluid with conductive properties inside a circular duct.

1. Introduction. Mechanical systems involving the motion of aggregates of small solid ferromagnetic nanoparticles relative to viscous magnetic fluids such as water, blood, hydrocarbons and many other liquids, either conducting or not, in which they are immersed, covers a wide range of heat and mass transfer areas of invaluable importance

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in physical and mathematical applications [1]. This class of ferrofluids, as they are named, exhibit a number of interesting behaviors when subjected to spatially varying or oscillating magnetic fields, where these characteristics can be readily utilized in numerous technological and biomechanical applications [1]–[3]. Modeling purposes permit the assumption of spherical magnetic particles, due to their small size and volumetric concentration in the generally Newtonian conductive carrier liquid. The Brownian motion of the particles is the mechanism that controls the stability of the suspension and the micro–particles act like rigid magnetic dipoles [1] that can respond to any applied magnetic field, which perturbs the flow and affects the phenomenon by preventing the rotation of each small particle, increasing the effective viscosity and causing the emergence of an additional magnetic pressure term. The induced magnetic field, which is produced by the electric current in the ferrofluid, can be neglected in several applications. However, in the general consideration, the total magnetic field (referred to as magnetic field) comprises the superposition of both the applied and the induced fields.

The system of partial differential equations for micropolar magnetic fluids flow [1]–[3] includes the equations of momentum (Navier–Stokes), continuity, Maxwell, energy (when thermomechanics of magnetic fluids [2] is considered), angular momentum and magnetization, all coupled with each other and expressed in terms of the velocity, the pressure and the magnetic field, as well as several constant hydrodynamic and magnetic parameters. To this end, principles of both ferrohydrodynamics and of magnetohydrodynamics serve as platforms, where the first one is concerned with the mechanics of motion of the micropolar fluids that is influenced by strong forces of magnetic polarization, while the latter deals with the current distribution of the electrically conductive carrier liquid. However, the necessity for analytical and theoretical models with fitting in real–life physical regimes, demand the simplification of the above system of equations in terms of practical explanations. For instance, a very important limiting case is based on the assumption of the creeping magnetic flow for small Reynolds numbers [4], leading to Stokes magnetic equations [5] with many important applications. On the other hand, the angular momentum equation can be readily manipulated [1], [5] and absorbed into the already simplified momentum equation, while the generalized form of the equation of magnetization [1], [5] is reduced to a collinear relation with the magnetic field, which is actually true as a fair approximation of nil relaxation time for magnetization and very slow motion [5]. Consequently, the combination of the hydrodynamic flow [4] with the properties of such micropolar conducting fluids [1]–[3], [5], illuminated by a magnetic field, provides us with the most appropriate tools for developing and solving boundary value magnetohydrodynamic and ferrohydrodynamic problems that are equipped with high mathematical and technical complexity, especially when time is encapsulated.

Those situations strongly require the development of analytical and mathematical techniques, which capture the essential features of the transport or fluid process under consideration in an analytical formula (before implementing a possibly inevitable numerical code) that incorporates properly the geometrical and physical characteristics with the minimum of simplifications. The general idea arises from the question of how far an analytical modeling of a physical problem may go. Since the cross–line of such applications is the retaining or the omission of the convection terms in the Navier–Stokes equations for magnetic fluids, we wish to push this line to the limit, where analytical procedures are not enough and the introduction of computational analysis becomes necessary. Within this precise aspect, the potential representation theory for the Stokes flow, which has evolved rapidly during the past [6]–[11], offers the perfect environment for tackling with analytical models, incorporating solutions in the form of differential representations for Stokes flows, which provide the velocity and the total pressure fields in terms of easy–to–handle potentials. Beyond the aim of pure hydrodynamic perspective, recently [12], a semi–analytical approach for the steady and incompressible flow of a Newtonian, as well as an electrically conducting carrier liquid, including a small concentration of magnetic particles, under the effect of an applied magnetic field, has been studied, which was the improvement of a first attempt in [5], where there the magnetization was hypothesized by the constant saturation expression.

Our main purpose is to extend further the analytical method highlighted in [12] and develop a quite comprehensive semi–analytical general solution of the magnetic ferrofluid flow equations, where the introduction of the catholic time–dependence inherits the novelty to this work. Although differential solutions of the unsteady homogeneous or non–homogeneous hydrodynamic Stokes flows have already appeared in the literature [13], [14], in this paper we imply magnetic ferrofluids that react in the presence of an arbitrarily orientated 3–D time–varying magnetic field, taking under consideration both the magnetization and the Lorentz forces (that correspond to the electric currents) in the momentum equation. Then, we construct a reliable time–dependent model independent of any geometry, which provides us with the flow fields in ready–to–use elegant form, in view of known potentials, comprising solutions of inhomogeneous diffusive equations. To accomplish that, we are working in the applied mathematical analysis framework using classical bibliography [15], [16] and we prove the completeness of this integro–differential representation of the flow fields for the aforementioned physical system, which can allow for further computational handling regarding the resulting integral formations [17]. In the absence of a magnetic field or in the case of a non–magnetic normal fluid, our solution reduces to the already known differential general solution for unsteady hydrodynamic Stokes flow [13].

The analytical section of this manuscript is followed by the application of the obtained representation to the computation of the velocity and total pressure fields of the creeping flow within a duct of circular cross–section in three (3–D) dimensions, simulating a micropolar liquid with electrically conducting nature, where the flow fields suppose easily amenable integral equations.

2. Physical and mathematical formulation. Practical physical applications of mathematical nature in engineering technology and biomechanics involve the low–Reynolds number flow of micropolar conducting ferrofluids under the effect of magnetic fields. In order to encounter such applications, we assume the time–dependent creeping flow of a viscous (constant dynamic viscosity η) and incompressible (constant mass density ρ) Newtonian magnetic fluid of kinematic viscosity $\nu = \eta/\rho$ within a smooth three–dimensional environment $V(\mathbb{R}^3) \equiv V$, which could be either bounded with surrounding boundary surface $\partial V(\mathbb{R}^3) \equiv S$ or unbounded, taking as $S \to +\infty$, when the case might be. However, problems in two dimensions $\mathbb{R}^2$ are also considered and treated in the same sense. The ferromagnetic particles of the micropolar colloidal fluid–solution are considered spherical, due to their small size, with radius r_p and density ρ_p, while the conductive carrier liquid possesses a constant electric conductivity σ. Henceforth, since the proceeding method is independent of the geometry of the 3–D flow, every field quantity is written in terms of the spatial position vector $\mathbf{r} = x_1\hat{\boldsymbol{x}}_1 + x_2\hat{\boldsymbol{x}}_2 + x_3\hat{\boldsymbol{x}}_3$, expressed via the Cartesian basis $\hat{\boldsymbol{x}}_j$ for $j = 1, 2, 3$ in Cartesian coordinates (x_1, x_2, x_3) and the time variable $t > t_0$, t_0 being an initial observation point, where dependence $(\mathbf{r}, t)$ is omitted for writing convenience. The Brownian motion controls the stability of the suspension and the particles act like rigid magnetic dipoles, whose rotation is prevented under the application of an arbitrary external 3–D magnetic field $\mathbf{H}$ (of measure $H = |\mathbf{H}|$), increasing the effective viscosity of the magnetic fluid and introducing a magnetic pressure, which affects the total pressure. Practically, the field $\mathbf{H}$ is given by the summation of the applied magnetic and the demagnetizing field in the ferrofluid, where the latter is frequently neglected in several applications, permitting $\mathbf{H}$ to stand for the applied magnetic field thereafter, where in general, it depends on time as well.

Based on the fundamentals of both ferrohydrodynamics and magnetohydrodynamics, the governing equation that relates the velocity field $\mathbf{v}$ (giving $v = |\mathbf{v}|$) with the total pressure field $P = P_t + \rho g h$ (P_t denoting the thermodynamic pressure, whereas $\mathbf{g} = -g\hat{\boldsymbol{h}}$ defines the gravity acceleration of measure g and $\rho g h$ referring to the hydrostatic pressure force, which corresponds to a basic height of reference h in the $\hat{\boldsymbol{h}}$ direction) for Reynolds number $\mathrm{Re} \ll 1$, resulting in the vanishing of the convection terms $\rho\mathbf{v} \cdot (\nabla \otimes \mathbf{v})$, is appropriately given in terms of the time derivative $\partial/\partial t \equiv \partial_t$, the invariant gradient ∇ and Laplacian Δ differential operators. Otherwise, it is conveyed via the time-dependent momentum equation [1]–[3], [5]

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \eta(1+\delta)\Delta\mathbf{v} - \nabla P + \mu_0(\nabla \otimes \mathbf{H}) \cdot \mathbf{M} + \mathbf{j} \times \mathbf{B} \tag{1}$$

for a micropolar electrically conducting ferrofluid. On the other hand, the continuity equation [4]

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

secures the mass conservation of the incompressible magnetic fluid. Once the velocity field is obtained, the vorticity field $\boldsymbol{\Omega}$ (of measure $\Omega = |\boldsymbol{\Omega}|$) is defined as $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$. The dyadic symbol "$\otimes$" stands for a juxtaposition connection between vectors ($\tilde{\mathbf{I}}$ yielding the unit dyadic), while as it is demonstrated in [5],

$$\eta_f \equiv \eta\delta = \frac{\tau_B \mu_0 M_0 H}{4[1 + (\tau_S/I)\tau_B \mu_0 M_0 H]}, \text{ where } M_0 = nm(\coth\xi - \xi^{-1}) \text{ with } \xi = \frac{m\mu_0 H}{KT}, \tag{3}$$

defines the additional effective viscosity caused by the interaction between the magnetic particles and the applied magnetic field (note $\delta = \delta(H)$), while $M_0 = M_0(H)$ is the equilibrium magnetization, reading in terms of the Langevin function [1], where m specifies the magnetic moment of a particle. Providing some useful notation, $\tau_S = r_p^2\rho_p/15\eta_0$ is the relaxation time of a single particle rotation (η_0 corresponding to the rotational viscosity), $I = 8\pi r_p^5\rho_p n/15$ comprises the sum of moments of inertia of the spherical–type ferromagnetic particles per unit volume (n being the number of particles per unit volume and $\phi = (4/3)\pi r_p^3 n$ referring to the volumetric concentration of particles), $\tau_B = 4\pi\eta r_p^3/KT$ is the relaxation time of Brownian rotation (K being the Boltzmann's constant and T denoting the constant temperature for our isothermal problem) and μ_0 is the magnetic permeability of the free space (vacuum). The rest of the fields implicated in (1) are as follows. The magnetic induction $\mathbf{B}$ of measure $B = |\mathbf{B}|$ is summed by the applied and the induced one, caused by the ferromagnetic material, while the magnetization $\mathbf{M}$ (measure $M = |\mathbf{M}|$) refers to the total magnetization of both the colloidal suspension of particles and the carrier liquid, which in fact does not have substantial magnetic properties. For a linear, homogeneous and isotropic medium of magnetic permeability $\mu = \mu(H)$ (zero magnetization relaxation time is presumed), both are formulated by the proceeding constitutive and linearly dependent relationships [1]

$$\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu\mathbf{H} \text{ with } \mu = \mu_0\left(1+\frac{M_0}{H}\right), \text{ since } \mathbf{M} = \frac{M_0}{H}\mathbf{H} \text{ (for Re} \ll 1 \text{ then } \Omega\tau_B \ll 1), \tag{4}$$

where [5] reveals all the logical steps to obtain a collinear relation of $\mathbf{M}$ and $\mathbf{H}$, due to slow motion. Additionally, $\mathbf{j}$ is the conduction current density of the carrier fluid, satisfying the continuity equation $\nabla \cdot \mathbf{j} = 0$ (the contribution of the electrical currents of the ferromagnetic particles is considered small, due to their small concentration), where Maxwell's equations, which are reduced in a simpler form for the case where the displacement currents are absent and no electric field is present, supplement (1)–(4) and they are rendered by the Ohm's law and the Gauss's magnetism law [5], i.e.

$$\mathbf{j} = \sigma(\mathbf{v}\times\mathbf{B}) \text{ and } \nabla\cdot\mathbf{B} = 0. \tag{5}$$

Obviously in (1), $\eta(1+\delta)\Delta\mathbf{v}$ corresponds to the viscous terms, since $1+\delta = (\eta + \eta\delta)/\eta = (\eta+\eta_f)\eta$ is the ratio of the viscosity of the micropolar fluid in the presence of magnetic particles over the viscosity of the fluid in the absence of them, $\nabla P_f \equiv -\mu_0(\nabla\otimes\mathbf{H})\cdot\mathbf{M}$ yields the additional pressure ferromagnetic force gradient, leading to an apparent total pressure field of gradient $\nabla P - \mu_0(\nabla\otimes\mathbf{H})\cdot\mathbf{M}$ and the Lorentz force of the conductive fluid is exhibited by the term $\mathbf{j}\times\mathbf{B}$. If there exist no magnetic particles and the fluid is non–conducting, we arrive at the well–known Stokes equations (for creeping hydrodynamic flow), which is the same result we attain in the case of zero applied magnetic field.

The system of the momentum equation (1), accompanied by (3) and (4), the continuity equation (2) and the Maxwell's equations (5) constitute a complete system for solving general ferrohydrodynamic problems of micropolar electrically conducting fluids, assuming the unsteady Stokes flow restriction and imposing the proper boundary and initial conditions, depending on the requirements of the physical problem.

3. Integro–differential general solution – Completeness. A semi–analytical differential solution, interfered with integral equations, of the ferromagnetic Stokes equations of conducting liquids is displayed and provides the three–dimensional time–dependent flow fields, which occupy any non–axisymmetric area, perturbed by an arbitrarily orientated 3–D magnetic field. It is given in an integral–type closed form in terms of differential operators acting on potential functions, which are solutions of amenable diffusion partial differential equations and contains the applied magnetic field, the effective viscosity, the magnetic pressure and several constant characteristic parameters of the particular flow. To this aim, we introduce the main theorem of this article.

Theorem. The integro–differential representation of solutions for (1) and (2) with (3)–(5), assuming

$$\mathbf{v} = \nabla\times(\mathbf{r}A)+\nabla\times\nabla\times(\mathbf{r}B) \quad \text{and} \quad P = P_0 - nKT\ln\frac{e\xi}{\sinh\xi} - \eta(1+\delta)(\varphi-\Psi-\mathbf{r}\cdot\nabla\Psi) \tag{6}$$

is complete, where P_0 is a constant reference pressure, while the scalar potentials A and B satisfy the following non–homogeneous diffusive equations, in terms of the harmonic potential Ψ, those being

$$(\Delta - \nu^{-1}\partial_t)A = \chi \text{ and } (\Delta - \nu^{-1}\partial_t)B = \psi + \Psi \text{ with } \Delta\Psi = 0 \tag{7}$$

with $\partial/\partial t \equiv \partial_t$. The functions χ and ψ are provided as solutions of the partial differential equations

$$L\chi = -\mathbf{r}\cdot\nabla\times\mathbf{f} \text{ and } L\psi = -\mathbf{r}\cdot(\mathbf{f}-\nabla\varphi), \tag{8}$$

where L is the dimensionless transverse part of the Laplacian operator and if $E(\mathbf{r},\mathbf{r}')$ is the Laplace's fundamental solution with prime denoting definition with respect to position $\mathbf{r}'$, then the integral–type function φ admits

$$\varphi = \iiint_V E(\mathbf{r},\mathbf{r}')(\nabla'\cdot\mathbf{f}')\,dV' \text{ with } \mathbf{f} = \frac{1}{\eta(1+\delta)}\left[\frac{\nabla\delta}{1+\delta}\left(P+nKT\ln\frac{e\xi}{\sinh\xi}\right)+\sigma\mu^2H^2\left(\tilde{\mathbf{I}}-\frac{\mathbf{H}\otimes\mathbf{H}}{H^2}\right)\cdot\mathbf{v}\right]. \tag{9}$$

Proof. Our first task will be the simplification of (1), with the aid of (3)–(5), so as to appear in a more elegant form for further analytically mathematical processing. Bearing this in mind, we perform some calculations, based on vector analysis and we expand $\mathbf{H}$ and $\mathbf{M}$ in the Cartesian system, to obtain

$$\begin{aligned}\mu_0(\nabla\otimes\mathbf{H})\cdot\mathbf{M} &= \mu_0\sum_{i,j=1}^{3}(\nabla H_i)\otimes\hat{\boldsymbol{x}}_i\cdot M_j\hat{\boldsymbol{x}}_j = \mu_0\sum_{j=1}^{3}M_j\nabla H_j \\ &= \mu_0\frac{M_0}{H}\sum_{j=1}^{3}H_j\nabla H_j = \mu_0 M_0\nabla\sqrt{\sum_{j=1}^{3}H_j^2} = \mu_0 M_0\nabla H \\ &= nKT\left(\coth\xi-\frac{1}{\xi}\right)\nabla\xi = -\nabla P_f, \text{ where } P_f = nKT\ln\frac{e\xi}{\sinh\xi} \text{ and } \xi = \frac{m\mu_0 H}{KT},\end{aligned} \tag{10}$$

where the additional pressure term P_f, due to the magnetic particles, has been recovered, while the Lorentz forces yield

$$\begin{aligned}\mathbf{j}\times\mathbf{B} &= \sigma(\mathbf{v}\times\mathbf{B})\times\mathbf{B} = -\sigma\mu^2\mathbf{H}\times(\mathbf{v}\times\mathbf{H}) = -\sigma\mu^2(H^2\mathbf{v}-(\mathbf{v}\cdot\mathbf{H})\mathbf{H})\\ &= -\sigma\mu^2H^2\mathbf{v}\cdot\left(\tilde{\mathbf{I}}-\frac{\mathbf{H}\otimes\mathbf{H}}{H^2}\right)\\ &= -\eta(1+\delta)\mathbf{v}\cdot\tilde{\mathbb{S}},\ \text{where } \tilde{\mathbb{S}}=\tilde{\mathbb{S}}^{\mathsf{T}}=\frac{\sigma\mu^2H^2}{\eta(1+\delta)}\left(\tilde{\mathbf{I}}-\frac{\mathbf{H}\otimes\mathbf{H}}{H^2}\right),\end{aligned} \tag{11}$$

projecting the velocity field on the plane that is perpendicular to the direction of the magnetic field. The symbol "T" denotes transposition that provides an inverted symmetric ($\mathbf{v}\cdot\tilde{\mathbb{S}}=\tilde{\mathbb{S}}\cdot\mathbf{v}$) dyadic. Finally, straightforward application of a trivial vector identity, implicating the gradient operator, leads to

$$\begin{aligned}\frac{\nabla(P+P_f)}{\eta(1+\delta)} &= \nabla\left[\frac{P+P_f}{\eta(1+\delta)}\right]-(P+P_f)\nabla\left[\frac{1}{\eta(1+\delta)}\right]\\ &= \nabla\left[\frac{P+P_f}{\eta(1+\delta)}\right]+\frac{P+P_f}{\eta(1+\delta)}\nabla\left[\ln(\eta(1+\delta))\right]\\ &= \nabla p+\mathbf{s}p,\ \text{where } p=\frac{P+P_f}{\eta(1+\delta)} \text{ and } \mathbf{s}=\nabla\left[\ln(\eta(1+\delta))\right],\end{aligned} \tag{12}$$

which manages efficiently the pressure terms. Collecting the outcomes (10)–(12) with definition (3), we substitute them within (1), arriving at

$$\frac{1}{\nu(1+\delta)}\partial_t\mathbf{v}\cong\nu^{-1}\partial_t\mathbf{v}=\Delta\mathbf{v}-\Delta p-\mathbf{f},\ \text{where } \mathbf{f}=\mathbf{s}p+\mathbf{v}\cdot\tilde{\mathbb{S}},\ \text{provided that } \nabla\cdot\mathbf{v}=0, \tag{13}$$

which is the unsteady Stokes ferrohydrodynamic and magnetohydrodynamic equation for the creeping motion and it is accompanied by the necessary continuity equation (2). The first approximate equality in (13) rises from the fact that the left–hand side of it is not affected by the increase of the effective viscosity in comparison with the rest of the terms, i.e. the viscous ones. Yet, function $\mathbf{f}$ contains the two fields under computation (velocity and pressure) in an algebraic fashion for further elaboration.

At this stage we shall work within the framework of the spherical geometry [16] to simplify the steps of our proof, but the final result will be independent of the geometry, as stated in (6)–(9), hence the generality of the theorem is kept untouched. In spherical coordinates, the transverse part of Laplacian operator excluding the factor $1/r^2$ is $L=[r^2\Delta-\partial_r(r^2\partial_r)]$, in terms of the spherical radial $r=|\mathbf{r}|$, when $\partial/\partial r\equiv\partial_r$. By virtue of the amenable expression (13) and since $\mathbf{f}$ is sufficiently smooth on the bounded simply connected domain V, we adopt the technique from [13] to express this function as

$$\mathbf{f}=\nabla\varphi+\nabla\times(\mathbf{r}\chi)+\nabla\times\nabla\times(\mathbf{r}\psi), \tag{14}$$

where ϕ, χ and ψ are the given scalar functions, which are solutions of the trivial system (see [10])

$$\Delta\varphi = \nabla\cdot\mathbf{f},\ \left[r^2\Delta - \partial_r(r^2\partial_r)\right]\chi = -\mathbf{r}\cdot\nabla\times(\mathbf{f}-\nabla\varphi) \text{ and } \left[r^2\Delta - \partial_r(r^2\partial_r)\right]\psi = -\mathbf{r}\cdot(\mathbf{f}-\nabla\varphi), \tag{15}$$

respectively, Such decomposition has been proved in [10] for bounded domains with the aid of the principal reference [6] and has been extended to unbounded regions, as demonstrated in [11], [13]. Since the solutions for functions χ and ψ depend on the φ potential that satisfies a Poisson's partial differential equation, then we may use the fundamental solution of the Laplace's operator

$$E(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \text{ for } \mathbf{r},\mathbf{r}' \in V(\mathbb{R}^3) \text{ or } E(\mathbf{r},\mathbf{r}') = -\frac{1}{2\pi}\ln\frac{1}{|\mathbf{r}-\mathbf{r}'|} \text{ for } \mathbf{r},\mathbf{r}' \in V(\mathbb{R}^2), \tag{16}$$

providing $\Delta E(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$ according to the delta function, in order to reveal that

$$\varphi = \iiint_V E(\mathbf{r},\mathbf{r}')(\nabla'\cdot\mathbf{f}')dV', \text{ where } \nabla' \equiv \nabla_{\mathbf{r}'}, dV' \equiv dV(\mathbf{r}') \text{ and } \mathbf{f} \equiv \mathbf{f}(\mathbf{r}',t), \tag{17}$$

the same notation being followed in the sequel for any $\mathbf{r}' \in V$ with $\mathbf{r} \neq \mathbf{r}'$, much as if V is unbounded ($S \to +\infty$) and $\mathbf{f}$ vanishes sufficiently fast at infinity.

Inserting (14) into (13), we render

$$(\Delta - \nu^{-1}\partial_t)\mathbf{v} = \nabla(p+\varphi) + \nabla\times(\mathbf{r}\chi) + \nabla\times\nabla\times(\mathbf{r}\psi) \text{ with } \nabla\cdot\mathbf{v} = 0, \tag{18}$$

while operating divergence on (18), commuting properly and using condition (2), we are concluding to

$$\Delta(p+\varphi) = 0, \tag{19}$$

which implies that function $p+\varphi$ is harmonic, i.e.,

$$p+\varphi = \Phi \text{ or } p = -\varphi + \Phi, \text{ where } \Delta\Phi = 0. \tag{20}$$

Herein, let us recall that φ satisfies (17), while p includes the total pressure field under calculation through the definition (12), utilizing (10), which completes our proof for P, given within (6)–(9), where Φ admits a specific form, as will be shown next. That way we accomplished the decoupling of the velocity field from the total pressure, where the latter is proved to satisfy (20). We proceed by substituting $p+\varphi$ from (20) into the equation (18), which reads as

$$(\Delta - \nu^{-1}\partial_t)\mathbf{v} = \nabla\Phi + \nabla\times(\mathbf{r}\chi) + \nabla\times\nabla\times(\mathbf{r}\psi) \text{ with } \nabla\cdot\mathbf{v} = 0. \tag{21}$$

The key method to our work is based on the manipulation of (21) in order to recover the velocity $\mathbf{v}$. To this end we recall the theorem developed in [6] and since $\mathbf{v}$ is a divergence–free ($\nabla\cdot\mathbf{v} = 0$) vector field that possesses partial derivatives of orders up to two, which are Hölder continuous on V, there exist scalar functions A and B, such that

$$\mathbf{v} = \nabla\times(\mathbf{r}A) + \nabla\times\nabla\times(\mathbf{r}B), \tag{22}$$

which satisfy the partial differential equations

$$[r^2\Delta - \partial_r(r^2\partial_r)]A = -\mathbf{r}\cdot\nabla\times\mathbf{v} \text{ and } [r^2\Delta - \partial_r(r^2\partial_r)]B = -\mathbf{r}\cdot\mathbf{v}. \tag{23}$$

The aforementioned theorem is valid for a bounded V but its proof has been diversified for solenoidal fields in an infinite domain [11], hence it is general. Our aim is to find these scalar potentials A and B that are solutions of (23), by making use of (21)–(22). Primarily, we have to withdraw a note following a lemma from [9], which addresses the fact that the operators Δ, $[r^2\Delta - \partial_r(r^2\partial_r)]$ and ∂_t commute. In that sense, following similar steps as those in [13] for the inhomogeneous unsteady hydrodynamic Stokes flow, we work as follows. Initially, we concentrate on B, hence we operate $\Delta(\Delta - \nu^{-1}\partial_t)$ on both sides of the second relationship of (23), we use the above comment and we calculate

$$[r^2\Delta - \partial_r(r^2\partial_r)]\{\Delta(\Delta - \nu^{-1}\partial_t)B\} = -\Delta(\Delta - \nu^{-1}\partial_t)(\mathbf{r}\cdot\mathbf{v}) = -\mathbf{r}\cdot\Delta(\Delta - \nu^{-1}\partial_t)\mathbf{v}, \quad (24)$$

in view of the identities $\Delta(\mathbf{r}\cdot\mathbf{v}) = \mathbf{r}\cdot\Delta\mathbf{v}$ (note that $\Delta\mathbf{r} = \mathbf{0}, (\nabla\otimes\mathbf{r})^{\mathsf{T}} : \nabla\otimes\mathbf{v} = \tilde{\mathbf{I}} : \nabla\otimes\mathbf{v} = \nabla\cdot\mathbf{v} = 0$) and $\Delta^2(\mathbf{r}\cdot\mathbf{v}) = \mathbf{r}\cdot\Delta^2\mathbf{v}$, since $\partial_t\mathbf{r} = \mathbf{0}$. On the other hand, aiming to expunge $\mathbf{v}$ from (24), we apply the Laplace's operator Δ on (21), we consider the harmonic character of Φ from (20) and we execute some analytical calculations with identities of vector differential analysis such as above (e.g. $\Delta\mathbf{r} = \mathbf{0}, \nabla\times\mathbf{r} = \mathbf{0}$, $\nabla\cdot\mathbf{r} = 3$ and $\nabla\otimes\mathbf{r} = \tilde{\mathbf{I}}$) to obtain

$$\begin{aligned}\Delta(\Delta - \nu^{-1}\partial_t)\mathbf{v} &= \nabla\times\Delta(\mathbf{r}\chi) + \nabla\times\nabla\times\Delta(\mathbf{r}\psi)\\ &= \nabla\times(\mathbf{r}\Delta\chi + 2\nabla\chi) + \nabla\times\nabla\times(\mathbf{r}\Delta\psi + 2\nabla\psi)\\ &= \nabla\times(\mathbf{r}\Delta\chi) + \nabla\times\nabla\times(\mathbf{r}\Delta\psi) = \nabla(\Delta\chi)\times\mathbf{r} + \nabla\nabla\cdot(\mathbf{r}\Delta\psi) - \Delta(\mathbf{r}\Delta\psi)\\ &= \nabla(\Delta\chi)\times\mathbf{r} + \nabla[3\Delta\psi + \mathbf{r}\cdot\nabla(\Delta\psi)] - \mathbf{r}\Delta^2\psi - 2\nabla(\Delta\psi)\\ &= \nabla(\Delta\chi)\times\mathbf{r} + \nabla[\Delta\psi + r\partial_r(\Delta\psi)] - \mathbf{r}\Delta^2\psi, \end{aligned} \quad (25)$$

where the alternation of the differential operators $\nabla, \nabla\cdot, \nabla\times$ and $\nabla\times\nabla\times$ has been extensively used, while $\mathbf{r}\cdot\nabla = r\partial_r$. We invoke (25) into (24)and attaining $\mathbf{r}\cdot\nabla(\Delta\chi)\times\mathbf{r} = 0$ and $\mathbf{r}\cdot\mathbf{r} = r^2$, we end up with

$$\begin{aligned}[r^2\Delta - \partial_r(r^2\partial_r)]\{\Delta(\Delta - \nu^{-1}\partial_t)B\} &= -\mathbf{r}\cdot\{\nabla[\Delta\psi + r\partial_r(\Delta\psi)] - \mathbf{r}\Delta^2\psi\}\\ &= -r\partial_r[\Delta\psi + r\partial_r(\Delta\psi)] + \{\partial_r(r^2\partial_r) + [r^2\Delta - \partial_r(r^2\partial_r)]\}(\Delta\psi) \quad (26)\\ &= [r^2\Delta - \partial_r(r^2\partial_r)](\Delta\psi)\end{aligned}$$

or similarly

$$[r^2\Delta - \partial_r(r^2\partial_r)]\{\Delta(\Delta - \nu^{-1}\partial_t)B - \Delta\psi\} = 0 \Rightarrow \Delta[(\Delta - \nu^{-1}\partial_t)B - \psi] = f(r) \equiv 0, \quad (27)$$

where f comprises an arbitrary function of r, which has been taken equal to zero without loss of generality of the method (see [9] for the proof). Therein, partial differential equation (27) enjoys

$$(\Delta - \nu^{-1}\partial_t)B - \psi = \Psi \text{ or } (\Delta - \nu^{-1}\partial_t)B = \psi + \Psi, \text{ where } \Delta\Psi = 0, \quad (28)$$

which is an inhomogeneous diffusive equation for the unknown function B. Similarly, we handle the case for evaluating function A, by acting henceforth as follows. We reinforce a direct application of $(\Delta - \nu^{-1}\partial_t)$ on the first part of equation (23), we commute the corresponding operators [9], we use the previous techniques and results, adding that

$\nabla \cdot \nabla \times \mathbf{v} = 0$ and we take the rotation on (18), to reach

$$\begin{aligned}
[r^2\Delta - \partial_r(r^2\partial_r)]\{(\Delta - \nu^{-1}\partial_t)A\} &= -(\Delta - \nu^{-1}\partial_t)(\mathbf{r} \cdot \nabla \times \mathbf{v}) = -\mathbf{r} \cdot (\Delta - \nu^{-1}\partial_t)(\nabla \times \mathbf{v}) \\
&= -\mathbf{r} \cdot [\nabla \times \nabla \times (\mathbf{r}\chi) + \nabla \times \nabla \times \nabla \times (\mathbf{r}\psi)] \\
&= -\mathbf{r} \cdot [\nabla\nabla \cdot (\mathbf{r}\chi) - \Delta(\mathbf{r}\chi)] + \mathbf{r} \cdot \nabla \times [\mathbf{r}\Delta\psi + 2\nabla\psi] \\
&= -\mathbf{r} \cdot [\nabla(3\chi + \mathbf{r} \cdot \nabla\chi) - \mathbf{r}\Delta\chi - 2\nabla\chi] \\
&= -r\partial_r(\chi + r\partial_r\chi) + \{\partial_r(r^2\partial_r) + [r^2\Delta - \partial_r(r^2\partial_r)]\}\chi \\
&= [r^2\Delta - \partial_r(r^2\partial_r)]\chi \qquad (29)
\end{aligned}$$

or equivalently

$$[r^2\Delta - \partial_r(r^2\partial_r)]\{(\Delta - \nu^{-1}\partial_t)A - \chi\} = 0 \Rightarrow (\Delta - \nu^{-1}\partial_t)A - \chi = g(r) \equiv 0, \qquad (30)$$

where for the same reason as sketched above the arbitrary function $g(r)$ is set to nil [9]. Therefore,

$$(\Delta - \nu^{-1}\partial_t)A - \chi = 0 \text{ or } (\Delta - \nu^{-1}\partial_t)A = \chi, \qquad (31)$$

which stands for an inhomogeneous diffusive equation with respect to the unknown function A. Hence, we were able to obtain the potentials A and B in a diffusive fashion from (28) and (31), respectively, when functions φ, χ and ψ are retrieved from (15) if $[r^2\Delta - \partial_r(r^2\partial_r)] = L$. Then, the velocity field is provided through (22) and, consequently, we ended up with the completeness for $\mathbf{v}$, summarized in (6)–(9)

To complete our proof we are obliged to put together the analytical tools in (21), use decomposition (22) and non–homogeneous diffusion equations (28) and (31) so as to derive an interrelation between the harmonic potentials Φ and Ψ, appearing within (20) and (28), respectively. This is doable, since acting this way and performing similar analytical assumptions as previously, we find

$$\begin{aligned}
\nabla\Phi &= (\Delta - \nu^{-1}\partial_t)[\nabla \times (\mathbf{r}A) + \nabla \times \nabla \times (\mathbf{r}B)] - \nabla \times (\mathbf{r}\chi) - \nabla \times \nabla \times (\mathbf{r}\psi) \\
&= \nabla \times \{\mathbf{r}[(\Delta - \nu^{-1}\partial_t)A - \chi] + 2\nabla A\} + \nabla \times \nabla \times \{\mathbf{r}[(\Delta - \nu^{-1}\partial_t)B - \psi] + 2\nabla B\} \\
&= \nabla \times \nabla \times (\mathbf{r}\Psi) = \nabla\nabla \cdot (\mathbf{r}\Psi) - \Delta(\mathbf{r}\Psi) = \nabla(\Psi + \mathbf{r} \cdot \nabla\Psi)
\end{aligned} \qquad (32)$$

so that, in terms of a constant $c \in \mathbb{R}$,

$$\Phi = (\Psi + \mathbf{r} \cdot \nabla\Psi) + c, \text{ where } \Delta\Phi = 0 \text{ and } \Delta\Psi = 0. \qquad (33)$$

Result (33) is actually a compatibility relation between Φ and Ψ, which is immediately verified if we practice the Laplacian operator onto (33), but it also secures the validity of our general representation. In addition, relationship (33) allows us to use only the harmonic potential Ψ in our solution, since Φ is related to Ψ. Consequently, if we use (33) into (20) and apply the replacement $P_0 = \eta(1 + \delta)c$, which stands for a constant pressure of reference, chosen accordingly to the physical problem, then we obtain the pressure field in (6) of our theorem. The integro–differential representation (6)–(9) is proved then to be complete and independent of any orthogonal curvilinear geometry, since the final formulae that are depicted, utilize invariant operators. This terminates the proof of the theorem.

The essence of the general semi–analytical solution (6)–(9) has assimilated into a mixed type integro–differential form with respect to function $\mathbf{f}$. There exist several ways to solve this system, involving either analytical or numerical methods [17], in view of an initial properly adjusted guess for $\mathbf{v}$ and P. For instance, an analytical way to deal with (6)–(9) is to pursue a solution of the integral equation (9) such as the Nyström or Neumann series method. On the contrary, if there is no field ($\mathbf{H} = \mathbf{0}$) or in the case of absence of magnetic particles ($\phi = 0$) and non–conducting fluid ($\sigma = 0$), our representation (6)–(9) reduces to the already known general unsteady Stokes flow solution [13].

4. Application and demonstration of the integro-differential representation. In order to illustrate our semi–analytical integro–differential solution (6)–(9), we address a particular boundary value problem, which is drawn from the library with physical problems in hydrodynamics. In detail, we consider the unsteady and creeping micropolar flow of a conducting ferrofluid in a straight circular cylindrical tube, which is perturbed by a time–dependent magnetic field and we formulate the problem with respect to an infinite circular cylinder of a fixed radius α. In terms of the cylindrical variables $\rho \in [0, +\infty)$ (actually it is $\rho \in [0, \alpha)$ in our case), $\varphi \in [0, 2\pi)$ and $z \in (-\infty, +\infty)$, we define the implemented to our application (for any $t > t_0$) circular cylindrical coordinate system via

$$\mathbf{r} = \sum_{j=1}^{3} x_j \hat{\boldsymbol{x}}_j = z\hat{\boldsymbol{x}}_1 + \rho\cos\varphi\hat{\boldsymbol{x}}_2 + \rho\sin\varphi\hat{\boldsymbol{x}}_3 = \rho\hat{\rho} + z\hat{z}, \tag{34}$$

where the unit normal coordinate vectors of this system $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\varphi}}, \hat{\boldsymbol{z}}$, as written to denote the (ρ, φ, z) right–handed system, assume the form

$$\hat{\boldsymbol{\rho}} = -\frac{\partial\hat{\boldsymbol{\varphi}}}{\partial\varphi} = \cos\varphi\hat{\boldsymbol{x}}_2 + \sin\varphi\hat{\boldsymbol{x}}_3, \hat{\boldsymbol{\varphi}} = \frac{\partial\hat{\boldsymbol{\rho}}}{\partial\varphi} = -\sin\varphi\hat{\boldsymbol{x}}_2 + \cos\varphi\hat{\boldsymbol{x}}_3 \text{ and } \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}_1, \tag{35}$$

respectively. The gradient and the Laplacian differential operators yield

$$\nabla = \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j \frac{\partial}{\partial x_j} = \hat{\boldsymbol{\rho}}\frac{\partial}{\partial\rho} + \frac{\hat{\boldsymbol{\varphi}}}{\rho}\frac{\partial}{\partial\varphi} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \text{ and } \Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}, \tag{36}$$

while it can be easily verified that

$$\nabla \otimes \hat{\boldsymbol{\rho}} = \frac{1}{\rho}\hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}}, \nabla \otimes \hat{\boldsymbol{\varphi}} = -\frac{1}{\rho}\hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\rho}} \text{ and } \nabla \otimes \hat{\boldsymbol{z}} = \tilde{\mathbf{0}}, \tag{37}$$

whereas $\tilde{\mathbf{0}}$ stands for the zero dyadic, while in this geometry, the unit dyadic admits

$$\nabla \otimes \mathbf{r} = \tilde{\mathbf{I}} = \sum_{j=1}^{3} \hat{\boldsymbol{x}}_j \otimes \hat{\boldsymbol{x}}_j = \hat{\boldsymbol{\rho}} \otimes \hat{\boldsymbol{\rho}} + \hat{\boldsymbol{\varphi}} \otimes \hat{\boldsymbol{\varphi}} + \hat{\boldsymbol{z}} \otimes \hat{\boldsymbol{z}}. \tag{38}$$

The physical problem that we are about to solve, is mathematically adjusted to this type of circular cylindrical geometry, where the x_1–axis is the axis of symmetry of an infinite circular cylinder and the other two axes are located properly so as to obtain the (ρ, φ, z) clockwise system. All the information described in (34)–(38) is trivial and can be found among other details in [16], but we inserted it here just to make this work complete and independent. Next and during the forthcoming analysis every field is defined for

$(\mathbf{r},t) \equiv (\rho,\varphi,z,t)$ in $V(\mathbb{R}^3) \equiv W = \{(\rho,\varphi,z) : \rho \in [0,\alpha), \varphi \in [0,2\pi), z \in (-\infty,+\infty)\}$ for any $t > 0$. Nevertheless, in practice, the area of observation is delimited for $z \in (0,\ell)$, where ℓ could be the length of the duct, where the magnetic field is applied.

The boundary value problem for the evaluation of the flow fields $\mathbf{v}$ and P is supplemented by the appropriate boundary conditions fixed on the boundaries of the cylinder and the initial condition at the beginning of the phenomenon, where we assume that coincides with $t = t_0 \equiv 0$. Correspondingly, those are the non–slip conditions on the wall of the duct,

$$\mathbf{v}(\mathbf{r},t) = \mathbf{0} \text{ at } \rho = \alpha \text{ for } \varphi \in [0,2\pi), z \in (-\infty,+\infty) \text{ and } t > 0, \tag{39}$$

as well as the requirement that the velocity field at the supposed infinite inlet and outlet of the duct, where there exists no magnetic field, obtains the fully developed 1–D parabolic profile in the classical hydrodynamics, where components perpendicular to the main flow vanish ($\hat{\boldsymbol{\rho}} \cdot \mathbf{v}(\mathbf{r},t) = \hat{\boldsymbol{\varphi}} \cdot \mathbf{v}(\mathbf{r},t) = 0$), while

$$\hat{\boldsymbol{z}} \cdot \mathbf{v}(\mathbf{r},t) = \hat{\boldsymbol{z}} \cdot \mathbf{v}_{\mathrm{P}}(\rho) \text{ at } z \to \pm\infty \text{ for } \rho \in [0,\alpha), \varphi \in [0,2\pi) \text{ and } t > 0, \tag{40}$$

where the pair $(\mathbf{v}_p(\rho), P_{\mathrm{P}}(z))$ for every $\rho \in [0,\alpha)$ and $z \in (-\infty,+\infty)$, which is given by

$$\lim_{z\to\pm\infty} \mathbf{v}(\mathbf{r},t) \equiv \mathbf{v}_{\mathrm{P}}(\rho) = \frac{A}{4\eta}(\alpha^2 - \rho^2)\hat{\boldsymbol{z}} \text{ with } \frac{dP_{\mathrm{P}}(z)}{dz} = -A, \text{ resulting in } P_{\mathrm{P}}(z) = -Az + P_{\mathrm{P},0} \tag{41}$$

correspond to the classical Poiseuille flow that satisfies the system of the well–known steady Stokes equations $\eta\Delta\mathbf{v}_{\mathrm{P}}(\rho) = \nabla P_{\mathrm{P}}(z)$ and $\nabla \cdot \mathbf{v}_{\mathrm{P}}(\rho) = 0$ in the case where the magnetic field is absent. Here, the term $-A \equiv dP_{\mathrm{P}}/dz < 0$ is the constant axial pressure gradient of the known Poiseuille flow, while the arbitrary constant of integration $P_{\mathrm{P},0}$ is chosen as much to obtain consistency at infinity. Though it does not affect the uniqueness of the solution, sine pressure field enters momentum equation (1) under the gradient operation. Finally, the initial condition that actually allows us to initiate the procedure of the flow mechanism at $t > 0$, is set at some point when the fluid is assumed to be moving as a Poiseuille fluid and just before the magnetic field is activated, declaring that

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_{\mathrm{P}}(\rho) \text{ for } t = 0 \text{ with } \rho \in [0,\alpha),\ \varphi \in [0,2\pi) \text{ and } z \in (-\infty,+\infty), \tag{42}$$

which concludes a well–posed problem.

Before we proceed to the imposition of the derived general solution (6)–(9) and attach it to the conditions (39)–(40) and (42), we can further manipulate the physical state. Primarily, we decompose suitably the velocity and the total pressure as

$$\mathbf{v}(\mathbf{r},t) = \mathbf{v}_{\mathrm{P}}(\rho) + \mathbf{v}_g(\mathbf{r},t) \text{ and } P(\mathbf{r},t) = P_{\mathrm{P}}(z) + P_g(\mathbf{r},t) \tag{43}$$

in terms of $\mathbf{v}_{\mathrm{P}}(\rho)$ and $P_{\mathrm{P}}(z)$ (see (41)), where the velocity $\mathbf{v}_g(\mathbf{r},t)$ and the total pressure $P_g(\mathbf{r},t)$ encounter the unknown fields that must be retrieved via our complete and general integro–differential representation (6)–(9). The reason for this splitting lies on the fact that at the limit when the length of the cylinder tends to infinity $z \to \pm\infty$, we presume condition (40), which now is converted to

$$\lim_{z\to\pm\infty} \mathbf{v}_g(\mathbf{r},t) = \mathbf{0} \text{ with } \lim_{z\to\pm\infty} P_g(\mathbf{r},t) = 0 \text{ for } \rho \in [0,\alpha), \varphi \in [0,2\pi) \text{ and } t > 0, \tag{44}$$

a condition that must be automatically satisfied as soon as the flow fields $(\mathbf{v}_g, P_g)$ are calculated via the basic theorem (6)–(9).

Summarizing all of the above, we have to construct the velocity and the total pressure fields (43), given (6)–(9), accompanied by the appropriate boundary (39)–(40) and initial (42) conditions, whereas the conducting micropolar fluid is excited by a hypothesized constant and generally time–dependent uniform magnetic field

$$\mathbf{H}_c(t) = \sum_{j=1}^{3} H_{j,c}(t)\hat{\boldsymbol{x}}_j \text{ with measure } H_c(t) = \sqrt{\sum_{j=1}^{3} H_{j,c}^2(t)}, \tag{45}$$

as is taken for simplicity, which is known and agrees with the imposed restrictions in section 2. Its form implies

$$\delta(t) = \frac{\tau_B \mu_0 M_{0,c}(t) H_c(t)}{4\eta\left(1 + \frac{\tau_S}{I}\tau_B \mu_0 M_{0,c}(t) H_c(t)\right)}, \text{ where } M_{0,c}(t) = nm\left(\coth\frac{m\mu_0 H_c(t)}{KT} - \frac{KT}{m\mu_0 H_c(t)}\right), \tag{46}$$

which is definition (3) for the present illumination, yielding $\nabla\delta(t) = \mathbf{0}$. Therefore, function $\mathbf{f}(\mathbf{r}, t; \mathbf{v}_g)$ within (9) (see also (4) and (11)) is affected such as

$$\mathbf{f}(\mathbf{r}, t; \mathbf{v}_g) = \mathbf{v}_g(\mathbf{r}, t)\cdot\tilde{\mathbb{S}}(t) = \tilde{\mathbb{S}}(t)\cdot\mathbf{v}_g(\mathbf{r}, t) \text{ with } \tilde{\mathbb{S}}(t) = \frac{\sigma\mu_0^2(H_c(t) + M_{0,c}(t))^2}{\eta(1 + \delta(t))}\left(\tilde{\mathbf{I}} - \frac{\mathbf{H}_c(t)\otimes\mathbf{H}_c(t)}{H_c^2(t)}\right), \tag{47}$$

while, for this dependence, the magnetic pressure entering (6) assumes

$$P_f(t) \equiv nKT\ln\frac{e\xi}{\sinh\xi} = nKT\left(1 + \ln\frac{m\mu_0 H_c(t)}{KT\sinh\frac{m\mu_0 H_c(t)}{KT}}\right) \tag{48}$$

and those are the terms appearing in our theorem that include the applied magnetic field (45).

Another crucial factor is the determination of the potential functions $\varphi(\mathbf{r}, t; \mathbf{v}_g)$ and $\Psi(\mathbf{r})$ (actually it is independent of the time), which are required in the generalized representation. As far as $\varphi(\mathbf{r}, t; \mathbf{v}_g)$ is concerned and since we work within a three–dimensional environment, we make use of the 3–D fundamental solution of Laplace's operator from (16), i.e.

$$E(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi}\frac{1}{|\mathbf{r} - \mathbf{r}'|} \text{ with } |\mathbf{r} - \mathbf{r}'| = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \varphi') + (z - z')^2} \text{ for } \mathbf{r} \neq \mathbf{r}' \in W \tag{49}$$

and with the aim of (9) and (47) (see also (36)) it holds

$$\varphi(\mathbf{r}, t, \mathbf{v}_g) = \iiint_W E(\mathbf{r}, \mathbf{r}')[\nabla_{r'}\cdot\mathbf{f}(\mathbf{r}', t; \mathbf{v}_g(\mathbf{r}', t))]dW(\mathbf{r}') \text{ with } \nabla_{\mathbf{r}'} \equiv \nabla|_{\mathbf{r}\to\mathbf{r}'}, \tag{50}$$

which defines this Poisson's potential in an integral form. On the other hand, $\Psi(\mathbf{r})$ is harmonic and since it belongs to the kernel space of Δ, it could be written in the cylindrical coordinate system in terms of the Bessel $J_n(\mu\rho)$ [16] and basic trigonometric and hyperbolic functions as

$$\Psi(\mathbf{r})=\sum_{n=0}^{\infty}\int\sum_{\mu}J_n(\mu\rho)[\cos n\varphi(\alpha_n^\mu\cosh(\mu z)+b_n^\mu\sinh(\mu z))+\sin n\varphi(c_n^\mu\cosh(\mu z)+d_n^\mu\sinh(\mu z))], \tag{51}$$

where the parameter $\mu\in\mathbb{R}$, comes from the method of separation of variables of Laplace's equation and the unknown constant coefficients $a_n^\mu, b_n^\mu, c_n^\mu$ and d_n^μ for $n\geq 0$ must be determined from the imposed conditions. Since our case involves an interior flow problem, we use regular solutions on the axis of symmetry ($\rho=0$), which means that the Neumann functions $N_n(\mu\rho)$ [16] are excluded from the harmonic expansion (51). Moreover, the introduced symbol "$\int\sum_\mu ...$" denotes integration if μ takes continuous values or summation in the case where μ is a parameter with discrete values. For instance, if $\Psi(\mathbf{r})$ cancel at $\rho=\alpha$, then we must set $\mu\equiv\mu_n^m=r_n^m/\alpha$ for $n\geq 0$ and $m\geq 1$, where r_n^m is the m–root ($m\geq 1$) of order $n\geq 0$ of the Bessel function ($J_n(r_n^m)=0$). Consequently, the defined symbol "$\int\sum_\mu\ldots$" is substituted by "$\sum_{m=1}^{\infty}\ldots$" inside (51), which stands for the standard series symbol.

Once functions (47) and (50) are calculated, the second order partial differential equations (8) easily provide us with $\chi(\mathbf{r},t;\mathbf{v}_g)$ and $\psi(\mathbf{r},t;\mathbf{v}_g)$. Indeed, since in the cylindrical geometry.

$$L\equiv\rho^2\Delta-\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)=\frac{\partial^2}{\partial\varphi^2}+\rho^2\frac{\partial^2}{\partial z^2}, \tag{52}$$

then, in view of (36), we have to solve the inhomogeneous equations

$$\frac{\partial^2\chi(\mathbf{r},t;\mathbf{v}_g)}{\partial\varphi^2}+\rho^2\frac{\partial^2\chi(\mathbf{r},t;\mathbf{v}_g)}{\partial z^2}=-\mathbf{r}\cdot\nabla\times\mathbf{f}(\mathbf{r},t;\mathbf{v}_g)\text{ with }\mathbf{r}=\rho\hat{\boldsymbol{\rho}}+z\hat{\boldsymbol{z}} \tag{53}$$

and

$$\frac{\partial^2\psi(\mathbf{r},t;v_g)}{\partial\varphi^2}+\rho^2\frac{\partial^2\psi(\mathbf{r},t;\mathbf{v}_g)}{\partial z^2}=-\mathbf{r}\cdot(\mathbf{f}(\mathbf{r},t;\mathbf{v}_g)-\nabla\varphi(\mathbf{r},t;\mathbf{v}_g))\text{ with }\mathbf{r}=\rho\hat{\boldsymbol{\rho}}+z\hat{\boldsymbol{z}}, \tag{54}$$

whose solutions can be derived with several methods (e.g. method of separation of variables).

The basic potentials $A(\mathbf{r},t;\mathbf{v}_g)$ and $B(\mathbf{r},t;\mathbf{v}_g)$ admit the solutions of the non–homogenous diffusive partial differential equations (7), as long as $\chi(\mathbf{r},t;\mathbf{v}_g)$, $\psi(\mathbf{r},t;\mathbf{v}_g)$ and $\Psi(\mathbf{r})$ are sketched. Here, we choose to deploy them, by accounting the corresponding fundamental solution

$$G(\mathbf{r},\mathbf{r}',t,t')=\frac{e^{-\frac{|\mathbf{r}-\mathbf{r}'|^2}{4\nu(t-t')}}}{\sqrt{\nu[4\pi(t-t')]^3}}\text{ with }|\mathbf{r}-\mathbf{r}'|^2=\rho^2+\rho'^2-2\rho\rho'\cos(\varphi-\varphi')+(z-z')^2,\quad t>t'>0 \tag{55}$$

and therefore

$$A(\mathbf{r},t;\mathbf{v}_g)=\int_0^t\iiint_W G(\mathbf{r},\mathbf{r}',t,t')\chi(\mathbf{r}',t';\mathbf{v}_g(\mathbf{r}',t'))dW(\mathbf{r}')\,dt', \tag{56}$$

while

$$B(\mathbf{r},t;\mathbf{v}_g)=\int_0^t\iiint_W G(\mathbf{r},\mathbf{r}',t,t')[\psi(\mathbf{r}',t';\mathbf{v}_g(\mathbf{r}',t'))+\Psi(\mathbf{r}')]\,dW(\mathbf{r}')\,dt',\ \text{ where } \Delta\Psi(\mathbf{r})=0, \tag{57}$$

which are integral representations of the unknown velocity $\mathbf{v}_g(\mathbf{r},t)$.

Now, we are ready to obtain the set $(\mathbf{v}_g(\mathbf{r},t),P_g(\mathbf{r},t))$ of flow fields from our integro–differential representation (6), considering (56) and (57) with (55), given all information in between (45)–(54). For the velocity (see proof steps of the main theorem for more technical details of vector analysis) we obtain

$$\begin{aligned}\mathbf{v}_g(\mathbf{r},t) &= \nabla\times(\mathbf{r}A(\mathbf{r},t;\mathbf{v}_g))+\nabla\times\nabla\times(\mathbf{r}B(\mathbf{r},t;\mathbf{v}_g))\\ &= \nabla A(\mathbf{r},t;\mathbf{v}_g)\times\mathbf{r}+\nabla\nabla\cdot(\mathbf{r}B(\mathbf{r},t;\mathbf{v}_g))-\Delta(\mathbf{r}B(\mathbf{r},t;\mathbf{v}_g))\\ &= -\mathbf{r}\times\nabla A(\mathbf{r},t;\mathbf{v}_g)+\nabla[B(\mathbf{r},t;\mathbf{v}_g)+\mathbf{r}\cdot\nabla B(\mathbf{r},t;\mathbf{v}_g)]-\mathbf{r}\Delta B(\mathbf{r},t;\mathbf{v}_g)\\ &= -\mathbf{r}\times\nabla A(\mathbf{r},t;\mathbf{v}_g)+2\nabla B(\mathbf{r},t;\mathbf{v}_g)+\mathbf{r}\cdot\nabla\otimes\nabla B(\mathbf{r},t;\mathbf{v}_g)-\mathbf{r}\Delta B(\mathbf{r},t;\mathbf{v}_g)\end{aligned} \tag{58}$$

and for the corresponding total pressure

$$P_g(\mathbf{r},t)=P_0-P_f(t)-\eta(1+\delta(t))[\varphi(\mathbf{r},t;\mathbf{v}_g)-\Psi(\mathbf{r})-\mathbf{r}\cdot\nabla\Psi(\mathbf{r})], \tag{59}$$

where all functions have been defined. Our final task is to put all these analytical tools together into the primary relation for the flow fields (43) and satisfy all conditions, bearing in mind the Poiseuille flow (41). Hence, substituting (56) and (57) into (58) we arrive at the velocity field of the fluid

$$\begin{aligned}\mathbf{v}(\mathbf{r},t)=\mathbf{v}_{\mathrm{P}}(\rho)-\int_0^t\iiint_W\Big\{&\chi(\mathbf{r}',t';\mathbf{v}(\mathbf{r}',t')-\mathbf{v}_{\mathrm{P}}(\rho'))\mathbf{r}\times\nabla G(\mathbf{r},\mathbf{r}',t,t')\\ &+[\psi(\mathbf{r}',t';\mathbf{v}(\mathbf{r}',t')-\mathbf{v}_{\mathrm{P}}(\rho'))+\Psi(\mathbf{r}')+\Psi(\mathbf{r}')](2\nabla+\mathbf{r}\cdot\nabla\otimes\nabla-\mathbf{r}\Delta)G(\mathbf{r},\mathbf{r}',t,t')\Big\}\,dW(\mathbf{r}')\,dt',\end{aligned} \tag{60}$$

whilst recalling (50), the total pressure field (59) of the solution assumes

$$\begin{aligned}P(\mathbf{r},t) &= P_0+P_{\mathrm{P}}(z)-P_f(t)\\ &-\eta(1+\delta(t))\left\{\iiint_W E(\mathbf{r},\mathbf{r}')[\nabla_{\mathbf{r}'}\cdot\mathbf{f}(\mathbf{r}',t;\mathbf{v}(\mathbf{r}',t')-\mathbf{v}_{\mathrm{P}}(\rho'))]\,dW(\mathbf{r}')-(1+\mathbf{r}\cdot\nabla)\Psi(\mathbf{r})\right\},\end{aligned} \tag{61}$$

where we may assimilate the two constant pressures appearing in (41) and (59) in a single reference pressure $P_c\equiv P_{\mathrm{P},0}+P_0$, while both expressions (60) and (61) are defined for any $\mathbf{r}\in W$ and $t>0$.

In order to complete our solution we are obliged to satisfy all conditions inferred. An immediate consequence of the kind of integral involved into (60) with respect to time, is that for $t=0$ it vanishes, hence the initial condition reading (42) is automatically satisfied. Proceeding to the boundary conditions now, we observe that as $z\to\pm\infty$. the fundamental form $G(\mathbf{r},\mathbf{r}',t,t')$ goes to zero very fast, as an immediate result from the exponential function. This behavior also does not change at all for the factors

$\mathbf{r}\times\nabla G(\mathbf{r},\mathbf{r}',t,t')$ and $(2\nabla+\mathbf{r}\cdot\nabla\otimes\nabla-\mathbf{r}\Delta)G(\mathbf{r},\mathbf{r}',t,t)$ into the velocity (60), which are set to zero as we move far away from the critical domain of magnetic reaction. Then, the integral within (60) vanishes and we recover the limiting condition (40), which is exactly the same result obtained via (44). On the contrary, things are not so easy when we deal with the boundary condition (39) on the walls of the cylinder. Employing this final condition to the velocity field (60) and accounting limiting condition (41), we get

$$\int_0^t \iiint_W \Big\{\chi(\mathbf{r}',t';\mathbf{v}(\mathbf{r}',t')-\mathbf{v}_{\mathrm{P}}(\rho'))\mathbf{r}\times\nabla G(\mathbf{r},\mathbf{r}',t,t') \\ +[\psi(\mathbf{r}',t';\mathbf{v}(\mathbf{r}',t')-\mathbf{v}_{\mathrm{P}}(\rho'))+\Psi(\mathbf{r}')](2\nabla+\mathbf{r}\cdot\nabla\otimes\nabla-\mathbf{r}\Delta)G(\mathbf{r},\mathbf{r}',t,t')\Big\}dW(\mathbf{r}')\,dt'=\mathbf{0} \tag{62}$$

at $\rho=\alpha$ with every $\varphi\in[0,2\pi), z\in(-\infty,+\infty)$ and $t>0$, which comprises the three different integral relations, which are needed to calculate the unknown constant coefficients and the Bessel parameter, appearing into the expansion (51), in a semi–analytical fashion. Then, the resulting integral equations can be solved with standard methods such as the Nyström or Neumann series method. However, since this is a very cumbersome task with many analytical manipulations, it stands beyond the purpose of the present article.

An alternative approach (we just explain the algorithm) would be the pure numerical implementation of the flow as it is, regarding relation (60) with condition (62) and in view of the harmonic–type function $\Psi(\mathbf{r})$, where an iterative procedure for every $k=0,1,2,\ldots$ must be followed for the velocity field $\mathbf{v}^{(k)}(\mathbf{r},t)$, which is based on a first suitable ansatz, where a preferable estimation is the fully developed 1–D parabolic profile of the classical Poiseuille flow (41), i.e. $\mathbf{v}^{(0)}(\mathbf{r},t)=\mathbf{v}_{\mathrm{P}}(\rho)$. The total pressure (61) is not involved with this method, since the primary function (9) that embodies the fields under consideration, contains only the velocity field for this particular application (see for example (47)). Sketching the basic steps, one may solve numerically the Laplace's equation $\Delta\Psi(\mathbf{r})=0$, using any of the classical ways in literature, e.g. the Gauss–Seidel method and applying boundary condition (62) for evaluating numerically $\Psi(\alpha,\varphi,z)$ for every $\varphi\in[0,2\pi)$ and $z\in(-\infty,+\infty)$. The convergence criterion for this internal iterative solution is

$$\left|\frac{\Psi^{(t+1)}(\mathbf{r})-\Psi^{(l)}(\mathbf{r})}{\Psi^{(l+1)}(\mathbf{r})}\right|_{L_2}<\varepsilon \text{ for every } \rho\in[0,\alpha),\varphi\in[0,2\pi) \text{ and } z\in(-\infty,+\infty), \tag{63}$$

where l is the number of the internal iteration, $|\cdot|_{L_2}$ is the Euclidean norm and $\varepsilon\ll 1$ is a very small number, required for achieving convergence, whose value depends on the method. When a converged solution for the internal procedure has been obtained, potential $\Psi(\mathbf{r})$ is substituted into relationship (60), where using required information through (45)–(54) and for $\mathbf{v}^{(0)}(\mathbf{r},t)=\mathbf{v}_{\mathrm{P}}(\rho)$, the new estimation for the velocity $\mathbf{v}^{(1)}(\mathbf{r},t)$ can be computed. This completes one iteration of the solution procedure, which continues until convergence is reached. The convergence criterion for the overall procedure is therein

$$\left|\frac{\mathbf{v}^{(k+1)}(\mathbf{r},t)-\mathbf{v}^{(k)}(\mathbf{r},t)}{\mathbf{v}^{(k+1)}(\mathbf{r},t)}\right|_{L_2}<\varepsilon \text{ for every } \rho\in[0,\alpha),\varphi\in[0,2\pi),z\in(-\infty,+\infty) \text{ and } t>0, \tag{64}$$

where k denotes the iteration number. When the numerical repeating procedure shows convergence for the velocity field (60), the corresponding functions, attributed to the particular number of iteration, are then inserted into the total pressure field (61) in order to provide us with the solution.

Under the aim to partly validate the aforementioned numerical procedure, we specialize the type of the conducting ferrofluid of our application to be comprised by ferromagnetic particles of iron oxides embedded in the carrier fluid of blood (organic solvent), so as to compare the results with those from reference [12], wherein a very similar demonstration has been presented for steady state situations. To do that, we consider in our case a very large time–scale, approximating $t\to+\infty$, whereas the applied magnetic field (45), in view of (3) and (4), becomes independent of time, that is,

$$\lim_{t\to+\infty}\mathbf{H}_c(t)=\mathbf{H}_{c,\infty}=\frac{\mathbf{B}_{c,\infty}}{\mu_\infty} \text{ with measure } H_{c,\infty}=\frac{B_{c,\infty}}{\mu_\infty} \tag{65}$$

where

$$\mu_\infty=\mu_0\left(1+\frac{M_{0,c,\infty}}{H_{c,\infty}}\right) \text{ with } M_{0,c,\infty}=\frac{3m\phi}{4\pi r_p^3}\left(\coth\frac{m\mu_0 H_{c,\infty}}{KT}-\frac{KT}{m\mu_0 H_{c,\infty}}\right), \tag{66}$$

approaching the case of the applied field used in [12]. The comparison between the two cases is shown in Figure 1 depicting the radial variation of the axial velocity $\hat{\boldsymbol{z}}\cdot\lim_{t\to+\infty}\mathbf{v}(\mathbf{r},t)\equiv\hat{\boldsymbol{z}}\cdot\mathbf{v}_\infty(\mathbf{r})=v_{z,\infty}(\rho,\varphi,z)$ for every $\rho\in[0,\alpha)$, $\varphi\in[0,2\pi)$ and $z\in(-\infty,+\infty)$ from (60) with the corresponding velocity component from reference [12] for various values of the applied magnetic induction field $B_{c,\infty}$ in (65) (actually, such fields are practically measured) that range between $B_{c,\infty}=0.0\,$Tesla and $B_{c,\infty}=0.1\,$Tesla.

For completeness, we share here the same implicated properties and conditions with those presented in [12], which were used for our simulation purposes. Hence, we consider a constant room temperature $T=311K$, while the blood–fluid stands for the conducting liquid of conductivity $\sigma=0.8$S/m, density $\rho_b=1050$kg/m^3 and dynamic viscosity $\eta\cong\eta_0=3.2\times10^{-3}$kg/ms. On the other hand, the iron oxide–type ferromagnetic particles have density $\rho_p=5240$kg/m^3 and radius $r_p=10^{-8}$m, occupying a volumetric percentage $\phi=10\%$ of the mixed–fluid, so that the assumptions, made for the derivation of Stokes equations are consistent. Then, the micropolar mixed–fluid density is given proportionally by $\rho=0.9\rho_p=1469$kg/m^3, the kinematic viscosity being $\nu=2.178\times10^{-6}$m^2/s, while the magnetic moment of each particle is $m=2\times10^{-18}$Am^2, the magnetic permeability of the free space (classical vacuum) is $\mu_0=4\pi\times10^{-7}$N/A^2 and the Boltzmann's constant is $K=1.3807\times10^{-23}$J/K. Proceeding to the geometrical characteristics that affect the magnetic flow, the circular cylindrical duct is assumed to have a radius of $\alpha=10^{-3}$m to simulate a blood artery and its length ℓ is adequate enough to provide a fully developed flow. Therefore, by definition of the characteristic velocity U as the mean axial velocity $\overline{v}_z=U=0.125\times10^{-3}$m/s and in terms of the characteristic

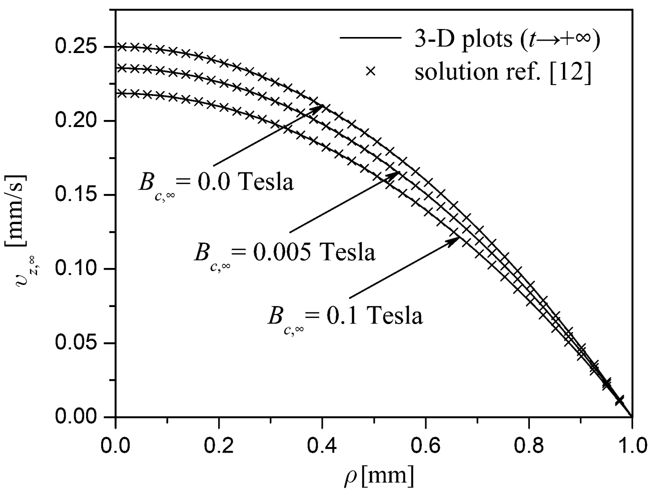

Fig. 1. Axial velocity distribution along the radial direction of the duct for various magnetic field strengths as $t \to +\infty$, within relation (60). Comparison with the results from reference [12].

diameter $D = 2\alpha = 2 \times 10^{-3}$m, the Reynolds number is then $\mathrm{Re} = \frac{\rho U D}{\eta} = 0.115$, which confines the boundaries of the creeping flow and this result complies with the physics we describe in our theory and this application. The final results depicted in Figure 1, reveal complete accordance with those forms [12], even though we have used the limiting case of infinite time observation, providing though a reliable tool of validating our theory via the present application.

Concluding, the velocity and the total pressure fields (60) and (61) of the conducting ferrofluid under the uniform effect of a known and only time–dependent magnetic field, offer the semi–analytical form of a mixed solution, given in terms of integral representations in the three–dimensional space (3–D). For the reduced case in which either there is no magnetic field ($\mathbf{H} = \mathbf{0}$) or the magnetic particles are absent ($\phi = 0$) and we deal with a non–conducting fluid ($\sigma = 0$), the fluid fields (60) and (61) obey the unsteady Stokes law situation in hydrodynamics, whereas the corresponding results [4] are recovered after some trivial analytical reduction method.

5. Conclusions and discussion. In this paper, we examined analytically how a 3–D arbitrarily orientated magnetic field perturbs an unsteady three–dimensional creeping motion (Stokes flow) of a viscous incompressible micropolar ferrofluid of generally non–zero electrical conductivity, where the magnetization of the carrier liquid was approximated by its equilibrium expression, while Lorentz forces were also counted.

We employed a semi–analytical method to evaluate the velocity and the total pressure fields for such flows. This technique was drawn from the classical potential representation theory for hydrodynamic flows and was extended properly to our case by constructing a novel and complete integro–differential representation of magnetic Stokes flow of conducting ferrofluids, valid for any non–axisymmetric geometry and provides in an analytical fashion the flow fields in terms of easy–to–find harmonic–type and diffusive–type potentials. The general representation degenerates to the well–known differential solutions for Stokes flow in the time–dependent regime, met in the literature.

We demonstrated the usefulness and the applicability of our integro–differential general solution by considering a particular physical boundary value problem of the Stokes flow of a micropolar fluid with electrically conducting properties, moving inside a circular tube under the influence of an otherwise space constant but time–dependent magnetic field. We computed the velocity and the total pressure fields in a closed analytical form for the special case of a 3–D creeping flow, proceeding to an adequate presentation via simple integral expressions. Future work involves intensive numerical implementation of applications and meddling with more complicated geometries, where cumbersome manipulations are required.

References

[1] B. Berkovski and V. Bashtovoy, *Magnetic Fluids and Applications Handbook*, Begell House, New York, 1996.

[2] V. G. Bashotovoy, B. M. Berkovski and A. N. Vislovich, *Introduction to Thermomechanics of Magnetic Fluids*, Hemisphere Publishing Corporation, New York, 1988.

[3] R.E. Rosensweig, *Ferrohydrodynamics*, Dover Publications, New York, 1997.

[4] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Prentice Hall: Englewood Cliffs, NJ, 1965 and Martinus Nijholl Publishers: Dordrecht, 1986.

[5] P. M. Hatzikonstantinou and P. Vafeas, *A general theoretical model for the magnetohydrodynamic flow of micropolar magnetic fluids. Application to Stokes flow*, Math. Methods Appl. Sci. **33** (2010), no. 2, 233–248, DOI 10.1002/mma.1170. MR2597203

[6] P. Chadwick and E. A. Trowbridge, *Elastic wave fields generated by scalar wave functions*, Proc. Cambridge Philos. Soc. **63** (1967), 1177–1187. MR0218047

[7] Xin Sheng Xu and Min Zhong Wang, *General complete solutions of the equations of spatial and axisymmetric Stokes flow*, Quart. J. Mech. Appl. Math. **44** (1991), no. 4, 537–548, DOI 10.1093/qjmam/44.4.537. MR1144982

[8] D. Palaniappan, S. D. Nigam, T. Amaranath, and R. Usha, *Lamb's solution of Stokes's equations: a sphere theorem*, Quart. J. Mech. Appl. Math. **45** (1992), no. 1, 47–56, DOI 10.1093/qjmam/45.1.47. MR1154762

[9] B. S. Padmavathi, G. P. Raja Sekhar, and T. Amaranath, *A note on complete general solutions of Stokes equations*, Quart. J. Mech. Appl. Math. **51** (1998), no. 3, 383–388, DOI 10.1093/qjmam/51.3.383. MR1639020

[10] G. D. McBain, *Convection in a horizontally heated sphere*, J. Fluid Mech. **438** (2001), 1–10, DOI 10.1017/S0022112001003913. MR1849873

[11] B. Sri Padmavati and T. Amaranath, *A note on decomposition of solenoidal fields*, Appl. Math. Lett. **15** (2002), no. 7, 803–805, DOI 10.1016/S0893-9659(02)00045-9. MR1920978

[12] Panayiotis Vafeas, Polycarpos K. Papadopoulos, and Pavlos M. Hatzikonstantinou, *Analytical integro-differential representation of flow fields for the micropolar Stokes flow of a conducting ferrofluid*, IMA J. Appl. Math. **80** (2015), no. 3, 839–864, DOI 10.1093/imamat/hxu016. MR3394306

[13] A. Venkatlaxmi, B. S. Padmavathi, and T. Amaranath, *Unsteady Stokes equations: some complete general solutions*, Proc. Indian Acad. Sci. Math. Sci. **114** (2004), no. 2, 203–213, DOI 10.1007/BF02829854. MR2062400

[14] D. Palaniappan, *On some general solutions of transient Stokes and Brinkman equations*, Journal of Theoretical and Applied Mechanics **52** (2014), 405–415.

[15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics,* Volumes I and II, McGraw–Hill, New York, 1953.

[16] E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Chelsea Publishing Company, New York, 1955. MR0064922

[17] L. M. Delves and J. Walsh, *Numerical Solutions of Integral Equations*, Clarendon, Oxford, 1974.