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ERRATA TO "ENERGETIC VARIATIONAL APPROACHES FOR INCOMPRESSIBLE FLUID SYSTEMS ON AN EVOLVING SURFACE"

By

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Abstract. There are two minor flaws in our 2017 paper. The first flaw is in the proof of Lemma 2.7, which relates a generalization of Helmholtz-Weyl decomposition on a closed surface. The second one is in Appendix (I), where we compare our model to Taylor's model when the surface does not move. We give a full proof of Lemma 2.7 as well as a correct comparison of our model with Taylor's model (1992). It will be properly interpreted.

1. On errata for Koba-Liu-Giga [6].

Generalized Helmholtz-Weyl decomposition and comparison with Taylor's model. There are two minor flaws in Koba-Liu-Giga [6]. The first one is in the proof of Lemma 2.7 in [6]. The proof of the sufficient condition of Lemma 2.7 in [6] is incomplete. Therefore,

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we give a detailed proof of Lemma 2.7 in Subsection 1.1. The second one is some miscalculation with misinterpretation in the Appendix (I) in [6]. We give some explanation to clarify the points in Subsection 1.2. We follow the notation in [6].

1.1. *Proof of Lemma* 2.7 *in* [6].

Let Γ_0 be a closed C^∞-surface, and let $H = H(x,t)$ be the mean curvature of Γ_0 in the direction of $n = n(x,t) = {}^t(n_1, n_2, n_3)$ which is the unit outer normal vector of Γ_0.

THEOREM 1.1 (Lemma 2.7 in [6]). Set

$$E := \left\{ f \in [L^2(\Gamma_0)]^3; \int_{\Gamma_0} f \cdot \varphi \; d\mathcal{H}^2_x = 0 \text{ for all } \varphi \in [C^\infty(\Gamma_0)]^3 \text{ with } \mathrm{div}_\Gamma \varphi = 0 \right\}.$$

Then $f \in E$ if and only if there is $\mathfrak{p} \in W^{1,2}(\Gamma_0)$ such that

$$f = \nabla^{tan}\mathfrak{p} + \mathfrak{p} H n.$$

Moreover, if f is continuous, then $\mathfrak{p} \in C^1(\Gamma_0)$.

Note that $C^\infty(\Gamma_0) = C_0^\infty(\Gamma_0)$ since Γ_0 is a closed surface. Note also that one can decompose an L^2-vector field on a surface into a surface divergence part, surface gradient part, and mean curvature part by Theorem 1.1. This is interpreted as a generalized Helmholtz-Weyl decomposition on a surface.

To prove Theorem 1.1, we prepare one proposition.

PROPOSITION 1.2. Let $f \in [L^2(\Gamma_0)]^3$ and $\mathfrak{p} \in L^2(\Gamma_0)$. Assume that for every $\varphi \in [C^\infty(\Gamma_0)]^3$ satisfying $\mathrm{div}_\Gamma \varphi = 0$,

$$\int_{\Gamma_0} (f \cdot n - H\mathfrak{p})(\varphi \cdot n) \; d\mathcal{H}^2_x = 0.$$

Then there is a $c \in \mathbb{R}$ such that

$$f \cdot n - H\mathfrak{p} = cH.$$

To prove Proposition 1.2, we prepare two lemmas.

LEMMA 1.3. Let $g, h \in L^1(\Gamma_0)$. Assume that for all $\psi \in C^\infty(\Gamma_0)$ satisfying $\int_{\Gamma_0} h\psi \; d\mathcal{H}^2_x = 0$,

$$\int_{\Gamma_0} g\psi \; d\mathcal{H}^2_x = 0.$$

Then there is $c \in \mathbb{R}$ such that

$$g = ch.$$

LEMMA 1.4. Let $\chi \in C^\infty(\Gamma_0)$ such that

$$\int_{\Gamma_0} \chi H \; d\mathcal{H}^2_x = 0.$$

Then there is $\varphi \in [C^\infty(\Gamma_0)]^3$ such that $\mathrm{div}_\Gamma \varphi = 0$ and $\varphi \cdot n = \chi$.

Proof of Lemma 1.3. When $h = 0$, we easily see that $g = 0$. Assume that $h \neq 0$. Let $\varphi \in C^\infty(\Gamma_0)$ such that

$$\int_{\Gamma_0} h\varphi \; d\mathcal{H}^2_x = 1.$$

Fix $\phi \in C^\infty(\Gamma_0)$. Set

$$\psi = \phi - \left(\int_{\Gamma_0} h\phi \ d\mathcal{H}^2_x \right) \varphi.$$

It is clear that $\psi \in C^\infty(\Gamma_0)$ and

$$\int_{\Gamma_0} h\psi \ d\mathcal{H}^2_x = 0.$$

By assumption, we observe that

$$\begin{aligned} 0 &= \int_{\Gamma_0} g\psi \ d\mathcal{H}^2_x \\ &= \int_{\Gamma_0} g\phi \ d\mathcal{H}^2_x - \left(\int_{\Gamma_0} g\varphi \ d\mathcal{H}^2_x \right) \left(\int_{\Gamma_0} h\phi \ d\mathcal{H}^2_x \right). \end{aligned}$$

Therefore, we see that for all $\phi \in C^\infty(\Gamma_0)$

$$\int_{\Gamma_0} (g - ch)\phi \ d\mathcal{H}^2_x = 0,$$

where $c = \int_{\Gamma_0} g\varphi \ d\mathcal{H}^2_x$. From fundamental lemmas of calculus of variations, we conclude that

$$g = ch.$$

Note that $C^\infty(\Gamma_0) = C_0^\infty(\Gamma_0)$. Therefore the lemma follows. □

Proof of Lemma 1.4. Fix $\chi \in C^\infty(\Gamma_0)$ such that

$$\int_{\Gamma_0} \chi H \ d\mathcal{H}^2_x = 0.$$

We consider the elliptic equation:

$$\Delta_\Gamma U = -\chi H,$$

where U is an unknown function. Since Γ_0 is a closed surface and

$$\int_{\Gamma_0} \chi H \ d\mathcal{H}^2_x = 0,$$

there is a weak solution $U \in W^{1,2}(\Gamma_0)$ such that $\langle \nabla^{tan} U, \nabla^{tan} \Phi \rangle = \langle \chi H, \Phi \rangle$ for $\Phi \in W^{1,2}(\Gamma_0)$. Moreover, we see that $U \in C^\infty(\Gamma_0)$ from the elliptic regularity theory. See Aubin [2, Section 4] and Jost [4, Appendix A] for the existence and regularity of solutions to the elliptic equation: $-\Delta_\Gamma U = F$. Set

$$\varphi = \nabla^{tan} U + \chi n.$$

We easily check that $\varphi \cdot n = \chi$ and that

$$\mathrm{div}_\Gamma \varphi = \Delta_\Gamma U - \chi H = 0.$$

Therefore the lemma follows. □

Proof of Proposition 1.2. Let $\chi \in C^\infty(\Gamma_0)$ such that

$$\int_{\Gamma_0} \chi H \ d\mathcal{H}_x^2 = 0.$$

From Lemma 1.4 there is a $\varphi \in C^\infty(\Gamma_0)$ such that $\mathrm{div}_\Gamma \varphi = 0$ and $\varphi \cdot n = \chi$. By assumption, we see that

$$\int_{\Gamma_0} (f \cdot n - H\mathfrak{p})(\varphi \cdot n) \ d\mathcal{H}_x^2 = 0.$$

Therefore we find that

$$\int_{\Gamma_0} (f \cdot n - H\mathfrak{p})\chi \ d\mathcal{H}_x^2 = 0$$

for all $\chi \in C^\infty(\Gamma_0)$ such that

$$\int_{\Gamma_0} \chi H \ d\mathcal{H}_x^2 = 0.$$

Lemma 1.3 implies that there is $c \in \mathbb{R}$ such that

$$f \cdot n - H\mathfrak{p} = cH.$$

Therefore Proposition 1.2 is proved. □

Proof of Theorem 1.1. We first show the necessary condition $\Leftarrow$). Let $\mathfrak{p} \in W^{1,2}(\Gamma_0)$. Set

$$f = \mathrm{div}_\Gamma (P_\Gamma \mathfrak{p}) = \nabla^{tan}\mathfrak{p} + \mathfrak{p}Hn.$$

It is clear that $f \in [L^2(\Gamma_0)]^3$. Fix $\varphi \in [C^\infty(\Gamma_0)]^3$ with $\mathrm{div}_\Gamma \varphi = 0$. Using integration by parts, we check that

$$\begin{aligned} \int_{\Gamma_0} f \cdot \varphi \ d\mathcal{H}_x^2 &= \int_{\Gamma_0} \mathrm{div}_\Gamma (P_\Gamma \mathfrak{p}) \cdot \varphi \ d\mathcal{H}_x^2 \\ &= - \int_{\Gamma_0} \mathfrak{p}(\mathrm{div}_\Gamma \varphi) \ d\mathcal{H}_x^2 = 0. \end{aligned}$$

Here we used the fact that $n_j \partial_j^{tan} = 0$. Therefore we see $f \in E$.

Next we prove the sufficient condition $\Rightarrow$). Let $f \in E$. By definition of E, we see that

$$\int_{\Gamma_0} f_{tan} \cdot \varphi_{tan} \ d\mathcal{H}_x^2 = 0 \text{ for all } \varphi \in [C^\infty(\Gamma_0)]^3 \text{ with } \mathrm{div}_\Gamma \varphi_{tan} = 0.$$

Here $f_{tan} := P_\Gamma f$ and $\varphi_{tan} := P_\Gamma \varphi$. Note that $f_{tan} \cdot \varphi_{tan} = f \cdot \varphi_{tan}$ and $f = f_{tan} + (f \cdot n)n$. We easily check that for every circle $\mathcal{C}$ in Γ_0

$$\int_{\mathcal{C}} f_{tan} \ d\mathcal{H}_x^1 = 0.$$

From Weyl's Theorem, there is a $\tilde{p} \in W^{1,2}(\Gamma_0)$ such that $f_{tan} = \nabla^{tan}\tilde{p}$. Therefore we have

$$f = \nabla^{tan}\tilde{p} + (f \cdot n)n.$$

Fix $\varphi \in [C^\infty(\Gamma_0)]^3$ with $\mathrm{div}_\Gamma \varphi = 0$. By definition of E, we have

$$0 = \int_{\Gamma_0} f \cdot \varphi \ d\mathcal{H}_x^2 = - \int_{\Gamma_0} Hn \cdot (\tilde{p}\varphi) \ d\mathcal{H}_x^2 + \int_{\Gamma_0} (f \cdot n)n \cdot \varphi \ d\mathcal{H}_x^2.$$

Here we used the fact that

$$\int_{\Gamma_0} (\nabla^{tan} \tilde{p}) \cdot \varphi \, d\mathcal{H}^2_x = \int_{\Gamma_0} \mathrm{div}_\Gamma(\tilde{p}\varphi) \, d\mathcal{H}^2_x = - \int_{\Gamma_0} Hn \cdot (\tilde{p}\varphi) \, d\mathcal{H}^2_x.$$

Since φ is arbitrary, it follows from Proposition 1.2 to see that there is $c \in \mathbb{R}$ such that

$$f \cdot n = \tilde{p}H + cH.$$

Set $\mathfrak{p} = \tilde{p} + c$. We find that $f = \nabla^{tan}\mathfrak{p} + \mathfrak{p}Hn$. Moreover, we see that $\mathfrak{p} \in C^1(\Gamma_0)$ when f is continuous since Γ_0 is a smooth surface. □

1.2. *Comparison of Koba-Liu-Giga's model with Taylor's.*

Let us first clarify one misinterpretation in the Appendix (I) in [6]. Let $\mathcal{M}$ be a closed 2-dimensional Riemannian manifold.

(i): Taylor [9] did not use $P_\Gamma D^{tan}(u)$ but $\{(\nabla_{\mathcal{M}} u) + {}^t(\nabla_{\mathcal{M}} u)\}/2$, where $D^{tan}(u) = \{(\nabla^{tan} u) + {}^t(\nabla^{tan} u)\}/2$ and $\nabla_{\mathcal{M}}$ is the covariant derivative. Note that in general $P_\Gamma D^{tan}(u)$ is different from $(\nabla_{\mathcal{M}} u) + {}^t(\nabla_{\mathcal{M}} u)$ even if u is a 1-form on $\mathcal{M}$. This is one interpretation in the Appendix (I) in [6]. Recall that Mitsumatsu-Yano [7] and Arnaudon-Cruzeiro [1] used Taylor's tensor $\{(\nabla_{\mathcal{M}} u) + {}^t(\nabla_{\mathcal{M}} u)\}/2$.

(ii) The equality: $P_\Gamma \mathrm{div}_\Gamma(P_\Gamma D^{tan}(v)) = \Delta_B v + Kv$ in the Appendix (I) in [6] is not right even if $\mathrm{div}_\Gamma v = 0$ and $v \cdot n = 0$. The following equality is correct: under the conditions that $\mathrm{div}_\Gamma v = 0$ and $v \cdot n = 0$,

$$2P_\Gamma \mathrm{div}_\Gamma D_\Gamma(v) = \Delta_B v + Kv$$

when we consider v as a 1-form on the surface $\Gamma_0 = \mathcal{M}$. See Jankuhn-Olshanskii-Reusken [3], Miura [8], and Koba [5] for details. Note that differential operators on a 1-form are different from the differential operators in [6].

Conclusion: The tangential incompressible fluid system in [6] is the same as Taylor's [9] when we consider v as a 1-form. Note that both systems in Mitsumatsu-Yano [7] and Arnaudon-Cruzeiro [1] agree with Taylor's system. For a more detailed comparison of our model and Taylor's model, see Miura [8, Lemma 2.5, Remark 4.2, and Remark 4.3].

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