QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXVI, NUMBER 1
MARCH 2018, PAGES 113–145
http://dx.doi.org/10.1090/qam/1487
Article electronically published on October 4, 2017

ON THE MOTION OF A LIQUID-FILLED HEAVY BODY AROUND A FIXED POINT

By

GIOVANNI P. GALDI (*Department of Mechanical Engineering and Materials Science, University of Pittsburgh, Pittsburgh, PA 15261*),

GIUSY MAZZONE (*Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240*),

AND

MAHDI MOHEBBI (*Department of Mechanical Engineering, SUNY Korea 119 Songdo Moonhwa-ro, Yeonsu-gu, Incheon, Korea 21985*)

Abstract. We study the motion of the coupled system $\mathcal{S}$ constituted by a heavy rigid body, $\mathcal{B}$, with an interior cavity entirely filled with a Navier-Stokes liquid. We suppose that $\mathcal{B}$ is constrained to move around a fixed and frictionless point, O, belonging to one of the central axes of inertia, $\mathbf{a}$, of $\mathcal{S}$. We then show, in a very general class of solutions, that the terminal motion of $\mathcal{S}$ must be a uniform rigid rotation around the vertical axis, $\mathbf{e}$, passing through O with $\mathbf{a}$ either parallel to $\mathbf{e}$ or, more generally, forming a (constant) non-zero angle, in which case the angular velocity must be sufficiently large. These results are in sharp contrast with the (classical) analogous ones when the cavity is empty, and show a remarkable stabilizing influence exerted by the liquid on the motion of $\mathcal{B}$. In order to point out these compelling differences, we apply our results to the two significant cases of the spherical pendulum and the heavy top. We show, among other things, the somehow unexpected property that a (frictionless) spherical pendulum with a cavity entirely filled with a viscous liquid may eventually reach the rest configuration with its center of mass in the lowest position.

1. Introduction. Problems involving the motion of a rigid body with a cavity filled with a viscous liquid are of fundamental interest in several applied areas of research, including dynamics of flight [12, 30], space technology [1, 2], and geophysical problems [27, 34].

Received April 28, 2017.
2010 *Mathematics Subject Classification.* Primary 35Q35, 35Q30, 76D05.
The work of the first author is partially supported by NSF DMS Grant-1614011.
E-mail address: galdi@pitt.edu
E-mail address: giusy.mazzone@vanderbilt.edu
E-mail address: mahdi.mohebbi@sunykorea.ac.kr

Besides its important role in physical and engineering disciplines, the motion of these coupled systems generates a number of mathematical questions, which are intriguing and challenging at the same time. They are principally due to the occurrence of different and coexisting dynamic properties, such as the *dissipative* nature of the liquid, and the *conservative* character of some components of the angular momentum of the coupled system *as a whole*. Thus, it is not surprising that a plethora of mathematical works has been devoted to the study of several fundamental aspects of the problem, beginning with the early contributions of Stokes [35], Zhoukovski [41], Hough [11], and Sobolev [33] to more recent papers [5, 16, 19, 20, 25, 29, 31, 32, 38–40] and comprehensive monographs [4, 15, 24], to cite a few.[1]

However, it must be emphasized that, for the most part, the above authors perform their analysis under a number of simplifying assumptions that may involve the shape of the cavity, viscosity of the liquid and "smallness" of the motion, the latter possibly leading to the disregard of significant non-linear effects. As a consequence, on the one hand, the findings are rarely of an exact nature, and, on the other hand and more importantly, these simplifications may induce one to overlook or even obscure certain important physical aspects of the problem.

Also motivated by these considerations, over the past few years the first and second author, jointly with their collaborators, have initiated a systematic rigorous study of the motion of the coupled system, $\mathcal{S}$, constituted by a rigid body with a liquid-filled cavity [7–10, 21, 22]. In particular, in [8] a rather complete analysis of the *inertial motions* of $\mathcal{S}$, characterized by $\mathcal{S}$ moving in *absence* of external forces is furnished. The analysis shows, among other things, that for very general distributions of mass and provided the initial (kinetic) energy of $\mathcal{S}$ is finite, all corresponding motions of $\mathcal{S}$ about its center of mass (in a quite large functional class) must eventually reduce to a uniform rotation around one of the central axes of inertia of $\mathcal{S}$. This shows, in particular, a stabilizing influence of the liquid on the motion of the body.

The objective of the present paper is to continue and, to an extent, complement and complete the analysis initiated in [8]. More precisely, we are interested in studying the motion of $\mathcal{S}$ when subject to the *action of gravity*. We shall perform this investigation assuming that $\mathcal{S}$ is constrained to move around a fixed (and frictionless) point O that belongs to one of its central axes of inertia, a. The reason for this choice is two-fold. For one thing, because this type of constrained motion is among the most studied in classical rigid-body mechanics (e.g. [17]), and, for another thing, because it thus allows us to compare our results with those when the cavity is liquid-empty. In doing this, we shall prove that the presence of the liquid may dramatically change the terminal dynamics of the body.

More specifically, in Sections 4 through 6 we begin to show that, for a very general class of motions (solutions à la Leray-Hopf) and for initial conditions of arbitrary "size", $\mathcal{S}$ must eventually tend to a steady-state where $\mathcal{S}$ moves as a single rigid-body (i.e. the liquid comes to a relative rest). The class of these rigid motions is characterized in Theorem 4.7. Precisely, denoting by e the axis directed along the gravity and passing

[1] See also the vast bibliography reported therein.

through O, the terminal motion will be a *uniform* rigid rotation around $\mathbf{e}$ with $\mathbf{a}$ either parallel to $\mathbf{e}$, or, more generally, forming with $\mathbf{e}$ a non-zero (constant) angle (*steady precession*; see [37, Section 10.10]). However, the latter may occur only if the terminal angular velocity is sufficiently large; see Proposition 6.3. The eventual rigid-body motion attained by the whole system is then shown to depend on the mass distribution of the system $\mathcal{S}$ and, in particular, its principal moments of inertia with respect to O; see Theorem 6.4 and Remark 6.5.

If we compare the above results with the analogous, classical ones when the cavity is liquid-empty and where the motion is generally "chaotic" at all times [17, 18, 26], we may then conclude that *the presence of the liquid has a strong stabilizing effect*, also in the presence of external forces. In order to analyze the latter in a more quantitative fashion, we apply the general results to two significant particular cases, namely, the spherical pendulum and the heavy top. In the first case (Section 7) we prove that, for a large class of initial conditions, the terminal motion must be a uniform rotation with $\mathbf{a}$ *parallel* to $\mathbf{e}$. This is in sharp contrast with the case of an empty cavity where, as is well known, the motion of the center of mass of the body is, in general, very complicated and not even time-periodic; e.g. [26, Section 5.3]. Moreover, in some relevant instances, the uniform rotation may also reduce to rest. More specifically, we show the somehow unexpected property that *a frictionless spherical pendulum, with a cavity filled with a viscous liquid, may reach the rest configuration with its center of mass at its lowest point.* Interestingly enough, this is the *same* terminal state that the pendulum would reach when the cavity is liquid-empty, and the pendulum moves in a viscous liquid ([36]).

Our second application (Section 8) regards the motion of a heavy hollow top with its interior entirely filled with viscous liquid. There are no restrictions on the distribution of mass in the body nor on the shape of the cavity, so that, in particular the top can be asymmetric. We focus our attention on the stability (in the sense of Lyapunov) of the steady-state motion, s_0, when the coupled system $\mathcal{S}$ rotates, as a single rigid-body, around $\mathbf{a}$ with constant angular velocity $\boldsymbol{\omega}_0$, and $\mathbf{a}$ parallel to $\mathbf{e}$. We first consider the interesting situation where the center of mass, G, of $\mathcal{S}$ is (initially) above O. We then show that if the moment of inertia of $\mathcal{S}$ along $\mathbf{a}$, C, is greater than those along the other two principal axes, say A and B, and $\boldsymbol{\omega}_0$ is sufficiently large in the sense of (8.5), not only is s_0 stable but also that the terminal state *must coincide with* s_0, namely, the top will eventually return to rotate uniformly around $\mathbf{e}$, with G above O; see part (a) in Theorem 8.2. This result needs some comments. In the first place, to our knowledge, it is the first *rigorous* and *non-linear* treatment of this stability problem. Previous contributions, in fact, either restrict their analysis to the stability of the rigid body only [24, 28], or else investigate the behavior of the perturbation fields by neglecting non-linear effects [16, 19, 20].[2] In the second place, our findings are in stark contrast with the (classical) analogous ones obtained when the top is liquid-empty. Actually, in such a case, under the same condition (8.5) the axis $\mathbf{a}$, already in the symmetric case, moves in a neighborhood of $\mathbf{e}$ with the top performing an *unsteady precession* around $\mathbf{e}$ (e.g. [26, Chapter 8]),

[2]Since, to date, a "linearization principle" for the relevant system of equations is not known –and likely, very difficult to prove, if at all– these results are of no avail in the non-linear context.

whereas, as mentioned earlier, the presence of the liquid forces a to remain parallel to e, thus exerting a substantial stabilizing influence regardless of the symmetry of the top. On the other side, if $\boldsymbol{\omega}_0$ is not sufficiently large, in the sense of (8.8), then s_0 is unstable and, in particular, there is an initial perturbation that will bring the top, eventually, to perform a *uniform rotation* again with a parallel to e but with G in its lowest position below O; see part (b) in Theorem 8.2. It is worth remarking that our instability condition (8.8) is more stringent than that requested by the linearized theory [16], probably due to the presence, in our analysis, of non-linear effects that are entirely disregarded in [16]; see also part (ii) in Remark 7. Finally, in parts (c) and (d) of Theorem 8.2 we provide similar results when, in the unperturbed state s_0, G is (initially) below O, under different assumptions on the relative magnitude of A, B, and C.

The plan of the paper is the following. After recalling some classical notation in Section 2, in Section 3 we state the formulation of the problem. In Section 4, we furnish a full characterization of the set of admissible steady-state solutions, whereas in Section 5, we define the functional class within which we will study the generic motion of $\mathcal{S}$. This class, basically, is constituted by weak solutions à la Leray-Hopf. Furthermore, we derive (formally) the important equations of energy balance and conservation of the vertical component of the *total* angular momentum, K_V. One important property is to show that every weak solution becomes regular for sufficiently large times. This is not completely obvious at the outset, since K_V can be arbitrarily large at all times and may possibly induce some turbulent features in the motion of the liquid. Following the arguments of [8], in Proposition 5.5 we show, however, that this is not the case. In Section 6 we study the asymptotic behavior of a generic weak solution by means of classical tools from Dynamical Systems. More specifically, we give a full characterization of the corresponding Ω-limit set (which may depend on the particular solution, due to the lack of uniqueness) and show, in particular, that it is not empty and is contained in the set of all possible steady-state solutions; see Proposition 6.3. In the subsequent Theorem 6.4 we then show which of these steady-state solutions is indeed attained by $\mathcal{S}$, under very general assumptions on its distribution of mass. Significant applications of these findings to the case of spherical pendulum and heavy top are then carried out in Sections 7 and 8, respectively. The paper ends with an Appendix dedicated to the proof of a Gronwall-like lemma and a simple stability result.

2. Basic notation. The notation we shall use throughout this paper is quite standard. By $\mathbb{N}$ we denote the set of positive integers, and by $\mathbb{R}$ that of real numbers, so that $\mathbb{R}^3$ is the Euclidean three-dimensional space. Also, S^2 indicates the unit sphere in $\mathbb{R}^3$. Vectors in $\mathbb{R}^3$ are denoted by boldfaced letters, and the canonical basis in $\mathbb{R}^3$ by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. The components of a vector v with respect to the canonical basis are $(\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3)$, whereas $|\mathsf{v}|$ represents the magnitude of v.

Let $\mathcal{O}$ be an open set of $\mathbb{R}^3$. $L^2(\mathcal{O})$, and $W^{k,2}(\mathcal{O})$, $W_0^{k,2}(\mathcal{O})$, denote the usual Lebesgue and Sobolev spaces, with norms $\|\cdot\|_2$ and $\|\cdot\|_{k,2}$, respectively. Moreover, the usual inner product in $L^2(\mathcal{O})$ will be indicated with $(\cdot, \cdot)$. We shall use the same symbol for spaces of scalar, vector and tensor functions.

For a bounded, Lipschitz domain, $\mathcal{A}$, with outward unit normal $\boldsymbol{n}$, we define

$$H(\mathcal{A}) := \{\boldsymbol{u} \in L^2(\mathcal{A}) : \operatorname{div} \boldsymbol{u} = 0 \text{ and } \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\mathcal{A}} = 0\},$$

where $\operatorname{div} \boldsymbol{u}$ and $\boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\mathcal{A}}$ are understood in the sense of distributions. We also set $\mathcal{D}_0^{1,2}(\mathcal{A}) := H(\mathcal{A}) \cap W_0^{1,2}(\mathcal{A})$.

If X is a Banach space with norm $\|\cdot\|_X$, and $I \subset \mathbb{R}$ an interval, we denote by $L^q(I;X)$ (resp. $W^{k,q}(I;X)$, $k \in \mathbb{N}$), the space of functions f from I to X such that $\left(\int_I \|f(t)\|_X^q \, dt\right)^{1/q} < \infty$ (resp. $\sum_{\ell=0}^k \left(\int_I \|\partial_t^\ell f(t)\|_X^q \, dt\right)^{1/q} < \infty$). Similarly, $C^k(I;X)$ indicates the space of functions which are k-times differentiable with values in X and having $\max_{t\in I} \|\partial_t^\ell \cdot\|_X < \infty$, for all $\ell = 0, 1, ..., k$. Moreover, $f \in C_w(I;X)$ iff the map $t \in I \mapsto \phi(f(t)) \in \mathbb{R}$ is continuous for all bounded linear functionals ϕ defined on X. Finally, if $X \equiv \mathbb{R}^d$, $d \geq 1$, in the above notation we shall omit the letter X, and denote the Euclidean norm simply by $|\cdot|$.

3. Formulation of the problem. Let $\mathcal{S} := \mathcal{B} \cup \mathcal{L}$, be the coupled system constituted by a heavy rigid-body, $\mathcal{B}$, with an interior cavity, $\mathcal{C}$, completely filled with a viscous liquid $\mathcal{L}$. Suppose that a point O of $\mathcal{B}$ is constrained to be fixed, at all times, with respect to an inertial frame, and that the center of mass, G, of $\mathcal{S}$ lies on the principal axis of inertia, $\mathbf{a}$, of $\mathcal{S}$ with respect to O.

Our main objective is to investigate the motions of $\mathcal{S}$ about the fixed point O, under the action of gravity.

In mathematical terms, we have $\mathcal{B} := \Omega_1 \setminus \overline{\Omega}_2$, $\mathcal{C} := \Omega_2$, where Ω_i, $i = 1, 2$, are simply connected bounded domains in $\mathbb{R}^3$ with $\overline{\Omega}_2 \subset \Omega_1$. Throughout this paper, we will assume that $\mathcal{C}$ is of class C^2.

Let $\mathcal{F} \equiv \{O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the body-fixed frame with origin at O, $\boldsymbol{e}_3 \equiv \mathbf{a}$, oriented from O to G, and $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ directed along the other principal axes of the inertia tensor $\boldsymbol{I}$ of $\mathcal{S}$ with respect to O. Thus, $\boldsymbol{I} = A\boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + B\boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + C\boldsymbol{e}_3 \otimes \boldsymbol{e}_3$ where $A, B, C > 0$ are constants representing, in the order, the moments of inertia of $\mathcal{S}$ with respect to $\boldsymbol{e}_1, \boldsymbol{e}_2$, and $\boldsymbol{e}_3$.

As in analogous problems of liquid-solid interaction [8, 10], it is convenient to study the motion of $\mathcal{S}$ when referred to a body-fixed frame. Thus, assuming that the viscous liquid $\mathcal{L}$ is of the Navier-Stokes type, one can show that the generic motion of $\mathcal{S}$ in the frame $\mathcal{F}$ is governed by the following set of equations (see [22] for details):

$$\begin{aligned}
&\left.\begin{aligned}
&\rho\left(\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + (\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}})\times\boldsymbol{x} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{v}\right)\\
&\qquad\qquad\qquad\qquad\qquad = \mu\Delta\boldsymbol{v} - \nabla p + \rho g\boldsymbol{\gamma}\\
&\operatorname{div}\boldsymbol{v} = 0
\end{aligned}\right\} \text{ in } \mathcal{C}\times(0,\infty),\\
&\boldsymbol{I}\cdot\dot{\boldsymbol{\omega}}_\infty + (\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty = \beta^2\boldsymbol{e}_3\times\boldsymbol{\gamma} \qquad \text{in } (0,\infty),\\
&\dot{\boldsymbol{\gamma}} + (\boldsymbol{\omega}_\infty + \boldsymbol{a})\times\boldsymbol{\gamma} = \boldsymbol{0},\\
&\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial\mathcal{C}\,.
\end{aligned} \tag{3.1}$$

Here $\boldsymbol{v}, \mu$ and ρ denote relative velocity of the liquid, its shear viscosity coefficient and (constant) density, respectively. Moreover, we set

$$\beta^2 := M\ell g, \;\; \boldsymbol{\omega}_\infty := \boldsymbol{\omega} - \boldsymbol{a}, \;\; \boldsymbol{a} := -\rho \boldsymbol{I}^{-1} \cdot \int_{\mathcal{C}} \boldsymbol{x} \times \boldsymbol{v}, \;\; p := \pi - \frac{\rho}{2} |\boldsymbol{\omega} \times \boldsymbol{x}|^2, \tag{3.2}$$

where π is the Eulerian pressure of the liquid, $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{B}$, while M is the total mass of $\mathcal{S}$, g is the magnitude of the acceleration of gravity and $\ell > 0$ is the distance between the fixed point O and G (which is constant in time by assumption). Finally, $\boldsymbol{\gamma}$ is a unit vector denoting the direction of the gravity, which is time-dependent, since the equations of motion are written in the non-inertial frame $\mathcal{F}$.

It is worth observing that, from the physical viewpoint, the generic motion of $\mathcal{S}$ consists of a combination of *dissipative*, *excited* and *conservative* ingredients, with the *dissipative* role played by the viscous liquid (see equations $(3.1)_{1,2,5}$), whereas the *excited* features are related to the dynamics of the solid (see equations $(3.1)_{3,4}$). Moreover, as we will show later on (see equation (5.6)), the component of the *total* angular momentum (with respect to O) along the direction of the gravity must be *conserved* in every motion of $\mathcal{S}$.

4. Steady-state solutions. One of the main goals of this paper is to analyze the asymptotic behavior in time of the generic motion of the coupled system $\mathcal{S}$. In particular, we will investigate under which conditions the ultimate motion of $\mathcal{S}$ reduces to a steady-state, namely, a time-independent solution of (3.1) (see Theorem 6.4). As a matter of fact, the class of such solutions is fairly rich, and the objective of the current section is to provide a corresponding complete and detailed characterization of steady states.

We begin to observe that from (3.1) it follows that all steady-state motions must be solutions to the following boundary value problem:

$$\begin{aligned} &\left.\begin{aligned} &\boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{v} = \nu \Delta \boldsymbol{v} - \nabla \tilde{p} \\ &\operatorname{div} \boldsymbol{v} = 0 \end{aligned}\right\} \text{ in } \mathcal{C}, \\ &(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_3 \times \boldsymbol{\gamma}, \\ &(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{\gamma} = \boldsymbol{0}, \\ &\boldsymbol{v} = \boldsymbol{0} \qquad \text{on } \partial C, \end{aligned} \tag{4.1}$$

where $\nu := \mu/\rho$ is the coefficient of kinematic viscosity, and $\tilde{p} := p/\rho - g\boldsymbol{\gamma} \cdot \boldsymbol{x}$ is the"modified" pressure.

We want to characterize solutions to (4.1) in the (a priori) very general class of *weak solutions*. To this end, we give the following.

DEFINITION 4.1. The triple $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma}) \in \mathcal{D}_0^{1,2}(\mathcal{C}) \times \mathbb{R}^3 \times S^2$ is a *weak steady-state solution* to (3.1), if it satisfies the following system of equations:

$$\begin{cases} \nu \displaystyle\int_{\mathcal{C}} \nabla \boldsymbol{v} : \nabla \boldsymbol{\varphi} = \int_{\mathcal{C}} \{(\boldsymbol{v} \cdot \nabla \boldsymbol{\varphi}) \cdot \boldsymbol{v} - 2[(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{v}] \cdot \boldsymbol{\varphi}\}, \text{ all } \boldsymbol{\varphi} \in \mathcal{D}_0^{1,2}(\mathcal{C}), \\ (\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_3 \times \boldsymbol{\gamma}, \\ (\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{\gamma} = \boldsymbol{0}. \end{cases} \tag{4.2}$$

REMARK 4.2. Let $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak steady-state solution. By standard arguments (see [6, Lemma IX.1.2]), one can show the existence of a corresponding pressure field $\tilde{p} \in L^2(\mathcal{C})$, $\int_{\mathcal{C}} \tilde{p} = 0$, such that $(\boldsymbol{v}, \tilde{p})$ is a weak solution to the Stokes problem

$$\left.\begin{aligned} &\nu \Delta \boldsymbol{v} = \nabla \tilde{p} + \boldsymbol{f}, \\ &\operatorname{div} \boldsymbol{v} = 0, \end{aligned}\right\} \text{ in } \mathcal{C},$$
$$\boldsymbol{v} = \mathbf{0} \text{ on } \partial\mathcal{C},$$

where $\boldsymbol{f} := \boldsymbol{v} \cdot \nabla \boldsymbol{v} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{v}$. Therefore, by using well-known results (see [6, Theorems IV.4.1, IV.4.2 and IV.6.1]) along with a classical boot-strap procedure, we may deduce that $\boldsymbol{v} \in W^{2,2}(\mathcal{C}) \cap C^\infty(\mathcal{C})$, $\tilde{p} \in W^{1,2}(\mathcal{C}) \cap C^\infty(\mathcal{C})$, and equations $(4.1)_{1,2}$ are satisfied in the ordinary sense.

We have the following simple but important result.

LEMMA 4.3. $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ is a weak steady-state solution to (3.1) if and only if it satisfies

$$\begin{cases} \boldsymbol{v} \equiv \mathbf{0} & \text{in } \mathcal{C}, \\ \boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_3 \times \boldsymbol{\gamma}, \\ \boldsymbol{\omega}_\infty \times \boldsymbol{\gamma} = \mathbf{0}, \end{cases} \tag{4.3}$$

with corresponding pressure field $p = \rho\, g\, \boldsymbol{\gamma} \cdot \boldsymbol{x}$.

Proof. Let $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak steady-state solution to (3.1). In particular, $\boldsymbol{v} \in \mathcal{D}_0^{1,2}(\mathcal{C})$, and we can use it in place of $\boldsymbol{\varphi}$ in the equation $(4.2)_1$. By a simple and easily justified integration by parts on the right-hand side of the resulting equation leads to $\|\nabla \boldsymbol{v}\|_2^2 = 0$. Thus, by Poincaré inequality,

$$\boldsymbol{v} = \boldsymbol{a} \equiv \mathbf{0}.$$

Replacing this information back in $(4.2)_{2,3}$, we at once derive (4.3). The reverse implication is obvious. $\square$

REMARK 4.4. The result just shown tells us, in particular, that *in every steady-state motion* the liquid is *at rest relative to* $\mathcal{B}$, which means that *the coupled system* $\mathcal{S}$ *moves like a single rigid-body.*

We now observe that any solution to (4.3) has the form $(\boldsymbol{v} \equiv \mathbf{0}, \boldsymbol{\omega}_\infty = \lambda\boldsymbol{\gamma}, \boldsymbol{\gamma})$, where λ and $\boldsymbol{\gamma} = \gamma_1 \boldsymbol{e}_1 + \gamma_2 \boldsymbol{e}_2 + \gamma_3 \boldsymbol{e}_3$ satisfy

$$\begin{cases} \lambda^2 (C - B)\gamma_2\gamma_3 = -\beta^2 \gamma_2, \\ \lambda^2 (A - C)\gamma_1\gamma_3 = \beta^2 \gamma_1, \\ \lambda^2 (B - A)\gamma_1\gamma_2 = 0, \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{cases} \tag{4.4}$$

Note that from the above equations, we have $\gamma_3 \neq 0$ since $\beta^2 \neq 0$.

In order to discuss and classify all possible solutions to (4.4), it is convenient to introduce appropriate classes of steady-state solutions. More precisely, let

$$\mathsf{PR} := \left\{ (\boldsymbol{u} \equiv \mathbf{0}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{D}_0^{1,2}(\mathcal{C}) \times \mathbb{R}^3 \times S^2 : \ \boldsymbol{q} \times \boldsymbol{e}_3 = \mathbf{0}, \ \boldsymbol{\omega} = \lambda \boldsymbol{q} \text{ for some } \lambda \in \mathbb{R} \right\}.$$

Clearly, each element of PR represents a *permanent (uniform) rotation* of $\mathcal{S}$ around $\boldsymbol{e}_3$.

Remark 4.5. Permanent rotations can only occur around the principal axis of inertia of $\mathcal{S}$ that passes through O and G, with the latter parallel to $\boldsymbol{q}$ (i.e., the direction of the gravity).

Next, define

$$\mathsf{SP} := \{(\boldsymbol{u} \equiv \mathbf{0}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{D}_0^{1,2}(\mathcal{C}) \times \mathbb{R}^3 \times S^2 : \ \boldsymbol{\omega} = \lambda \boldsymbol{q}, \ q_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \\ \text{for some } \lambda \in \mathbb{R} - \{0\}, \ 0 < q_3 < 1\},$$

along with

$$\mathsf{SP}_1 := \{(\boldsymbol{u} \equiv \mathbf{0}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{D}_0^{1,2}(\mathcal{C}) \times \mathbb{R}^3 \times S^2 : \ \boldsymbol{\omega} = \lambda \boldsymbol{q}, \text{ some } \lambda \in \mathbb{R} - \{0\}, \\ q_2 \equiv 0, \ \ q_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \ 0 < q_3 < 1\,\},$$

$$\mathsf{SP}_2 := \{(\boldsymbol{u} \equiv \mathbf{0}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{D}_0^{1,2}(\mathcal{C}) \times \mathbb{R}^3 \times S^2 : \ \boldsymbol{\omega} = \lambda \boldsymbol{q}, \ \text{ some } \lambda \in \mathbb{R} - \{0\}, \\ q_1 \equiv 0, \ \ q_3 = -\frac{\beta^2}{\lambda^2(C-B)}, \ 0 < q_3 < 1\,\}.$$

Elements of SP_i, $i = 1, 2$, represent *steady precessions*, namely, motions that when observed from an inertial frame, are characterized by having the $\boldsymbol{e}_3$-axis rotating with constant angular velocity about $\boldsymbol{q}$ (i.e. the direction of the gravity) and describing a cone of constant (non-zero) angle; see, e.g., [37, Section 10.10].

Remark 4.6. For fixed A, B and C, the class SP_i, $i = 1, 2$, is not empty if and only if $|\boldsymbol{\omega}|$ is sufficiently large; see equations (4.5), (4.6).

We are now in a position to provide a complete and detailed description of the class of steady-state solutions to (3.1) in terms of principal moments of inertia of $\mathcal{S}$.

Theorem 4.7. The set S of all steady-state solutions to (3.1) is not empty and satisfies

$$\mathsf{S} \subseteq \mathsf{PR} \cup \mathsf{SP}\,.$$

More precisely, the following characterization holds.

If $\lambda = 0$, then

$$\mathsf{S} = \{(\boldsymbol{u} \equiv \mathbf{0}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{D}_0^{1,2}(\mathcal{C}) \times \mathbb{R}^3 \times S^2 : \boldsymbol{\omega} = \mathbf{0}\,, \ \boldsymbol{q} = \pm \boldsymbol{e}_3\}\,.$$

If $\lambda \neq 0$, we have, instead:

(1) If $A = B = C$, then $\mathsf{S} \equiv \mathsf{PR}$.
(2) If $A = B \neq C$, then $\mathsf{S} \equiv \mathsf{PR} \cup \mathsf{SP}$.
(3) If $A \neq B = C$, then $\mathsf{S} \equiv \mathsf{PR} \cup \mathsf{SP}_1$.
(4) If $A = C \neq B$, then $\mathsf{S} \equiv \mathsf{PR} \cup \mathsf{SP}_2$.
(5) If $A \neq B \neq C$, then $\mathsf{S} \equiv \mathsf{PR} \cup \mathsf{SP}_1 \cup \mathsf{SP}_2$.

In all cases, the corresponding pressure field is given by $p = \rho g \boldsymbol{\gamma} \cdot \boldsymbol{x}$, for all $\boldsymbol{x} \in \mathcal{C}$.

Proof. Let us denote by $\mathsf{s} := (\boldsymbol{v} \equiv \mathbf{0}, \boldsymbol{\omega}_\infty = \lambda\boldsymbol{\gamma}, \boldsymbol{\gamma})$, a generic solution to (4.3), for some $\lambda \in \mathbb{R}$.

If $\lambda = 0$, then $\gamma_1 = \gamma_2 = 0$ and $\gamma_3 = \pm 1$. Thus, $(\boldsymbol{v} \equiv \mathbf{0}, \boldsymbol{\omega}_\infty \equiv \mathbf{0}, \boldsymbol{\gamma} = \pm\boldsymbol{e}_3)$ which proves the first claim. From now on, we assume $\lambda \neq 0$.

(1) Let $A = B = C$. Then, (4.4) furnishes

$$\begin{cases} \gamma_2 = 0, \\ \gamma_1 = 0, \\ \gamma_3 = \pm 1. \end{cases}$$

Thus, $\mathsf{s} = (\boldsymbol{v} \equiv \mathbf{0}, \boldsymbol{\omega}_\infty = \pm\lambda\boldsymbol{e}_3, \boldsymbol{\gamma} = \pm\boldsymbol{e}_3)$.

(2) Let $A = B \neq C$. Then, (4.4) becomes

$$\begin{cases} \lambda^2(C - A)\gamma_2\gamma_3 = -\beta^2\gamma_2, \\ \lambda^2(A - C)\gamma_1\gamma_3 = \beta^2\gamma_1, \\ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1. \end{cases}$$

As remarked earlier, $\gamma_3 \neq 0$ otherwise, since $\beta \neq 0$, from the first two equations we would infer $\gamma_1 = \gamma_2 = \gamma_3 = 0$, which is at odds with the third equation. Thus, only the following cases (a)–(c) may occur.

(a) $\gamma_1 = \gamma_2 = 0$. So, $\gamma_3 = \pm 1$, and $\mathsf{s} = (\boldsymbol{v} \equiv \mathbf{0}, \boldsymbol{\omega}_\infty = \pm\lambda\boldsymbol{e}_3, \boldsymbol{\gamma} = \pm\boldsymbol{e}_3)$, namely, $\mathsf{s} \in \mathsf{PR}$.

(b) $\gamma_1, \gamma_2 \neq 0$. In this case,

$$\gamma_3 = -\frac{\beta^2}{\lambda^2(C - A)}, \quad \gamma_1^2 + \gamma_2^2 = 1 - \frac{\beta^4}{\lambda^4(C - A)^2}.$$

(c) $\gamma_1 = 0$, $\gamma_2 \neq 0$. In this case,

$$\gamma_3 = -\frac{\beta^2}{\lambda^2(C - A)}, \quad \gamma_2 = \pm\sqrt{1 - \frac{\beta^4}{\lambda^4(C - A)^2}}.$$

(d) $\gamma_1 \neq 0$, $\gamma_2 = 0$. In this case,

$$\gamma_3 = -\frac{\beta^2}{\lambda^2(C - A)}, \quad \gamma_1 = \pm\sqrt{1 - \frac{\beta^4}{\lambda^4(C - A)^2}}.$$

Thus, in all cases (b)–(d), $\mathsf{s} \in \mathsf{SP}$, provided, of course,

$$\beta^2 < \lambda^2|C - A| \tag{4.5}$$

(3) Let $A \neq B = C$. From $(4.4)_1$, it follows that $\gamma_2 = 0$. Moreover, $(4.4)_2$ implies that only the following two cases may occur.

(a) $\gamma_1 = 0$, then $\gamma_3 = \pm 1$ and $\mathsf{s} \in \mathsf{PR}$.

(b) $\gamma_1 \neq 0$, then

$$\gamma_3 = -\frac{\beta^2}{\lambda^2(C - A)}, \quad \gamma_1 = \pm\sqrt{1 - \frac{\beta^4}{\lambda^4(C - A)^2}},$$

and $\mathsf{s} \in \mathsf{SP}_1$, provided (4.5) holds.

(4) Let $A = C \neq B$. From $(4.4)_2$, it follows that $\gamma_1 = 0$. Moreover, $(4.4)_1$ implies that we have only the following two cases:
 (a) $\gamma_2 = 0$, then $\gamma_3 = \pm 1$ and $\mathsf{s} \in \mathsf{PR}$.
 (b) $\gamma_2 \neq 0$, then

$$\gamma_3 = -\frac{\beta^2}{\lambda^2(C-B)}, \quad \gamma_2 = \pm\sqrt{1 - \frac{\beta^4}{\lambda^4(C-B)^2}},$$

 and $\mathsf{s} \in \mathsf{SP}_2$, provided

$$\beta^2 < \lambda^2 |C - B| \,. \tag{4.6}$$

(5) Let $A \neq B \neq C$. From $(4.4)_3$, it follows that at least one of the components γ_1, γ_2 is zero. Thus, we have the following cases:
 (a) $\gamma_1 = \gamma_2 = 0$, then $\gamma_3 = \pm 1$, and $\mathsf{s} \in \mathsf{PR}$.
 (b) $\gamma_1 \neq 0, \gamma_2 = 0$, then

$$\gamma_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \quad \gamma_1 = \pm\sqrt{1 - \frac{\beta^4}{\lambda^4(C-A)^2}},$$

 and $\mathsf{s} \in \mathsf{SP}_1$, provided (4.5) holds.
 (c) $\gamma_1 = 0, \gamma_2 \neq 0$, then

$$\gamma_3 = -\frac{\beta^2}{\lambda^2(C-B)}, \quad \gamma_2 = \pm\sqrt{1 - \frac{\beta^4}{\lambda^4(C-B)^2}},$$

 and $\mathsf{s} \in \mathsf{SP}_2$, provided (4.6) holds □

Our next result furnishes sufficient conditions for a permanent rotation around $\boldsymbol{e}_3$ to be an *axially isolated* steady-state. More precisely, set $\mathcal{H} := H(\mathcal{C}) \times \mathbb{R}^3 \times S^2$ endowed with its natural topology; the following proposition holds.

Proposition 4.8. The following two properties hold:
(1) Let $m_0 = (\mathbf{0}, r_0\, \boldsymbol{e}_3, -\boldsymbol{e}_3) \in \mathsf{PR}$. If

$$r_0^2 \neq \frac{\beta^2}{C-A}, \frac{\beta^2}{C-B}, \tag{4.7}$$

 then there exists a neighborhood of m_0, $\mathcal{I}(m_0) \subset \mathcal{H}$, such that

$$m \in \mathcal{I}(m_0) \cap \mathsf{S} \quad \Rightarrow \quad m \in \mathsf{PR}.$$

(2) Let $m_1 = (\mathbf{0}, r_0\, \boldsymbol{e}_3, \boldsymbol{e}_3) \in \mathsf{PR}$. If

$$r_0^2 \neq -\frac{\beta^2}{C-A}, -\frac{\beta^2}{C-B}, \tag{4.8}$$

 then there exists a neighborhood of m_1, $\mathcal{I}(m_1) \subset \mathcal{H}$, such that

$$m \in \mathcal{I}(m_1) \cap \mathsf{S} \quad \Rightarrow \quad m \in \mathsf{PR}.$$

Proof. Consider case (1) and let us argue by contradiction. By Theorem 4.7, we know $\mathsf{S} \subseteq \mathsf{PR} \cup \mathsf{SP}$. Thus, assume that there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ with $m_n := (\mathbf{0}, \lambda_n\boldsymbol{\gamma}_n, \boldsymbol{\gamma}_n) \in \mathsf{SP}$ for all $n \in \mathbb{N}$, such that $m_n \to m_0$ as $n \to \infty$; precisely

$$|\lambda_n\boldsymbol{\gamma}_n - r_0\boldsymbol{e}_3| \to 0 \quad \text{and} \quad |\boldsymbol{\gamma}_n + \boldsymbol{e}_3| \to 0 \text{ as } n \to \infty.$$

So, as $n \to \infty$,

$$(\boldsymbol{\gamma}_n \cdot \boldsymbol{e}_1) \to 0,\ (\boldsymbol{\gamma}_n \cdot \boldsymbol{e}_2) \to 0,\ (\boldsymbol{\gamma}_n \cdot \boldsymbol{e}_3) \to -1 \text{ and } \lambda_n \to -r_0\,. \tag{4.9}$$

On the other side, since $m_n \in \mathsf{SP}$ for all $n \in \mathbb{N}$, either

$$\boldsymbol{\gamma}_n \cdot \boldsymbol{e}_3 = -\frac{\beta^2}{\lambda_n^2(C-A)}$$

or

$$\boldsymbol{\gamma}_n \cdot \boldsymbol{e}_3 = -\frac{\beta^2}{\lambda_n^2(C-B)}.$$

Letting $n \to \infty$ on both sides of the last two equations, by (4.9), we find a result in contradiction with the hypothesis (4.7).

The proof of the second statement is carried out by an entirely analogous argument and will be therefore omitted. □

5. Weak solutions and preliminary results. The study of the motion of a liquid-filled rigid-body constrained to move around a fixed point will be carried out in a considerably large class of solutions to (3.1) characterized, basically, by having finite total energy (weak solutions *à la Leray-Hopf*). Before introducing this class, let us *formally* derive the balance of the total energy, together with an important first integral of motion for $\mathcal{S}$.

To this end, we recall the following result for whose proof we refer to [15, Chapter 1, Section 7.2.2 and 7.2.3]: for any bounded Lipschitz domain $\mathcal{C}$, the functional

$$\begin{aligned}\langle\cdot,\cdot\rangle :\ &(\boldsymbol{u},\boldsymbol{v}) \in H(\mathcal{C}) \times H(\mathcal{C})\\ &\mapsto \langle\boldsymbol{u},\boldsymbol{v}\rangle := \rho(\boldsymbol{u},\boldsymbol{v}) - \left(\int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \boldsymbol{u}\right) \cdot \boldsymbol{I}^{-1} \cdot \left(\int_{\mathcal{C}} \rho\, \boldsymbol{x} \times \boldsymbol{v}\right)\end{aligned} \tag{5.1}$$

defines an inner product in $L^2(\mathcal{C})$ with associated norm $\|\boldsymbol{u}\| := \langle\boldsymbol{u},\boldsymbol{u}\rangle^{1/2}$, which is equivalent to the norm $\|\cdot\|_2$. In particular, there exists a positive constant $c < 1$ such that

$$c\,\|\boldsymbol{u}\|_2^2 \le \|\boldsymbol{u}\|^2 \le \|\boldsymbol{u}\|_2^2, \qquad \text{all } u \in H(\mathcal{C}). \tag{5.2}$$

We shall now (formally) derive the *energy balance* equation:

$$\frac{d}{dt}(\mathcal{E} + \mathcal{U}) + 2\mu\|\nabla\boldsymbol{v}\|_2^2 = 0, \tag{5.3}$$

where

$$\mathcal{E}(t) := \rho\|\boldsymbol{v}\|^2 + \boldsymbol{\omega}_\infty \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty, \qquad \mathcal{U}(t) := -2\beta^2\boldsymbol{\gamma}\cdot\boldsymbol{e}_3,$$

are *kinetic* and *potential* energy of $\mathcal{S}$, respectively. To obtain (5.3), let us formally dot-multiply $(3.1)_1$ by $\boldsymbol{v}$, and integrate the resulting equation over $\mathcal{C}$. By a simple integration by parts we show

$$\frac{1}{2}\rho\frac{d}{dt}\|\boldsymbol{v}\|_2^2 + \rho\int_{\mathcal{C}}[(\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}})\times\boldsymbol{x}]\cdot\boldsymbol{v} + \mu\|\nabla\boldsymbol{v}\|_2^2 = 0. \tag{5.4}$$

Using $(3.1)_3$ and $(3.1)_4$, one can deduce that

$$\rho\int_{\mathcal{C}}[(\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}})\times\boldsymbol{x}]\cdot\boldsymbol{v} = \frac{1}{2}\frac{d}{dt}(\boldsymbol{\omega}_\infty\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty) - \beta^2\frac{d}{dt}(\boldsymbol{\gamma}\cdot\boldsymbol{e}_3) - \frac{1}{2}\frac{d}{dt}(\boldsymbol{a}\cdot\boldsymbol{I}\cdot\boldsymbol{a}). \tag{5.5}$$

Taking into account (5.4) and (5.5) along with (5.1), we at once infer (5.3).

We also notice that the projection of the total angular momentum along $\boldsymbol{\gamma}$ has to be conserved:

$$\frac{d}{dt}(\boldsymbol{\gamma}\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty) = 0. \tag{5.6}$$

Indeed, let us dot-multiply both sides of $(3.1)_3$ by $\boldsymbol{\gamma}$, and of $(3.1)_4$ by $\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty$. Summing up the resulting equations, we find

$$\begin{aligned}\frac{d}{dt}(\boldsymbol{\gamma}\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty) &= -\boldsymbol{\gamma}\cdot[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty] - (\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty)\cdot[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{\gamma}]\\ &= -\boldsymbol{\gamma}\cdot[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty] + \boldsymbol{\gamma}\cdot[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times(\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty)] = 0.\end{aligned}$$

The *weak formulation* for the problem (3.1) is obtained in the usual way, namely, by dot-multiplying both sides of $(3.1)_1$ by $\boldsymbol{\psi}\in\mathcal{D}_0^{1,2}(\mathcal{C})$, and integrating (by parts) the resulting equation over $\mathcal{C}\times(0,t)$; this leads to the following:

$$\begin{aligned}(\rho\,\boldsymbol{v}(t),\boldsymbol{\psi}) &+ \rho(\boldsymbol{\omega}_\infty(t)+\boldsymbol{a}(t))\cdot\int_{\mathcal{C}}\boldsymbol{x}\times\boldsymbol{\psi}\\ &+\int_0^t\{\rho(\boldsymbol{v}\cdot\nabla\boldsymbol{v},\boldsymbol{\psi}) + 2\rho((\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{v},\boldsymbol{\psi}) + \mu(\nabla\boldsymbol{v},\nabla\boldsymbol{\psi})\}\\ =&(\rho\,\boldsymbol{v}(0),\boldsymbol{\psi}) + \rho(\boldsymbol{\omega}_\infty(0)+\boldsymbol{a}(0))\cdot\int_{\mathcal{C}}\boldsymbol{x}\times\boldsymbol{\psi}\,,\\ &\text{for all } \boldsymbol{\psi}\in\mathcal{D}_0^{1,2}(\mathcal{C}) \text{ and all } t\in(0,\infty).\end{aligned} \tag{5.7}$$

Moreover, integrating $(3.1)_3$ and $(3.1)_4$ over $(0,t)$ we get

$$\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty(t) = \boldsymbol{I}\cdot\boldsymbol{\omega}_\infty(0) - \int_0^t\left[(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times(\boldsymbol{I}\cdot\boldsymbol{\omega}_\infty) - \beta^2(\boldsymbol{e}_3\times\boldsymbol{\gamma})\right], \qquad \text{for all } t\in(0,\infty) \tag{5.8}$$

and

$$\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(0) - \int_0^t(\boldsymbol{\omega}_\infty+\boldsymbol{a})\times\boldsymbol{\gamma}\,, \qquad \text{for all } t\in(0,\infty). \tag{5.9}$$

Definition 5.1. The triple $(\boldsymbol{v},\boldsymbol{\omega}_\infty,\boldsymbol{\gamma})$ is a *weak solution* to (3.1) if it meets the following requirements:

(a) $\boldsymbol{v}\in C_w([0,\infty);H(\mathcal{C}))\cap L^\infty(0,\infty;H(\mathcal{C}))\cap L^2(0,\infty;W_0^{1,2}(\mathcal{C}))$;
(b) $\boldsymbol{\omega}_\infty\in C^0([0,\infty))\cap C^1((0,\infty))$, $\boldsymbol{\gamma}\in C^1([0,\infty);S^2)$;
(c) $(\boldsymbol{v},\boldsymbol{\omega}_\infty,\boldsymbol{\gamma})$ satisfies (5.7), (5.8) and (5.9);

(d) $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ obeys the "strong" energy inequality:

$$\mathcal{E}(t) + \mathcal{U}(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}(\tau)\|_2^2 \, d\tau \leq \mathcal{E}(s) + \mathcal{U}(s) \tag{5.10}$$

for all $t \geq s$ and a.a. $s \geq 0$ including $s = 0$.

The class of weak solutions to (3.1) is not empty, provided the initial motion imparted to the system has finite total energy. Moreover, since by (b) $\boldsymbol{\omega}_\infty \in C^1((0,\infty))$ and $\boldsymbol{\gamma} \in C^1([0,\infty))$, we may use the same argument employed earlier to conclude that weak solutions satisfy indeed also (5.6). These properties are summarized in the following proposition, whose proof is quite standard and, therefore, will only be sketched (see [21, 22] for details).

Proposition 5.2. For every $\boldsymbol{v}_0 \in H(\mathcal{C}), \boldsymbol{\omega}_{\infty 0} \in \mathbb{R}^3$ and $\boldsymbol{\gamma}_0 \in S^2$, there exists at least one weak solution to (3.1) such that

$$\lim_{t\to 0^+} \|\boldsymbol{v}(t) - \boldsymbol{v}_0\|_2 = \lim_{t\to 0^+} |\boldsymbol{\omega}_\infty(t) - \boldsymbol{\omega}_{\infty 0}| = \lim_{t\to 0^+} |\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}_0| = 0.$$

Proof. The proof of the statement can be accomplished with a combination of the classical Galerkin method with a priori estimates of the energy. Let $\{\boldsymbol{\psi}_n\}_{\in\mathbb{N}}$ be a subset of smooth functions from $\mathcal{D}_0^{1,2}(\mathcal{C})$ whose linear hull is dense in $\mathcal{D}_0^{1,2}(\mathcal{C})$, and let us normalize it with respect to the inner product $\langle\cdot,\cdot\rangle$ defined in (5.1). We look for "approximate" solutions

$$\boldsymbol{v}_n(\boldsymbol{x},t) = \sum_{k=1}^n c_{nk}(t)\boldsymbol{\psi}_k(\boldsymbol{x}), \quad \boldsymbol{\omega}_{\infty,n}(t) = \sum_{i=1}^3 \tilde{c}_{ni}(t)\boldsymbol{e}_i, \quad \boldsymbol{\gamma}_n(t) = \sum_{j=1}^3 \hat{c}_{nj}(t)\boldsymbol{e}_j,$$

satisfying (5.7), (5.8) and (5.9) for all $n \in \mathbb{N}$. This leads to a system of first order ordinary differential equations in the unknowns c_{nk}, $\tilde{c}_{ni}$ and $\hat{c}_{nj}$. Thanks to the balance of energy equation (5.3) and (5.2), we get

$$\begin{aligned}\rho\|\boldsymbol{v}_n\|^2(t) + \boldsymbol{\omega}_{\infty,n}(t)\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_{\infty,n}(t) - 2\beta^2\boldsymbol{e}_3\cdot\boldsymbol{\gamma}_n(t)&\\ \leq \rho\|\boldsymbol{v}_n\|^2(0) + \boldsymbol{\omega}_{\infty,n}(0)\cdot\boldsymbol{I}\cdot\boldsymbol{\omega}_{\infty,n}(0) - 2\beta^2\boldsymbol{e}_3\cdot\boldsymbol{\gamma}_n(0),&\end{aligned}$$

which allows us to show that this system admits a unique global (in time) solution $(c_{nk}, \tilde{c}_{ni}, \hat{c}_{nj})$ corresponding to initial data

$$(c_{nk}(0) = \langle\boldsymbol{v}_0, \boldsymbol{\psi}_k\rangle, \tilde{c}_{ni}(0) = \boldsymbol{\omega}_{\infty 0}\cdot\boldsymbol{e}_i, \hat{c}_{nj} = \boldsymbol{\gamma}_0\cdot\boldsymbol{e}_j)$$

for all $n \in \mathbb{N}$, $k = 1,\ldots,n$, $i,j = 1,2,3$. We omit the details of the previous argument and the proof of the convergence properties of these approximate solutions as the treatment is standard and analogous (up to some minor changes and adaptations) to the one given in [21], Chapter 3. □

Remark 5.3. Using standard arguments, it can be shown that if $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ is sufficiently smooth to allow for integration by parts on $\mathcal{C} \times (0,\infty)$, then there exists a corresponding *pressure field* $p = p(\boldsymbol{x},t)$ such that $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma}, p)$ satisfies $(3.1)_{1,2,3,4}$ a.e. in space and time.

As in the case of the "classical" Navier-Stokes problem, also for the problem at hand, the uniqueness and continuous dependence upon initial data of weak solutions is not known. However, as in the classical Navier-Stokes case, one can show that the above properties hold for any weak solution possessing a further regularity, as stated in the next proposition, and for whose proof we refer to [22], Proposition 5.1.4, and [21], Theorem 3.4.2.

PROPOSITION 5.4. Let $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$, $(\boldsymbol{v}^*, \boldsymbol{\omega}_\infty^*, \boldsymbol{\gamma}^*)$ be two weak solutions corresponding to initial data $(\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0)$ and $(\boldsymbol{v}_0^*, \boldsymbol{\omega}_0^*, \boldsymbol{\gamma}_0^*)$, respectively. Suppose that there exists a time $T > 0$ such that

$$\boldsymbol{v}^* \in L^p(0,T;L^q(\mathcal{C})), \quad \frac{2}{p} + \frac{3}{q} = 1, \text{ for some } q > 3. \tag{5.11}$$

Then, there exists a constant $c > 0$, depending only on $\operatorname{ess\,sup}_{t\in[0,T]}\|\boldsymbol{v}^*(t)\|_2$, $\|\boldsymbol{v}^*\|_{L^p(0,T;L^q(\mathcal{C}))}$ and $\max_{t\in[0,T]}|\boldsymbol{\omega}_\infty^*(t)|$, such that

$$\begin{aligned}\|\boldsymbol{v}(t) - \boldsymbol{v}^*(t)\|_2 + |\boldsymbol{\omega}_\infty(t) - \boldsymbol{\omega}_\infty^*(t)| + |\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}^*(t)| \\ \le c\left(\|\boldsymbol{v}_0 - \boldsymbol{v}_0^*\|_2 + |\boldsymbol{\omega}_0 - \boldsymbol{\omega}_0^*| + |\boldsymbol{\gamma}_0 - \boldsymbol{\gamma}_0^*|\right).\end{aligned}$$

In particular, if $(\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0) \equiv (\boldsymbol{v}_0^*, \boldsymbol{\omega}_0^*, \boldsymbol{\gamma}_0^*)$, then $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma}) \equiv (\boldsymbol{v}^*, \boldsymbol{\omega}_\infty^*, \boldsymbol{\gamma}^*)$ a.e. in $[0,T] \times \mathcal{C}$.

Moreover, the *energy equality* holds:

$$\mathcal{E}(t) + \mathcal{U}(t) + 2\mu \int_s^t \|\nabla \boldsymbol{v}\|_2^2 = \mathcal{E}(s) + \mathcal{U}(s) \;\text{ for all } 0 \le s \le t \le T.$$

We next show the fundamental property that weak solutions become *strong* after a *sufficiently large time*. As we expect from the classical Navier-Stokes theory, this will happen if the weak solution becomes eventually "small" in appropriate norms. In the case at hand, however, such a property, at the outset, does not seem obvious, because of conservation of the projection of the *total* angular momentum in the direction of the gravity (see (5.6)), whose size can be made *arbitrarily large*. However, we can show the following.

PROPOSITION 5.5. Let $\mathsf{s} := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution corresponding to initial data with finite energy, in the sense of Proposition 5.2. Then, there exists $t_0 = t_0(\mathsf{s}) > 0$ such that for all $T > 0$

$$\begin{aligned}&\boldsymbol{v} \in C^0([t_0, t_0+T]; W_0^{1,2}(\mathcal{C})) \cap L^2(t_0, t_0+T; W^{2,2}(\mathcal{C})), \\ &\qquad \boldsymbol{v}_t \in L^2(t_0, t_0+T; H(\mathcal{C})), \; \boldsymbol{\omega}_\infty \in W^{1,\infty}(t_0, t_0+T), \; \boldsymbol{\gamma} \in W^{1,\infty}(t_0, t_0+T).\end{aligned}$$

Moreover, there exists $p \in L^2(t_0, t_0+T; W^{1,2}(\mathcal{C}))$, all $T > 0$; such that $(\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma}, p)$ satisfies (3.1) a.e. in (t_0, ∞). Finally,

$$\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_{1,2} = 0. \tag{5.12}$$

Proof. The stated properties are obtained by suitably adapting to the current situation the arguments employed in the proofs of [8, Theorem 1 and Proposition 1] and

[22, Proposition 5.1.5] in a slightly different context. Due to their importance in accomplishing the main results, for the reader's sake we find it appropriate to sketch a proof here, by limiting ourselves to derive the necessary fundamental estimates, at least formally. For all missing technical details, we refer to the references cited above. Now, by the requirement (d) of weak solution, it easily follows that for any given $\varepsilon, \eta > 0$ there is $t_0 = t_0(\varepsilon, \eta, \mathsf{s}) > 0$ such that

$$\|\nabla \boldsymbol{v}(t_0)\|_2 < \varepsilon\,, \quad \int_{t_0}^{\infty} \|\nabla \boldsymbol{v}(\tau)\|_2^2\, d\tau < \eta\,. \tag{5.13}$$

We now use $(\boldsymbol{v}(t_0), \boldsymbol{\omega}_\infty(t_0), \boldsymbol{\gamma}(t_0))$ as initial data to construct a corresponding strong solution $\tilde{\mathsf{s}} \equiv (\tilde{\boldsymbol{v}}, \tilde{\omega}, \tilde{\boldsymbol{\gamma}})$ in the interval $[t_0, T^*)$, for *some* $T^* > t_0$. Such a solution can be obtained by using the Galerkin method described in Proposition 5.2 with $\{\psi_n\}_{n\in\mathbb{N}}$ eigenvectors of the Stokes operator $-P\,\Delta$,[3] and coupling it with suitable "energy" estimates that we are (formally) about to derive. By dot-multiplying both sides of $(3.1)_1$ by $\boldsymbol{v}_t$, integrating by parts over $\mathcal{C}$, and employing $(3.1)_2$ and $(3.1)_5$, we deduce (with $\nu := \mu/\rho$)

$$\begin{aligned} \frac{\nu}{2}\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + \|\boldsymbol{v}_t\|_2^2 + \quad & \dot{\boldsymbol{a}} \cdot \int_{\mathcal{C}} \boldsymbol{x} \times \boldsymbol{v}_t = -(\boldsymbol{v}\cdot\nabla\boldsymbol{v}, \boldsymbol{v}_t) \\ & -\dot{\boldsymbol{\omega}}_\infty \cdot \int_{\mathcal{C}} \boldsymbol{x} \times \boldsymbol{v}_t - 2(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \cdot \int_{\mathcal{C}} \boldsymbol{v} \times \boldsymbol{v}_t\,, \end{aligned} \tag{5.14}$$

where we used $g\,\boldsymbol{\gamma} = \nabla(g\,\boldsymbol{\gamma}\cdot\boldsymbol{x})$. Notice that by (5.1) and (5.2) we have

$$\|\boldsymbol{v}_t\|_2^2 + \dot{\boldsymbol{a}} \cdot \int_{\mathcal{C}} \boldsymbol{x} \times \boldsymbol{v}_t \geq c_0 \|\boldsymbol{v}_t\|_2^2$$

so that, with the help of Schwarz inequality, from (5.14) we easily deduce

$$\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + c_1\|\boldsymbol{v}_t\|_2^2 \leq c_2\big[|\dot{\boldsymbol{\omega}}_\infty|^2 + (|\boldsymbol{a}|^2 + |\boldsymbol{\omega}_\infty|^2)\|\boldsymbol{v}\|_2^2 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2\big]\,. \tag{5.15}$$

Moreover, by dot-multiplying $(3.1)_1$ by $P\Delta\boldsymbol{v}$ (see Footnote 3), it follows that

$$\nu\|P\Delta\boldsymbol{v}\|_2^2 = \Big(\big[\boldsymbol{v}_t + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + (\dot{\boldsymbol{\omega}}_\infty + \dot{\boldsymbol{a}}) \times \boldsymbol{x} + 2(\boldsymbol{\omega}_\infty + \boldsymbol{a}) \times \boldsymbol{v}\big], P\Delta\boldsymbol{v}\Big)\,,$$

which, in turn, again by the Schwarz inequality, delivers

$$\|P\,\Delta\boldsymbol{v}\|_2^2 \leq c_3\,\left(\|\boldsymbol{v}_t\|_2^2 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 + |\dot{\boldsymbol{\omega}}_\infty|^2 + (|\boldsymbol{a}|^2 + |\boldsymbol{\omega}_\infty|^2)\|\boldsymbol{v}\|_2^2\right)\,. \tag{5.16}$$

Here and in the following $c_i > 0$, $i \in \mathbb{N}$, denote constants depending at most on the physical parameters and the data, s_0, of the weak solution s at $t = 0$. We now observe that since $|\boldsymbol{\gamma}(t)| = 1$ for all $t \geq t_0$, by the strong energy inequality (5.10) and $(3.1)_3$ we infer

$$\|\boldsymbol{v}(t)\|_2 + |\boldsymbol{\omega}(t)| + |\dot{\boldsymbol{\omega}}_\infty(t)| + |\boldsymbol{a}(t)| \leq c_4\,, \quad \text{all } t \geq t_0\,. \tag{5.17}$$

As a result, from (5.15), (5.16) and (5.17) it follows that

$$\begin{aligned} \frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 + c_1\|\boldsymbol{v}_t\|_2^2 \leq \quad & c_5\big(\|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 + 1\big)\,, \\ \|P\Delta\boldsymbol{v}\|_2^2 \leq \quad & c_6\big(\|\boldsymbol{v}_t\|_2^2 + \|\boldsymbol{v}\cdot\nabla\boldsymbol{v}\|_2^2 + 1\big)\,, \end{aligned}$$

[3] P is the (Helmholtz-Weyl) projection of $L^2(\mathcal{C})$ onto $H(\mathcal{C})$.

which, in turn, implies

$$\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + c_7\|\boldsymbol{v}_t\|_2^2 + c_8\|P\Delta \boldsymbol{v}\|_2^2 \le c_9 \left(\|\boldsymbol{v}\cdot\nabla \boldsymbol{v}\|_2^2 + 1\right) . \tag{5.18}$$

Recalling that $\mathcal{C}$ is of class C^2, we have (e.g. [6, Theorem IV.6.1])

$$\|\boldsymbol{v}\|_{2,2}^2 \le c_{10}\|P\Delta \boldsymbol{v}\|_2^2 , \tag{5.19}$$

and also, by well-known embedding results and Cauchy-Schwarz inequality we deduce with arbitrary $\zeta > 0$,

$$\|\boldsymbol{v}\cdot\nabla \boldsymbol{v}\|_2^2 \le \|\boldsymbol{v}\|_\infty^2\|\nabla \boldsymbol{v}\|_2^2 \le c_{11}\,\|\nabla \boldsymbol{v}\|_2^3\|\boldsymbol{v}\|_{2,2} \le c_{12}\|\nabla \boldsymbol{v}\|_2^6 + \zeta\,\|\boldsymbol{v}\|_{2,2}^2 , \tag{5.20}$$

where $c_{12} \to \infty$ as $\zeta \to 0$. Therefore, choosing ζ small enough, from (5.18)–(5.20) we may conclude

$$\frac{d}{dt}\|\nabla \boldsymbol{v}\|_2^2 + c_7\|\boldsymbol{v}_t\|_2^2 + c_{13}\|\boldsymbol{v}\|_{2,2}^2 \le c_{14} \left(\|\nabla \boldsymbol{v}\|_2^6 + 1\right) . \tag{5.21}$$

From this differential inequality we derive that there are continuous functions G_1 and G_2 defined on $[t_0, t_0 + T^*)$ for some $T^* = T^*(\mathsf{s}_0) > 0$, such that

$$\|\boldsymbol{v}(t)\|_{1,2} \le G_1(t)\,, \quad \int_{t_0}^{t} \left(\|\boldsymbol{v}_\tau(\tau)\|^2 + \|\boldsymbol{v}(\tau)\|_{2,2}^2\right) d\tau \le G_2(t)\,. \tag{5.22}$$

By combining the latter with the classical Galerkin method, we can then show the existence of a solution $\tilde{\mathsf{s}} \equiv (\tilde{\boldsymbol{v}}, \tilde{\omega}, \tilde{\boldsymbol{\gamma}})$ corresponding to the initial data s_0, and such that, setting $I_{t_0,\tau} := (t_0, t_0 + \tau)$,

$$\begin{aligned} &\tilde{\boldsymbol{v}} \in C^0(\overline{I_{t_0,\tau}}; W_0^{1,2}(\mathcal{C})) \cap L^\infty(I_{t_0,\tau}; W_0^{1,2}(\mathcal{C})) \cap L^2(I_{t_0,\tau}; W^{2,2}(\mathcal{C}))\,, \\ &\tilde{\boldsymbol{v}}_t \in L^2(I_{t_0,\tau}; H(\mathcal{C})), \quad \tilde{\omega} \in W^{1,\infty}(I_{t_0,\tau})\,, \quad \tilde{\boldsymbol{\gamma}} \in W^{2,\infty}(I_{t_0,\tau})\,, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } \tau \in (0, T^*)\,, \end{aligned} \tag{5.23}$$

with t_0 as in (5.13). By virtue of the Sobolev embedding theorem, we check at once that $\tilde{\boldsymbol{v}}$ satisfies (5.11) for some $r > 3$, so that $\tilde{\mathsf{s}} = \mathsf{s}$ on the interval $[t_0, t_0 + T^*)$. Now, by proceeding exactly as in the proof of [8, Theorem 3], one shows that if $T^* < \infty$ necessarily

$$\lim_{t\to T^{*-}} \|\nabla \boldsymbol{v}(t)\|_2 = \infty\,. \tag{5.24}$$

However, this condition cannot hold. Actually, by choosing in (5.13) ε and η appropriately, from (5.21) we see that $\|\nabla \boldsymbol{v}\|_2^2$ satisfies all assumptions of the Gronwall-like Lemma shown in the Appendix. Therefore, by that lemma we deduce that, on the one hand, $\|\nabla \boldsymbol{v}(t)\|_2 \in L^\infty(t_0, \infty)$ and, on the other hand, also using Poincaré inequality, $\|\boldsymbol{v}(t)\|_{1,2}$ obeys (5.12), which ends the (formal) proof of the proposition. □

From the previous propositions, we can easily deduce the following result.

COROLLARY 5.6. Let $\mathsf{s} := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution to (3.1). Then, there exists $t_0 > 0$ such that

(1) s is unique in the class of weak solutions to (3.1) in $[t_0, \infty)$;
(2) s depends continuously upon the data in $[t_0, \infty)$, in the class of weak solutions, in the sense of Proposition 5.4.

6. Asymptotic behavior of weak solutions. For the proof of our main theorem (Theorem 6.4) concerning the long-time behavior of the coupled system $\mathcal{S}$, we will use some tools from classical theory of the Dynamical Systems. In particular, we will give a complete characterization of the Ω-limit set corresponding to a weak solution and, for a large class of liquid-solid configurations, we will show that this weak solution indeed converges to a point of the corresponding Ω-limit set. To this end, let us recall some definitions and well-known results. Consider $\mathcal{H} = H(\mathcal{C}) \times \mathbb{R}^3 \times S^2$, endowed with its natural topology. We define the Ω-limit set of a weak solution $\mathsf{s} := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ as follows:

$$\Omega(\mathsf{s}) := \{(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{H} : \text{ there exists } t_k \geq 0,\ t_k \nearrow \infty \text{ s.t.}$$
$$\lim_{k\to\infty} \|\boldsymbol{v}(t_k) - \boldsymbol{u}\|_2 = \lim_{k\to\infty} |\boldsymbol{\omega}_\infty(t_k) - \boldsymbol{\omega}| = \lim_{k\to\infty} |\boldsymbol{\gamma}(t_k) - \boldsymbol{q}| = 0\}.$$

We also use the notation $w(t; z)$ to denote a weak solution to (3.1) corresponding to the initial data $z \in \mathcal{H}$, in the sense of Proposition 5.2.

Definition 6.1. $\Omega(\mathsf{s})$ is positively invariant if the following implication holds:

$$y \in \Omega(\mathsf{s}) \quad \Rightarrow \quad w(t; y) \in \Omega(\mathsf{s}), \qquad \text{all } t \geq 0,$$

and for all weak solutions $w(t; y)$.

In [10], Proposition 1.4.2 (see also [22], Proposition 5.1.8), it has been proved that $\Omega(\mathsf{s})$ is positively invariant in the class of weak solutions, if $\mathsf{s}(t; \mathsf{s}_0)$ is *asymptotically regular*. We recall the statement here, for completeness.

Proposition 6.2. Let $\mathsf{s}(t; \mathsf{s}_0)$ be a weak solution to (3.1). Suppose there exists $t_0 > 0$ such that the following properties hold:

(i) *Asymptotic uniqueness*:

$$\mathsf{s}(t + \tau; \mathsf{s}_0) = \mathsf{s}(t; s(\tau; \mathsf{s}_0)), \qquad \text{for all } \tau \geq t_0, \text{ and } t \geq 0.$$

(ii) *Asymptotic continuous data dependence*:

$$\{t_k\}_{k\in\mathbb{N}} \subset [t_0, +\infty) \quad \text{with} \quad \mathsf{s}(t_k; \mathsf{s}_0) \to y \quad \text{in } \mathcal{H}$$
$$\Rightarrow \mathsf{s}(t; \mathsf{s}(t_k; \mathsf{s}_0)) \to w(t, y) \quad \text{in } \mathcal{H}, \quad \text{all } t \geq 0.$$

Then, $\Omega(\mathsf{s})$ is positively invariant.

In the next proposition, we show a full characterization of the Ω-limit set of any weak solution to (3.1). As expected, such a characterization depends on the mass distribution in the system $\mathcal{S}$.

Proposition 6.3. Let $\mathsf{s}(t; \mathsf{s}_0) := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution to (3.1) corresponding to an initial data $s_0 := (\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0) \in \mathcal{H}$, in the sense of Proposition 5.2. Then, $\Omega(s)$ is non-empty, compact, connected, and positively invariant in the class of weak solutions to (3.1). Moreover, $\Omega(\mathsf{s}) \subseteq \mathsf{S}$ where S is given in Theorem 4.7, and it has the following characterization:

(1) If $A = B = C$, then either

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = (\boldsymbol{\gamma}_0 \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = \boldsymbol{e}_3)\} \subset \mathsf{PR}$$

or

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = -(\boldsymbol{\gamma}_0 \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = -\boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(2) If $A = B \neq C$, then we have the following possibilities:

(a)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = \boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(b)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = -\frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = -\boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(c)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \\ \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \;\; \bar{\gamma}_1^2 + \bar{\gamma}_2^2 = 1 - \frac{\beta^4}{\lambda^4(C-A)^2}\} \subset \mathsf{SP},$$

where λ is a (real) solution to the following fourth order algebraic equation:

$$(C-A)A\lambda^4 - (C-A)(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\lambda^3 + \beta^4 = 0. \tag{6.1}$$

(3) If $A \neq B = C$, then we have the following cases:

(a)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = \boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(b)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = -\frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = -\boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(c)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \\ \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \;\; \bar{\gamma}_2 = 0, \;\; \bar{\gamma}_1 = \sqrt{1 - \frac{\beta^4}{\lambda^4(C-A)^2}}\} \subset \mathsf{SP}_1.$$

(d)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \\ \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \;\; \bar{\gamma}_2 = 0, \;\; \bar{\gamma}_1 = -\sqrt{1 - \frac{\beta^4}{\lambda^4(C-A)^2}}\} \subset \mathsf{SP}_1.$$

In the last two cases, λ satisfies (6.1).

(4) If $A = C \neq B$, then we have the following cases:

(a)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = \boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(b)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = -\frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = -\boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(c)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-B)}, \;\; \bar{\gamma}_1 = 0, \;\; \bar{\gamma}_2 = \sqrt{1 - \frac{\beta^4}{\lambda^4(C-B)^2}}\} \subset \mathsf{SP}_2.$$

(d)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-B)}, \;\; \bar{\gamma}_1 = 0, \;\; \bar{\gamma}_2 = -\sqrt{1 - \frac{\beta^4}{\lambda^4(C-B)^2}}\} \subset \mathsf{SP}_2.$$

In the last two cases, λ satisfies

$$(C-B)B\lambda^4 - (C-B)(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\lambda^3 + \beta^4 = 0. \tag{6.2}$$

(5) If $A \neq B \neq C$, then we have the following cases:

(a)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = \boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(b)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = -\frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \bar{\boldsymbol{\gamma}} = -\boldsymbol{e}_3)\} \subset \mathsf{PR}.$$

(c)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \;\; \bar{\gamma}_2 = 0, \;\; \bar{\gamma}_1 = \sqrt{1 - \frac{\beta^4}{\lambda^4(C-A)^2}}\} \subset \mathsf{SP}_1.$$

(d)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-A)}, \;\; \bar{\gamma}_2 = 0, \;\; \bar{\gamma}_1 = -\sqrt{1 - \frac{\beta^4}{\lambda^4(C-A)^2}}\} \subset \mathsf{SP}_1.$$

(e)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-B)}, \;\; \bar{\gamma}_1 = 0, \;\; \bar{\gamma}_2 = \sqrt{1 - \frac{\beta^4}{\lambda^4(C-B)^2}}\} \subset \mathsf{SP}_2.$$

(f)

$$\Omega(\mathsf{s}) = \{(\boldsymbol{v} \equiv \boldsymbol{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \\ \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C-B)}, \;\; \bar{\gamma}_1 = 0, \;\; \bar{\gamma}_2 = -\sqrt{1 - \frac{\beta^4}{\lambda^4(C-B)^2}}\} \subset \mathsf{SP}_2.$$

In the cases (c) and (d), λ satisfies (6.1). While, in the cases (e) and (f), λ satisfies (6.2).

Proof. The strong energy inequality (5.10) and Proposition 5.5 show that, for sufficiently large times, the trajectory becomes uniformly bounded, continuous in time and belongs to a compact subset of $\mathcal{H}$. Thus, $\Omega(\mathsf{s})$ is non-empty, connected and compact. In addition, from Corollary 5.6 and Proposition 6.2, we infer that $\Omega(\mathsf{s})$ is invariant in the class of weak solutions to (3.1). Thus, by (5.12) and (3.1), the dynamics on $\Omega(\mathsf{s})$ is completely described by the following system of equations:

$$\begin{aligned}
&\boldsymbol{v} \equiv \boldsymbol{0}, \qquad \dot{\boldsymbol{\omega}}_\infty \times \boldsymbol{x} = -\nabla p + \rho g \boldsymbol{\gamma}, \\
&\boldsymbol{I} \cdot \dot{\boldsymbol{\omega}}_\infty + \boldsymbol{\omega}_\infty \times \boldsymbol{I} \cdot \boldsymbol{\omega}_\infty = \beta^2 \boldsymbol{e}_3 \times \boldsymbol{\gamma}, \\
&\dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}_\infty \times \boldsymbol{\gamma} = \boldsymbol{0}.
\end{aligned} \tag{6.3}$$

Taking the curl of the second equation in $(6.3)_1$, we find $\dot{\boldsymbol{\omega}}_\infty = \boldsymbol{0}$, which implies $\boldsymbol{e}_3 \times \boldsymbol{\gamma} =$ const. Moreover, by $(6.3)_3$ we get

$$(\boldsymbol{\gamma} \dot{\cdot} \boldsymbol{e}_3) = \boldsymbol{\gamma} \times \boldsymbol{\omega}_\infty \cdot \boldsymbol{e}_3$$

and so dot-multiplying by $\boldsymbol{\omega}_\infty$ both sides of $(6.3)_2$ (with $\dot{\boldsymbol{\omega}}_\infty \equiv \boldsymbol{0}$), we deduce also $\boldsymbol{\gamma} \cdot \boldsymbol{e}_3 =$ const., and conclude $\dot{\boldsymbol{\gamma}} = \boldsymbol{0}$. Therefore, $\Omega(\mathsf{s}) \subseteq \mathsf{S}$, and the generic motion over $\Omega(\mathsf{s})$ must thus be a solution to the system of equations (4.3) (or equivalently (4.4)), and whose characterization is given by Theorem 4.7. In particular, in view of (5.6), on $\Omega(\mathsf{s})$ we must have

$$\boldsymbol{\omega}_\infty = \lambda \boldsymbol{\gamma}, \qquad \lambda \boldsymbol{\gamma} \cdot \boldsymbol{I} \cdot \boldsymbol{\gamma} = \boldsymbol{\omega}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\gamma}_0 . \tag{6.4}$$

From $(6.4)_2$ and the expression of the $\boldsymbol{\gamma}$'s given in parts 4(c) and 4(d), and 5(c)–5(f) of this proposition, one can easily derive that λ has to satisfy, equations (6.1) and (6.2), as claimed. □

It is important to emphasize that in all the cases above (except the case $A = B \neq C$), $\Omega(\mathsf{s})$ is the union of disjoint points in $\mathcal{H}$. The next theorem exploits this fact along with the topological properties of $\Omega(\mathsf{s})$ to provide a rather complete description of the asymptotic behavior of weak solutions.

Theorem 6.4. Let $\mathcal{C}$ be a bounded domain in $\mathbb{R}^3$ of class C^2, and $\mathsf{s} := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution to (3.1) corresponding to an initial data $(\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0) \in \mathcal{H}$, in the sense of Proposition 5.2. Then,

$$\lim_{t \to \infty} \|\boldsymbol{v}(t)\|_{1,2} = 0.$$

Moreover, the following properties hold:

(i) In each of the cases $A = B = C$, $A \neq B = C$, $A = C \neq B$ and $A \neq B \neq C$, there exists $(\mathbf{0}, \bar{\lambda}\,\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) \in \Omega(\mathsf{s})$, with

$$\bar{\lambda} = \frac{\boldsymbol{\omega}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\gamma}_0}{\bar{\boldsymbol{\gamma}} \cdot \boldsymbol{I} \cdot \bar{\boldsymbol{\gamma}}}\,, \tag{6.5}$$

such that

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \bar{\lambda}\,\bar{\boldsymbol{\gamma}}, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = \bar{\boldsymbol{\gamma}}.$$

(ii) In the case $A = B \neq C$, let $K := \min\{A, C\}$. If

$$|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2 < \frac{K^2\beta^2}{|C - A|}, \tag{6.6}$$

then either

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = \boldsymbol{e}_3,$$

or

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = -\frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = -\boldsymbol{e}_3.$$

Proof. In Proposition 5.5, we have shown that

$$\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_{1,2} = 0.$$

Moreover, by Proposition 6.3, in the cases $A = B = C$, $A \neq B = C$, $A = C \neq B$ and $A \neq B \neq C$, the corresponding Ω-limit set reduces to a point set in S; as a matter of fact $\Omega(\mathsf{s})$ is connected, and equations (6.1) and (6.2) are fourth order algebraic equations, which admits at most four (real) roots. Thus, taking into account (6.4), for the above mentioned cases, we can conclude that there exists $(\mathbf{0}, \bar{\lambda}\,\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) \in \Omega(\mathsf{s})$ with $\bar{\lambda}$ given in (6.5), such that

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \bar{\lambda}\,\bar{\boldsymbol{\gamma}}, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = \bar{\boldsymbol{\gamma}}.$$

Let us show part (ii) of the statement. Because of Proposition 6.3 2(c), we have to show that, under the stated assumptions, the case

$$\Omega(\mathsf{s}) = \left\{(\boldsymbol{v} \equiv \mathbf{0}, \bar{\boldsymbol{\omega}} = \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) : \bar{\gamma}_3 = -\frac{\beta^2}{\lambda^2(C - A)},\ \bar{\gamma}_1^2 + \bar{\gamma}_2^2 = 1 - \frac{\beta^4}{\lambda^4(C - A)^2}\right\}$$

with λ a real solution to (6.1), is not allowed. We argue by contradiction. By the connectedness property of $\Omega(\mathsf{s})$ and the uniquely determined values of $\bar{\gamma}_3$ and $\bar{\gamma}_1^2 + \bar{\gamma}_2^2$ (for each fixed value of λ), we deduce that the following limits exist:

$$\lim_{t\to\infty} \gamma_3(t) = -\frac{\beta^2}{\lambda^2(C - A)}, \quad \lim_{t\to\infty} (\gamma_1^2(t) + \gamma_2^2(t)) = 1 - \frac{\beta^4}{\lambda^4(C - A)^2}\,.$$

However, by (6.4),

$$|\lambda| \leq \frac{|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|}{K}.$$

Thus, from the latter and the assumption (6.6), we infer

$$|\lim_{t\to\infty} \gamma_3(t)| = \frac{\beta^2}{\lambda^2|C - A|} \geq \frac{\beta^2K^2}{|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2\ |C - A|} > 1,$$

which is at odds with the constraint $|\gamma_3(t)| \leq 1$, for all $t > 0$. $\square$

Remark 6.5. We find it appropriate to rephrase the results of Theorem 6.4 in more physical terms as follows. Let $\mathcal{B}$ be a heavy rigid-body, constrained to move around a fixed point, O, and with an interior, sufficiently smooth cavity, $\mathcal{C}$, completely filled with a viscous liquid. We assume that the center of mass, G, of the coupled system $\mathcal{S} := \mathcal{B} \cup \mathcal{C}$ lies on a principal axis of inertia, a, of $\mathcal{S}$ relative to O. Then, if the principal moments of inertia of $\mathcal{S}$ satisfy the assumptions in (i), all motions of $\mathcal{S}$ that initially possess a finite kinetic energy must eventually converge to a steady state characterized by the property that $\mathcal{S}$ moves, as a whole rigid-body, either by a constant rotation around the direction of gravity, e, passing through O, or else by a motion of steady precession, where a rotates uniformly around e, and forms with e a non-zero, constant angle. If, instead, the principal moments of inertia of $\mathcal{S}$ satisfy the assumptions in (ii), then the final state reduces to a rotation around e, provided the projection of the initial angular momentum along e is not too large, in the sense of (6.6).

7. Application to a spherical pendulum with a liquid-filled cavity. The finding shown in Theorem 6.4 and commented in Remark 6.5 implies that the presence of the liquid in the cavity may affect dramatically the "terminal" motion of the rigid-body with an empty cavity and, in particular, may produce a strong stabilizing influence by forcing the coupled system to perform a specific steady motion. The objective of this and the next section is to provide two significant examples of such a stabilization, in the cases of a spherical pendulum and a spinning top, respectively.

To this purpose, we premise the following general result.

Proposition 7.1. Let $\mathsf{s}(t;\mathsf{s}_0) := (\boldsymbol{v}, \boldsymbol{\omega}_\infty, \boldsymbol{\gamma})$ be a weak solution to (3.1) corresponding to initial data $\mathsf{s}_0 := (\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0) \in \mathcal{H}$, in the sense of Proposition 5.2. Suppose that

$$|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2 < \min\left\{\frac{K^2\beta^2}{|C-A|}, \frac{K^2\beta^2}{|C-B|}\right\}, \tag{7.1}$$

where $K := \min\{A, B, C\}$. Then, in the cases $A = B \neq C$, $A \neq B = C$, $A = C \neq B$, $A \neq B \neq C$,[4] either

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = \boldsymbol{e}_3 \tag{7.2}$$

or

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = -\frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = -\boldsymbol{e}_3. \tag{7.3}$$

Moreover, if (7.1) is augmented with the following condition:

$$C\left[\rho\|\boldsymbol{v}_0\|^2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0 - 2\beta^2(\boldsymbol{\gamma}_0 \cdot \boldsymbol{e}_3 + 1)\right] \le |\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2, \tag{7.4}$$

[4] If $A = C$ or $B = C$, one of the fractions in (7.1) becomes undefined. However, by the characterization of the Ω-limit, in these two cases, (7.1) reduces to

$$|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2 < \frac{K^2\beta^2}{|C-B|} \text{ or } |\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2 < \frac{K^2\beta^2}{|C-A|},$$

respectively.

then necessarily[5]

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = \boldsymbol{e}_3.$$

Finally, if $A = B = C$, the same conclusion holds under the sole assumption (7.4)

Proof. We begin to observe that, by Theorem 6.4, it follows that if $A = B = C$, either condition (7.2) or condition (7.3) is valid, for arbitrary initial data $s_0 \in \mathcal{H}$. By the same theorem, either (7.2) or (7.3) holds also when $A = B \neq C$ as (7.1) coincides with (6.6). For all the remaining cases, $A \neq B = C$, $A = C \neq B$, $A \neq B \neq C$, we will argue by contradiction. By Theorem 6.4 and Proposition 6.3, we infer that there exists $(\mathbf{0}, \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) \in \Omega(\mathsf{s}) \subset \mathsf{SP}_1 \cup \mathsf{SP}_2$ such that

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = \lambda\bar{\boldsymbol{\gamma}}, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = \bar{\boldsymbol{\gamma}},$$

where, by (6.5),

$$\lambda\bar{\boldsymbol{\gamma}} \cdot \boldsymbol{I} \cdot \bar{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0.$$

Thus,

$$|\lambda| \leq \frac{|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|}{K}.$$

Since $(\mathbf{0}, \lambda\bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\gamma}}) \in \Omega(\mathsf{s}) \subset \mathsf{SP}_1 \cup \mathsf{SP}_2$, and by (7.1), we find that

$$\begin{aligned} |\lim_{t\to\infty} \gamma_3(t)| &\geq \frac{\beta^2}{\lambda^2} \min\left\{\frac{1}{|C-A|}, \frac{1}{|C-B|}\right\} \\ &\geq \frac{K^2\beta^2}{|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2} \min\left\{\frac{1}{|C-A|}, \frac{1}{|C-B|}\right\} > 1, \end{aligned}$$

and this is at odd with the constraint $|\gamma_3(t)| \leq 1$, for all times. The proof of the last part of the theorem is an immediate consequence of the strong energy inequality. Indeed, if the following holds:

$$\lim_{t\to\infty} \boldsymbol{\omega}_\infty(t) = -\frac{1}{C}(\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0)\boldsymbol{e}_3, \quad \lim_{t\to\infty} \boldsymbol{\gamma}(t) = -\boldsymbol{e}_3,$$

taking the limit as $t \to \infty$ in (5.10), we would get[6]

$$\frac{1}{C}|\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0|^2 + 2\beta^2 < \rho\|\boldsymbol{v}_0\|^2 + \boldsymbol{\omega}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0 - 2\beta^2\boldsymbol{\gamma}_0 \cdot \boldsymbol{e}_3,$$

but the latter displayed inequality contradicts (7.4). □

We now wish to apply the previous proposition to the case when $\mathcal{B}$ is a *spherical pendulum.* As is well known, by the latter is meant the system constituted by a massive rigid body $\mathcal{P}$ which is attached to a frictionless spherical joint, whose center is placed at a fixed point O, via a rod whose mass can be neglected. In such a case, the motion of the center of mass, $\hat{G}$, of $\mathcal{P}$ can be very complicated. In fact, in general, it is not necessarily periodic, with $\hat{G}$ describing a trajectory that lies in the zone between two horizontal concentric circles centered at points of the vertical axis that passes through O; see, e.g., [26, Section 5.3]. Suppose now that we entirely fill the hollow cavity $\mathcal{C}$ in $\mathcal{P}$ with a viscous

[5]We exclude that $(\boldsymbol{v}_0, \boldsymbol{\omega}_0, \boldsymbol{\gamma}_0) \equiv (\mathbf{0}, r_0\boldsymbol{e}_3, -\boldsymbol{e}_3)$, since, in this case, the corresponding motion (weak solution) will then reduce simply to a rigid rotation of $\mathcal{S}$ around $\boldsymbol{e}_3$.

[6]See Footnote 5.

liquid $\mathcal{L}$. Then, Proposition 7.1 ensures that for an open set of initial data (satisfying, in general, condition (7.1)), the coupled system $\mathcal{S} := \mathcal{P} \cup \mathcal{L}$ must, eventually, perform a *uniform rotation* around the vertical axis passing through O, $\boldsymbol{a}_O$, with its center of mass G being either in its lowest, G_ℓ, or highest, G_h, position. This uniform rotation can even *reduce to the rest* if the initial data produce zero angular momentum along $\boldsymbol{a}_O$ (namely, $\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0 = \mathbf{0}$). Furthermore, if the initial data obey also (7.4), then the terminal steady-state motion of the pendulum will have $G \equiv G_\ell$. In order to exemplify the latter, suppose $\mathcal{P}$ is a hollow homogeneous sphere entirely filled with a viscous liquid. We then have $A = B = C$, so that, by Proposition 7.1, we know that the terminal state of $\mathcal{S}$ will be a uniform rotation around $\boldsymbol{a}_O$, or even the rest, with either $G = G_\ell$ or $G = G_h$. Now take $\boldsymbol{v}_0 \equiv \mathbf{0}$ (liquid initially at rest relative to $\mathcal{P}$), and $\boldsymbol{\omega}_0 = \omega_0 \boldsymbol{\gamma}_0$ (initial spin around the vertical direction passing through O; see Fig 1 (a)). Under these circumstances, (7.4) becomes

$$C[C\,\omega_0^2 - 2\beta^2(\boldsymbol{\gamma}_0 \cdot \boldsymbol{e}_3 + 1)] \leq C^2\,\omega_0^2\,,$$

a condition that is always satisfied.[7] As a consequence, the sphere will eventually perform a *uniform rotation* about $\boldsymbol{a}_O$, with $G = G_\ell$. If we instead choose $\boldsymbol{\omega}_0 \cdot \boldsymbol{\gamma}_0 = \mathbf{0}$ (see Fig. 1 (b)), we get $\boldsymbol{\gamma}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\omega}_0 = \mathbf{0}$ and, again by Proposition 7.1, we conclude that the terminal motion of the pendulum is the rest state with $G = G_\ell$.

This latter circumstance provides the rather unforeseen property that *a spherical pendulum, with a cavity filled with a viscous liquid, may reach the equilibrium configuration with its center of mass at its lowest point*, similarly to what happens when the cavity is empty and the pendulum is immersed in a viscous liquid ([36]).

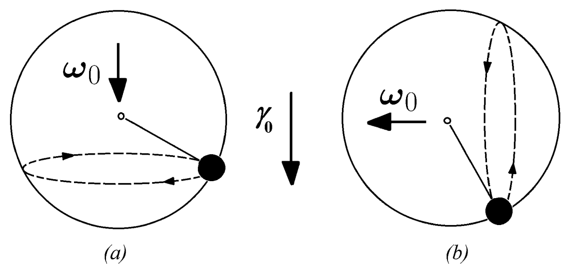

Figure 1.

8. Application to a top with a liquid-filled cavity. The main objective of this section is to investigate the motion of a (non-necessarily symmetric) top with a liquid-filled cavity that is initially spinning around the $\boldsymbol{e}_3$-axis, with the latter slightly off the vertical direction. It is worth emphasizing that this type of problems has been investigated by numerous authors for the last forty years; see, e.g., [3, 4, 13, 15, 16, 20, 24, 39]. However, these results are mostly obtained by making ad hoc assumptions such as neglecting the non-linear effects and/or imposing suitably symmetry restrictions on

[7]See Footnote 5.

the shape of the cavity. In contrast, the study we shall perform here will not only be of completely rigorous nature, but will also show certain significant features that were not envisaged by previous authors, also because of their simplifying hypotheses.

To achieve all the above, we begin with some general considerations about the stability of steady-state solutions.

Let $\mathsf{m}_0 = (\mathbf{0}, k\boldsymbol{\gamma}, \boldsymbol{\gamma}) \in \mathsf{S}$ be a given steady-state configuration of $\mathcal{S}$, and let $\mathsf{m} := \tilde{\mathsf{m}} + \mathsf{m}_0 = (\boldsymbol{v}, \tilde{\boldsymbol{\omega}}_\infty + k\boldsymbol{\gamma}, \boldsymbol{z} + \boldsymbol{\gamma})$ be a corresponding "perturbed" motion with initial data $(\boldsymbol{v}_0, k\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_{\infty 0}, \boldsymbol{\gamma} + \boldsymbol{z}_0) \in \mathcal{H}$.

We collect a number of fundamental properties that the "perturbation" $(\boldsymbol{v}(t), \tilde{\boldsymbol{\omega}}_\infty(t), \boldsymbol{z}(t))$ must satisfy at all times $t \geq 0$. By Proposition 5.2, the perturbed motion obeys the strong energy inequality (5.10), which in particular furnishes

$$\begin{aligned}\rho\|\boldsymbol{v}(t)\|^2 &+ (k\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_\infty(t)) \cdot \boldsymbol{I} \cdot (k\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_\infty(t)) - 2\beta^2(\boldsymbol{\gamma} + \boldsymbol{z}(t)) \cdot \boldsymbol{e}_3 \\ &\leq \rho\|\boldsymbol{v}_0\|^2 + (k\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_{\infty 0}) \cdot \boldsymbol{I} \cdot (k\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_{\infty 0}) - 2\beta^2(\boldsymbol{\gamma} + \boldsymbol{z}_0) \cdot \boldsymbol{e}_3\,, \text{ for all } t \geq 0\,,\end{aligned}$$

as well as conservation of axial angular momentum (5.6):

$$(\boldsymbol{\gamma} + \boldsymbol{z}(t)) \cdot \boldsymbol{I} \cdot (k\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_\infty(t)) = (\boldsymbol{\gamma} + \boldsymbol{z}_0) \cdot \boldsymbol{I} \cdot (k\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_{\infty 0}), \quad \text{for all } t \geq 0.$$

The above implies that the perturbation must satisfy

$$\begin{aligned}\rho\|\boldsymbol{v}(t)\|^2 &+ \tilde{\boldsymbol{\omega}}_\infty(t) \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty(t) + 2k\boldsymbol{\gamma} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty(t) - 2\beta^2 z_3(t) \\ &\leq \rho\|\boldsymbol{v}_0\|^2 + \tilde{\boldsymbol{\omega}}_{\infty 0} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_{\infty 0} + 2k\boldsymbol{\gamma} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_{\infty 0} - 2\beta^2 \boldsymbol{z}_0 \cdot \boldsymbol{e}_3\,, \text{ for all } t \geq 0\,, \quad (8.1)\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\gamma} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty(t) &+ k\boldsymbol{z}(t) \cdot \boldsymbol{I} \cdot \boldsymbol{\gamma} + \boldsymbol{z}(t) \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty(t) \\ &= \boldsymbol{\gamma} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_{\infty 0} + k\boldsymbol{z}_0 \cdot \boldsymbol{I} \cdot \boldsymbol{\gamma} + \boldsymbol{z}_0 \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_{\infty 0}, \quad \text{for all } t \geq 0. \quad (8.2)\end{aligned}$$

Moreover, we have the following integral of motion:

$$(\boldsymbol{\gamma} + \boldsymbol{z}(t)) \cdot (\boldsymbol{\gamma} + \boldsymbol{z}(t)) = 1, \qquad \text{for all } t \geq 0$$

which, together with the constraint $\boldsymbol{\gamma} \cdot \boldsymbol{\gamma} = 1$, gives the following:

$$\boldsymbol{z}(t) \cdot \boldsymbol{z}(t) + 2\boldsymbol{\gamma} \cdot \boldsymbol{z}(t) = 0, \qquad \text{for all } t \geq 0. \quad (8.3)$$

Following classical literature, we shall say that m_0 is *stable* (in the sense of Lyapunov) if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|\boldsymbol{v}_0\|_2 + |\tilde{\boldsymbol{\omega}}_{\infty 0}| + |\boldsymbol{z}_0| < \delta \implies \|\boldsymbol{v}(t)\|_2 + |\tilde{\boldsymbol{\omega}}_\infty(t)| + |\boldsymbol{z}(t)| < \varepsilon\,, \text{ for all } t > 0.$$

Otherwise, m_0 is *unstable*.

The following result provides a general stability criterion. Very probably, its (simple) proof can be found in the existing literature; however, for completeness, we shall give our own in the Appendix.

LEMMA 8.1. Let $\mathsf{m}_0 \in \mathsf{S}$ and $\tilde{\mathsf{m}} = (\boldsymbol{v}, \boldsymbol{y})$, $\boldsymbol{y} \equiv (\tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{z})$, be a corresponding perturbation. Moreover, let $F : H(\mathcal{C}) \to [0, \infty)$ be such that

$$c_1\|\boldsymbol{v}\|_2 \leq F(\boldsymbol{v}) \leq c_2\|\boldsymbol{v}\|_2\,, \quad (8.4)$$

and let $U: \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be continuous and, in addition, positive definite in a neighborhood $\mathcal{I}(\mathbf{0})$ of the origin in $\mathbb{R}^3 \times \mathbb{R}^3$, namely,

(i) $U(\mathbf{0}) = 0$,

(ii) $U(\boldsymbol{y}) > 0$ for all $\boldsymbol{y} \in \mathcal{I}(\mathbf{0}) \setminus \{\mathbf{0}\}$.

Then, if $V(t) := F(\boldsymbol{v}(t)) + U(\boldsymbol{y}(t))$ satisfies $V(t) \leq V(0)$ for all $t \geq 0$, m_0 is stable.

We are now in a position to analyze in detail some important aspects of the motion of a spinning (possibly asymmetric) top with a liquid-filled cavity. Precisely, we have:

THEOREM 8.2. Let $\mathsf{m}_0 = (\mathbf{0}, r_0\boldsymbol{e}_3, -\boldsymbol{e}_3)$, that is, the top is spinning with G in the highest position. Then, if $C > A, B$ the following properties hold:

(a) If

$$r_0^2 > \frac{\beta^2}{C - M}, \qquad M := \max\{A, B\}, \tag{8.5}$$

then m_0 is stable. Moreover, there exists $\delta > 0$ such that if

$$\|(\boldsymbol{v}_0, \tilde{\boldsymbol{\omega}}_{\infty 0}, \boldsymbol{z}_0)\|_{\mathcal{H}} < \delta\,, \tag{8.6}$$

then

$$\begin{aligned}
&\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_{1,2} = 0\,,\\
&\lim_{t\to\infty} \tilde{\boldsymbol{\omega}}_\infty(t) = -\frac{1}{C}\Big\{(\boldsymbol{z}_0 - \boldsymbol{e}_3)\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_{\infty 0} - r_0 \boldsymbol{z}_0 \cdot \boldsymbol{I}\cdot \boldsymbol{e}_3\Big\}\boldsymbol{e}_3\,,\\
&\lim_{t\to\infty} \boldsymbol{z}(t) = \mathbf{0}\,.
\end{aligned} \tag{8.7}$$

(b) If [8]

$$r_0^2 < \min\left\{\frac{\beta^2}{C}, \frac{\mu^2}{C^2}\frac{\beta^2}{C - M}\right\}, \qquad \mu := \min\{A, B\}\,, \tag{8.8}$$

then m_0 is unstable. More precisely, there is an initial perturbation such that

$$\lim_{t\to\infty} \boldsymbol{z}(t) = 2\boldsymbol{e}_3\,. \tag{8.9}$$

Let $\mathsf{m}_0 = (\mathbf{0}, r_0\boldsymbol{e}_3, \boldsymbol{e}_3)$, that is, the top is spinning with G in its lowest position. The following properties hold:

(c) If $C > A, B$, then m_0 is stable.

(d) If $A, B > C$ and

$$r_0^2 < \frac{\beta^2}{M - C}, \qquad M := \max\{A, B\}, \tag{8.10}$$

then m_0 is stable.

[8] Notice that if $C \geq [(1+\sqrt{5})/2]M$, condition (8.8) becomes

$$r_0^2 < \frac{\mu^2}{C^2}\frac{\beta^2}{C - M}.$$

In fact, to fix the ideas, suppose $B \geq A$. It is at once verified that if C satisfies the further restriction, then

$$B^2 < C(C - B)\,,$$

which proves our claim. Conversely, if $M < C \leq [(1+\sqrt{5})/2]\mu$, condition (8.8) becomes

$$r_0^2 < \frac{\beta^2}{C}.$$

Moreover, in both cases (c), (d) there exists $\delta > 0$ such that, if condition (8.6) is met, then

$$\begin{aligned} &\lim_{t\to\infty} \|\boldsymbol{v}(t)\|_{1,2} = 0\,, \\ &\lim_{t\to\infty} \tilde{\boldsymbol{\omega}}_\infty(t) = \frac{1}{C}\Big\{(\boldsymbol{z}_0 + \boldsymbol{e}_3)\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_{\infty 0} + r_0 \boldsymbol{z}_0 \cdot \boldsymbol{I}\cdot \boldsymbol{e}_3\Big\}\,\boldsymbol{e}_3\,, \\ &\lim_{t\to\infty} \boldsymbol{z}(t) = \boldsymbol{0}\,. \end{aligned} \tag{8.11}$$

Proof. We will begin to show the stability properties stated in (a), (c) and (d). To this end, consider the perturbed fields $m = (\boldsymbol{0} + \tilde{\boldsymbol{v}}, r_0\boldsymbol{\gamma} + \tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{\gamma} + \boldsymbol{z})$, where

$$\boldsymbol{\gamma} := \begin{cases} -\boldsymbol{e}_3 & \text{in case (a)}, \\ \ \ \boldsymbol{e}_3 & \text{in cases (c) and (d)}\,. \end{cases}$$

By multiplying both sides of (8.2) by $2r_0$, we get

$$\begin{aligned} 2r_0\boldsymbol{\gamma}\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty(t) = {}& -2r_0^2 \boldsymbol{z}(t)\cdot \boldsymbol{I}\cdot \boldsymbol{\gamma} - 2r_0\boldsymbol{z}(t)\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty(t) \\ & + 2r_0\boldsymbol{\gamma}\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_{\infty 0} + 2r_0^2\boldsymbol{z}_0\cdot \boldsymbol{I}\cdot \boldsymbol{\gamma} + 2r_0\boldsymbol{z}_0\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_{\infty 0}, \end{aligned} \tag{8.12}$$

Next, consider the following scalar function:

$$\begin{aligned} U(\tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{z}) := {}& \tilde{\boldsymbol{\omega}}_\infty\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty - 2r_0^2\boldsymbol{z}\cdot \boldsymbol{I}\cdot \boldsymbol{\gamma} - 2r_0\boldsymbol{z}\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty - 2\beta^2 z_3 \\ & + \eta\left[(\boldsymbol{z}\cdot\boldsymbol{z})^2 + 4(\boldsymbol{\gamma}\cdot\boldsymbol{z})^2 + 4(\boldsymbol{z}\cdot\boldsymbol{z})(\boldsymbol{\gamma}\cdot\boldsymbol{z})\right], \end{aligned} \tag{8.13}$$

where $\eta \geq 0$ is a parameter that we will determine later. Replacing (8.12) in (8.1), and taking into account (8.3) and (8.13), we find that

$$\rho\|\boldsymbol{v}(t)\|^2 + U(\tilde{\boldsymbol{\omega}}_\infty(t), \boldsymbol{z}(t)) \leq \rho\|\boldsymbol{v}_0\|^2 + U(\tilde{\boldsymbol{\omega}}_\infty(0), \boldsymbol{z}(0)). \tag{8.14}$$

Notice that (8.13), in a more explicit form, is

$$\begin{aligned} U(\tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{z}) = {}& \tilde{\boldsymbol{\omega}}_\infty\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty - 2r_0^2(Az_1\gamma_1 + Cz_3\gamma_3) - 2r_0\boldsymbol{z}\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty - 2\beta^2 z_3 \\ & + \eta\left[(\boldsymbol{z}\cdot\boldsymbol{z})^2 + 4(\boldsymbol{z}\cdot\boldsymbol{z})(\boldsymbol{z}\cdot\boldsymbol{\gamma}) + 4(\gamma_1^2 z_1^2 + \gamma_3^2 z_3^2 + 2\gamma_1\gamma_3 z_1 z_3)\right]. \end{aligned}$$

Thus, by multiplying both sides of (8.3) by $r_0^2 A$, and replacing the resulting equation in the previous one, we infer

$$\begin{aligned} U(\tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{z}) = {}& \tilde{\boldsymbol{\omega}}_\infty\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty + (Ar_0^2 + 4\eta\gamma_1^2)z_1^2 + Ar_0^2 z_2^2 + (Ar_0^2 + 4\eta\gamma_3^2)z_3^2 \\ & - 2[(r_0^2(C-A) - 4\eta\gamma_1 z_1)\gamma_3 + \beta^2]z_3 - 2k\boldsymbol{z}\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty \\ & + \eta\left[(\boldsymbol{z}\cdot\boldsymbol{z})^2 + 4(\boldsymbol{z}\cdot\boldsymbol{z})(\boldsymbol{z}\cdot\boldsymbol{\gamma})\right]. \end{aligned}$$

To prove the theorem, it is enough to show that the following quadratic form is positive definite:

$$\begin{aligned} Q(\tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{z}) := {}& \tilde{\boldsymbol{\omega}}_\infty\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty + (Ar_0^2 + 4\eta\gamma_1^2)z_1^2 + Ar_0^2 z_2^2 + (Ar_0^2 + 4\eta\gamma_3^2)z_3^2 \\ & - 2[(r_0^2(C-A) - 4\eta\gamma_1 z_1)\gamma_3 + \beta^2]z_3 - 2r_0\boldsymbol{z}\cdot \boldsymbol{I}\cdot \tilde{\boldsymbol{\omega}}_\infty. \end{aligned} \tag{8.15}$$

In fact, if Q is positive definite, then U is positive definite in a suitable neighborhood of the origin, and we can apply Theorem 8.1 with $F := \rho\,\|\boldsymbol{v}\|^2$ and employ (8.14), to conclude our proof. Now, consider the cases in the statement.

(a) In this case, $\boldsymbol{\gamma} = -\boldsymbol{e}_3$. Then (8.3) becomes

$$2z_3 = z_1^2 + z_2^2 + z_3^2.$$

Replacing this in (8.15), we find that

$$Q(\tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{z}) = \tilde{\boldsymbol{\omega}}_\infty \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty \\ + (Cr_0^2 - \beta^2)(z_1^2 + z_2^2) + (Cr_0^2 - \beta^2 + 4\eta)z_3^2 - 2r_0 \boldsymbol{z} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty.$$

Choosing $\eta > \beta^2/4$, we at once deduce that Q is positive definite provided

$$r_0^2 > \frac{\beta^2}{C - M}, \quad C > M, \quad M := \max\{A, B\}.$$

(c)-(d) In both cases, $\boldsymbol{\gamma} = \boldsymbol{e}_3$. Then, (8.3) reads as follows:

$$2z_3 = -z_1^2 - z_2^2 - z_3^2.$$

Using the latter displayed equation in (8.15) and choosing $\eta \equiv 0$, we find that

$$Q(\tilde{\boldsymbol{\omega}}_\infty, \boldsymbol{z}) = \tilde{\boldsymbol{\omega}}_\infty \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty + (Cr_0^2 + \beta^2)(z_1^2 + z_2^2 + z_3^2) - 2r_0 \boldsymbol{z} \cdot \boldsymbol{I} \cdot \tilde{\boldsymbol{\omega}}_\infty. \tag{8.16}$$

From this it easily follows that if $C > A, B$, then Q is a positive definite quadratic form, whereas if $C < A, B$, then Q enjoys the same property provided condition (8.10) is satisfied.

We shall next prove the asymptotic properties stated in (a), (c) and (d). We begin to observe that, in all cases, the decay property for the velocity field $\boldsymbol{v}$ follows from Theorem 6.4. Furthermore, once the stated asymptotic condition on $\boldsymbol{z}$ is proved, the asymptotic expression for the angular velocity follows from Proposition 7.1. Therefore, we only have to show $(8.7)_3$ and $(8.11)_3$, and begin with case (a). Since condition (8.5) holds and $C > A, B$, in particular

$$r_0^2 \neq \frac{\beta^2}{C - A}, \frac{\beta^2}{C - B}.$$

Thus, we can apply Proposition 4.8 to infer the existence of a neighborhood $\mathcal{I}(\mathsf{m}_0)$ such that $\mathcal{I}(\mathsf{m}_0) \cap \mathsf{S} \subset \mathsf{PR}$. Fix $\varepsilon > 0$ such that

$$B_\varepsilon(\mathsf{m}_0) := \{(\boldsymbol{u}, \boldsymbol{\omega}, \boldsymbol{q}) \in \mathcal{H} : \ \|\boldsymbol{u}\|_2 + |\boldsymbol{\omega} - r_0\boldsymbol{e}_3| + |\boldsymbol{q} + \boldsymbol{e}_3| < \varepsilon\} \subset \mathcal{I}(\mathsf{m}_0).$$

Since m_0 is stable, corresponding to $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|\boldsymbol{v}(0)\|_2 + |\tilde{\boldsymbol{\omega}}_\infty(0)| + |\boldsymbol{z}(0)| < \delta \Rightarrow \|\boldsymbol{v}(t)\|_2 + |\tilde{\boldsymbol{\omega}}_\infty(t)| + |\boldsymbol{z}(t)| < \varepsilon \text{ for all } t \geq 0,$$

in other words, $\mathsf{m} := \tilde{\mathsf{m}} + \mathsf{m}_0 \in B_\varepsilon(\mathsf{m}_0)$ at all times. The latter along with Proposition 6.3 imply that the Ω-limit set of the weak solution m satisfies $\Omega(\mathsf{m}) \subset B_\varepsilon(\mathsf{m}_0) \cap \mathsf{S} \subset \mathsf{PR}$. Therefore, we have the following two possibilities: either

$$\Omega(\mathsf{m}) = \left\{ \left(\boldsymbol{u} \equiv \boldsymbol{0}, \bar{\boldsymbol{\omega}} = -\frac{1}{C}(\boldsymbol{z}(0) - \boldsymbol{e}_3) \cdot \boldsymbol{I} \cdot (\tilde{\boldsymbol{\omega}}_\infty(0) + r_0\boldsymbol{e}_3), \bar{\boldsymbol{\gamma}} = -\boldsymbol{e}_3 \right) \right\}$$

or

$$\Omega(\mathsf{m}) = \left\{ \left(\boldsymbol{u} \equiv \boldsymbol{0}, \bar{\boldsymbol{\omega}} = \frac{1}{C}(\boldsymbol{z}(0) - \boldsymbol{e}_3) \cdot \boldsymbol{I} \cdot (\tilde{\boldsymbol{\omega}}_\infty(0) + r_0\boldsymbol{e}_3), \bar{\boldsymbol{\gamma}} = \boldsymbol{e}_3 \right) \right\}.$$

However, the second choice cannot occur since, otherwise,

$$\lim_{t \to \infty} \boldsymbol{z}(t) = 2\boldsymbol{e}_3$$

and this would contradict the stability property. Therefore, $\boldsymbol{z}(t) \to \mathbf{0}$ as $t \to \infty$, as claimed. An entirely similar argument can be used to show that the statement holds also in the cases (c) and (d). In particular, condition (4.8) is automatically satisfied if $C > A, B$.

It remains to show the instability property claimed in part (b). We choose as initial conditions $\boldsymbol{v}_0 = \tilde{\boldsymbol{\omega}}_{\infty 0} = \mathbf{0}$ and $\boldsymbol{z}_0 \neq \mathbf{0}$. This means that we are just applying a small disorientation (tilt) to the vertical axis. By (8.3) with $\boldsymbol{\gamma} = -\boldsymbol{e}_3$, it follows

$$z_{30} = \frac{1}{2}(z_{10}^2 + z_{20}^2 + z_{30}^2). \tag{8.17}$$

We now show that we can choose z_{30} so small as to satisfy both conditions (7.1), (7.4) in Proposition 7.1. In fact, under the given assumptions on A, B and C, and our choice of initial perturbations (7.1) becomes

$$|(\boldsymbol{z}_0 - \boldsymbol{e}_3) \cdot \boldsymbol{I} \cdot (r_0 \boldsymbol{e}_3)|^2 < \frac{\mu^2 \beta^2}{(C - M)},$$

which, in view of hypothesis (8.8) and (8.17), is certainly satisfied by taking z_{30} sufficiently small. As for (7.4), with our choice of initial perturbations it becomes

$$C^2 r_0^2 - 2\beta^2 C\, z_{30} \leq |(\boldsymbol{z}_0 - \boldsymbol{e}_3) \cdot \boldsymbol{I} \cdot (r_0 \boldsymbol{e}_3)|^2,$$

namely,

$$C\, r_0^2 \leq \beta^2 + O(z_3),$$

which, again by the assumption (8.8), is also satisfied. We may then conclude that, when condition (8.8) is met, both requirements (7.1) and (7.4) are satisfied, so that Proposition 7.1 ensures the validity of (8.17) and the claimed instability follows. The proof of the theorem is completed. □

REMARK 8.3. The following important comments about the results just proved are in order.

(i) A classical result of Rumyantsev [28] (see also [23]) ensures that condition of the type (8.5) guarantees the stability of the "upright" rotation of the top also when the cavity is liquid-empty. Thus, in such a case, the perturbed motion will occur with $\boldsymbol{e}_3$ in a neighborhood of the vertical axis through O, e. In particular, if the top is symmetric ($A = B$) the top will eventually perform an unsteady motion of precession around e. The fundamental difference with a liquid-filled cavity consists in the fact that, in the latter circumstance, the *only* terminal state that the top can reach is a uniform rotation around e, with $\boldsymbol{e}_3 \equiv \mathsf{e}$, and G in its highest position. This shows, one more time, the strong *stabilizing influence of the liquid*. A similar stabilizing effect occurs for the "downright" rotation discussed in cases (c) and (d), where in the presence of liquid, the terminal state is again a uniform rotation around e, with $\boldsymbol{e}_3$ parallel to e, and G in its lowest position.

(ii) The condition for the instability of the "upright" rotation presented in (8.8) is somehow stronger than the one found by a linearized instability analysis. In fact, if one linearizes (3.1) around $\mathsf{m}_0 = (\mathbf{0}, r_0\, \boldsymbol{e}_3, -\boldsymbol{e}_3)$ one can show [16] that m_0 is unstable if

$$r_0^2 < \frac{\beta^2}{(C - M)},$$

which is the "strict" negation of (8.5). Seemingly, our analysis is not able to confirm this result at the *non-linear* level. However, we may guess the following. For simplicity, let us assume that $C \geq (1+\sqrt{5})M/2$ so that (8.10) becomes (see Footnote 8)

$$r_0^2 < \frac{\mu^2}{C^2}\frac{\beta^2}{(C-M)}\,. \tag{8.18}$$

By Theorem 8.2, we know that if (8.5) holds, then m_0 is stable and the terminal motion is a uniform "upright" rotation around $\mathbf{e}$, while if (8.18) holds, then m_0 is unstable and the terminal state is a uniform "downright" rotation around $\boldsymbol{a}_O$. The question is then what will the terminal state, t_S, be if

$$\frac{\mu^2}{C^2}\frac{\beta^2}{(C-M)} \leq r_0^2 \leq \frac{\beta^2}{(C-M)}\,.$$

If we expect that m_0 is unstable, then t_S cannot be a uniform rotation around $\mathbf{e}$ with G in its highest position. On the other hand, especially for values of r_0^2 close to $\beta^2/(C-M)$, the system could still have enough kinetic energy to overbalance the potential energy and thus sustain a motion different from a uniform rotation around $\boldsymbol{a}_O$ with G in its lowest position. Thus, according to parts 2 and 5 in Proposition 6.3, the only remaining possibility is that t_S is a steady precession with G above O. Proving or disproving this statement is still far from our current analytical reach. As a matter of fact, we do not even know if these steady precessions are stable or not. In this respect, it is worth noticing that, instead, one can show that steady precessions with G below O are indeed stable.[9] For example, following the same argument used in the proof of Theorem 8.2 (a), we may prove that if $A > B, C$, then every element of SP_1 is stable, whereas if $B > A, C$, then every element of SP_2, is stable. In conclusion, we believe that a targeted numerical simulation could prove useful insight on the whole issue.

Appendix.

LEMMA A.1 (A Gronwall-like Lemma). Let $y : [t_0, t_1) \to [0,\infty)$, $t_1 > t_0 \geq 0$, be an absolutely continuous function satisfying for some $a, b, c, \delta > 0$[10] and $\alpha > 1$,

(i) $y' \leq -a\,y + b\,y^\alpha + c$ in (t_0, t_1);

(ii) $\displaystyle\int_{t_0}^{t_1} y(\tau)\,d\tau < \frac{\delta^2}{4c}\,,\quad y(t_0) < \frac{\delta}{\sqrt{2}}\,.$

Then, if $k := -a + b\,\delta^{\alpha-1} < 0$, we have

$$y(t) < \delta\,,\quad \text{for all } t \in [t_0, t_1)\,. \tag{A.1}$$

Moreover, if $t_1 = \infty$ we have also

$$\lim_{t\to\infty} y(t) = 0\,. \tag{A.2}$$

Proof. Setting $Y := y^2$, from (i) we get

$$Y' \leq -2a\,Y + 2b\,Y^\beta + F\,,\quad t \in (t_0, t_1)\,, \tag{A.3}$$

[9]Notice that such steady motions can only exist if either $C < A$ or $C < B$.

[10]For the sake of completeness, we would like to remark that, as is well known, the lemma continues to hold if $c = 0$, even without assuming the first condition in (ii).

where $\beta := (\alpha+1)/2$, $F := 2c\,y$. In view of the second condition in (ii), contradicting (A.1) means that there exists $t^* \in (t_0, t_1)$ such that

$$Y(t) < \delta^2\,, \quad \text{for all } t \in [t_0, t^*)\,; \quad Y(t^*) = \delta^2\,. \tag{A.4}$$

Using this information back in (A.3) we find for all $t \in (t_0, t^*)$

$$Y'(t) \le 2(-a + b\delta^{\alpha-1})Y(t) + F(t)\,,$$

which in view of the assumptions, after integration from t_0 to t^*, furnishes

$$Y(t^*) < \frac{\delta^2}{2} + \int_{t_0}^{t_1} F(t)\,dt < \delta^2\,.$$

However, the latter is at odds with (A.4), and we thus conclude the proof of the first part of the lemma. In order to show the second part, we observe that from (A.1) and (A.3) we have

$$Y' \le -2a\,Y + 2(b\delta^\alpha + c)y := -2a\,Y + \alpha\,y\,.$$

By integrating the latter and using (A.1) we get, for all $t \ge 2t_0$,

$$Y(t) \le Y(t/2)\,\mathrm{e}^{-2a(t/2)} + \alpha \int_{t/2}^{t} y(\tau)d\tau < \delta\,\mathrm{e}^{-2a(t/2)} + \alpha \int_{t/2}^{t} y(\tau)d\tau\,,$$

from which (A.2) follows. □

Proof of Lemma 8.1. Denote by $|||\boldsymbol{y}||| := \sqrt{|\tilde{\boldsymbol{\omega}}_\infty|^2 + |\boldsymbol{z}|^2}$ the Euclidean norm on $\mathbb{R}^3 \times \mathbb{R}^3$, and let $\varepsilon_0 > 0$ be such that

$$B_{\varepsilon_0} := \{\boldsymbol{y} \in \mathbb{R}^3 \times \mathbb{R}^3 : |||\boldsymbol{y}||| \le \varepsilon_0\} \subset \mathcal{I}(\boldsymbol{0}).$$

Fix $\varepsilon \in (0, \varepsilon_0)$ and define

$$\xi := \min_{|||\boldsymbol{y}|||=\varepsilon/2} U(\boldsymbol{y}) > 0.$$

The minimum exists since U is continuous on B_{ε_0} and the sphere of radius ε is compact in $\mathbb{R}^3 \times \mathbb{R}^3$; moreover, ξ is strictly positive because of condition (ii) in the lemma. Again by the continuity of U, we find $\delta_0 > 0$ such that $|||\boldsymbol{y}||| < \delta_0$ implies $U(\boldsymbol{y}) < \frac{1}{2}\min\{\xi,\ \varepsilon,\ \frac{1}{2}c_1\varepsilon\}$, with c_1 defined in (8.4). Choose $\delta < \min\{\delta_0, \frac{1}{2c_2}\xi, \frac{1}{4}\frac{c_1}{c_2}\varepsilon\}$ with c_2 defined in (8.4). We want to show that

$$|||\boldsymbol{y}(t)||| < \varepsilon/2\,, \quad \text{for all } t \ge 0. \tag{A.5}$$

Suppose, on the contrary, that $\bar{t}$ is the first instant of time when $|||\boldsymbol{y}(\bar{t})||| = \varepsilon/2$. Thus, since F is positive definite and $V(t) \le V(0)$ for all $t \ge 0$, we deduce with the help of (8.4)

$$\xi \le V(\bar{t}) = F(\boldsymbol{v}(\bar{t})) + U(\boldsymbol{y}(\bar{t})) \le c_2\|\boldsymbol{v}(0)\|_2 + U(\boldsymbol{y}(0)) < \frac{\xi}{2} + \frac{\xi}{2} = \xi\,,$$

which shows a contradiction. Thus, (A.5) holds and, in addition, for all $t \ge 0$,

$$c_1\|\boldsymbol{v}(t)\|_2 \le F(\boldsymbol{v}(t)) \le V(0) \le c_2\|\boldsymbol{v}(0)\|_2 + U(\boldsymbol{y}(0)) < \frac{1}{4}c_1\varepsilon + \frac{1}{4}c_1\varepsilon = \frac{1}{2}c_1\varepsilon$$

which completes the proof. □

References

[1] K. T. Alfriend and T. M. Spencer, *Comparison of filled and partly filled nutation dampers*, J. Astronaut. Sci. **31** (1983), 189–202.

[2] P. G. Bhuta and L. R. Koval, *A viscous ring damper for a freely precessing satellite*, Intern. J. Mech. Sci. **8** (1966), 5–21.

[3] F. L. Chernousko, *Rotational motion of a rigid body with a cavity filled with fluid*, Prikl. Mat. Mekh. **31** (1967), 416–432.

[4] F. L. Chernousko, *Motion of a Rigid Body with Cavities Containing a Viscous Fluid*, NASA Technical Translations, Moscow (1972).

[5] M. Z. Dosaev and V. A. Samsonov, *On the stability of the rotation of a heavy body with a viscous filling* (Russian, with Russian summary), Prikl. Mat. Mekh. **66** (2002), no. 3, 427–433, DOI 10.1016/S0021-8928(02)00051-5; English transl., J. Appl. Math. Mech. **66** (2002), no. 3, 419–424. MR1937458

[6] G. P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations*, 2nd ed., Springer Monographs in Mathematics, Springer, New York, 2011. Steady-state problems. MR2808162

[7] G. P. Galdi, G. Mazzone, and P. Zunino, *Inertial motions of a rigid body with a cavity filled with a viscous liquid*, Comptes Rendus Mécanique **341** (2013), 760–765.

[8] K. Disser, G. P. Galdi, G. Mazzone, and P. Zunino, *Inertial motions of a rigid body with a cavity filled with a viscous liquid*, Arch. Ration. Mech. Anal. **221** (2016), no. 1, 487–526, DOI 10.1007/s00205-016-0966-2. MR3483900

[9] G. P. Galdi, G. Mazzone, and M. Mohebbi, *On the motion of a liquid-filled rigid body subject to a time-periodic torque*, Recent developments of mathematical fluid mechanics, Adv. Math. Fluid Mech., Birkhäuser/Springer, Basel, 2016, pp. 233–255. MR3524188

[10] G. P. Galdi and G. Mazzone, *On the motion of a pendulum with a cavity entirely filled with a viscous liquid*, Recent progress in the theory of the Euler and Navier-Stokes equations, London Math. Soc. Lecture Note Ser., vol. 430, Cambridge Univ. Press, Cambridge, 2016, pp. 37–56. MR3497686

[11] S. S. Hough, *On the oscillations of a rotating ellipsoidal shell containing fluid*, Phil. Trans. Roy. Soc. London **186** (1895), 469–506.

[12] B. G. Karpov, *Dynamics of liquid-filled shell: Resonance and effect of viscosity*, Ballistic Research Laboratories, Report no. 1279, 1965.

[13] T. V. Khomyak, *Stabilization of the unstable spinning of a Lagrange top filled with a fluid*, Internat. Appl. Mech. **51** (2015), no. 6, 702–709, DOI 10.1007/s10778-015-0728-0. Translation of Prikl. Mekh. **51** (2015), no. 6, 119–127. MR3432181

[14] N. N. Kolesnikov, *On the stability of a free rigid body with a cavity filled with an incompressible viscous fluid*, J. Appl. Math. Mech. **26** (1963), 914–923, DOI 10.1016/0021-8928(62)90155-7. MR0153206

[15] N. D. Kopachevsky and S. G. Krein, *Operator approach to linear problems of hydrodynamics. Vol. 2*, Operator Theory: Advances and Applications, vol. 146, Birkhäuser Verlag, Basel, 2003. Nonself-adjoint problems for viscous fluids. MR2002951

[16] A. G. Kostyuchenko, A. A. Shkalikov, and M. Yu. Yurkin, *On the stability of a top with a cavity filled with viscous fluid* (Russian, with Russian summary), Funktsional. Anal. i Prilozhen. **32** (1998), no. 2, 36–55, 95–96, DOI 10.1007/BF02482596; English transl., Funct. Anal. Appl. **32** (1998), no. 2, 100–113. MR1647828

[17] E. Leimanis, *The general problem of the motion of coupled rigid bodies about a fixed point*, Springer-Verlag, 1965.

[18] D. Lewis, T. Ratiu, J. C. Simo, and J. E. Marsden, *The heavy top: a geometric treatment*, Nonlinearity **5** (1992), no. 1, 1–48. MR1148788

[19] A. A. Lyashenko, *On the instability of a rotating body with a cavity filled with viscous liquid*, Japan J. Indust. Appl. Math. **10** (1993), no. 3, 451–469, DOI 10.1007/BF03167284. MR1247877

[20] A. A. Lyashenko and S. J. Friedlander, *A sufficient condition for instability in the limit of vanishing dissipation*, J. Math. Anal. Appl. **221** (1998), no. 2, 544–558, DOI 10.1006/jmaa.1998.5914. MR1621750

[21] G. Mazzone, *A mathematical analysis of the motion of a rigid body with a cavity containing a newtonian fluid*, Ph.D. thesis, Università del Salento, 2012.

[22] G. Mazzone, *On the dynamics of a rigid body with cavities completely filled by a viscous liquid*, ProQuest LLC, Ann Arbor, MI, 2016. Thesis (Ph.D.)–University of Pittsburgh. MR3597782

[23] D. J. McGill and L. S. Long III, *Stability regions for an unsymmetrical rigid body spinning about an axis through a fixed point* (English, with German summary), Acta Mech. **22** (1974), no. 1-2, 91–112, DOI 10.1007/BF01170620. MR0375868

[24] N. N. Moiseyev and V. V. Rumyantsev, *Dynamic stability of bodies containing fluid*, Springer, New York, 1968.

[25] Ngo ZuĭKan, *The free rotation of a rigid body having a cavity filled with viscous liquids* (Russian), Dokl. Akad. Nauk SSSR **246** (1979), no. 4, 823–827. MR543540

[26] L. A. Pars, *A treatise on analytical dynamics*, John Wiley & Sons, Inc., New York, 1965. MR0208873

[27] P. H. Roberts and K. Stewartson, *On the motion of a liquid in a spheroidal cavity of a precessing rigid body. II*, Proc. Cambridge Philos. Soc. **61** (1965), 279–288. MR0173437

[28] V. V. Rumyancev, *Stability of permanent rotations of a heavy rigid body* (Russian), Prikl. Mat. Meh. **20** (1956), 51–66. MR0085717

[29] V. V. Rumyantsev, *Dynamics and stability of rigid bodies*, C.I.M.E. Summer Schools, Volume 56, Springer-Verlag, 1971, 165–271.

[30] W. E. Scott, *The free flight stability of liquid-filled shell*, Ballistic Research Laboratories, Report no. 120, part 1a, 1960.

[31] A. L. Silvestre and T. Takahashi, *On the motion of a rigid body with a cavity filled with a viscous liquid*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), no. 2, 391–423, DOI 10.1017/S0308210510001034. MR2911173

[32] E. P. Smirnova, *Stabilization of free rotation of an asymmetric top with cavities completely filled with a liquid*, Prikl. Mat. Meh. **38** (1974), 980–985.

[33] S. L. Sobolev, *On a new problem of mathematical physics* (Russian), Izv. Akad. Nauk SSSR. Ser. Mat. **18** (1954), 3–50. MR0069382

[34] K. Stewartson and P. H. Roberts, *On the motion of a liquid in a spheroidal cavity of a precessing rigid body*, J. Fluid Mech. **17** (1963), 1–20, DOI 10.1017/S0022112063001063. MR0161533

[35] G. G. Stokes, *Mathematical and physical papers. Volume 1*, Cambridge Library Collection, Cambridge University Press, Cambridge, 2009. Reprint of the 1880 original. MR2858161

[36] G. G. Stokes, *On the effect of the internal friction of fluids on the motion of pendulums*, Trans. Cambridge Phil. Soc. **9**, part II (1850), 8–106.

[37] T. T. Taylor, *Mechanics: classical and quantum*, Pergamon Press, Oxford-New York-Toronto, Ont., 1976. International Series in Natural Philosophy, Vol. 82. MR0391743

[38] S. C. Xu, *On the nonlinear stability theory of spinning solid bodies with liquid-filled cavities and its application to the Columbus problem*, Sci. Sinica Ser. A **25** (1982), no. 6, 601–614. MR670884

[39] M. Yu. Yurkin, *On the stability of small oscillations of a rotating asymmetric top with fluid inside* (Russian), Dokl. Akad. Nauk **362** (1998), no. 2, 170–173. MR1673786

[40] I. G. Zagnibeda, *The solvability of the problem of the dynamics of a solid body with a cavity containing a viscous fluid* (Russian), Dokl. Akad. Nauk SSSR **238** (1978), no. 5, 1075–1078. MR0502848

[41] N. I. Zhoukovski, *On the motion of a rigid body having cavities filled with a homogeneous liquid drop*, Russian Journal of Physical and Chemical Society **17** (1885), 31–152.