

OSCILLATORY TRAVELING WAVE SOLUTIONS FOR COAGULATION EQUATIONS

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Abstract. We consider Smoluchowski's coagulation equation with kernels of homogeneity one of the form $K_\varepsilon(\xi, \eta) = (\xi^{1-\varepsilon} + \eta^{1-\varepsilon})(\xi\eta)^{\frac{\varepsilon}{2}}$. Heuristically, in suitable exponential variables, one can argue that in this case the long-time behaviour of solutions is similar to the inviscid Burgers equation and that for Riemann data solutions converge to a traveling wave for large times. Numerical simulations in a work by Herrmann and the authors indeed support this conjecture, but also reveal that the traveling waves are oscillatory and the oscillations become stronger with smaller ε . The goal of this paper is to construct such oscillatory traveling wave solutions and provide details of their shape via formal matched asymptotic expansions.

1. Introduction.

1.1. *The coagulation equation.* The classical Smoluchowski coagulation equation is given by

$$\partial_t f(t, \xi) = \frac{1}{2} \int_0^\xi K(\xi-\eta, \eta) f(t, \xi-\eta) f(t, \eta) d\eta - f(t, \xi) \int_0^\infty K(\xi, \eta) f(t, \eta) d\eta, \quad (1.1)$$

where $f(t, \xi)$ denotes the number density of clusters of size $\xi \in (0, \infty)$ at time $t > 0$ and it describes the formation of larger clusters by binary coagulation of smaller ones. The rate kernel K is a nonnegative, symmetric function that contains all the information about the microscopic details of the coagulation process. For example, in [12] Smoluchowski derived the kernel $K(\xi, \eta) = (\xi^{1/3} + \eta^{1/3})(\xi^{-1/3} + \eta^{-1/3})$ for clusters that move according

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to Brownian motion and merge if they come close to each other. Various other kernels from different application areas can be found in the survey articles [6], [1] and [3].

Most kernels that one encounters in applications are homogeneous and we will assume this from now on. It is well known that for kernels of homogeneity larger than one gelation occurs, that is, the loss of mass at finite time, while for kernels of homogeneity smaller than or equal to one, solutions conserve the mass if it is initially finite. A topic of particular interest in this latter case is whether the large-time behaviour is described by self-similar solutions. While this issue is well understood for the constant and the additive kernel [10], for other kernels only few results are available. In the case of kernels with homogeneity strictly smaller than one, existence results for self-similar solutions have been established for a large class of kernels [5], [4], while uniqueness and convergence to self-similar form has only recently been proved for some special cases [11], [8].

1.2. *Kernels with homogeneity one.* In this article we are going to consider kernels with homogeneity one. Van Dongen and Ernst [13] already noticed that one needs to distinguish two cases. In the first case, called class II kernels in [13], one has $K(\xi, 1) \rightarrow c_0 > 0$ as $\xi \rightarrow 0$. The most prominent example is the additive kernel $K(\xi, \eta) = \xi + \eta$, where it is possible to solve the equation with the Laplace transform. In fact, for the additive kernel there exists a whole family of self-similar solutions with finite mass [10]. One of them has exponential decay, the others decay like a power law in a way such that the second moment is infinite. Recently, as the first mathematical result for nonsolvable kernels with homogeneity one, existence of self-similar solutions for a range of class II kernels has been obtained in [2].

On the other hand, kernels that satisfy $\lim_{\xi \rightarrow 0} K(\xi, 1) = 0$, are called class-I kernels in [13] and we also sometimes call them *diagonally dominant*. In this case it is known that self-similar solutions with finite mass cannot exist, but a suitable change of variables reveals that one can expect that the long-time behaviour of solutions is similar as in the case of the inviscid Burgers equation. To explain the idea it is useful to rewrite (1.1) in conservative form, that is, as

$$\partial_t(\xi f) + \partial_\xi \left(\int_0^\xi \int_{\xi-\eta}^\infty K(\eta, \rho) \eta f(\eta) f(\rho) d\rho d\eta \right) = 0. \quad (1.2)$$

Then we make the change of variables

$$\xi = e^x \quad \text{and} \quad u(t, x) = \xi^2 f(t, \xi) \quad (1.3)$$

such that (1.2) becomes

$$\partial_t u(t, x) = -\partial_x \left(\int_{-\infty}^x dy \int_{x+\ln(1-e^{y-x})}^\infty dz K(e^{y-z}, 1) u(t, y) u(t, z) \right). \quad (1.4)$$

Notice also that $\int_\infty \xi f(t, \xi) d\xi = \int_{\mathbb{R}} u(t, x) dx$. Hence, mass conserving solutions to (1.1) correspond to nonnegative integrable solutions of (1.4) with conserved L^1 -norm.

As in [7], we consider the rescaled function $u_\varepsilon(\tau, \tilde{x}) = \frac{1}{\varepsilon} u(\frac{\tau}{\varepsilon^2}, \frac{\tilde{x}}{\varepsilon})$, where $0 < \varepsilon \ll 1$, to understand the large-time behaviour of solutions to (1.4). We find

$$\begin{aligned} \partial_\tau u_\varepsilon(\tau, \tilde{x}) &= -\partial_{\tilde{x}} \left(\frac{1}{\varepsilon^2} \int_{-\infty}^{\tilde{x}} \int_{\tilde{x}+\varepsilon \ln(1-e^{-\frac{y-\tilde{x}}{\varepsilon}})}^{\infty} K(e^{\frac{y-z}{\varepsilon}}, 1) u_\varepsilon(\tau, y) u_\varepsilon(\tau, z) dz dy \right) \\ &= -\partial_{\tilde{x}} \left(\frac{1}{\varepsilon^2} \int_{-\infty}^0 \int_{\varepsilon \ln(1-e^{-\frac{y}{\varepsilon}})}^{\infty} K(e^{\frac{y-z}{\varepsilon}}, 1) u_\varepsilon(\tau, \tilde{x}+y) u_\varepsilon(\tau, \tilde{x}+z) dz dy \right) \\ &\approx -c_0 \partial_{\tilde{x}} (u_\varepsilon(\tau, \tilde{x})^2) \end{aligned} \quad (1.5)$$

with $c_0 = \int_{-\infty}^0 \int_{\ln(1-e^y)}^{\infty} K(e^{y-z}, 1) dy dz < \infty$. Hence, we conclude that u_ε approximately solves the Burgers equation. Notice that in the second step in (1.5) we have used the translation invariance of the integral which is specific to the property that the kernel has homogeneity one. Recall that for integrable nonnegative data with mass M the solutions to the Burgers equation $\partial_t u + \partial_x(u^2) = 0$ converge in the long-time limit to an N -wave with the same mass, i.e. $u(t, x) \sim \frac{1}{\sqrt{t}} N(x/\sqrt{t})$ with $N(x) = \frac{x}{2} \chi_{[0, 2\sqrt{M}]}$. On the other hand, if one starts with data such that $u_0(x) \rightarrow b > 0$ as $x \rightarrow -\infty$ the solution converges to a traveling wave with height b and speed related to b .

In order to investigate, whether the long-time behaviour of solutions to (1.4) is indeed the same as for the Burgers equation we performed in [7] numerical simulations of (1.4) for a family of kernels

$$K_\alpha(\xi, \eta) = c_\alpha (\xi + \eta)^{1-2\alpha} \xi^\alpha \eta^\alpha, \quad \alpha > 0, \quad (1.6)$$

with a suitable normalization constant $c_\alpha > 0$. Figure 1 shows the simulations for several values of α for smooth initial data with compact support.

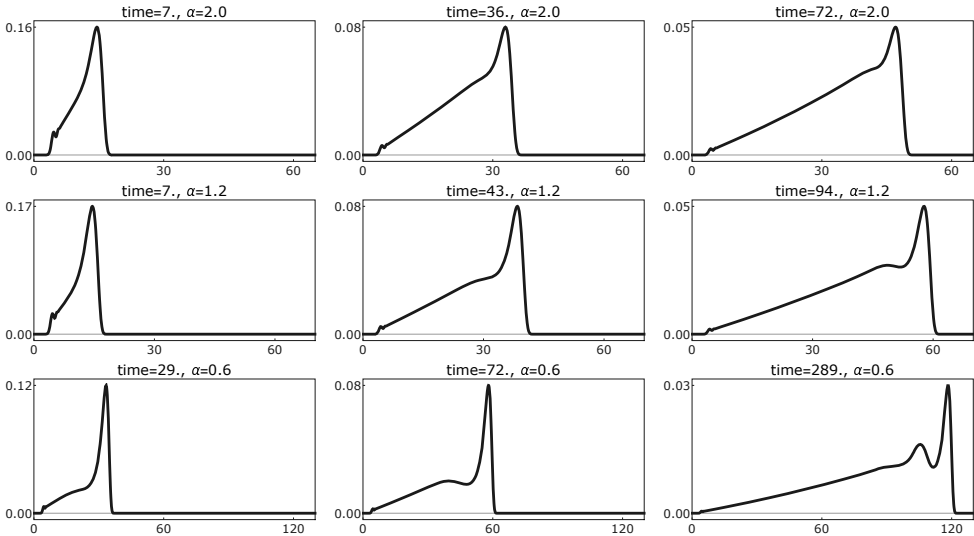


FIG. 1. Convergence to the N-wave for $\alpha = 2$ (top), $\alpha = 1.2$ (middle) and $\alpha = 0.6$ (bottom).

We see that solutions indeed converge to an N -wave in the long-time limit, but that the transition at the shock front is oscillatory, with oscillations becoming stronger when α becomes smaller. One expects that the transition at the shock front is given by a rescaled traveling wave profile and simulations of (1.4) for Riemann data (see Figure 2) indeed confirm that the traveling wave profiles are oscillatory for small α .

It is the purpose of this paper to establish the existence of such a traveling wave and compute details of its shape via formal matched asymptotic expansions. We will do this for kernels of the form

$$K_\varepsilon(\xi, \eta) = (\xi^{1-\varepsilon} + \eta^{1-\varepsilon})(\xi\eta)^{\frac{\varepsilon}{2}}, \tag{1.7}$$

where $\varepsilon > 0$ is a small parameter. We choose to consider kernels of the form (1.7) since the computations become slightly simpler. For small ε and α we do not expect that the solutions for kernels as in (1.6) and (1.7) respectively behave very differently.

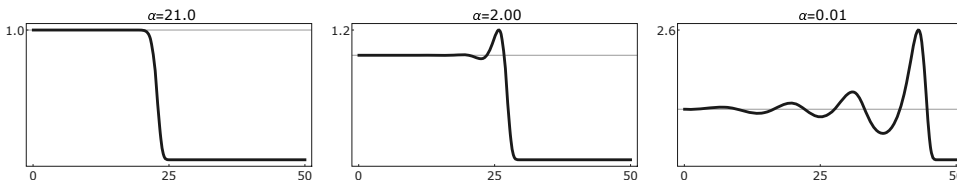


FIG. 2. The shape of the traveling wave for different values of α (against x).

Before we proceed to an informal description of construction of oscillatory traveling waves, we also mention that we discuss in [7] in detail also the situation for large α . While for sufficiently large α , the traveling wave appears to be monotone, in this case the constant solution is unstable which leads to additional phenomena; we refer to [7] for details.

1.3. *Traveling waves for small ε .* The goal of this paper is to construct traveling wave solutions of (1.4). The ansatz $u(t, x) = bG(x - bt)$ indeed leads to the equation

$$G(x) = \int_{-\infty}^x dy \int_{x+\ln(1-e^{y-x})}^{\infty} dz K(e^{y-z}, 1) G(y) G(z) \tag{1.8}$$

for the profile G with the boundary condition $G(x) \rightarrow 0$ as $x \rightarrow \infty$. Such solutions $G(x)$ of (1.8) correspond to special solutions f of (1.2) of the form $\xi^2 f(t, \xi) = bG(\ln \xi - bt)$, hence to traveling waves in the variable $\ln \xi$.

1.3.1. *Basic considerations.* We recall that for $\varepsilon = 0$ in (1.7) we obtain the additive kernel, for which a whole family of mass conserving self-similar solutions exist. One of them with exponential decay in the self-similar variable $\zeta = e^{-bt}\xi$, the other ones with power law decay $\zeta^{-(2+\rho)}$ as $x \rightarrow \infty$ for $\rho \in (0, 1)$ [10]. In the variables (1.3) this means that $G(x) \sim e^{-\rho x}$ as $x \rightarrow \infty$. We also have that $G(x) \rightarrow 0$ as $x \rightarrow -\infty$ with the asymptotics $e^{-\frac{\rho}{1+\rho}x}$.

In contrast, solutions to (1.8) with diagonally dominant kernel must satisfy

$$\lim_{x \rightarrow -\infty} G(x) = G_\infty > 0,$$

hence, the integral of G cannot be finite. This is the reason that self-similar solutions with finite mass cannot exist in this case. Nevertheless, nonnegative solutions to (1.8) in the case of diagonally dominant kernels are relevant as well, since we expect that they describe the long-time behaviour of solutions if one starts with Riemann data in the variable x , and they describe the transition at the shock front of the N -wave that appears if the initial data are integrable. Notice also, that by varying the parameter b , we can change the step height of the traveling wave. For the following analysis it is convenient to normalize $b = 1$. The numerical simulations on the other hand were carried out by normalizing G such that $G(x) \rightarrow 1$ as $x \rightarrow -\infty$.

Our goal in this paper is to study how solutions G to (1.8) behave for kernels as in (1.7) in the regime $\varepsilon \ll 1$.

1.3.2. *Construction of a traveling wave by shooting.* We now describe in an informal way the main strategy of our construction of a traveling wave solution.

Shooting from $-\infty$. We try to obtain the solution by a shooting argument starting at $x \rightarrow -\infty$. In Sections 2.1-2.3 we obtain that as $x \rightarrow -\infty$ a solution G of (1.8) is oscillatory with

$$G(x) \sim G_\infty + \operatorname{Re}\left(Le^{i\varphi}e^{\mu^+x}\right) + \Psi(x; L, \varphi) + C_*e^{\mu_*x} \quad \text{as } x \rightarrow -\infty \quad (1.9)$$

and

$$G_\infty \approx \frac{\varepsilon^2}{4}, \quad \mu^+ \approx \frac{\varepsilon^2}{8} + i\frac{\varepsilon}{2} \quad \text{and} \quad \mu_* \approx 1 - \frac{\varepsilon}{2} \quad \text{as } \varepsilon \rightarrow 0.$$

The constants $L > 0, \varphi \in [0, 2\pi)$ and $C_* \in \mathbb{R}$ can be chosen freely, while the function $\Psi(X; L, \varphi)$ yields the contribution of the nonlinear terms induced by the term $\operatorname{Re}(Le^{i\varphi}e^{\mu^+x})$ which are larger than the contribution due to the term $C_*e^{\mu_*x}$. As we will explain in detail in the end of Section 2.3 we can restrict ourselves, due to the translation invariance of (1.8), to $\varphi = 0$ and $L \in [1, e^{\frac{\pi\varepsilon}{2}})$. Then (1.9) still describes a two-dimensional manifold. It turns out that the solution that we are seeking lies close to the submanifold given by $C_* = 0$. The reason is that otherwise instabilities kick in, the solution becomes negative and is not a nonnegative solution of (1.8) with the correct behaviour as $x \rightarrow \infty$.

We remark here that our strategy is similar to our construction of self-similar solutions to the coagulation equation with kernel $K(\xi, \eta) = (\xi\eta)^\lambda$ with $\lambda \in (0, 1/2)$ in [9]. Also in this case the solution develops oscillations that become more extreme with smaller λ . However, the details are somewhat different and in particular, there we do not have the additional term $C_*e^{\mu_*x}$ in (1.9), but instead can do the shooting in a one-dimensional manifold.

Reformulation as a Volterra-type system. For simplicity of the argument assume for the moment that the terms with C_* can be neglected and that we can use L as a single shooting parameter. The main technical difficulty in the analysis of equation (1.8) is to control the nonlocal integral terms. A key idea in our approach is to reformulate equation (1.8) as a Volterra-type system of equations for which the solution at ‘time’ x depends only on values $y \leq x$. Of course, not every solution to this Volterra-type system corresponds to a nonnegative solution of (1.8). We have to choose the shooting

parameter L in such a way such that the corresponding G remains positive and converges to zero as $x \rightarrow \infty$.

The next key insight is, that as long as $G \ll 1$ we can approximate the Volterra-type system by a nonlinear ODE system that is a perturbation of a Lotka-Volterra system; see Section 2.4. We use an adiabatic approximation to compute the increase of an associated energy E along the trajectories. This approximation is valid as long as $E \ll \frac{1}{\varepsilon}$. When $E \sim \frac{1}{\varepsilon}$ we enter what we call the intermediate regime; see Section 2.5. In this regime G develops on intervals of length of order $O(\ln \frac{1}{\varepsilon})$ peaks of height of order one that can be approximated by self-similar solutions to the coagulation equation with additive kernel; see Section 2.5.2. These peaks are connected by regions of length $O(\varepsilon^{-2})$ in which G is of order $O(e^{-\frac{1}{\varepsilon}})$. In the regime where G is small we can again approximate by simple ODE systems. Still, we have to distinguish three different regimes that have to be coupled appropriately as described in Sections 2.5.4-2.5.7. A key result in order to justify the approximation by ODEs is Lemma 2.4 in Section 2.5.3 that allows us to control, if G is small and in a certain sense regular, the nonlocal terms by pointwise estimates.

For each $L \in [1, e^{\frac{\pi\varepsilon}{2}})$ we can construct in this way a solution to (1.8). However, in general they become negative. By a continuity argument, explained in Section 2.6, we show that there exists a parameter L such that the corresponding solution satisfies $G(x) \rightarrow 0$ as $x \rightarrow \infty$ with the asymptotics as given in (2.62) of Lemma 2.5.

Solution stays close to sub-manifold with $C_* = 0$. In Section 3 we then address the additional parameter C_* . We are going to show that C_* must be very small and carefully chosen in order to obtain a nonnegative solution to (1.8) that tends to zero as $x \rightarrow \infty$. More precisely, we consider the different regions explained above separately and show that if the term associated with C_* becomes of a size comparable to the solution of the Volterra-like problem, the corresponding solution will not be admissible as a solution to the coagulation equation.

General remark on techniques. We emphasize that the construction of our solution is based on a matched asymptotic expansion. However, since we deal with an integral equation we can in general not use standard tools for asymptotic problems in ODEs, but have to estimate carefully the nonlocal terms corresponding to the integrals.

1.3.3. *The main result.* For completeness we now briefly summarize the main properties of the solution that we construct. We obtain a solution G to (1.8) which satisfies

$$G(x) \sim \frac{\varepsilon^2}{4} + L_* e^{\frac{\varepsilon^2}{8}x} \cos\left(\frac{\varepsilon x}{2}\right) + h.o.t \quad \text{as } x \rightarrow -\infty, \quad (1.10)$$

where L_* is chosen such that G remains positive and satisfies $G(x) \rightarrow 0$ as $x \rightarrow \infty$.

The approximation (1.10) is valid, as long as an associated energy E (see (2.28)) satisfies $E \ll \frac{1}{\varepsilon}$. The intermediate regime starts when $E \sim \frac{1}{\varepsilon}$ and is characterized by well-separated peaks of increasing height. More precisely, G develops peaks of increasing height (but of order one in ε), with width of order $O(\ln \frac{1}{\varepsilon})$. These peaks can be described by fat-tailed solutions of the coagulation equation with additive kernel. The peaks are separated by regions of length $O(\frac{1}{\varepsilon})$ in which G is very small, that is, $O(e^{-\frac{1}{\varepsilon}})$ (see also Figure 3).

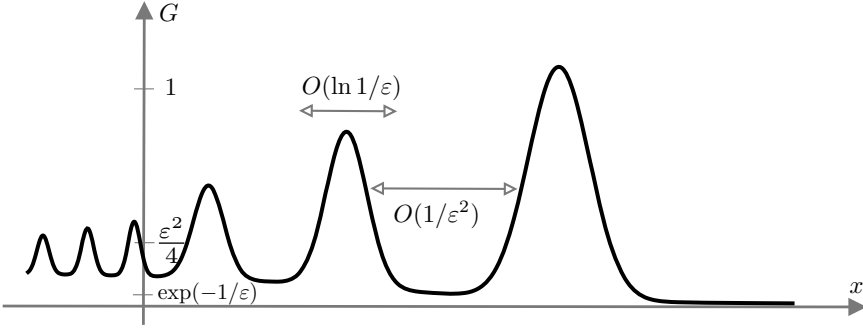


FIG. 3. Sketch of the traveling wave profile.

Finally, G enters the last regime where it converges to zero like a double exponential (see (2.62) for detailed asymptotics).

2. Construction of an oscillatory traveling wave.

2.1. *The behaviour for $x \rightarrow -\infty$.* As a first step we derive the asymptotics of a solution as $x \rightarrow -\infty$. For this we note

$$K_\varepsilon(e^y, 1) = (e^{(1-\varepsilon)y} + 1)e^{\frac{\varepsilon y}{2}} = e^{(1-\frac{\varepsilon}{2})y} + e^{\frac{\varepsilon y}{2}}$$

and it follows that $\int_{-\infty}^0 dy \int_{\ln(1-e^y)}^\infty K_\varepsilon(e^{y-z}, 1) dz < \infty$. We are interested in the asymptotics of the solution of (1.8) satisfying $\lim_{x \rightarrow -\infty} G(x) = G_\infty$ where G_∞ is given by

$$G_\infty \int_{-\infty}^0 dy \int_{\ln(1-e^y)}^\infty dz K_\varepsilon(e^{y-z}, 1) = 1. \tag{2.1}$$

A simple explicit computation gives $G_\infty = \frac{(1-\frac{\varepsilon}{2})\frac{\varepsilon}{2}}{\Gamma(1-\frac{\varepsilon}{2})\Gamma(\frac{\varepsilon}{2})}$ and using the asymptotics $\Gamma(z)z \rightarrow 1$ as $z \rightarrow 0$ we also find

$$G_\infty = \frac{\varepsilon^2}{4} \left(1 - \frac{\varepsilon}{2} + O(\varepsilon^2)\right) \quad \text{as } \varepsilon \rightarrow 0. \tag{2.2}$$

2.2. *Reformulation as a Volterra-like problem.* Suppose that $G(\cdot)$ is a solution of (1.8). We rewrite this equation as

$$\begin{aligned} G(x) = & \int_{-\infty}^0 dy \int_0^\infty dz \left(e^{(1-\frac{\varepsilon}{2})(y-z)} + e^{\frac{\varepsilon}{2}(y-z)} \right) G(y+x)G(z+x) \\ & + \int_{-\infty}^0 dy \int_{\ln(1-e^y)}^0 dz \left(e^{(1-\frac{\varepsilon}{2})(y-z)} + e^{\frac{\varepsilon}{2}(y-z)} \right) G(y+x)G(z+x) \end{aligned} \tag{2.3}$$

and define the following functions:

$$A(x) = \frac{1}{2} \int_{-\infty}^0 G(y+x)e^{\frac{\varepsilon}{2}y} dy, \quad B(x) = \frac{1}{2} \int_0^{\infty} e^{-\frac{\varepsilon}{2}z} G(z+x) dz, \quad (2.4)$$

$$P(x) = \int_{-\infty}^0 G(y+x)e^{(1-\frac{\varepsilon}{2})y} dy, \quad Q(x) = \int_0^{\infty} e^{-(1-\frac{\varepsilon}{2})z} G(z+x) dz, \quad (2.5)$$

$$J[G](x) = \int_{-\infty}^0 dY \int_{\ln(1-e^y)}^0 dz \left(e^{(1-\frac{\varepsilon}{2})(y-z)} + e^{\frac{\varepsilon}{2}(y-z)} \right) G(y+x)G(z+x).$$

Then

$$\frac{dA(x)}{dx} = -\frac{\varepsilon}{2}A(x) + \frac{1}{2}G(x), \quad \frac{dB(x)}{dx} = \frac{\varepsilon}{2}B(x) - \frac{1}{2}G(x), \quad (2.6)$$

$$\frac{dP(x)}{dx} = -\left(1 - \frac{\varepsilon}{2}\right)P(x) + G(x), \quad \frac{dQ(x)}{dx} = \left(1 - \frac{\varepsilon}{2}\right)Q(x) - G(x), \quad (2.7)$$

and rewrite (2.3) as

$$G(x) = 4A(x)B(x) + P(x)Q(x) + J[G](x). \quad (2.8)$$

Notice that $J[G](x)$ is determined by the values of $G(y)$ with $y \leq x$ and the functions A, B, P and Q solve the ODEs in (2.6) and (2.7). Hence, we will call (2.6)-(2.8) a Volterra-like problem since the values of the functions A, B, P, Q and G at x depend only on the values of these functions for $y \leq x$.

A solution of (2.6)-(2.8) is however not necessarily a solution of (1.8). More precisely, suppose that G is a global solution of (2.6)-(2.8). Then, the differential equations for B and Q in (2.6) and (2.7) suggest that the functions B and Q behave as $C_B e^{\frac{\varepsilon x}{2}}$ and $C_Q e^{(1-\frac{\varepsilon}{2})x}$ respectively as $x \rightarrow \infty$. However, the formulas for $B(x)$ and $Q(x)$ in (2.4) and (2.5) imply that these two functions are bounded for large x if G is bounded. This suggests that some suitable shooting parameters need to be adjusted in order to obtain bounded solutions of (2.6)-(2.8).

2.3. *Linearization of the Volterra-like problem as $x \rightarrow -\infty$.* We begin by constructing solutions of (2.6)-(2.8) for $x \leq -x_0$ with sufficiently large $x_0 > 0$. These solutions depend on two parameters and satisfy

$$G(x) \rightarrow G_\infty \sim \frac{\varepsilon^2}{4} \text{ as } x \rightarrow -\infty \quad (2.9)$$

where $G_\infty > 0$ is as in (2.1). Moreover, we also assume that

$$\begin{aligned} A(x) \rightarrow A_\infty &= \frac{G_\infty}{\varepsilon} \sim \frac{\varepsilon}{4}, & B(x) \rightarrow B_\infty &= \frac{G_\infty}{\varepsilon} \sim \frac{\varepsilon}{4}, \\ P(x) \rightarrow P_\infty &= \frac{2G_\infty}{2-\varepsilon} \sim \frac{\varepsilon^2}{4}, & Q(x) \rightarrow Q_\infty &= \frac{2G_\infty}{2-\varepsilon} \sim \frac{\varepsilon^2}{4}, \end{aligned} \quad (2.10)$$

as $x \rightarrow -\infty$. We now look for solutions that behave asymptotically as

$$(G, A, B, P, Q) = (G_\infty, A_\infty, B_\infty, P_\infty, Q_\infty) + \left(G_\infty^1, A_\infty^1, B_\infty^1, P_\infty^1, Q_\infty^1 \right) e^{\mu x} \quad \text{as } x \rightarrow -\infty.$$

We linearize in (2.6)-(2.8), to obtain after some rearrangements that

$$A_\infty^1 = \frac{1}{2} \frac{G_\infty^1}{\left(\mu + \frac{\varepsilon}{2}\right)}, \quad B_\infty^1 = \frac{1}{2} \frac{G_\infty^1}{\left(\frac{\varepsilon}{2} - \mu\right)}, \quad P_\infty^1 = \frac{G_\infty^1}{\left(\mu + 1 - \frac{\varepsilon}{2}\right)}, \quad Q_\infty^1 = \frac{G_\infty^1}{\left(1 - \frac{\varepsilon}{2} - \mu\right)},$$

and

$$G_\infty^1 = \frac{2G_\infty}{\varepsilon} \frac{G_\infty^1}{\left(\frac{\varepsilon}{2} - \mu\right)} + \frac{2G_\infty}{\varepsilon} \frac{G_\infty^1}{\left(\mu + \frac{\varepsilon}{2}\right)} + \frac{2G_\infty}{(2-\varepsilon)} \frac{G_\infty^1}{\left(1 - \frac{\varepsilon}{2} - \mu\right)} \\ + \frac{2G_\infty}{(2-\varepsilon)} \frac{G_\infty^1}{\left(\mu + 1 - \frac{\varepsilon}{2}\right)} + G_\infty G_\infty^1 J(\mu, \varepsilon)$$

with

$$J(\mu, \varepsilon) := \int_{-\infty}^0 dy \int_{\ln(1-e^y)}^0 dz \left(e^{(1-\frac{\varepsilon}{2})(y-z)} + e^{\frac{\varepsilon}{2}(y-z)} \right) \left(e^{\mu y} + e^{\mu z} \right). \quad (2.11)$$

We are looking for solutions with $\operatorname{Re}(\mu) \geq 0$. Then $G_\infty^1 \neq 0$ since otherwise $\operatorname{Re}(\mu) < 0$ or $A_\infty^1 = B_\infty^1 = P_\infty^1 = Q_\infty^1 = 0$. Then, if $G_\infty^1 \neq 0$ we obtain after some lengthy but elementary rearrangements and integrations by parts, that

$$\frac{\Gamma\left(1 - \frac{\varepsilon}{2}\right)\Gamma\left(\frac{\varepsilon}{2}\right)\Gamma(1 + \mu)}{\left(1 - \frac{\varepsilon}{2}\right)\frac{\varepsilon}{2}} = \Gamma\left(1 - \frac{\varepsilon}{2} + \mu\right)\Gamma\left(\frac{\varepsilon}{2}\right) \frac{(1 - \mu)}{\left(1 - \frac{\varepsilon}{2}\right)\left(\frac{\varepsilon}{2} - \mu\right)} \\ + \Gamma\left(\frac{\varepsilon}{2} + \mu\right)\Gamma\left(1 - \frac{\varepsilon}{2}\right) \frac{(1 - \mu)}{\frac{\varepsilon}{2}\left(1 - \frac{\varepsilon}{2} - \mu\right)}. \quad (2.12)$$

We state here a result about the roots of (2.12), whose proof is given in the Appendix.

LEMMA 2.1. Equation (2.12) has exactly three roots with $\operatorname{Re}(\mu) \geq 0$, denoted by μ^\pm and μ_* respectively. Their asymptotics are given by

$$\mu^\pm = \frac{\varepsilon^2}{8} \pm i\left(\frac{\varepsilon}{2} + \frac{\varepsilon^2}{8}\right) + O(\varepsilon^3) \quad \text{and} \quad \mu_* \simeq 1 - \frac{\varepsilon}{2} \quad \text{as } \varepsilon \rightarrow 0. \quad (2.13)$$

The linearization of the problem (2.6)-(2.8) suggests that there exists a three-dimensional family of solutions which indeed can be characterized by the asymptotics (1.9) with free constants $L > 0$, $\varphi \in [0, 2\pi)$ and $C_* \in \mathbb{R}$.

For any choice of φ , L and C_* the solution of (2.6)-(2.8) is uniquely determined. In order to find a traveling wave solution to our original problem, we need to find φ , L and C_* such that $B(x) \rightarrow 0$ and $Q(x) \rightarrow 0$ as $x \rightarrow \infty$. However, due to the translation invariance of (1.8) we have for any $a \in \mathbb{R}$ that $\tilde{G}_a(x) := G(x+a)$ gives the same solution up to translations, such that, denoting $\mu^+ = \mu_1 + i\mu_2$,

$$\tilde{G}_a(x) \sim G_\infty + \operatorname{Re}\left(Le^{\mu_1 a} e^{i(\varphi + \mu_2 a)} e^{\mu^+ x}\right) + \tilde{\psi}(x) + C_* e^{\mu_* a} e^{\mu_* x} \quad \text{as } x \rightarrow -\infty.$$

Hence, we can identify solutions of (2.6)-(2.8) up to translations with the set of real numbers contained between two consecutive intersections of the spiral $\{Le^{\mu^+ a} \mid a \in \mathbb{R}\}$ with the real axis. If $L = 1$, the next larger point on the real axis is $e^{\frac{2\pi\mu_1}{\mu_2}}$. Therefore we can assume that $\varphi = 0$ and $L \in [1, e^{\frac{2\pi\mu_1}{\mu_2}}) \approx [1, e^{\frac{\pi\varepsilon}{2}})$. Notice that if the value of L is modified from L to $Le^{\frac{2\pi\mu_1}{\mu_2}}$ the value of C_* is modified to $C_* e^{\frac{2\pi\mu_*}{\mu_2}}$.

Thus, finding a nonnegative solution $G(x)$ to the original problem (1.8) with $G(x) \rightarrow 0$ as $x \rightarrow \infty$ is equivalent to finding a solution to (2.6)-(2.8) with the asymptotics (1.9) with $\varphi = 0$, $L \in [1, e^{\frac{2\pi\mu_1}{\mu_2}})$ and $C_* \in \mathbb{R}$ such that $B(x), Q(x) \rightarrow 0$ as $x \rightarrow \infty$.

Notice that in the formula (1.9) the second term changes on scales of order $\frac{1}{\varepsilon}$, whereas the last term changes on scales of order one. This is a consequence of the fact that the

characteristic length scales in the equations for A and B in (2.6) are different from the ones for P and Q in (2.7). In particular, the equation for the variable Q indicates that Q separates in scales of order one from G . This has the consequence that C_* must be very small, because otherwise the fast growth of Q will produce a change of sign either for Q or B as we will see later. Even though C_* is very small it must be tuned in a careful way because otherwise the above-mentioned instability in (2.7) will cause the solution to change sign.

We will postpone the detailed analysis of the effect of the term $C_*e^{\mu_*x}$ to Section 3 and consider now solutions to (2.6) and (2.7) with $(1 - \frac{\varepsilon}{2})Q(x) \approx G(x)$. This reduces the two-dimensional shooting problem to a one-dimensional one with shooting parameter L .

2.4. *The Lotka-Volterra regime.* It is possible to derive an ODE approximation of (1.8) which is valid as long as G remains much smaller than one. Since G_∞ is of order ε^2 we introduce the following rescaling:

$$\begin{aligned} G &= G_\infty g, & A &= A_\infty a, & B &= B_\infty b, & P &= P_\infty p, & Q &= Q_\infty q, \\ x &= \frac{u}{\varepsilon}, & y &= \frac{v}{\varepsilon}, & z &= \frac{w}{\varepsilon}, \end{aligned} \quad (2.14)$$

where $A_\infty, B_\infty, P_\infty$ and Q_∞ are as in (2.10). Then (2.6)-(2.8) become

$$\frac{da}{du} = \frac{1}{2}(a + g), \quad \frac{db}{du} = \frac{1}{2}(b - g), \quad (2.15)$$

$$\frac{dp}{du} = \left(\frac{1}{\varepsilon} - \frac{1}{2}\right)(-p + g), \quad \frac{dq}{du} = \left(\frac{1}{\varepsilon} - \frac{1}{2}\right)(q - g), \quad (2.16)$$

and

$$g = \frac{4G_\infty}{\varepsilon^2}ab + \frac{4G_\infty}{(2 - \varepsilon)^2}pq + \frac{G_\infty}{\varepsilon^2}\bar{J}[g](u), \quad (2.17)$$

where

$$\bar{J}[g](u) = \int_{-\infty}^0 dv \int_{\varepsilon \ln(1 - e^{-\frac{v}{\varepsilon}})}^0 dw \left(e^{(\frac{1}{\varepsilon} - \frac{1}{2})(v-w)} + e^{\frac{1}{2}(v-w)} \right) g(v+u)g(w+u). \quad (2.18)$$

LEMMA 2.2. Under the assumption that $g \rightarrow 1$ as $u \rightarrow -\infty$ and

$$|a(u) - 1| + |b(u) - 1| + |p(u) - 1| + |q(u) - 1| \leq C\|g - 1\|_u, \quad \left| \frac{dg}{du} \right| \leq C\|g - 1\|_u, \quad (2.19)$$

where $\|f\|_u := \sup_{s \leq u} |f(s)|$, we can rewrite (2.17) as the system of ODEs (2.15) together with

$$g(u) = a(u)b(u) - \frac{\varepsilon}{2}(a(u)b(u) - 1) + \frac{\varepsilon}{2}(g(u)a(u) - 1) + R(u) \quad (2.20)$$

with $|R(u)| \leq C\varepsilon^2(1 + \|g\|_u)\|g - 1\|_u$.

Proof. To approximate g we use (2.4) and (2.1) in (2.17) to find

$$\begin{aligned} g(u) &= a(u)b(u) + \left(\frac{4G_\infty}{\varepsilon^2} - 1\right)(a(u)b(u) - 1) + \frac{4G_\infty}{(2 - \varepsilon)^2}(p(u)q(u) - 1) \\ &\quad + \frac{G_\infty}{\varepsilon^2} \int_{-\infty}^0 dv \int_{\varepsilon \ln(1 - e^{-\frac{v}{\varepsilon}})}^0 dw \left(e^{(\frac{1}{\varepsilon} - \frac{1}{2})(v-w)} + e^{\frac{1}{2}(v-w)} \right) (g(v+u)g(w+u) - 1) \\ &=: a(u)b(u) + (I) + (II) + (III)_a + (III)_b. \end{aligned}$$

We first note that

$$|g(v+u)g(w+u)-1| \leq C(1+\|g\|_u)\|g-1\|_u. \quad (2.21)$$

With the assumption (2.19) we have $|(II)| \leq C\varepsilon^2(1+\|g\|_u)\|g-1\|_u$. Furthermore, recalling (2.2), we have $\frac{G_\infty}{\varepsilon^2} - \frac{1}{4} = -\frac{\varepsilon}{8} + (\varepsilon^2)$, whence as a consequence we find

$$(I) = -\frac{\varepsilon}{2}(a(u)b(u)-1) + O(\varepsilon^2\|g-1\|_u(1+\|g\|_u)).$$

In addition we can estimate

$$\begin{aligned} |(III)_b| &\leq C(1+\|g\|_u)\|g-1\|_u \int_{-\infty}^0 dv \int_{\varepsilon \ln(1-e^{\frac{w}{\varepsilon}})}^0 dw e^{\frac{1}{2}(v-w)} \\ &= C\varepsilon^2(1+\|g\|_u)\|g-1\|_u \int_{-\infty}^0 dv \int_{\ln(1-e^v)}^0 dw e^{\frac{\varepsilon}{2}(v-w)} \\ &\leq C\varepsilon^2(1+\|g\|_u)\|g-1\|_u \end{aligned}$$

since $\int_{-\infty}^0 dv \int_{\ln(1-e^v)}^0 dw e^{\frac{\varepsilon}{2}(v-w)} = \int_0^1 dy y^{\frac{\varepsilon}{2}-1} \int_{1-y}^1 dz z^{-(1+\frac{\varepsilon}{2})} \leq C$.

We finally claim that

$$(III)_a = \frac{\varepsilon}{2}(ga-1) + O(\varepsilon^2(1+\|g\|_u)\|g-1\|_u). \quad (2.22)$$

Indeed, after changing the order of integration we obtain

$$(III)_a = \frac{G_\infty}{\varepsilon^2} \int_{-\infty}^0 dw \int_{\varepsilon \ln(1-e^{\frac{w}{\varepsilon}})}^0 dv e^{(\frac{1}{\varepsilon}-\frac{1}{2})(v-w)} (g(v+u)g(w+u)-1).$$

In the region $w \geq -\varepsilon$ we use (2.21) and the fact that $\int_{-\varepsilon}^0 dw \int_{\varepsilon \ln(1-e^{\frac{w}{\varepsilon}})}^0 dv e^{(\frac{1}{\varepsilon}-\frac{1}{2})(v-w)} \leq C\varepsilon^2$ which follows from scaling. Therefore we can replace the integral $\int_{-\infty}^0 dw$ by $\int_{-\varepsilon}^{-\varepsilon} dw$ introducing an error of order $C\varepsilon^2(1+\|g\|_u)\|g-1\|_u$.

Furthermore, in the remaining integral, we can, due to (2.2), replace the prefactor $\frac{G_\infty}{\varepsilon^2}$ by $\frac{1}{4}$ and this yields an error of order $C\varepsilon^2(1+\|g\|_u)\|g-1\|_u$ since $\int_{-\varepsilon}^{-\varepsilon} dw \int_{\varepsilon \ln(1-e^{w/\varepsilon})}^0 dv e^{(\frac{1}{\varepsilon}-\frac{1}{2})(v-w)} \leq C\varepsilon$.

In the remaining integral we can also replace $e^{(\frac{1}{\varepsilon}-\frac{1}{2})(v-w)}$ by $e^{-(\frac{1}{\varepsilon}-\frac{1}{2})w}$, using $-\varepsilon \leq v \leq 0$, Taylor expansion and the fact that $\int_{-\infty}^{-\varepsilon} \int_{\varepsilon \ln(1-e^{w/\varepsilon})}^0 |v| e^{-(\frac{1}{\varepsilon}-\frac{1}{2})w} \leq C\varepsilon^3$. This gives an additional error of the order $C\varepsilon^3(1+\|g\|_u)\|g-1\|_u$. Finally, due to (2.19) we have $|g(u+v)-g(u)| \leq C\|g-1\|_u|v|$, such that we can replace $g(u+v)$ by $g(u)$ introducing another error of the order $C\varepsilon^3(1+\|g\|_u)\|g-1\|_u$. Thus, we have found

$$(III)_a = \frac{1}{4} \int_{-\infty}^{-\varepsilon} dw \int_{\varepsilon \ln(1-e^{w/\varepsilon})}^0 dv e^{-(\frac{1}{\varepsilon}-\frac{1}{2})w} (g(u)g(w+u)-1) + O(\varepsilon^2(1+\|g\|_u)\|g-1\|_u).$$

We can now integrate over v and use $-e^{-\frac{w}{\varepsilon}} \ln(1-e^{\frac{w}{\varepsilon}}) = 1 + O(e^{\frac{w}{\varepsilon}})$ for $w \leq -\varepsilon$ to find

$$\begin{aligned} (III)_a &= \frac{\varepsilon}{4} \int_{-\infty}^{-\varepsilon} dw e^{\frac{w}{2}} (g(u)g(u+w)-1) + O(\varepsilon^2(1+\|g\|_u)\|g-1\|_u) \\ &= \frac{\varepsilon}{4} \int_{-\infty}^0 dw e^{\frac{w}{2}} (g(u)g(u+w)-1) + O(\varepsilon^2(1+\|g\|_u)\|g-1\|_u). \end{aligned}$$

Since $g \rightarrow 1$ as $u \rightarrow -\infty$ and since a is bounded, we find $a(u) = \frac{1}{2} \int_{-\infty}^0 dwe^{\frac{w}{2}} g(u+w)$ and the claim (2.22) follows from the previous formula. \square

Lemma 2.2 suggests to approximate problem (2.15)-(2.17) by the system of ODEs

$$\frac{da}{du} = \frac{1}{2} \left(-a + ab - \frac{\varepsilon}{2} ab(1-a) \right), \quad (2.23)$$

$$\frac{db}{du} = \frac{1}{2} \left(b - ab + \frac{\varepsilon}{2} ab(1-a) \right), \quad (2.24)$$

$$\frac{dp}{du} = \left(\frac{1}{\varepsilon} - \frac{1}{2} \right) \left(-p + ab - \frac{\varepsilon}{2} ab(1-a) \right), \quad (2.25)$$

$$\frac{dq}{du} = \left(\frac{1}{\varepsilon} - \frac{1}{2} \right) \left(q - ab + \frac{\varepsilon}{2} ab(1-a) \right). \quad (2.26)$$

Linearizing this system around the solution $a = b = p = q = 1$ we find a three-dimensional unstable manifold characterized by the eigenvalues $\pm \frac{i}{2} + \varepsilon \left(\frac{1}{8} \pm \frac{i}{2} \right)$ and $\left(\frac{1}{\varepsilon} - \frac{1}{2} \right)$. Recalling the rescaling (2.14), this agrees with the results of Lemma 2.1.

Notice that the third eigenvalue is much larger than the other two, which is related to the fact, that the solutions to (2.25) and (2.26) change on a faster scale. As discussed before, we restrict ourselves for the moment to solutions for which $p = g + O(\varepsilon)$ and $q = g + O(\varepsilon)$. In particular this implies, that p and q satisfy the assumptions of Lemma 2.2 if a and b do so. Therefore we consider for the moment solutions of (2.23) and (2.24).

2.4.1. *Adiabatic increase of the amplitude.* We have seen that with increasing u the trajectories follow almost ellipses, but spiral outwards in each round. This property does not only hold for a and b close to one, but also when $|a-1| + |b-1| = O(1)$.

We compute the increase of the amplitude of the spiral by using an adiabatic approximation. More precisely, we recall that the leading order of the system (2.23)-(2.24) is

$$\frac{da}{du} = -\frac{a}{2} + \frac{ab}{2} = \frac{a}{2}(b-1), \quad \frac{db}{du} = \frac{b}{2} - \frac{ab}{2} = \frac{b}{2}(1-a), \quad (2.27)$$

and the energy

$$E = a + b - \ln(ab) - 2 \quad (2.28)$$

is conserved by (2.27).

We compute now the change of energy for the perturbed problem. We have $\frac{dE}{du} = -\left(\frac{1-a}{a}\right) \frac{da}{du} + \left(\frac{1-b}{b}\right) \frac{db}{du}$ and using (2.23) and (2.24) we obtain

$$\frac{dE}{du} = \frac{\varepsilon}{4}(b-a)(1-a).$$

We need to estimate the change of energy in a cycle where we use the approximation (2.27). The change of the energy in a period is given by $\frac{\varepsilon}{4} D(E)$ where

$$D(E) = \int_0^{T(E)} (b-a)(1-a) du = 2 \int_{C(E)} (b-a) \frac{db}{b}, \quad (2.29)$$

where we used that $(1-a)du = \frac{2}{b}db$ and where $C(E)$ denotes the contour in the plane (a, b) associated to the trajectory with energy E . We can parameterize it by two curves

$\{(a_+(b), b) : b \in (b_{\max}(E), b_{\min}(E))\} \cup \{(a_-(b), b) : b \in (b_{\min}(E), b_{\max}(E))\}$. Then

$$\begin{aligned} D(E) &= -2 \int_{b_{\min}(E)}^{b_{\max}(E)} (b - a_+(b)) \frac{db}{b} + 2 \int_{b_{\min}(E)}^{b_{\max}(E)} (b - a_-(b)) \frac{db}{b} \\ &= 2 \int_{b_{\min}(E)}^{b_{\max}(E)} (a_+(b) - a_-(b)) \frac{db}{b}. \end{aligned}$$

Since $a_+(b) > a_-(b)$ for any $b \in (b_{\min}(E), b_{\max}(E))$ we find $D(E) > 0$ and the energy increases. We now compute the asymptotics of the solutions as $E \rightarrow 0$ and $E \rightarrow \infty$. We recall that the contours $C(E)$ are defined by (2.28).

We first consider the limit $E \rightarrow 0$. In this regime we obtain the asymptotics $a_+(b) \sim 1 + \sqrt{2E - (b-1)^2}$ and $a_-(b) \sim 1 - \sqrt{2E - (b-1)^2}$, as well as $b_{\min}(E) \sim 1 - \sqrt{2E}$ and $b_{\max}(E) \sim 1 + \sqrt{2E}$. Furthermore we obtain $a_+(b) - a_-(b) \sim 2\sqrt{2E - (b-1)^2}$. Therefore

$$\begin{aligned} D(E) &= 2 \int_{b_{\min}(E)}^{b_{\max}(E)} (a_+(b) - a_-(b)) \frac{db}{b} \sim 4 \int_{b_{\min}(E)}^{b_{\max}(E)} \sqrt{2E - (b-1)^2} db \\ &\sim 4 \int_{-\sqrt{2E}}^{\sqrt{2E}} \sqrt{2E - x^2} dx = 8E \int_{-1}^1 \sqrt{1 - x^2} dx = 4\pi E \end{aligned}$$

since $\int_{-1}^1 \sqrt{1 - x^2} dx = \frac{1}{2}\pi$. Notice that we thus recover the increase of the amplitude of the oscillations for a and b that follow from (1.9) and (2.13).

To examine the limit $E \rightarrow \infty$, we will use the expression (2.29) and estimate the different regions of $C(E)$ separately. This ensures, that the energy does not change too much during the cycle of length $T(E)$.

We first consider the region where $b \leq 1$ and $a \in (a_*, E)$ with $1 \ll a_* \ll E$. We denote by b_* the value of b such that $a_+(b_*) = a_*$. Since in this region $E \approx a_* - \ln b_*$ this yields $b_* \approx e^{a_* - E}$. Then the contribution to $D(E)$ in this region is

$$2 \int_1^{b_*} (b - a) \frac{db}{b} = 2 \int_{b_*}^1 (a - b) \frac{db}{b} \approx 2 \int_{b_*}^1 \frac{a}{b} db \approx 2 \int_{b_*}^1 \frac{E + \ln b}{b} db \approx 2 \int_{b_*}^1 \frac{\ln b}{b} db \approx E^2$$

as $E \rightarrow \infty$. We claim that this gives the main contribution to the change of the energy and are going to show that the remaining regions give smaller contributions as $E \rightarrow \infty$.

Indeed, consider now the part where $a \in (1, a_*)$. In this region $E \approx a - \ln b$ and if we denote by \hat{b} the value of b on the cycle that corresponds to $a = 1$ we have $b \approx e^{-E}$. Then

$$2 \int_{b_*}^{\hat{b}} (b - a) \frac{db}{b} \leq C \int_{\hat{b}}^{b_*} \frac{a}{b} db \leq C a_* \ln \hat{b} \leq a_* E \ll E^2.$$

If both a and b are small, then $E \approx \ln a + \ln b$ and the contribution of the rate of change of energy is estimated by

$$C \int_{b_{\min}}^1 \frac{db}{b} \leq C \ln b_{\min} \leq CE \ll E^2 .$$

If $b \geq 1$ and if a is small, then the contribution of the change of the energy on the other hand is estimated by

$$2 \int_1^{b_{\max}} (b - a) \frac{db}{b} \leq C b_{\max} \leq CE .$$

Finally, we consider the regime where $a, b \geq 1$, such that $E \approx a + b$. The corresponding part of $D(E)$ is given by

$$\int_E^1 (b - a) \frac{2}{b} db \approx 2 \int_E^1 (2b - E) \frac{db}{b} \approx E \ln E \quad \text{as } E \rightarrow \infty .$$

Hence we have derived the approximation

$$E_{n+1} - E_n \sim \frac{\varepsilon}{4} (E_n)^2 \quad \text{if } |E_n| \rightarrow \infty . \tag{2.30}$$

This approximation is valid as long as $\varepsilon E \ll 1$, that is, $a + b \ll \frac{1}{\varepsilon}$ and $a + b \gg e^{-\frac{1}{\varepsilon}}$.

2.5. *Intermediate regime.* The intermediate regime is characterized by the energy E becoming of order $\frac{1}{\varepsilon}$. The energy, as defined in (2.28), is given in terms of A and B as $E = \frac{A}{A_\infty} + \frac{B}{B_\infty} - \ln(AB) + \ln(A_\infty B_\infty) - 2$. Since $A_\infty = O(\varepsilon)$ and $B_\infty = O(\varepsilon)$ (recall (2.10)), the fact that the energy is of order $\frac{1}{\varepsilon}$ implies that either $A = O(1)$, or $B = O(1)$, or $AB \ll 1$.

The intermediate regime will be split in four different regions. When A and B are of order one, we can approximate the equation for G by (1.8) with K given by the additive kernel $K(\xi, \eta) = \xi + \eta$. We call this the *kinetic regime*. If A is of order one, but B small we can approximate the equation by a simple ODE system for A and B ; see (2.47), that gives an explicit expression for G . If both, A and B are small, we approximate again by the Lotka-Volterra equation (2.27), while if B is of order one, but A small, we approximate by another simple ODE system; see (2.58).

2.5.1. *Overview of different regimes.* If we make a plot of the solution in the A - B -plane (see Figure 4 for a graph), it turns out that also in the regime where $E = O(\frac{1}{\varepsilon})$ the curve $A(x), B(x)$ continues to spiral around the point (A_∞, B_∞) with increasing amplitude in the clockwise sense for increasing x . This curve intersects the line $A = A_\infty$ and $B = B_\infty$ at consecutive points $x = x_n$ and $x = \bar{x}_n$ where $B(x_n)$ and $A(\bar{x}_n)$ are of order one, respectively.

In the interval (x_n, \bar{x}_n) the solution can be approximated by a solution to the kinetic equation with additive kernel. In the interval (\bar{x}_n, x_{n+1}) the solution will be approximated by solutions of certain ODEs. We will see that the length of these intervals is $|x_n - \bar{x}_n| = O(\ln \frac{1}{\varepsilon})$, while $|\bar{x}_n - x_{n+1}| = O(\frac{1}{\varepsilon^2})$.

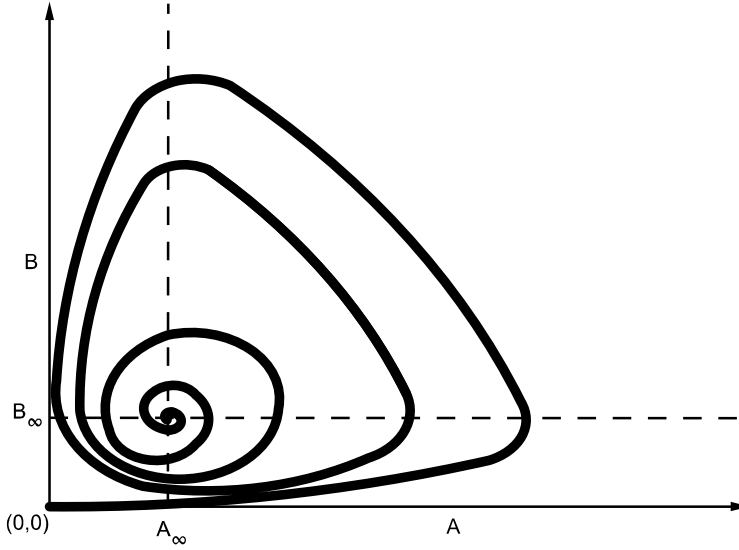


FIG. 4. Sketch of the solution in the $A - B$ -plane.

2.5.2. A, B of order one: *The kinetic regime.* We now consider equation (2.6)-(2.8) for $\varepsilon = 0$, that is,

$$\frac{dA(x)}{dx} = \frac{1}{2}G(x), \quad \frac{dB(x)}{dx} = -\frac{1}{2}G(x), \quad (2.31)$$

$$\frac{dP(x)}{dx} = -P(x) + G(x), \quad \frac{dQ(x)}{dx} = Q(x) - G(x), \quad (2.32)$$

and

$$G(x) = 4A(x)B(x) + P(x)Q(x) + J[G](x), \quad (2.33)$$

with $J[G](x) = \int_{-\infty}^0 dy \int_{\ln(1-ey)}^0 dz e^{y-z} G(y+x)G(z+x)$.

LEMMA 2.3. For any $\rho \in (0, 2)$ there exists a continuous solution $(A_\rho, B_\rho, P_\rho, Q_\rho)$ to (2.31)-(2.33) such that the corresponding G_ρ satisfies for $\rho \in (0, 2)$, $\rho \neq 1$, the asymptotics

$$G_\rho(x) \sim \frac{\rho}{1+\rho} \frac{\sin\left(\frac{\pi\rho}{1+\rho}\right)\Gamma\left(\frac{1}{1+\rho}\right)}{\pi(1+\rho)} e^{\frac{\rho}{1+\rho}x} \quad \text{as } x \rightarrow -\infty \quad (2.34)$$

and

$$G_\rho(x) \sim \frac{\rho}{1+\rho} \frac{\Gamma(1+\rho)\sin(\pi(1-\rho))}{\pi} e^{-\rho x} \quad \text{as } x \rightarrow \infty. \quad (2.35)$$

For $\rho \in (0, 1]$ the solution is positive. Furthermore, for $\rho = 1$, the solution is explicit

$$G_1(x) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{\frac{x}{2}} e^{-\frac{e^x}{2}}. \quad (2.36)$$

For these solutions we also have that $M_\rho := \int_{-\infty}^{\infty} G_\rho(x) dx = \frac{\rho}{1+\rho}$. Moreover it holds

$$A_\rho(x) = \frac{1}{2} \int_{-\infty}^x G_\rho(y) dy \quad \text{and} \quad B_\rho(x) = \frac{1}{2} \int_x^{\infty} G_\rho(y) dy. \quad (2.37)$$

We also have the representation formulas

$$P_\rho(x) = \int_{-\infty}^x G_\rho(y)e^{y-x} dy \quad \text{and} \quad Q_\rho(x) = \int_x^\infty G_\rho(y)e^{x-y} dy. \quad (2.38)$$

The above result was proved in [10] for $\rho \in (0, 1]$. However, inspection of the formula for the Laplace transform of the solutions, given in [10], reveals that the result also holds for $\rho \in (1, 2)$. It was proved in [10] that the functions $G_\rho(x)$ are nonnegative for $\rho \in (0, 1]$, while the asymptotics (2.35) imply that $G_\rho(x) < 0$ for large x if $\rho \in (1, 2)$. We also note that $M_1 = \frac{1}{2}$ while $M_\rho \rightarrow \frac{2}{3}$ as $\rho \rightarrow 2$.

We now explain how to use the solutions from Lemma 2.3 to approximate the function G in the intervals (x_n, \bar{x}_n) . Due to (2.37) we define $M_n = 2B(x_n)$ and approximate the solution in the interval (x_n, \bar{x}_n) by $G(x) = G_{\rho_n}(x - \hat{x}_n)$, where $M_n = \frac{\rho_n}{1+\rho_n}$. The point \hat{x}_n is determined by the condition

$$\frac{1}{2} \frac{\sin\left(\frac{\pi\rho}{1+\rho_n}\right)\Gamma\left(\frac{1}{1+\rho_n}\right)}{\pi(1+\rho_n)} e^{\frac{\rho_n}{1+\rho_n}(x_n-\hat{x}_n)} = A_\infty \approx \frac{\varepsilon}{4}. \quad (2.39)$$

This condition is obtained by matching the asymptotics of $G_{\rho_n}(x - \hat{x}_n)$, see (2.34), for $x < \hat{x}_n$, $|x - \hat{x}_n| \gg 1$ with $A(x_n) = A_\infty$, compare (2.37). Using a similar matching argument in the region $x - \bar{x}_n \gg 1$, we find

$$B(\bar{x}_n - \hat{x}_n) = \frac{1}{2} \frac{1}{1+\rho_n} \frac{\Gamma(1+\rho_n)\sin(\pi(1-\rho_n))}{\pi} e^{-\rho_n(\bar{x}_n-\hat{x}_n)} = B_\infty = \frac{\varepsilon}{4}. \quad (2.40)$$

Notice that the two matching conditions (2.39) and (2.40) determine \bar{x}_n as a function of x_n . We also see that $|x_n - \bar{x}_n| = O(\ln \frac{1}{\varepsilon})$. Furthermore we remark that (2.37) implies that $A(\bar{x}_n) = \frac{M_n}{2}$. This is not immediately obvious but follows a posteriori from the fact that the peaks have distance of order $\frac{1}{\varepsilon^2}$, hence, to leading order A sees only the mass of the last peak.

2.5.3. *An approximation for G if one of A or B is small.* We now derive the following key lemma, that, under certain regularity assumptions, controls the nonlocal effects by pointwise estimates, which in particular will allow us to approximate (2.6)-(2.8) by systems of ODEs in regions where G is small and where the kinetic approximation with $\varepsilon = 0$ fails.

LEMMA 2.4. Suppose that A, B, P and Q are solutions to (2.6) and (2.7) and G satisfies (2.9). Furthermore, assume that for some constants k_1, k_2 and k_3 with $k_1 \in (0, 1-2\delta]$, for some $\delta > 0$, the function G satisfies

$$\begin{aligned} \ln(G(x+y)) - \ln(G(x)) &\leq k_1|y| + k_2, & \text{for all } y \leq -1, \\ |\ln(G(x+y)) - \ln(G(x))| &\leq k_3|y|, & \text{for all } y \in (-1, 0), \end{aligned} \quad (2.41)$$

then

$$|G(x) - (4A(x)B(x) + P(x)Q(x) + 2G(x)A(x))| \leq C_\delta G(x)^2. \quad (2.42)$$

If furthermore

$$|P(x)| \leq 2|G(x)| \quad \text{and} \quad |Q(x)| \leq 2|G(x)|, \quad (2.43)$$

then we have

$$|G(x) - (4A(x)B(x) + 2G(x)A(x))| \leq C_\delta G(x)^2. \quad (2.44)$$

Proof. We rewrite (2.8) as

$$G(x) = 4A(x)B(x) + P(x)Q(x) + \mathcal{L}_1[G](x) + \mathcal{L}_2[G](x),$$

with

$$\begin{aligned} \mathcal{L}_1[G](x) &= \int_{-\infty}^0 dy \int_{\ln(1-e^y)}^0 dz e^{(1-\frac{\epsilon}{2})(y-z)} G(y+x)G(z+x), \\ \mathcal{L}_2[G](x) &= \int_{-\infty}^0 dy \int_{\ln(1-e^y)}^0 dz e^{\frac{\epsilon}{2}(y-z)} G(y+x)G(z+x). \end{aligned}$$

We first show that $|\mathcal{L}_2[G]| \leq C_\delta G(x)^2$. Using assumption (2.41) we obtain

$$\begin{aligned} \mathcal{L}_2[G](x) &= \int_{-\infty}^0 dz G(x+z) \int_{\ln(1-e^z)}^0 dY G(x+y) e^{\frac{\epsilon}{2}(y-z)} \\ &\leq CG(x)^2 \left(\int_{-\infty}^{-1} dz e^{-(k_1+\frac{\epsilon}{2})z} \int_{\ln(1-e^z)}^0 dy e^{-k_3y} \right. \\ &\quad \left. + \int_{-1}^0 dz e^{-(k_3+\frac{\epsilon}{2})z} \left(\int_{\ln(1-e^z)}^{-1} dy e^{-(k_1-\frac{\epsilon}{2})y} + 1 \right) \right) \\ &\leq CG(x)^2 \left(\int_{-\infty}^{-1} dz e^{-(1-\delta)z} \int_{\ln(1-e^z)}^0 dy e^{-k_3y} \right. \\ &\quad \left. + \int_{-1}^0 dz e^{-(k_3+\frac{\epsilon}{2})z} \left(\int_{\ln(1-e^z)}^{-1} dy e^{-(1-\delta)y} + 1 \right) \right) \leq C_\delta G(x)^2. \end{aligned}$$

With $R(x, z) := \int_{\ln(1-e^z)}^0 dy \frac{G(x+y)}{G(x)} e^{(1-\frac{\epsilon}{2})y}$ we rewrite $\mathcal{L}_1[G](x)$ as follows:

$$\begin{aligned} \mathcal{L}_1[G](x) &= G(x) \int_{-\infty}^0 dz G(x+z) e^{-(1-\frac{\epsilon}{2})z} \int_{\ln(1-e^z)}^0 dy \frac{G(x+y)}{G(x)} e^{(1-\frac{\epsilon}{2})y} \\ &= G(x) \int_{-\infty}^{-1} dz G(x+z) e^{-(1-\frac{\epsilon}{2})z} R(x, z) + G(x) \int_{-1}^0 dz G(x+z) e^{-(1-\frac{\epsilon}{2})z} R(x, z) \\ &=: \mathcal{L}_{1,1}[G](x) + \mathcal{L}_{1,2}[G](x). \end{aligned}$$

Using (2.41) we obtain $|R(x, z)| \leq Q(x, 0) \leq C_\delta$ and then also that $|\mathcal{L}_{1,2}[G](x)| \leq C_\delta G(x)^2$.

To compute $\mathcal{L}_{1,1}[G]$ note first that since $z \leq -1$, we have $y \in (-1, 0)$ in the integral for R . Hence, due to the second estimate in (2.41) we have $|R(x, z) - e^z| \leq Ce^{2z}$. Thus we find, using again (2.41), that

$$\left| \mathcal{L}_{1,1}[G](x) - G(x) \int_{-\infty}^{-1} G(x+z) e^{\frac{\epsilon}{2}z} \right| \leq C_\delta G(x)^2.$$

In the previous formula we can replace the integral $\int_{-\infty}^{-1}$ by $\int_{-\infty}^0$, which introduces, due to (2.41), an error of order $G(x)^2$. Since G satisfies (2.9), it turns out that the solution to the first equation in (2.6) is given by the first formula in (2.4). Thus, the above estimate implies (2.42). Estimate (2.44) follows then from (2.42) and the additional assumption (2.43). \square

Lemma 2.4 implies that, as long as assumptions (2.41) and (2.43) of the lemma are satisfied, we have

$$\left| \frac{dA}{dx} + \frac{\varepsilon}{2}A - \frac{2AB}{1-2A} \right| \leq C \frac{G^2}{1-2A}, \quad \left| \frac{dB}{dx} - \frac{\varepsilon}{2}B + \frac{2AB}{1-2A} \right| \leq C \frac{G^2}{1-2A}. \quad (2.45)$$

In particular, we can use the result of Lemma 2.4 to approximate (2.6)-(2.8) by the system of ODEs

$$\frac{dA}{dx} = \frac{2AB}{1-2A} - \frac{\varepsilon}{2}A, \quad \frac{dB}{dx} = \frac{\varepsilon}{2}B - \frac{2AB}{1-2A}, \quad (2.46)$$

for any value of x such that $G(x)$ is small. To make the argument self-consistent, we will have to check afterwards that the solutions of these approximations satisfy the assumptions of Lemma 2.4.

2.5.4. ODE regime 1: B small. We consider now the region where $x = \bar{x}_n + O(1)$. From the results of Section 2.5.2 we have that $A(\bar{x}_n) = \frac{M_n}{2} = \frac{\rho_n}{2(1+\rho_n)} < \frac{1}{4}$ and $B = O(\varepsilon)$. Due to (2.35) the assumption (2.41) of Lemma 2.2 holds with $k = \rho_n$ and we can use the approximation (2.46).

It turns out that A and B change on different scales. Notice that A increases as long as $\frac{2B}{1-2A} > \frac{\varepsilon}{2}$ and decreases for $\frac{2B}{1-2A} < \frac{\varepsilon}{2}$. As long as A is of order one we obtain that B decreases exponentially on the time scale for which x is of order one while A changes very slowly. Initially the term $\frac{2AB}{1-2A}$ is relevant. However as soon as B becomes significantly smaller than ε we obtain that A changes via the equation $\frac{dA}{dx} = -\frac{\varepsilon}{2}A$.

Without loss of generality we can assume, due to the translation invariance of the equation, that $\bar{x}_n = 0$. This assumption is made for notational convenience throughout this and the following Subsections 2.5.5-2.5.7. For the same reason we will also drop the index n in ρ_n and M_n respectively.

Hence we can assume $\frac{2B(0)}{1-2A(0)} = \frac{\varepsilon}{2}$. Notice that A is very close to M as long as $x \ll \frac{1}{\varepsilon}$. Given that B decreases for x of order one and A changes very little, we obtain the approximate equations

$$\frac{dA}{dx} = -\frac{\varepsilon}{2}A, \quad \frac{dB}{dx} = -\frac{2AB}{1-2A}, \quad (2.47)$$

whence

$$A(x) = \frac{M}{2} \exp\left(-\frac{\varepsilon}{2}x\right). \quad (2.48)$$

Since

$$\int_0^x \frac{2A(u)du}{1-2A(u)} = -\frac{2}{\varepsilon} \int_M^{M \exp(-\frac{\varepsilon}{2}x)} \frac{dt}{1-t} = \frac{2}{\varepsilon} \left(\ln(1 - M \exp(-\frac{\varepsilon}{2}x)) - \ln(1 - M) \right)$$

we obtain for B

$$B(x) = \frac{\varepsilon}{4} \frac{(1-M)(1-M)^{\frac{2}{\varepsilon}}}{(1 - M \exp(-\frac{\varepsilon}{2}x))^{\frac{2}{\varepsilon}}}. \quad (2.49)$$

Since G is small, we can use the approximation $G = \frac{4AB}{1-2A}$. We need to check that G satisfies assumption (2.41) of Lemma 2.2. With the above computations

$$G(x) = \frac{2MB(0)}{1 - M e^{-\frac{\varepsilon}{2}x}} \exp\left(-\frac{\varepsilon}{2}x - \int_0^x \frac{M e^{-\frac{\varepsilon}{2}\xi}}{1 - M e^{-\frac{\varepsilon}{2}\xi}} d\xi\right) \quad (2.50)$$

such that

$$\begin{aligned} |\ln G(x+y) - \ln G(x)| &\leq C\varepsilon|y| + \frac{\varepsilon}{2}|y| + \int_x^{x+y} \frac{Me^{-\frac{\varepsilon}{2}\xi}}{1 - Me^{-\frac{\varepsilon}{2}\xi}} d\xi \\ &\leq C\varepsilon|y| + \frac{M}{1-M}|y| \leq k_1|y|, \end{aligned}$$

where we can choose $k_1 < 1$ since due to $M < \frac{1}{2}$ we have $\frac{M}{1-M} < 1$. Hence assumption (2.41) of Lemma 2.4 is satisfied at least for $y > -x$. If $y \leq -x$ we can use the asymptotics of the solutions in the kinetic regime, see (2.35), which gives (2.41) since $\rho < 1$.

2.5.5. *A, B small: Matching ODE 1 regime with Lotka-Volterra.* We now consider the matching between the solutions of the previous subsection and the regime where A and B are small, and hence the solution behaves as a solution to the Lotka-Volterra system (2.27). For this purpose recall the relation between A, B and a, b ; see (2.14).

We enter the Lotka-Volterra regime if a becomes of order one, i.e. A of order ε . In this range b is very small. We define \tilde{x} to be the time when a becomes one, i.e. by $\frac{M}{2} \exp(-\frac{\varepsilon}{2}\tilde{x}) = \frac{2G_\infty}{\varepsilon} \sim \frac{\varepsilon}{2}$ and then introduce the new variable $t = \varepsilon(x - \tilde{x})$. Using (2.48) we then find

$$a(t) = \frac{\varepsilon}{2G_\infty} A(x) \sim \frac{\varepsilon M}{4G_\infty} \exp(-\frac{\varepsilon}{2}x) = \frac{\varepsilon M}{4G_\infty} \exp(-\frac{\varepsilon}{2}\tilde{x}) \exp(-\frac{t}{2}) = \exp(-\frac{t}{2})$$

as $t \rightarrow -\infty$, which gives the matching condition

$$a(t) \sim \exp(-\frac{t}{2}) \quad \text{as } t \rightarrow -\infty. \quad (2.51)$$

In order to compute the asymptotics of $b(t)$ we use

$$b(t) = \frac{\varepsilon}{2G_\infty} B(x) = \frac{\varepsilon^2}{8G_\infty} \frac{(1-M)^{\frac{2}{\varepsilon}+1}}{(1 - M \exp(-\frac{\varepsilon}{2}\tilde{x}) \exp(-\frac{t}{2}))^{\frac{2}{\varepsilon}}} \approx \frac{1}{2} \frac{(1-M)^{\frac{2}{\varepsilon}+1}}{(1 - \frac{2G_\infty}{\varepsilon} \exp(-\frac{t}{2}))^{\frac{2}{\varepsilon}}}.$$

In the limit $\varepsilon \rightarrow 0$ we obtain the approximation for large $t < 0$ (but of order one)

$$\begin{aligned} \frac{1}{(1 - \frac{2G_\infty}{\varepsilon} \exp(-\frac{t}{2}))^{\frac{2}{\varepsilon}}} &= \exp\left(-\frac{2}{\varepsilon} \ln\left(1 - \frac{2G_\infty}{\varepsilon} \exp(-\frac{t}{2})\right)\right) \\ &\approx \exp\left(\frac{2}{\varepsilon} \frac{2G_\infty}{\varepsilon} \exp(-\frac{t}{2})\right) \approx \exp\left(\exp(-\frac{t}{2})\right), \end{aligned}$$

which implies the matching condition

$$b(t) \sim \frac{1}{2}(1-M)^{\frac{2}{\varepsilon}+1} \exp\left(\exp(-\frac{t}{2})\right) \quad \text{as } t \rightarrow -\infty. \quad (2.52)$$

We recall that due to (2.45) and (2.14) the equations in this region are approximated to leading order by (2.27) which has the conserved energy $E = a + b - \ln(ab) - 2$. We obtain that in the matching region, given the smallness of b and the fact that a is large $E \sim a - \ln b$, whence

$$E \sim \exp(-\frac{t}{2}) - \frac{2}{\varepsilon} \ln(1-M) - \exp(-\frac{t}{2}) = -\frac{2}{\varepsilon} \ln(1-M). \quad (2.53)$$

2.5.6. *Lotka-Volterra; transition time.* In the region where $a + b = O(1)$ we approximate the dynamics of (2.6)-(2.8) by the Lotka-Volterra system (2.27) together with the matching conditions (2.51) and (2.52).

We need to estimate the time that the trajectory spends in the region where $a, b = O(1)$. Due to the invariance of the Lotka-Volterra equation under the transformation $(a, b, t) \rightarrow (b, a, -t)$ it suffices to compute the time that the trajectory needs to arrive to the line $\{a = b\}$. The key simplification is that in all the time required to bring the trajectory from the asymptotics (2.51), (2.52) to the line $\{a = b\}$ we have $b \ll 1$. Then, we have the approximations $\frac{da}{dt} = -\frac{a}{2}$ and $\frac{db}{dt} = \frac{b(1-a)}{2}$ such that $a(t) = \exp(-\frac{t}{2})$.

On the other hand the Lotka-Volterra equations and (2.53) imply

$$a - \ln(a) - \ln(b) = -\frac{2}{\varepsilon} \ln(1 - M) \tag{2.54}$$

and if $a = b$ this implies $a - 2 \ln a = -\frac{2}{\varepsilon} \ln(1 - M)$. As a consequence we obtain for a at the line $\{a = b\}$ the following asymptotics $a = (1 - M)^{\frac{1}{\varepsilon}}$.

This implies that a reaches an extremely small value. Using $a(t) = \exp(-\frac{t}{2})$ we obtain for the time \hat{t} to arrive to the line $\{a = b\}$ can be approximated as

$$\hat{t} = \frac{2}{\varepsilon} \ln\left(\frac{1}{1 - M}\right). \tag{2.55}$$

Due to (2.45), the original equation (2.6)-(2.8) can be approximated by the Lotka-Volterra equation with an error on the right-hand side of the order εab . Therefore it follows that $\frac{dE}{dt} = O(\varepsilon(a + b))$. Since $b < a$, and $a(t) = e^{-\frac{t}{2}}$, $b = O(e^{-\frac{C}{\varepsilon}})$ it follows from (2.54) that the change of the energy until \hat{t} is of order ε and since the energy is of order $\frac{1}{\varepsilon}$ we can assume that the energy is approximately constant.

We can now compute the asymptotics of the solution when $t - \hat{t} \gg 1$ in order to obtain the matching condition with the next region. Neglecting also the term $\ln b$ compared to b we obtain, if $b \gg 1$, the approximation $a \sim e^{-E} e^b$ and using (2.54) it follows that

$$a \sim (1 - M)^{\frac{2}{\varepsilon}} e^b, \quad b \gg 1. \tag{2.56}$$

In this range of values of a, b we can use the approximation $\frac{db}{dt} = \frac{b}{2}$, hence $b(t) = C e^{\frac{t}{2}}$. Since $b(\hat{t}) = a(\hat{t}) = e^{-\frac{\hat{t}}{2}}$ we have

$$b(t) = e^{\frac{t-2\hat{t}}{2}}. \tag{2.57}$$

Notice, that the assumption (2.41) of Lemma 2.4 is satisfied, since the function $\ln G(\cdot)$ is a function of t and hence the derivative of this function with respect to x is of order ε .

2.5.7. *ODE regime 2: A small.* We now describe the region where A is small and B increases up to values of order one.

We recall (2.45) and given that in this region $A \ll \varepsilon$ is small and $B \gg \varepsilon$ we obtain

$$\frac{dA}{dx} = AB, \quad \frac{dB}{dx} = \frac{\varepsilon}{2} B, \tag{2.58}$$

which implies $B = C_0 e^{\frac{\varepsilon x}{2}}$, where C_0 has to be determined by matching with (2.56) and (2.57). This gives $\frac{dA}{dx} = C_0 e^{\frac{\varepsilon x}{2}} A$ and hence $A(x) = C_1 \exp\left(\frac{2C_0}{\varepsilon} e^{\frac{\varepsilon x}{2}}\right)$. To determine C_0 and C_1 we recall (2.14), $G_\infty \approx \frac{\varepsilon^2}{4}$, the definition $x = \tilde{x} + \frac{t}{\varepsilon}$ and denote $x^* = \tilde{x} + \frac{2\hat{t}}{\varepsilon}$. We

obtain up to exponential accuracy that $B(x) \sim \frac{\varepsilon}{2} e^{\frac{\varepsilon(x-x^*)}{2}}$ and

$$A(x) \sim \frac{\varepsilon}{2}(1-M)^{\frac{2}{\varepsilon}} \exp\left(e^{\frac{\varepsilon(x-x^*)}{2}}\right) = \frac{\varepsilon}{2}(1-M)^{\frac{2}{\varepsilon}} \exp\left(\frac{2}{\varepsilon}B(x)\right). \quad (2.59)$$

Due to (2.46) the asymptotics above is valid as long as $A \ll \frac{\varepsilon}{2}$. We denote by x_{n+1} the point when $A(x_{n+1}) = A_\infty \approx \frac{\varepsilon}{2}$. Notice that $B(x_{n+1})$ is maximal and decreases afterwards. Using (2.59) simple rearrangements give that in the limit $\varepsilon \rightarrow 0$ we obtain

$$B(x_{n+1}) = -\ln(1-M). \quad (2.60)$$

This formula yields the desired iterative condition for the masses. Starting with the mass M we reach after one cycle the new value of the mass $\ln\left(\frac{1}{1-M}\right) > M$.

Notice also that the first inequality in (2.41) is satisfied since G is increasing in the region that we consider in this section. The second inequality follows from

$$\left| \ln G(x+y) - \ln G(x) \right| \leq \frac{2}{\varepsilon} \left| B(x)(e^{\frac{\varepsilon}{2}y} - 1) \right| + \frac{\varepsilon}{2} |y| \leq \left(B(x_{n+1}) + \frac{\varepsilon}{2} \right) |y|.$$

The analysis of this subsection is similar to the one in Subsection 2.5.4. However, there A was of order one, while here it is small. As a consequence the value $B(x_{n+1})$ of B at the end of this region is different from $A(0) = A(x_n)$ in Subsection 2.5.4. Due to this fact, the amplitude of the oscillations in the A - B -plane is increasing.

2.5.8. Summary of the intermediate regime. We have identified successive points x_n , \bar{x}_n and x_{n+1} such that $A(x_n) = A(x_{n+1}) = A_\infty \approx \frac{\varepsilon}{2}$ and $B(\bar{x}_n) = B_\infty \approx \frac{\varepsilon}{2}$. We found that $\bar{x}_n - x_n = O(\ln \frac{1}{\varepsilon})$ and $x_{n+1} - \bar{x}_n = O(\frac{1}{\varepsilon^2})$. In $[x_n, \bar{x}_n]$ the solution G is of order one and is of leading order given by a solution to the kinetic equation (2.31)-(2.33) given in Lemma 2.3 with ρ_n such that $2B(x_n) = M_n = \frac{\rho_n}{1+\rho_n}$. In $[\bar{x}_n, x_{n+1}]$, where G is small, the solution can be approximated by three different simple ODE systems. The key finding in the previous subsections is a formula for the increase of the amplitude of the oscillations in the A - B -plane characterized by the numbers M_n for which, due to (2.60), we have the recursive formula

$$M_{n+1} = \ln \frac{1}{1-M_n}. \quad (2.61)$$

We emphasize, that the value of M does not change during the kinetic regime nor the Lotka-Volterra regime, but that the change is due to the asymmetry in the equations that describe ODE regime 1 and ODE regime 2 respectively.

In the limit $M_n \rightarrow 0$, formula (2.61) implies $M_{n+1} - M_n \sim \frac{1}{2}M_n^2$. On the other hand, we have in this regime that $E \sim \frac{4}{\varepsilon}(A+B) \sim \frac{4B}{\varepsilon} \sim \frac{2}{\varepsilon}M$. Hence, (2.61) agrees with (2.30) in the limit $M_n \rightarrow 0$.

2.6. Shooting argument. We will argue in Section 3 that the constant C_* in (1.9) can be chosen such that no fast instabilities develop for any finite x . We will show now that under the assumption that no such ‘dormant’ instability occurs, the constant L can be selected in the interval $[1, e^{\frac{\pi\varepsilon}{2}})$ in such a way that the resulting function $G(x)$ is globally

positive and satisfies $G(x) \rightarrow 0$ as $x \rightarrow \infty$. More precisely we obtain

LEMMA 2.5. There exists a value $L_* \in [1, e^{\frac{\pi\varepsilon}{2}})$ and $\hat{x}_n \in \mathbb{R}$ such that the corresponding G satisfies

$$G(x) \sim \left(\frac{1}{2\sqrt{2}} e^{\frac{x-\hat{x}_n}{2}} + \frac{\varepsilon}{4} e^{x-\hat{x}_n} \right) e^{-\frac{\varepsilon(x-\hat{x}_n)}{2}} \tag{2.62}$$

for $x \gg \hat{x}_n$.

Proof. Due to (2.61) it follows that M_n becomes larger than $\frac{1}{2}$ for sufficiently large n . By making a translation of the variable x in (1.9) if needed, we can assume that for some values of $L \in [1, e^{\frac{\pi\varepsilon}{2}})$ the corresponding value of M_n is larger than $\frac{1}{2}$ and for other values $L \in [1, e^{\frac{\pi\varepsilon}{2}})$ we have $M_n < \frac{1}{2}$.

We recall that if $M_n < \frac{1}{2}$ we can approximate $G(x)$ arguing as in Subsections 2.5.4-2.5.7. Then $A(x), B(x)$ converge to values where $A(x) = A_\infty$ and $B(x)$ is of order one for $x = x_{n+1}$. On the other hand if $M_n > \frac{1}{2}$ Lemma 2.3 implies that $B(x)$ changes sign at some finite x . Given that the solutions of (2.6)-(2.8) depend continuously on the parameter L in (1.9) we have the existence of $L = L_* \in [1, e^{\frac{\pi\varepsilon}{2}})$ for which the corresponding solution of (2.6)-(2.8) neither changes sign nor arrives to values of $A(x) = A_\infty$ with $B(x)$ of order one. We will argue that for L_* we have $G(x) \rightarrow 0$ as well as $(A(x), B(x)) \rightarrow (0, 0)$ as $x \rightarrow \infty$.

To see this we first remark that due to the arguments above, M_n must take the value $\frac{1}{2}$ or is close to it for small ε . Then during the kinetic regime $G(x)$ is approximated by $G_1(x)$ (cf. Lemma 2.3) and at the end of that phase the asymptotics of $G(x)$ is given by the right-hand side of (2.36). Then $G(x)$ cannot be approximated by ODEs as in the case for $M_n < \frac{1}{2}$, but we will approximate $G(x)$ by a suitable integral equation. For notational simplicity we assume for the moment that $\hat{x}_n = 0$ and introduce the variables $\xi = e^x, \eta = e^y, \zeta = e^z, \xi^2 h(\xi) = G(x), \bar{a}(\xi) = A(x), \bar{b}(\xi) = B(x), \bar{p}(\xi) = P(x)$ and $\bar{q}(\xi) = Q(x)$. Then (2.6)-(2.8) becomes

$$\xi \frac{d\bar{a}}{d\xi} = -\frac{\varepsilon}{2} \bar{a} + \frac{1}{2} \xi^2 h, \quad \xi \frac{d\bar{b}}{d\xi} = \frac{\varepsilon}{2} \bar{b} - \frac{1}{2} \xi^2 h, \tag{2.63}$$

$$\xi \frac{d\bar{p}}{d\xi} = -(1 - \frac{\varepsilon}{2}) \bar{p} + \xi^2 h, \quad \xi \frac{d\bar{q}}{d\xi} = (1 - \frac{\varepsilon}{2}) \bar{q} - \xi^2 h, \tag{2.64}$$

$$\xi^2 h = 4\bar{a}\bar{b} + \bar{p}\bar{q} + J[h], \tag{2.65}$$

where

$$J[h](\xi) = \int_0^\xi \eta h(\eta) d\eta \int_{\xi-\eta}^\xi h(\zeta) (\eta\zeta)^{\frac{\varepsilon}{2}} (\eta^{1-\varepsilon} + \zeta^{1-\varepsilon}) d\zeta. \tag{2.66}$$

On the other hand, since $G(x)$ behaves like the right-hand side of (2.36) we obtain

$$h(\xi) \sim \frac{1}{2\sqrt{2\pi}} \frac{e^{-\frac{\xi}{2}}}{\xi^{\frac{3}{2}}} \quad \text{for } \xi^\varepsilon \sim 1, \text{ i.e. } \ln \xi \sim \frac{1}{\varepsilon}. \tag{2.67}$$

We remark that in the set of values ξ for which the approximation (2.67) holds we have that \bar{a} is of order one, \bar{p} is of order $\frac{1}{\xi}$ and \bar{b}, \bar{q} are of order $\frac{\varepsilon^{-\frac{\xi}{2}}}{\sqrt{\xi}}$. Therefore, the terms $4\bar{a}\bar{b}$ and $\bar{p}\bar{q}$ in (2.65) can be neglected and (2.65) can be approximated by

$$\xi^2 h = J[h]. \tag{2.68}$$

We now use the approximation (2.68) to describe how the asymptotics (2.67) is modified if ξ^ε becomes of order one or larger. To this end we will look for solutions of (2.68) of the form $h(\xi) = r(\xi)e^{-\frac{\xi}{2}}$, where we assume that $r(\xi)$ contains only functions that change algebraically for large ξ , i.e. $r(\xi + 1) \simeq r(\xi)$. Then (2.68) becomes

$$\begin{aligned} \xi^2 r(\xi) &= \int_0^\xi r(\zeta) d\zeta \int_{\xi-\zeta}^\xi \eta r(\eta) (\eta\zeta)^{\frac{\varepsilon}{2}} (\eta^{1-\varepsilon} + \zeta^{1-\varepsilon}) e^{\frac{1}{2}(\xi-\eta-\zeta)} d\eta \\ &= \int_0^\xi r(\zeta) d\zeta \int_0^\zeta (\xi-\zeta+\theta) Q(\xi-\zeta+\theta) ((\xi-\zeta+\theta)\zeta)^{\frac{\varepsilon}{2}} ((\xi-\zeta+\theta)^{1-\varepsilon} + \zeta^{1-\varepsilon}) e^{-\frac{1}{2}\theta} d\theta. \end{aligned} \quad (2.69)$$

For large values of ξ the function $r(\xi - \zeta + \theta)$ changes more slowly than $e^{-\frac{1}{2}\theta}$ such that

$$\begin{aligned} \xi^2 r(\xi) &= \int_0^\xi r(\zeta) d\zeta \int_0^\zeta (\xi-\zeta) r(\xi-\zeta) ((\xi-\zeta)\zeta)^{\frac{\varepsilon}{2}} ((\xi-\zeta)^{1-\varepsilon} + \zeta^{1-\varepsilon}) e^{-\frac{1}{2}\theta} d\theta \\ &= 2 \int_0^\xi r(\zeta) (\xi-\zeta) r(\xi-\zeta) ((\xi-\zeta)\zeta)^{\frac{\varepsilon}{2}} ((\xi-\zeta)^{1-\varepsilon} + \zeta^{1-\varepsilon}) (1 - e^{-\frac{1}{2}\zeta}) d\zeta. \end{aligned} \quad (2.70)$$

For large values of ξ and small values of ξ the main contribution to the integral in (2.70) is due to the values of ζ satisfying $\zeta \leq \delta\xi$, for δ small. This will be checked later ‘‘a posteriori’’. Thus, if $\zeta \leq \delta\xi$ we have the approximations $(\xi - \zeta)^{1+\frac{\varepsilon}{2}} \sim \xi^{1+\frac{\varepsilon}{2}}$, $(\xi - \zeta)^{1-\varepsilon} + \zeta^{1-\varepsilon} \sim \xi^{1-\varepsilon}$ such that

$$r(\xi) = 2\xi^{-\frac{\varepsilon}{2}} \int_0^{\delta\xi} r(\zeta) r(\xi - \zeta) (1 - e^{-\frac{1}{2}\zeta}) \zeta^{\frac{\varepsilon}{2}} d\zeta. \quad (2.71)$$

We recall that (2.67) yields a valid approximate solution of (2.71) if $\xi^\varepsilon \sim 1$. For these ξ we have

$$r(\xi) \sim \frac{1}{2\sqrt{2\pi}} \frac{1}{\xi^{\frac{3}{2}}}. \quad (2.72)$$

Indeed, plugging (2.72) into (2.71) and using $r(\xi - \zeta) \approx r(\xi)$ for large ξ we readily obtain that (2.71) reduces to checking the identity

$$1 \approx \frac{2\xi^{-\frac{\varepsilon}{2}}}{2\sqrt{2\pi}} \int_0^{\delta\xi} (1 - e^{-\frac{1}{2}\zeta}) \zeta^{\frac{\varepsilon}{2} - \frac{3}{2}} d\zeta \approx \frac{\xi^{-\frac{\varepsilon}{2}}}{\sqrt{2\pi}} \int_0^\infty (1 - e^{-\frac{1}{2}\zeta}) \zeta^{\frac{\varepsilon}{2} - \frac{3}{2}} d\zeta.$$

Using that $\int_0^\infty (1 - e^{-\frac{1}{2}\zeta}) \zeta^{-\frac{3}{2}} d\zeta = \sqrt{2\pi}$ this approximation follows due to the assumption $\xi^\varepsilon \sim 1$. For $\xi^\varepsilon \gg 1$ a scaling argument suggests the asymptotics $r(\xi) \sim \frac{\bar{C}}{\xi}$ for some $\bar{C} > 0$. Plugging this into (2.71) and using also that $r(\xi - \zeta) \approx r(\xi)$ we reduce (2.71) to $1 \approx 2\bar{C}\xi^{-\frac{\varepsilon}{2}} \int_0^{\delta\xi} (1 - e^{-\frac{1}{2}\zeta}) \zeta^{\frac{\varepsilon}{2} - 1} d\zeta \approx \frac{4\bar{C}}{\varepsilon}$, such that $\bar{C} = \frac{\varepsilon}{4}$. For $\xi^\varepsilon \gg 1$ we have thus obtained

$$r(\xi) \sim \frac{\varepsilon}{4\xi}. \quad (2.73)$$

Notice that the contributions due to the part of the integrals with $\zeta > \delta\xi$ which have been neglected in (2.70) in order to derive (2.71) give a small contribution if ξ is large and ε small for both solutions (2.72) and (2.73).

We can now combine the solutions (2.72) and (2.73) to obtain a global solution of (2.71) that is valid for arbitrarily large values of ξ . Indeed, it is easy to check that

$$r(\xi) \sim \frac{1}{2\sqrt{2\pi}} \frac{1}{\xi^{\frac{3}{2}}} + \frac{\varepsilon}{4\xi} \tag{2.74}$$

satisfies (2.71) approximately for arbitrary values of ξ and ε .

In order to check the consistency of the approximation (2.74) we must check that the terms $4\bar{a}\bar{b}$ and $\bar{p}\bar{q}$ in (2.65) are negligible. It follows from integrating (2.63)-(2.64), using the definition of r and (2.74), that $\bar{a}(\xi) \leq \frac{C}{\xi^{\frac{3}{2}}}$, $\bar{b}(\xi) \leq C\xi h(\xi)$, $\bar{p}(\xi) \leq \frac{C}{\xi^{1-\frac{3}{2}}}$ and $\bar{q}(\xi) \leq C\xi h(\xi)$. Therefore $4\bar{a}\bar{b} + \bar{p}\bar{q} \ll \xi^2 h$ for large ξ and the self-consistency of the argument follows. It is important to remark, however, that although the term $4\bar{a}\bar{b}$ can be neglected in (2.65), the equation for $\bar{b}(\xi)$ in (2.63) shows that for some solutions of (2.6)-(2.8) the function $B(x)$ can start an exponential growth taking place in lengths of x of order $\frac{1}{\varepsilon}$. In other words, the solutions of (2.6)-(2.8) might contain dormant instabilities that result either in a change of sign of B or in B becoming large enough and thus the solution entering another cycle in the (A, B) -plane. Thus, this instability corroborates the fact that the desired solution is obtained by a shooting argument and appears as the transition between the ODE behaviour described in Subsections 2.5.4-2.5.7 and B changing sign.

Finally, amending for the fact that we had assumed that $\hat{x}_n = 0$, we obtain from (2.74) the result (2.62). □

3. Fast dynamics. We have seen in the asymptotics of the function G in (1.9) that the term $C_* e^{\mu_* x}$ changes much faster than the other terms in the formula (1.9) due to the exponentially growing mode for Q in (2.7). If C_* is of order one, the instability will appear for x of order one. However, if C_* is very small, the exponentially growing instability will only become visible for large values of x .

For the solution computed in Section 2 the following holds. In all the regions where the solution can be approximated by ODEs the function $G(x)$ changes in length scales much larger than one and in particular, in all those regions we have $|Q - G| \ll Q$. However, if C_* is sufficiently small, there exists an $x_* \gg 1$

$$Q(x) = G(x_*) \left(1 \pm e^{x-x_*} \right) + \dots \tag{3.1}$$

for $x < x_*$ and $|x - x_*| \gg 1$. In this case the instability is triggered in $x = x_*$. We can obtain both, the positive and the negative sign in (3.1) by choosing $|C_*|$ in (1.9) sufficiently large and positive or negative respectively. Moreover, modifying the value of C_* we can tune the value of x_* . It turns out that if the instability is triggered in any region where the solution of (2.6)-(2.8) can be approximated by an ODE, then either Q , if the sign in (3.1) is minus, or B if otherwise, become negative and hence the solution cannot represent an admissible solution to the coagulation equation (cf. Sections 3.1 and 3.2.2).

It is also possible to choose the value of C_* such that the instability is triggered in the kinetic regime, described in Section 2.5.2. In this regime $|Q - G|$ is not small compared to G anymore. Here we have to take into account that the Volterra problem (2.6)-(2.8)

with $\varepsilon = 0$ has more solutions than those given by Lemma 2.3. The general solution has the form $Q_\rho^\alpha(x)$ with $Q_\rho^\alpha(x) - Q_\rho(x) \sim \alpha e^x$ as $x \rightarrow -\infty$ with arbitrary α and we denote by G_ρ^α the corresponding function G . Recall that for our special solutions we have $G(x) \sim G_\rho(x - \hat{x}_n)$ for $|x - \hat{x}_n|$ of order one. However, the difference $G_\rho - G_\rho^\alpha$ differs by an exponentially small amount for $x < \hat{x}_n$ and $|x - \hat{x}_n| \gg 1$, hence there is a priori no reason to choose G_ρ instead of G_ρ^α . We will see later, however, in Section 3.2.3 that if $\alpha \neq 0$, then either Q or B become negative and again we do not obtain an admissible solution.

Therefore, adjusting the value of C_* we can obtain a solution for which no instability is triggered either in the ODE regime, or in the kinetic regime.

3.1. *Fast instabilities appearing during the Lotka-Volterra regime.* We first assume that the instability is triggered in the Lotka-Volterra regime, see Section 2.4; that is, the regime with energy $E \ll \frac{1}{\varepsilon}$.

Suppose that the instability appears at a given x_* , for which the ODE approximation is still valid. This would mean that the asymptotics (3.1) holds for $x < x_*$ and $\frac{1}{\varepsilon} \gg |x - x_*| \gg 1$.

If (3.1) holds with a minus sign, we can approximate (2.6)-(2.8) as follows. Lemma 2.4 implies that $J[G] \leq C(G^2 + GA)$. In the Lotka-Volterra regime we have $G \ll 1$ and $A \ll 1$ and hence $J[G]$ can be neglected in (2.8). As long as G remains approximately constant, we can approximate Q by

$$\frac{dQ}{dx} = \left(1 - \frac{\varepsilon}{2}\right)Q - G, \quad Q(x) = G(x_*) \left(1 - e^{(1 - \frac{\varepsilon}{2})(x - x_*)}\right), \quad (3.2)$$

hence Q becomes negative for $x = x_*$. Since Q is decreasing, the term PQ in (2.8) remains negligible, the other variables essentially do not change and the approximation is self-consistent.

If (3.1) holds with a plus sign, the function Q becomes of order one if $x \sim x_1$ where

$$G(x_*)e^{(x_1 - x_*)} = 1. \quad (3.3)$$

Since $G(x_*)$ can be very small if the energy E is large, it might happen that $|x_1 - x_*| \rightarrow \infty$ as $\varepsilon \rightarrow 0$. During this part of the evolution the variables a and b are described by (2.27) and hence if $|x_1 - x_*|$ is large, we might have that the difference between $(a(x_1), b(x_1))$ and $(a(x_*), b(x_*))$ is large. This phenomenon can be particularly relevant if $E \sim \frac{1}{\varepsilon \ln \frac{1}{\varepsilon}}$ and $E \approx a + b$, since then the characteristic scale in the Lotka-Volterra regime is of order one up to logarithmic corrections. Therefore $|x_1 - x_*|$ might be much larger than this scale. However, we will prove an estimate that shows that the change of (a, b) in the interval $[x_*, x_1]$ is negligible in the part of the Lotka-Volterra cycle contained in $\{a \leq 1\} \cup \{b \leq 1\}$. Indeed, the first equation (2.27) implies that $\frac{da}{dx} \geq -\varepsilon a$ and hence, using (3.3), $G(x_*) \approx \frac{\varepsilon^2}{4} a(x_*) b(x_*)$ and $a(x_*) b(x_*) \geq C e^{-E}$, we find

$$a(x_1) \geq a(x_*) e^{\varepsilon \ln G(x_*)} = a(x_*) G(x_*)^\varepsilon \geq a(x_*) e^{\varepsilon \ln(C\varepsilon^2)} e^{-\varepsilon E} \geq \frac{1}{2} a(x_*).$$

Similarly, since $a(x_*), b(x_*) \leq 2E$, it follows that $b(x_1) \leq 2b(x_*)$. In particular a and b cannot make more than one cycle in the Lotka-Volterra phase plane.

In order to describe the regime when Q becomes of order one, we introduce the new variable $t = x - x_1$. Then equation (3.1) implies the matching condition

$$Q(t) = e^t \quad \text{as } t \rightarrow -\infty. \tag{3.4}$$

Denote by $A_1 := A(x_1)$, $B_1 := B(x_1)$ and $P_1 := 4A_1B_1$ and define $A = A_1\hat{a}$, $B = B_1\hat{b}$ and $P = P_1\hat{p}$. Then we obtain

$$\frac{d\hat{a}}{dt} = -\frac{\varepsilon}{2}\hat{a} + \frac{1}{2}\left(4B_1(\hat{a}\hat{b} + \hat{p}Q) + \frac{J[G]}{A_1}\right), \tag{3.5}$$

$$\frac{d\hat{b}}{dt} = \frac{\varepsilon}{2}\hat{b} - \frac{1}{2}\left(4A_1(\hat{a}\hat{b} + \hat{p}Q) + \frac{J[G]}{B_1}\right), \tag{3.6}$$

$$\frac{d\hat{p}}{dt} = -\ell + \left(\hat{a}\hat{b} + \hat{p}Q + \frac{J[G]}{P_1}\right), \tag{3.7}$$

$$\frac{dQ}{dt} = Q - (P_1(\hat{a}\hat{b} + \hat{p}Q) + J[G]). \tag{3.8}$$

Using Lemma 2.4 we can check that the terms involving $J[G]$ can be neglected in (3.5)-(3.8). Since furthermore A_1, B_1 and P_1 are small, we obtain for t of order one the following approximation of (3.5)-(3.8):

$$\frac{d\hat{a}}{dt} = \frac{d\hat{b}}{dt} = 0, \quad \frac{d\hat{p}}{dt} = -\hat{p} + \hat{a}\hat{b} + \hat{p}Q \quad \text{and} \quad \frac{dQ}{dt} = Q. \tag{3.9}$$

Using the matching condition (3.4) we obtain $\hat{a} \equiv 1, \hat{b} \equiv 1$,

$$\hat{p} = e^{Q(t)-t}\left(1 - e^{-Q(t)}\right) \quad \text{and} \quad Q(t) = e^t. \tag{3.10}$$

Assume now that for large t we have that $\hat{p}Q \gg \hat{a}\hat{b}$. We can then approximate (3.5) and (3.7), assuming for the moment that the nonlocal terms can still be neglected, by

$$\frac{d\hat{a}}{dt} = -\frac{\varepsilon}{2}\hat{a} + 2B_1\hat{p}Q, \quad \frac{d\hat{b}}{dt} = \frac{\varepsilon}{2}\hat{b} - 2A_1\hat{p}Q \quad \text{and} \quad \frac{d\hat{p}}{dt} = \hat{p}(Q - 1). \tag{3.11}$$

The equation for \hat{b} in (3.11) implies that

$$\hat{b}(t) = e^{\frac{\varepsilon}{2}t} - 2A_1 \int_0^t e^{\frac{\varepsilon}{2}(t-s)}(\hat{p}Q)(s) dx \approx e^{\frac{\varepsilon}{2}t} - 2A_1 \int_0^t e^{e^s} ds \approx e^{\frac{\varepsilon}{2}t} - 2A_1 e^{-t} e^{e^t}.$$

Then $\hat{b}(t_\varepsilon) = 0$ with t_ε as

$$e^{e^{t_\varepsilon}} e^{-t_\varepsilon} \sim \frac{1}{2A_1}, \quad \text{that is, } t_\varepsilon \sim \ln\left(\ln\left(\frac{1}{2A_1}\right)\right). \tag{3.12}$$

Since $A_1 \geq Ce^{-E}$ we also have $t_\varepsilon \leq \ln E \leq \ln\left(\frac{1}{\varepsilon}\right)$. Notice that this justifies a posteriori that one can neglect the term $\frac{\varepsilon}{2}\hat{b}$ in the equation for b even though A_1 might be very small.

We now check that the assumption $\hat{a}\hat{b} \ll \hat{p}Q$ indeed holds for all $t \in [0, t_\varepsilon]$, which is not clear a priori given that B_1 could be much larger than A_1 and this could produce a large growth of \hat{a} in this interval. Due to the first equation in (3.11) we have $\hat{a}(t) \leq 1 + 2B_1 \int_0^t (\hat{p}Q)(s) ds$ and the third equation in (3.11) give $\hat{p}(t) = e^{Q(t)-t}$. This gives $\hat{a}(t) \leq CB_1\hat{p}(t)$, whence, since \hat{b} is bounded, $\hat{a}(t)\hat{b}(t) \leq CB_1\hat{p}(t)$. Since $Q(t) \geq 1$ and since $B_1 \ll 1$ we indeed obtain $\hat{a}\hat{b} \ll \hat{p}Q$.

In order to show the consistency of the whole approximation, it remains to check that the nonlocal term $J[G]$ remains small. This is feasible, but requires some tedious estimates. In order not to interrupt too much the flow of the argument, we just state the main result.

The key remark in order to compute the variations of $J[G]$ is that $\varepsilon \hat{p}Q$ can be approximated by a mollification of the Dirac mass as follows:

$$\begin{aligned} 2A_1 \hat{p}Q &\approx 2A_1 e^{Q(t)} (1 - e^{-Q(t)}) \approx 2A_1 e^{e^t} \\ &= 2A_1 e^{e^{t_\varepsilon} e^{t-t_\varepsilon}} = 2A_1 e^{e^{t_\varepsilon} [e^{t-t_\varepsilon} - 1]} e^{e^{t_\varepsilon}} \approx \frac{1}{\delta_\varepsilon} e^{\frac{1}{\delta_\varepsilon} (t-t_\varepsilon)} \end{aligned} \tag{3.13}$$

where we used the definition of t_ε in (3.12) and where $\delta_\varepsilon = \frac{1}{e^{t_\varepsilon}} \approx \ln \frac{1}{2A_1}$.

By estimating carefully all the terms in $J[G]$ and using the previously obtained asymptotics we can obtain by lengthy but otherwise rather straightforward estimates that for sufficiently large $R > 0$

$$J[G](x) \leq B_1 \left(A_1 + \frac{e^{\frac{1}{\delta_\varepsilon} e^{x-x_1-t_\varepsilon}}}{\delta_\varepsilon} \chi_{\{t_\varepsilon - R \leq x - x_1 \leq t_\varepsilon\}} \right) \quad \text{for } x_1 \leq x \leq x_1 + t_\varepsilon. \tag{3.14}$$

We need to compare the size of this nonlocal term with the main terms in the equations (3.5)-(3.8) and check that the contribution of this nonlocal term is small. This requires the inequalities:

$$\frac{J[G]}{A_1} \ll B_1 \hat{p}Q, \quad \frac{J[G]}{B_1} \ll A_1 \hat{p}Q, \quad \frac{J[G]}{P_1} \ll \hat{p}Q \tag{3.15}$$

which are all equivalent. Moreover, the same inequality implies also that the term $J[G]$ in (3.8) gives a negligible contribution compared with $P_1 \hat{p}Q$ and it is possible to approximate this equation by an ODE.

In order to derive (3.15) we notice that (3.10) and (3.13) imply

$$2A_1 \hat{p}Q \geq C_1 \left(A_1 + \frac{e^{\frac{1}{\delta_\varepsilon} e^{x-x_1-t_\varepsilon}}}{\delta_\varepsilon} \chi_{\{t_\varepsilon - R \leq x - x_1 \leq t_\varepsilon\}} \right)$$

for some $C_1 > 0$ and for $x_1 \leq x \leq x_1 + t_\varepsilon$. Then (3.15) follows from (3.14).

We also remark that in the estimates of $J[G]$ we do not use Lemma 2.4 due to the fast growth of G that implies that the second inequality in (2.41) fails.

3.2. Instability is triggered for values of x such that $E \sim \frac{1}{\varepsilon}$. We now consider the case in which the instability is triggered at times in which the energy E defined in (2.28) is of order $\frac{1}{\varepsilon}$. We recall that for these values of E there are some values of x for which we describe the solution G, A, B, P, Q by solutions of the integro-differential equation (2.6)-(2.8) with $\varepsilon = 0$ (cf. Subsection 2.5.2), while for other values (2.6)-(2.8) is approximated by a system of ODEs (cf. Subsections 2.5.4, 2.5.5 and 2.5.7). We will denote the first set of values of x , where G is of order one, as values of Type I and the second set, where $G \ll 1$, as values of Type II. For x of Type II the values of the functions G, A, B, P, Q change in a scale much longer than one. Then, the triggering of the instabilities at some $x = x^*$ can be described by the asymptotic behaviour for $x < x^*$ and $|x - x^*| \gg 1$. For values x of Type I the characteristic length scale for the development of the instability given by the equation for Q in (2.7) is the same as for the variation of G . Then we cannot

obtain a definition like (3.1) for the value of x^* in which the instability is triggered. In this case we need to study in detail the solutions of the Volterra-like problem with additive kernel (2.31)-(2.33).

3.2.1. *Additional solutions of problem (2.31)-(2.33).* In Lemma 2.3 we obtained a family of solutions to (2.31)-(2.33), that correspond to solutions of the coagulation equation. These are however not the only solutions and in the following proposition, which is proved in the Appendix, we construct additional solutions with different higher order asymptotics as $x \rightarrow -\infty$.

PROPOSITION 3.1. Suppose that for $\rho \in (0, 2]$ we denote by $A_\rho, B_\rho, P_\rho, Q_\rho$ and G_ρ the solution in Lemma 2.3 that are characterized by the asymptotics (2.34). For any $\alpha \in \mathbb{R}$ we define

$$G_\rho^\alpha(x) = G_\rho(x)e^{\alpha e^x} \tag{3.16}$$

and, with $B_\infty := B_\rho(-\infty)$,

$$\begin{aligned} A_\rho^\alpha(x) &= \frac{1}{2} \int_{-\infty}^x G_\rho^\alpha(y) dy, & B_\rho^\alpha(x) &= B_\infty - \frac{1}{2} \int_{-\infty}^0 G_\rho^\alpha(y) dy, \\ P_\rho^\alpha(x) &= \int_{-\infty}^x G_\rho^\alpha(y)e^{y-x} dy, & Q_\rho^\alpha(x) &= Q_\rho(x)e^{\alpha e^x} + \alpha e^x \left(1 - \int_{-\infty}^x e^{\alpha e^y} Q_\rho(y) dy\right). \end{aligned} \tag{3.17}$$

Then $A_\rho^\alpha, \dots, G_\rho^\alpha$ solve (2.31)-(2.33).

Furthermore, for $\rho \in (0, 1]$, if $\alpha < 0$, we obtain that $Q_\rho^\alpha(x)$ becomes negative for some $x \in \mathbb{R}$, while if $\alpha > 0$, then B_ρ^α becomes negative for some $x \in \mathbb{R}$.

REMARK 3.2. The solutions constructed in Proposition 3.1 are the only solutions of (2.31)-(2.33) with the asymptotics $G(x) \rightarrow 0$ as $x \rightarrow -\infty$. Indeed, suppose that $G > 0$, $G(x) \sim e^{\beta x}$ and $B(x) \rightarrow B_\infty$ as $x \rightarrow -\infty$. We can assume that $B_\infty > 0$ since otherwise $A \rightarrow \infty$ and then also $G \rightarrow \infty$ as $x \rightarrow -\infty$. The equations in (2.32) imply that $Q \leq Ce^{\beta x}$, $P \leq Ce^{\beta x}$ and $J[G](x) \leq Ce^{2\beta x}$. Then (2.33) implies that $G(x) \sim 4B_\infty A(x)$ and (2.31) implies that $A(x) = c_1 e^{2B_\infty x}$ as $x \rightarrow -\infty$. Thus, $G(x) \sim k_1 e^{2B_\infty x}$ as $x \rightarrow -\infty$ with $\beta = 2B_\infty$. Using this asymptotic formula we can compute also the asymptotics of A, B, P, Q and $J[G]$ as $x \rightarrow -\infty$ to obtain $G(x) \sim K_1 e^{\beta x} + k_2 e^{2\beta x} + \dots + k_m e^{m\beta x} + \dots$ as $x \rightarrow -\infty$ with $m\beta \leq 1 + \beta < (m + 1)\beta$. (In some resonance cases the terms $e^{k\beta x}$ might have to be replaced by $x e^{k\beta x}$.)

However, computing the next order in this asymptotics we can include an additional parameter \bar{k} from the second equation in (2.32), since we have $Q(x) = \nu_1 e^{\beta x} + \nu_2 e^{2\beta x} + \dots + \nu_m e^{m\beta x} + \bar{k}e^x$ as $x \rightarrow -\infty$. Plugging this into (2.33) gives

$$G(x) \sim k_1 e^{\beta x} + k_2 e^{2\beta x} + \dots + k_m e^{m\beta x} + C\bar{k}e^{(1+\beta)x} + \dots \quad \text{as } x \rightarrow -\infty \tag{3.18}$$

with the free parameters k_1 and \bar{k} , such by varying α in Proposition 3.1 one indeed obtains all solutions.

Proposition 3.1 implies that we must have $\alpha = 0$ at the beginning of each of the ranges of x for which the solutions of (2.6)-(2.8) are approximately described by the solutions of (2.31)-(2.33).

3.2.2. *ODE regimes with $E \approx \frac{1}{\varepsilon}$.* We now examine the case in which the instability for Q in the second equation of (2.7) is triggered for values of x for which E is of order $\frac{1}{\varepsilon}$ but G is small. For those ranges of x the dynamics of $G(x)$ is described, before the instability is triggered, as in Subsections 2.5.4, 2.5.5 and 2.5.7. In all those cases the functions A, B, P, Q and G are approximated by ODEs. Given that the characteristic time scale for all these functions is much larger than one and the instability associated to $Q(x)$ is triggered in times of order one, we can describe the triggering by (3.1). We can describe the evolution after the instability is triggered similarly as in Subsection 3.1. By assumption $G(x_*)$ is small and due to Lemma 2.4 we can approximate (2.6)-(2.8) by a system of ODEs. Moreover, as long as $G(x)$ remains small we can neglect the contribution of the term $P(x)Q(x)$ in (2.8). Then, we can approximate the evolution of the functions A, B and G using the ODEs in Subsections 2.5.4, 2.5.5 and 2.5.7. This approximation is valid as long as $P(x)Q(x)$ is small compared to $G(x)$ and the evolution of $P(x)$ and $Q(x)$ is described by (2.7). If the sign in (3.1) is a minus, we can then conclude exactly as in Subsection 3.1 that Q becomes negative.

Suppose now that the sign in (3.1) is a plus sign. Using the same arguments as above, it follows that as long as $P(x)Q(x) \ll G(x)$ we can approximate $Q(x)$ as $G(x_*)(1 + e^{(1-\frac{\varepsilon}{2})(x-x_*)})$. We then define x_1 by (3.3). Notice that as with the fast instability described in Subsection 3.1 we might have that $x_1 - x_*$ is very large and therefore we might have relevant changes of the values of A and B during this phase. However, as long as we can use the ODE approximations in Subsections 2.5.4, 2.5.5 and 2.5.7 we can deduce, arguing as in Subsection 3.1, that either $a(x_1) \geq \frac{a(x_*)}{2}$ or $b(x_1) \leq 2b(x_*)$ and conclude the argument as before.

3.2.3. *Fast instability in the kinetic regime.* A difference to the argument in Subsection 3.1 is that in principle it might be possible for $G(x)$ to reach values of order one during the interval $[x_*, x_1]$. Then we recall the definition of the sequence $\{\hat{x}_n\}$ in Subsection 2.5.2 and that those are the characteristic times for which we have the approximation $G(x) \simeq G(x - \hat{x}_n)$. For $x < \hat{x}_n$ and $|x - \hat{x}_n| \gg 1$ we can match the asymptotics of the solutions for $x - x_* \gg 1$. With $t = x - \hat{x}_n$ we have the approximation

$$Q \simeq G(x_*)e^{(1-\frac{\varepsilon}{2})(x-x_*)} = G(x_*)e^{(1-\frac{\varepsilon}{2})(\hat{x}_n-x_*)}e^{(1-\frac{\varepsilon}{2})(x-\hat{x}_n)} = G(x_*)e^{(1-\frac{\varepsilon}{2})(\hat{x}_n-x_*)}e^{(1-\frac{\varepsilon}{2})t}$$

for $t < 0$ and $|t| \gg 1$. By assumption $G(x_*)e^{(1-\frac{\varepsilon}{2})(\hat{x}_n-x_*)}$ is at most of order one. If we have that $G(x_*)e^{(1-\frac{\varepsilon}{2})(\hat{x}_n-x_*)}$ is of order one we would have that G can be approximated for $t < 0$, $|t|$ large by a solution of the Volterra-like problem for the additive kernel (cf. (2.31)-(2.33)) with matching condition $G(t) \sim G_{\rho_n}^\alpha(t)$ as $t \rightarrow -\infty$ where $\alpha = G(x_*)e^{(1-\frac{\varepsilon}{2})(\hat{x}_n-x_*)}$. If $\alpha > 0$ is of order one it then follows that $B_{\rho_n}^\alpha(x)$ becomes negative for a finite value. Alternatively, we might have $\alpha \rightarrow 0$. In that case we can approximate $A(x), B(x), P(x), Q(x)$ and $G(x)$ using the functions $A_{\rho_n}^\alpha, \dots, G_{\rho_n}^\alpha$ with small α . Notice that for t of order one we cannot approximate any longer Q by the second equation in (3.2) but Q is close to $Q_{\rho_n}(t)$ for t of order one. However, using (3.17) as well as the fact that $x = e^t$ we obtain the following asymptotics for $\alpha \rightarrow 0$ and then $t \gg 1$:

$$Q \sim Q_{\rho_n}(t) + \alpha \left(1 - \int_{-\infty}^{\infty} Q_{\rho_n}(y) e^t \sim G_{\rho_n}(t) + \alpha(1 - M)e^t, \quad (3.19)$$

where we use the fact that for large t we have $Q_{\rho_n}(t) \sim G_{\rho_n}(t)$ and that $\int_{-\infty}^{\infty} Q_{\rho_n}(y)dy = \int_{-\infty}^{\infty} G_{\rho_n}(y)dy = \frac{\rho_n}{1+\rho_n} < 1$. Notice that (3.19) yields an interesting connection condition for the coefficient describing the exponential separation of Q from G . Indeed, during the kinetic regime this separation takes place according to αe^t for $t \rightarrow -\infty$ and it becomes $\alpha(1-M)e^t$ as $t \rightarrow \infty$. For sufficiently large t the term $\alpha(1-M)e^t$ becomes larger than $G_{\rho_n}(t)$. Using the asymptotics of $G_{\rho_n}(t)$ (cf. (2.35)) it follows that this happens for $e^{(1+\rho_n)t} \approx \frac{1}{\alpha}$ and $G_{\rho_n}(t)$ is of order $\alpha^{\frac{\rho_n}{1+\rho_n}}$. For large t we enter the regime where G is described as in Subsection 2.5.4. In particular $2A(t)$ approaches M for large t and arguing as in Subsection 2.5.1 it follows that A cannot decrease to values smaller than $\frac{M}{4}$ without arriving to values of $x = x_1$ where (3.3) takes place. We can then argue as in the rest of Subsection 2.5.1 to show that B changes sign. Indeed, the arguments used there only require that E is bounded by $\frac{C}{\varepsilon}$.

It is relevant to remark that the amplitude of the exponentially growing perturbation implicitly contained in the equation for Q (cf. (2.7)) cannot become arbitrarily small as $x \rightarrow \infty$ in spite of the fact that it is reduced by a factor $(1-M)$ each time that the solution passes by one of the intervals $[x_n, \bar{x}_n]$ where it is described by the solutions described in Lemma 2.3. Indeed, during the intervals (\bar{x}_n, x_{n+1}) in which the function $G(x)$ is described by means of the ODE approximations described in Subsections 2.5.4-2.5.7 the perturbation of $Q - G$ would continue growing exponentially and the length of the intervals (\bar{x}_n, x_{n+1}) is very large of order $\frac{1}{\varepsilon}$. This growth is much larger than the decrease by the factor $(1-M)$ during the kinetic regime.

3.3. Summary of the analysis of the instabilities. We have seen, using formal arguments, that an instability for Q yields a change of sign, either of Q or of B . To this end we have considered several cases. Initially the Volterra-like problem can be approximated by a Lotka-Volterra equation. In such a case, ODE arguments show that instabilities for Q yield changes of sign, either for Q or B .

It is possible to compute explicit formulas for the solutions of the Volterra-like problem associated to the additive kernel. The final conclusion is the same. We can put an additional parameter α in the solution as $x \rightarrow -\infty$. If $\alpha < 0$ we obtain that Q changes sign and it vanishes for finite s . If $\alpha > 0$ we obtain that B becomes negative for finite x .

The most delicate case corresponds to the ODE regimes in the case when the energy is large. In that case there are different possibilities. In some cases B or Q change sign quickly, because the integral terms are irrelevant and the problem can be approximated by an integral equation. In other cases the solutions must be approximated by the solutions of the additive kernel (the Volterra-like problem) discussed before.

4. Appendix.

4.1. Proof of Lemma 2.1. We assume from now on that $\varepsilon \leq \varepsilon_0$ for sufficiently small ε_0 . Then we claim that there exists $R \geq 3$ that is independent of ε such that all solutions of (2.12) are contained in the set $B_R(0) \cap \{\text{Re}(\mu) \geq 0\}$.

Indeed, if $|\mu| \geq R$ and $\text{Re}(\mu) > 0$ we have, using Stirling's formula,

$$\left| \frac{\Gamma(1 - \frac{\varepsilon}{2} + \mu)}{\Gamma(1 + \mu)} \right| \leq 2 \left| \frac{(1 - \frac{\varepsilon}{2} + \mu)^{(1 - \frac{\varepsilon}{2} + \mu)}}{(1 + \mu)^{(1 + \mu)}} \right| =: 2 \exp(T(\mu, \varepsilon)). \quad (4.1)$$

where

$$T(\mu, \varepsilon) = \operatorname{Re} \left(\left(1 - \frac{\varepsilon}{2} + \mu\right) \left(\ln \left|1 - \frac{\varepsilon}{2} + \mu\right| + i \arg \left(1 - \frac{\varepsilon}{2} + \mu\right) \right) \right) \\ - \operatorname{Re} \left((1 + \mu) \left(\ln |1 + \mu| + i \arg(1 + \mu) \right) \right).$$

Using that $|\mu| \geq R \geq 3$ we obtain

$$T(\mu, \varepsilon) \leq \operatorname{Re} \left((1 + \mu) \left(\ln \left|1 - \frac{\varepsilon}{2} + \mu\right| + i \arg \left(1 - \frac{\varepsilon}{2} + \mu\right) \right) \right) \\ - \operatorname{Re} \left((1 + \mu) \left(\ln |1 + \mu| + i \arg(1 + \mu) \right) \right) \\ - \operatorname{Re} \left((1 + \mu) \ln |1 + \mu| \right) - \operatorname{Re} \left(i(1 + \mu) \arg(1 + \mu) \right) \\ = \operatorname{Re} \left((1 + \mu) \ln \left(\frac{|1 - \frac{\varepsilon}{2} + \mu|}{|1 + \mu|} \right) \right) + \operatorname{Re} \left(i(1 + \mu) \left(\arg \left(1 - \frac{\varepsilon}{2} + \mu\right) - \arg(1 + \mu) \right) \right). \quad (4.2)$$

We have

$$\ln \left(\frac{|1 - \frac{\varepsilon}{2} + \mu|}{|1 + \mu|} \right) \leq 2 \left| \left| \frac{1 - \frac{\varepsilon}{2} + \mu}{1 + \mu} \right| - 1 \right| \leq \frac{2\varepsilon}{|1 + \mu|}$$

and choosing $\arg(\cdot) \in (-\pi, \pi)$ we obtain

$$\left| \arg \left(1 - \frac{\varepsilon}{2} + \mu\right) - \arg(1 + \mu) \right| = \left| \arg \left(1 - \frac{\varepsilon}{2} \frac{1}{1 + \mu}\right) \right| \leq \frac{\varepsilon}{|1 + \mu|}.$$

Combining these last two inequalities with (4.2) we find $T(\mu, \varepsilon) \leq \frac{3\varepsilon}{|1 + \mu|} |1 + \mu| \leq 3\varepsilon$ whence, using (4.1), it follows that $\left| \frac{\Gamma(1 - \frac{\varepsilon}{2} + \mu)}{\Gamma(1 + \mu)} \right| \leq 3$. Similarly we obtain $\left| \frac{\Gamma(\frac{\varepsilon}{2} + \mu)}{\Gamma(1 + \mu)} \right| \leq 3$.

Thus, if μ solves (2.12), we have

$$\frac{\Gamma(1 - \frac{\varepsilon}{2})}{(1 - \frac{\varepsilon}{2})^{\frac{\varepsilon}{2}}} = \frac{\Gamma(1 - \frac{\varepsilon}{2} + \mu)}{\Gamma(1 + \mu)} \frac{(1 - \mu)}{(1 - \frac{\varepsilon}{2})(\frac{\varepsilon}{2} - \mu)} + \frac{\Gamma(\frac{\varepsilon}{2} + \mu)}{\Gamma(1 + \mu)} \Gamma\left(1 - \frac{\varepsilon}{2}\right) \frac{(1 - \mu)}{\Gamma(\frac{\varepsilon}{2})^{\frac{\varepsilon}{2}}(1 - \frac{\varepsilon}{2} - \mu)}$$

and using the above inequalities and the fact that $\lim_{z \rightarrow 0} \Gamma(z)z = 1$ we obtain that for any root μ of (2.12) we have

$$\frac{\Gamma(1 - \frac{\varepsilon}{2})}{(1 - \frac{\varepsilon}{2})^{\frac{\varepsilon}{2}}} \leq 3 \left| \frac{(1 - \mu)}{(1 - \frac{\varepsilon}{2})(\frac{\varepsilon}{2} - \mu)} \right| + 4\Gamma\left(1 - \frac{\varepsilon}{2}\right) \left| \frac{(1 - \mu)}{(1 - \frac{\varepsilon}{2} - \mu)} \right|.$$

The right-hand side of this inequality is bounded by a constant independent of ε if $|\mu| \geq R \geq 3$. However, the left-hand side diverges as $\varepsilon \rightarrow 0$. This gives a contradiction. Therefore, all solutions of (2.12) satisfy $|\mu| < R$.

We now rewrite (2.12) as

$$0 = -\frac{\Gamma(1 - \frac{\varepsilon}{2})\Gamma(1 + \mu)}{(1 - \frac{\varepsilon}{2})^{\frac{\varepsilon}{2}}} + \frac{\Gamma(1 - \frac{\varepsilon}{2} + \mu)(1 - \mu)}{(1 - \frac{\varepsilon}{2})(\frac{\varepsilon}{2} - \mu)} + \frac{\Gamma(\frac{\varepsilon}{2} + \mu)\Gamma(1 - \frac{\varepsilon}{2})(1 - \mu)}{\Gamma(\frac{\varepsilon}{2})^{\frac{\varepsilon}{2}}(1 - \frac{\varepsilon}{2} - \mu)} =: \omega(\mu, \varepsilon). \quad (4.3)$$

Using that $\Gamma(\frac{\varepsilon}{2})^{\frac{\varepsilon}{2}} \rightarrow 1$ as $\varepsilon \rightarrow 0$ it follows that the right-hand side of (4.3) is bounded in the set $\{\operatorname{Re}(\mu) \geq 0, |\mu| \leq R, \min\{|\mu - 1|, |\mu|\} \geq \delta\}$ for each $\delta > 0$ if we take ε sufficiently small (depending on δ). However, since $|\Gamma(1 + \mu)| \geq C_* > 0$ in the set $\{\operatorname{Re}(\mu) \geq 0, |\mu| \leq R\}$ it follows that the left-hand side tends to infinity as $\varepsilon \rightarrow 0$ uniformly in this set. Then, all roots of (2.12) in $\{\operatorname{Re}(\mu) \geq 0\}$ are contained in the

set $\{\operatorname{Re}(\mu) \geq 0, |\mu| \leq R, \min\{|\mu - 1|, |\mu|\} < \delta\}$. We can compute the asymptotics of the roots of (2.12) approximating $\omega(\mu, \varepsilon)$ in (4.3) by suitable comparison functions and Rouché's theorem.

In the region $|\mu| \leq \delta$ with small δ , we use as the comparison function $\omega_1(\mu, \varepsilon) := -\frac{2}{\varepsilon} + \frac{\frac{\varepsilon^2}{4} - \mu^2}{(\frac{\varepsilon}{2} - \mu)}$, while in the region $|\mu - 1| \leq \delta$ we use $\omega_2(\mu, \varepsilon) := -\frac{2}{\varepsilon} + (1 - \mu) \frac{\frac{\varepsilon}{2} + \mu}{(1 - \frac{\varepsilon}{2} - \mu)}$. Using the asymptotics of the function $\Gamma(z)$ for $z \sim 1$ and $z \sim 0$ we obtain (2.13).

4.2. *Proof of Proposition 3.1.* We first state the following auxiliary lemma.

LEMMA 4.1. Suppose that $F, W \in L^1(0, R)$ satisfy

$$W(\xi) - \lambda \int_0^\xi W(\eta) d\eta = F(\xi) \quad \text{a.e. in } (0, R)$$

for some $\lambda \in \mathbb{R}$. Then

$$W(\xi) = AF(\xi) := \lambda e^{\lambda\xi} \int_0^\xi e^{-\lambda\eta} F(\eta) d\eta + F(\xi) \quad \text{a.e. in } (0, R). \quad (4.4)$$

Proof. We define $H(\xi) = \int_0^\xi W(\eta) d\eta$. The function H is absolutely continuous and satisfies $\frac{dH(\eta)}{d\eta} - \lambda H(\eta) = F(\eta)$ and $H(0) = 0$. Therefore $H(\xi) = e^{\lambda\xi} \int_0^\xi e^{-\lambda\eta} F(\eta) d\eta$ for $\xi \in (0, R)$, whence (4.4) follows. \square

For the proof of Proposition 3.1 it is convenient to go over to the variable $\xi = e^x$. For notational convenience we neglect the index ρ throughout this proof.

We first define functions a, b, p, q and h for $\xi > 0$ by

$$a(\xi) = A(x), \quad b(\xi) = B(x), \quad p(\xi) = P(x), \quad q(\xi) = Q(x), \quad \xi^2 h(\xi) = G(x). \quad (4.5)$$

We then define functions $a^\alpha, b^\alpha, p^\alpha, q^\alpha$ and G^α as

$$\begin{aligned} h^\alpha(\xi) &= h(\xi)e^{\alpha\xi}, & p^\alpha(\xi) &= \frac{1}{\xi} \int_0^\xi \eta^2 h^\alpha(\eta) d\eta, \\ a^\alpha(\xi) &= \frac{1}{2} \int_0^\xi \eta h^\alpha(\eta) d\eta, & b^\alpha(\xi) &= B_\infty - \frac{1}{2} \int_0^\xi \eta h^\alpha(\eta) d\eta, \\ q^\alpha(\xi) &= q(\xi)e^{\alpha\xi} + \alpha\xi \left(1 - \int_0^\xi e^{\alpha\eta} \frac{q(\eta)}{\eta} d\eta\right). \end{aligned} \quad (4.6)$$

Then it holds

$$\begin{aligned} A^\alpha(x) &= a^\alpha(\xi), & B^\alpha(x) &= b^\alpha(\xi), & P^\alpha(x) &= p^\alpha(\xi), \\ Q^\alpha(x) &= q^\alpha(\xi), & G^\alpha(x) &= \xi^2 h^\alpha(\xi), \end{aligned} \quad (4.7)$$

with $A^\alpha, \dots, G^\alpha$ as in (3.17).

Notice that the asymptotics of the functions A, \dots, G in Lemma 2.3 imply that all the integrals appearing in (4.6) are well defined. In particular notice that $0 \leq h(\xi) \leq C\xi^{b-2}$ and $0 \leq \frac{q(\xi)}{\xi} \leq C\xi^{b-1}$ for $\xi \leq 1$ with $b > 0$. This implies that all the integrals appearing on the right-hand side of (4.6) are convergent. One slight complication is due to the fact that $h(\xi)$ is not integrable and thus we cannot write directly $\frac{q(\xi)}{\xi} = -\int_0^\xi h(\eta) d\eta$. The singularity of $h(\xi)$ is like $\frac{1}{\xi^{2-\sigma}}$ with $\sigma > 0$ such that $\frac{q(\xi)}{\xi}$ is integrable and $q(\xi) \sim \xi^b$ as $\xi \rightarrow 0$.

It is worth noticing that the last equation in (4.6) can due to Lemma 4.1 be inverted to obtain

$$q(\xi) = n^\alpha(\xi)e^{-\alpha\xi} + \alpha\xi \int_0^\xi \frac{q^\alpha(\eta)}{\eta} e^{-\alpha\eta} d\eta - \alpha. \tag{4.8}$$

Using the change of variables $\xi = e^x$ and (4.5) in (2.31)-(2.33) we obtain that the functions a, b, p, q and h solve

$$\frac{da}{d\xi} = \frac{1}{2}\xi h(\xi), \quad \frac{db}{d\xi} = -\frac{1}{2}\xi h(\xi), \tag{4.9}$$

$$\frac{d}{dx}(\xi p(\xi)) = \xi^2 h(\xi), \quad \frac{d}{d\xi} \left(\frac{q(\xi)}{\xi} \right) = -h(\xi), \tag{4.10}$$

$$\xi^2 h(\xi) = 4a(\xi)b(\xi) + p(\xi)q(\xi) + j[h](\xi), \tag{4.11}$$

with $j[h](\xi) = \int_0^\xi \eta h(\eta) d\eta \int_{\xi-\eta}^\xi h(\zeta)(\eta + \zeta) d\zeta$.

Moreover, the definition of a^α and b^α in (4.6) implies that these two functions solve (4.9) with $h = h^\alpha$. Similarly p^α and h^α solve the first equation of (4.10). It remains to check that h^α and q^α satisfy the second equation of (4.10) and (4.11). In order to simplify the notation we define the functions $\varphi(\xi) = \frac{q(\xi)}{\xi}$ and $\varphi^\alpha(\xi) = \frac{q^\alpha(\xi)}{\xi}$. Using (4.8) we obtain

$$\varphi(\xi) = \varphi^\alpha(\xi)e^{-\alpha\xi} + \alpha \int_0^\xi \varphi^\alpha(\eta)e^{-\alpha\eta} d\eta - \alpha \tag{4.12}$$

and it is straightforward to check that q^α and h^α solve the second equation of (4.10).

We now check that (4.11) holds. To this end we first transform the equation to a more convenient form. Notice that both sets of functions a, \dots, h , as well as $a^\alpha, \dots, h^\alpha$ satisfy $b(\xi) = B_\infty - \int_0^\xi \eta h(\eta) d\eta$ and $p(\xi) = \frac{1}{\xi} \int_0^\xi \eta^2 h(\eta) d\eta$. Then (4.11) can be rewritten as

$$\begin{aligned} \xi^2 h(\xi) &= B_\infty \int_0^\xi \eta h(\eta) d\eta - \int_0^\xi \eta h(\eta) d\eta \int_0^{\xi-\eta} \zeta h(\zeta) d\zeta + \frac{q(\xi)}{\xi} \int_0^\xi \eta^2 h(\eta) d\eta \\ &\quad + \int_0^\xi \eta^2 h(\eta) d\eta \int_{\xi-\eta}^\eta h(\zeta) d\zeta. \end{aligned} \tag{4.13}$$

We now rewrite the last term, using the second equation of (4.10), as

$$\int_0^\xi \eta^2 h(\eta) d\eta \int_{\xi-\eta}^\xi h(\zeta) d\zeta = - \int_0^\xi \eta^2 h(\eta) \left(\frac{q(\xi)}{\xi} - \frac{q(\xi-\eta)}{\xi-\eta} \right) d\eta.$$

Plugging this in (4.13) we find, after some simplifications, and using the definition of the function $\varphi(\xi) = \frac{q(\xi)}{\xi}$ that

$$\xi^2 h(\xi) = B_\infty \int_0^\xi \eta h(\eta) d\eta - \int_0^\xi \eta h(\eta) d\eta \int_0^{\xi-\eta} \zeta h(\zeta) d\zeta + \int_0^\xi \eta^2 h(\eta) \varphi(\xi-\eta) d\eta. \tag{4.14}$$

Thus, (4.9)-(4.11) is equivalent to (4.9), (4.10), (4.14).

We will now check that, if φ and h solve (4.14), we have the same for φ^α and h^α . Using the definitions in (4.6) we obtain

$$\begin{aligned} & \xi^2 h^\alpha(xi) e^{-\alpha\xi} + \alpha \int_0^{xi} \eta^2 h^\alpha(\eta) e^{-\alpha\eta} d\eta \\ &= B_\infty \int_0^\xi \eta h^\alpha(\eta) e^{-\alpha\eta} d\eta - \int_0^\xi \eta h^\alpha(\eta) e^{-\alpha\eta} d\eta \int_0^{\xi-\eta} \zeta h^\alpha(\zeta) e^{-\alpha\zeta} d\zeta \\ & \quad + e^{-\alpha\xi} \int_0^\xi \eta^2 h^\alpha(\eta) \varphi^\alpha(\xi - \eta) d\eta + \alpha \int_0^\xi \eta^2 h^\alpha(\eta) e^{-\alpha\eta} d\eta \int_0^{\xi-\eta} \varphi^\alpha(\zeta) e^{-\alpha\zeta} d\zeta. \end{aligned} \quad (4.15)$$

We can now apply Lemma 4.1 with $W(\xi) = \xi^2 h^\alpha(\xi) e^{-\alpha\xi}$ to obtain

$$\xi^2 h^\alpha(\xi) e^{-\alpha\xi} = B_\infty A F_1(\xi) - A F_2(\xi) + A F_3(\xi) + \alpha A F_4(\xi), \quad (4.16)$$

where A is the operator defined in (4.4) with $\lambda = -\alpha$ and

$$\begin{aligned} F_1(\xi) &= \int_0^\xi \eta h^\alpha(\eta) e^{-\alpha\eta} d\eta, & F_2(\xi) &= \int_0^\xi \eta h^\alpha(\eta) e^{-\alpha\eta} d\eta \int_0^{\xi-\eta} \zeta h^\alpha(\zeta) e^{-\alpha\zeta} d\zeta, \\ F_3(\xi) &= e^{-\alpha\xi} \int_0^\xi \eta^2 h^\alpha(\eta) \varphi^\alpha(\xi - \eta) d\eta, & F_4(\xi) &= \int_0^\xi \eta^2 h^\alpha(\eta) e^{-\alpha\eta} d\eta \int_0^{\xi-\eta} \varphi^\alpha(\zeta) e^{-\alpha\zeta} d\zeta. \end{aligned}$$

We then have

$$\begin{aligned} A F_1(\xi) &= \int_0^\xi \eta h^\alpha(\eta) e^{-\alpha\eta} - \alpha e^{-\alpha\xi} \int_0^\xi e^{\alpha\eta} d\eta \left(\int_0^\eta \zeta h^\alpha(\zeta) e^{-\alpha\zeta} d\zeta \right) \\ &= \int_0^\xi \eta h^\alpha(\eta) e^{-\alpha\eta} d\eta - \alpha e^{-\alpha\xi} \int_0^\xi \zeta h^\alpha(\zeta) e^{-\alpha\zeta} d\zeta \int_\zeta^\xi e^{\alpha\eta} d\eta \\ &= \int_0^\xi \eta h^\alpha(\eta) e^{-\alpha\eta} d\eta - e^{-\alpha\xi} \int_0^\xi \zeta h^\alpha(\zeta) e^{-\alpha\zeta} (e^{\alpha\xi} - e^{\alpha\zeta}) d\zeta = e^{-\alpha\xi} \int_0^\xi \zeta h^\alpha(\zeta) d\zeta \end{aligned} \quad (4.17)$$

and similarly

$$A F_2(\xi) = e^{-\alpha\xi} \int_0^\xi \eta h^\alpha(\eta) d\eta \int_0^{\xi-\eta} \zeta h^\alpha(\zeta) d\zeta, \quad (4.18)$$

$$A F_3(\xi) = e^{-\alpha\xi} \int_0^\xi \eta^2 h^\alpha(\eta) \varphi^\alpha(\xi - \eta) d\eta - \alpha e^{-\alpha\xi} \int_0^\xi \zeta^2 h^\alpha(\zeta) d\zeta \int_0^{\xi-\zeta} \varphi^\alpha(\eta) d\eta \quad (4.19)$$

and

$$A F_4(\xi) = e^{-\alpha\xi} \int_0^\xi \eta^2 h^\alpha(\eta) d\eta \int_0^{\xi-\eta} \varphi^\alpha(\zeta) d\zeta. \quad (4.20)$$

Plugging (4.17)-(4.20) into (4.16) we obtain after some rearrangements that h^α and φ^α solve (4.14) and thus the first statement of Proposition 3.1 follows by elementary computations.

We now compute for $\rho \in (0, 1]$ the asymptotics of the solutions constructed above.

Suppose first that $\alpha < 0$.

We recall that $2B_\infty = \int_0^\infty \eta h_\rho(\eta) d\eta = \frac{\rho}{1+\rho} < 1$. Since $0 < h^\alpha < h$ we obtain that b^α is decreasing and $b^\alpha(\xi) > b^\alpha(\infty) > 0$. Since $\alpha < 0$ we have $\int_0^\xi \frac{q(\eta)}{\eta} e^{\alpha\eta} d\eta < \int_0^\infty \frac{q(\eta)}{\eta} e^{\alpha\eta} d\eta < \int_0^\infty \frac{q(\eta)}{\eta} d\eta = 2B_\infty < 1$. Therefore, if $\alpha < 0$ we have $\alpha\xi(1 - \int_0^\xi \frac{q(\eta)}{\eta} e^{\alpha\eta} d\eta) \rightarrow -\infty$ as

$\xi \rightarrow \infty$ and thus, since $q_\rho(\xi)$ is bounded, we have that q^α vanishes for some $\xi > 0$ if $C_\infty < 0$ while all the other functions remain positive.

We now consider the case $\alpha > 0$. Then h^α increases exponentially and therefore b^α vanishes for some $\xi > 0$. This is obvious for all functions but q^α . To show that q^α remains positive we rewrite the formula for q^α in (4.6). Integrating by parts in the last term we obtain, using also that $\frac{d}{d\xi}\left(\frac{q(\xi)}{\xi}\right) = -h(\xi)$, we obtain after some rearrangements $q^\alpha(\xi) = \alpha\xi + q(\xi) - \xi \int_0^\xi h(\eta)(e^{\alpha\eta} - 1)d\eta$. On the other hand $b^\alpha(\xi) = B_\infty - \frac{1}{2} \int_0^\xi \eta h(\eta)e^{\alpha\eta} d\eta$.

We now use the inequality $e^{\alpha\eta} - 1 \leq \alpha\eta e^{\alpha\eta}$ for $\eta \geq 0$. Then $\int_0^\xi h(\eta)(e^{\alpha\eta} - 1)d\eta \leq \alpha \int_0^\xi h(\eta)e^{\alpha\eta}d\eta$, whence

$$q^\alpha(\xi) \geq \alpha\xi + q(\xi) - \alpha\xi \int_0^\xi h(\eta)\eta e^{\alpha\eta}d\eta > \alpha\xi \left(1 - \int_0^\xi h(\eta)\eta e^{\alpha\eta}d\eta\right) \geq 2\alpha\xi b^\alpha(\xi),$$

where we used that $2B_\infty < 1$. Thus, b^α vanishes before q^α .

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