QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXXVI, NUMBER 3
SEPTEMBER 2018, PAGES 489–498
http://dx.doi.org/10.1090/qam/1486
Article electronically published on September 19, 2017

ON A NONLINEAR INTEGRAL EQUATION FOR THE OCEAN FLOW IN ARCTIC GYRES

By

JIFENG CHU

Department of Mathematics, Shanghai Normal University, Shanghai 200234, China

Abstract. We investigate the existence and uniqueness of bounded solutions to a nonlinear integral equation which models the ocean flow in arctic gyres.

1. Introduction. The modelling of ocean flows is constrained by the specific location where it occurs, with fundamental differences between equatorial ocean flows, mid-latitude ocean flows and flows in polar regions (see the discussion in [8], [9], [11], [12], [13], [16]). Moreover, the major topographical differences between the two poles makes the behaviour of the ocean flows in polar regions very different (see the discussion in [5], [10]). The ocean flow in arctic regions plays an important role for the global climate and studies of its aspects are found throughout the recent scientific literature. In this paper we will investigate a recent model for arctic gyres – very large ocean flows with a predominantly horizontal motion, which rotate slowly due mainly to the Coriolis effect, which occurs because of the Earth's rotation around its polar axis (see the discussion in [4], [11]). For a given continuous vorticity, we establish the existence of solutions using a fixed point approach. The obtained result does not require a Lipschitz-type condition (being thus of a wider applicability than the recent result obtained in [5]). On the other hand, by means of an example we show that mere continuity does not ensure the uniqueness of solutions. However, we prove that an Osgood-type condition on the vorticity function guarantees uniqueness. In particular, we are therefore able to ensure the existence and uniqueness of physically realistic solutions outside the class of Lipschitz-continuous vorticity functions.

2. Preliminaries. Let us briefly present the model for arctic gyres, discussed in more detail in the papers [4], [5], [11]. Using the stereographical projection from the South Pole to the plane of the Equator, the flow of an arctic gyre near the North Pole, which

---

Received July 26, 2017 and, in revised form, August 16, 2017.
2010 *Mathematics Subject Classification.* Primary 45G99, 58J32, 76B03.
This work was supported by the National Natural Science Foundation of China (Grant No. 11671118).
E-mail address: jifengchu@126.com, jchu@shnu.edu.cn

presents negligible azimuthal variations (so that the flow is only dependent on the polar angle) is modelled by the second-order ordinary differential equation

$$u''(t) = \frac{F(u(t))}{\cosh^2(t)} - \frac{2\omega\sinh(t)}{\cosh^3(t)}\,, \qquad t > t_0\,, \tag{2.1}$$

and physically relevant solutions should satisfy the asymptotic conditions

$$\lim_{t\to\infty}\{u(t)\} = \psi_0 \quad \text{and} \quad \lim_{t\to\infty}\{u'(t)\,\cosh(t)\} = 0\,, \tag{2.2}$$

for some constant ψ_0, which is the value of the stream function u at the North Pole. The second asymptotic condition in (2.2) expresses the fact that the flow is stagnant at the North Pole, which is the gyre's center. In (2.1) the function F is given and specifies the total vorticity of the flow (spin vorticity, due to the Earth's rotation, plus oceanic vorticity), $\omega > 0$ is the nondimensional form of the Coriolis parameter, while $t = -\ln\cot(\theta/2)$ is the relation between the independent variable and the polar angle $\theta \in [0, \pi/2)$ for the Earth's spherical coordinate system (with $\theta = 0$ corresponding to the North Pole). The gyre flow modelled by (2.1)-(2.2) has a vanishing azmiuthal velocity component, with the polar azimuthal velocity given by $\cosh(t)\,u'(t)$, with the arctic ocean region corresponding to values $t_0 \geq 2$.

One can easily show (see [5]) that the problem (2.1)-(2.2) is equivalent to the integral equation

$$u(t) = [\psi_0 - \omega] + \omega\tanh(t) + \int_t^\infty (s-t)\frac{F(u(s))}{\cosh^2(s)}\,ds\,, \qquad t \geq t_0. \tag{2.3}$$

Note that if $u(t)$ is a solution to (2.1) satisfying (2.2), then

$$-u'(t) = -\frac{\omega}{\cosh^2(t)} + \int_t^\infty \frac{F(u(s))}{\cosh^2(s)}\,ds\,, \qquad t \geq t_0.$$

The integral equation (2.3) is a convenient way to simplify (2.1) and (2.2). In this context, let us recall the following useful facts, which have been used in [4] and [5], but which we state and prove here for completion.

Lemma 2.1. The following equality and inequality hold:

$$\int_t^\infty (s-t)\frac{1}{\cosh^2 s}ds = \ln(1+e^{-2t}), \quad t \in \mathbb{R},$$

$$\int_t^\infty (s-t)\frac{1}{\cosh^2 s}ds \leq \frac{1}{\cosh t}, \quad t \geq 0. \tag{2.4}$$

Proof. First, by direct computations, we have

$$\begin{aligned}
\int_t^\infty \frac{s-t}{\cosh^2(s)}\,ds &= \int_t^\infty\int_s^\infty \frac{1}{\cosh^2(\tau)}\,d\tau\,ds \\
&= \int_t^\infty [1-\tanh(s)]\,ds \\
&= \int_t^\infty \frac{2e^{-2s}}{1+e^{-2s}}\,ds = \ln(1+e^{-2t})\,.
\end{aligned}$$

On the other hand, using the fact $\sinh s \geq s$ for all $s \geq 0$, we obtain

$$\int_t^\infty (s-t)\frac{1}{\cosh^2 s}ds \leq \int_t^\infty \frac{s}{\cosh^2 s}ds \leq \int_t^\infty \frac{\sinh s}{\cosh^2 s}ds = \frac{1}{\cosh t}.$$

□

3. Main results. We now prove an existence result for the solution to the integral equation (2.3) with a nonlinear vorticity function F.

THEOREM 3.1. Assume that $F : \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a nondecreasing continuous function $g : [0,\infty) \to [0,\infty)$ such that

$$|F(u)| \leq g(|u|), \qquad u \in \mathbb{R}. \tag{3.1}$$

Then for every $\psi_0 \in \mathbb{R}$, there exists some $T_0 \geq t_0$ such that the equation (2.3) has at least one bounded continuous solution $u : [T_0,\infty) \to \mathbb{R}$ satisfying $\lim_{t\to\infty} u(t) = \psi_0$.

Proof. Let us choose $T_0 \geq t_0$ large enough such that

$$\frac{g(|\psi_0| + \omega + 1)}{\cosh T_0} \leq 1.$$

Define the set

$$X = \Big\{u \in C([T_0,\infty),\mathbb{R}) : \lim_{t\to\infty} u(t) = \psi_0\Big\}.$$

Then X is a Banach space endowed with the supremum norm $\|u\| = \sup_{t \geq T_0}\{|u(t)|\}$. Set

$$\Omega = \Big\{u \in X : \psi_0 - \omega - 1 \leq u(t) \leq |\psi_0| + \omega + 1\Big\}.$$

Define the operator $\mathcal{F} : \Omega \to X$ as

$$[\mathcal{F}(u)](t) = [\psi_0 - \omega] + \omega \tanh(t) + \int_t^\infty (s-t)\frac{F(u(s))}{\cosh^2(s)}\, ds, \quad t \geq T_0\,. \tag{3.2}$$

Note that the inequality

$$\Big|\int_t^\infty (s-t)\frac{F(u(s))}{\cosh^2(s)}\, ds\Big| \leq \int_t^\infty \frac{s\,|F(u(s))|}{\cosh^2(s)}\, ds, \quad t \geq T_0\,,$$

confirms that $\mathcal{F} : \Omega \to X$. Moreover, for any $u \in \Omega$, we have $\lim_{t\to\infty}[\mathcal{F}(u)](t) = \psi_0$ since

$$\lim_{t\to\infty}\int_t^\infty \frac{s|F(u(s))|}{\cosh^2(s)}\, ds = 0.$$

We shall apply the Schauder fixed point theorem (see [17] and some applications for differential equations in [2], [3], [6]) to prove that there exists a fixed point for the operator $\mathcal{F}$ in the nonempty closed bounded convex set Ω. It is divided into three steps.

Step 1. We show that $\mathcal{F}(\Omega) \subset \Omega$.

It follows from the condition (3.1), the monotonicity property of g and the inequality (2.4) that, for each $u \in \Omega$ and $t \geq T_0$,

$$\begin{aligned} \Big|[\mathcal{F}(u)](t) - (\psi_0 - \omega) - \omega \tanh t\Big| &= \Big|\int_t^\infty (s-t)\frac{F(u(s))}{\cosh^2(s)}\,ds\Big| \\ &\leq \int_t^\infty (s-t)\frac{|F(u(s))|}{\cosh^2(s)}\,ds \\ &\leq \int_t^\infty (s-t)\frac{g(|u(s)|)}{\cosh^2(s)}\,ds \\ &\leq g(|\psi_0| + \omega + 1)\int_t^\infty \frac{s}{\cosh^2(s)}\,ds \\ &\leq g(|\psi_0| + \omega + 1)\frac{1}{\cosh T_0} \\ &\leq 1, \end{aligned}$$

in which Lemma 2.1 is used. Therefore, for $t \geq T_0$,

$$[\mathcal{F}(u)](t) \leq (\psi_0 - \omega) + \omega \tanh t + 1 \leq |\psi_0| + \omega + 1,$$

and

$$[\mathcal{F}(u)](t) \geq (\psi_0 - \omega) + \omega \tanh t - 1 \geq \psi_0 - \omega - 1.$$

Thus $\mathcal{F} : \Omega \to \Omega$ is well-defined.

Step 2. We show that $\mathcal{F}(\Omega)$ is relatively compact in X.

Differentiating two sides of (3.2) with respect to t we obtain

$$[\mathcal{F}(u)]'(t) = \frac{\omega}{\cosh^2(t)} - \int_t^\infty \frac{F(u(s))}{\cosh^2(s)}\,ds, \quad t \geq t_0. \tag{3.3}$$

Now by the condition (3.1) and equation (3.3), for all $t \geq T_0$, we obtain

$$\begin{aligned} \Big|[\mathcal{F}(u)]'(t)\Big| &\leq \Big|\frac{\omega}{\cosh^2 t}\Big| + \Big|\int_t^\infty \frac{F(u(s))}{\cosh^2(s)}\,ds\Big| \\ &\leq \frac{\omega}{\cosh^2 T_0} + \int_t^\infty \frac{|F(u(s))|}{\cosh^2(s)}\,ds \\ &\leq \frac{\omega}{\cosh^2 T_0} + \int_t^\infty \frac{g(|u(s)|)}{\cosh^2(s)}\,ds \\ &\leq \frac{\omega}{\cosh^2 T_0} + g(|\psi_0| + \omega + 1)\int_t^\infty \frac{1}{\cosh^2(s)}\,ds \\ &\leq \frac{\omega}{\cosh^2 T_0} + g(|\psi_0| + \omega + 1)[1 - \tanh t] \\ &\leq \frac{\omega}{\cosh^2 T_0} + \cosh T_0, \end{aligned}$$

which implies that for all $u \in \Omega$,

$$\Big|[\mathcal{F}(u)]'(t)\Big| \leq M, \qquad t \geq T_0,$$

with

$$M = \frac{\omega}{\cosh^2 T_0} + \cosh T_0.$$

Let $\{u_n\}$ be an arbitrary sequence in Ω. Then

$$\Big|[\mathcal{F}(u_n)]'(t)\Big| \leq M, \qquad t \geq T_0, \quad n \geq 1.$$

By using the mean value theorem, we obtain

$$\Big|[\mathcal{F}(u_n)](t_1) - [\mathcal{F}(u_n)](t_2)\Big| \leq M|t_1 - t_2|, \qquad t_1, t_2 \geq T_0, \quad n \geq 1,$$

which shows that $\{[\mathcal{F}(u_n)]\}$ is equicontinuous in X.

On the other hand, it is easy to verify that $\{[\mathcal{F}(u_n)]\}$ is uniformly bounded in X. In fact, for $t \geq T_0$,

$$\begin{aligned}
\Big|[\mathcal{F}(u_n)](t)\Big| &\leq |\psi_0| + \omega|1 - \tanh(t)| + \int_t^\infty (s-t)\Big|\frac{F(u_n(s))}{\cosh^2(s)}\,ds\Big| \\
&\leq |\psi_0| + \omega + g(|\psi_0| + \omega + 1)\int_t^\infty \frac{s-t}{\cosh^2(s)}\,ds \\
&\leq |\psi_0| + \omega + 1.
\end{aligned}$$

Moreover, since

$$\lim_{t\to\infty}\Big[\psi_0 - \omega + \omega\tanh(t) + \int_t^\infty (s-t)\frac{F(u_n(s))}{\cosh^2(s)}\,ds\Big] = \psi_0,$$

we obtain

$$\begin{aligned}
\Big|[\mathcal{F}(u_n)](t) - \psi_0\Big| &\leq \Big|\omega[1 - \tanh t]\Big| + \Big|\int_t^\infty (s-t)\frac{F(u_n(s))}{\cosh^2(s)}\,ds\Big| \\
&\leq \Big|\omega[1 - \tanh t]\Big| + g(|\psi_0| + \omega + 1)\int_t^\infty \frac{s}{\cosh^2(s)}\,ds,
\end{aligned}$$

which implies that for every $\varepsilon > 0$, there exists $t_\varepsilon > t_0$ such that

$$\Big|[\mathcal{F}(u_n)](t) - \psi_0\Big| \leq \varepsilon, \quad t \geq t_\varepsilon, \quad n \geq 1.$$

This shows that $\{[\mathcal{F}(u_n)]\}$ is equi-convergent in X.

Now by applying the Arzela-Ascoli theorem [17], we obtain that $\{[\mathcal{F}(u_n)]\}$ is relatively compact in X.

Step 3. We show that $\mathcal{F} : \Omega \to \Omega$ is continuous.

Given a fixed $\varepsilon > 0$, there exists some $T_* \geq T_0$ such that

$$g(|\psi_0| + \omega + 1)\frac{1}{\cosh T_*} < \frac{\varepsilon}{3}.$$

Since $F : [\psi_0 - \omega - 1, |\psi_0| + \omega + 1] \to \mathbb{R}$ is uniformly continuous, there exists a constant $\delta > 0$ such that if $u, v \in [\psi_0 - \omega - 1, |\psi_0| + \omega + 1]$ with $|u - v| < \delta$, then

$$|F(u) - F(v)| < \frac{\cosh T_0}{3}\varepsilon.$$

Therefore, for all $u_1, u_2 \in \Omega$ with $\|u_1 - u_2\| < \delta$, by direct computations we can obtain

$$\begin{aligned}
\Big|[\mathcal{F}(u_1)](t) - [\mathcal{F}(u_2)](t)\Big| &= \Big|\int_t^\infty (s-t)\frac{F(u_1(s)) - F(u_2(s))}{\cosh^2 s} ds\Big| \\
&\le \int_t^\infty (s-t)\frac{\Big|F(u_1(s)) - F(u_2(s))\Big|}{\cosh^2 s} ds \\
&\le \int_{T_0}^{T_*} (s-T_0)\frac{\Big|F(u_1(s)) - F(u_2(s))\Big|}{\cosh^2 s} ds \\
&+ \int_{T_*}^\infty (s-T_*)\frac{\Big|F(u_1(s)) - F(u_2(s))\Big|}{\cosh^2 s} ds = I_1 + I_2.
\end{aligned}$$

Note that

$$I_1 \le \frac{\cosh T_0}{3}\varepsilon \int_{T_0}^{T_*} \frac{s - T_0}{\cosh^2 s} ds \le \frac{\cosh T_0}{3}\varepsilon \frac{1}{\cosh T_0} = \frac{\varepsilon}{3},$$

$$\begin{aligned}
I_2 &\le \int_{T_*}^\infty \frac{s}{\cosh^2 s}\Big\{|F(u_1(s))| + |F(u_2(s))|\Big\} ds \\
&\le \int_{T_*}^\infty \frac{s}{\cosh^2 s}\Big\{g(|u_1(s)|) + g(|u_2(s)|)\Big\} ds \\
&\le 2g(|\psi_0| + \omega + 1)\int_{T_*}^\infty \frac{s}{\cosh^2 s} \\
&\le 2g(|\psi_0| + \omega + 1)\frac{1}{\cosh T_*} < \frac{2\varepsilon}{3}.
\end{aligned}$$

Therefore,

$$\Big\|[\mathcal{F}(u_1)] - [\mathcal{F}(u_2)]\Big\| \le \varepsilon.$$

Hence $\mathcal{F} : \Omega \to \Omega$ is a continuous operator.

We therefore showed that all assumptions of the Schauder fixed point theorem are satisfied. Therefore, there exists $u \in \Omega$ such that $[\mathcal{F}(u)] = u$, which corresponds to a bounded solution of (2.3) on $[T_0, \infty)$. □

THEOREM 3.2. Assume that assumptions in Theorem 3.1 are satisfied. Suppose further that F satisfies the Osgood condition:

$$|F(u) - F(v)| \le G(|u - v|),$$

where G is continuous and nondecreasing, $G(0) = 0$, $G(r) > 0$ for $r > 0$ and

$$\lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 \frac{1}{G(r)} dr = \infty.$$

Then for every $\psi_0 \in \mathbb{R}$, the equation (2.3) has a unique bounded continuous solution $u : [T_0, \infty) \to \mathbb{R}$ satisfying $\lim_{t\to\infty} u(t) = \psi_0$, where T_0 is the same as in Theorem 3.1.

Proof. It follows from Theorem 3.1 that equation (2.3) has at least one bounded continuous solution $u : [T_0, \infty) \to \mathbb{R}$ satisfying $\lim_{t\to\infty} u(t) = \psi_0$. Now suppose that $u_1(t)$ and $u_2(t)$ are two bounded solutions of (2.3) with

$$\lim_{t\to\infty} u_1(t) = \lim_{t\to\infty} u_2(t) = \psi_0. \tag{3.4}$$

We shall prove that $u_1(t) = u_2(t)$ for $t \ge T_0$. It follows from (2.3) and condition (3.1) that

$$\begin{aligned} |u_1(t) - u_2(t)| &= \Big| \int_t^\infty (s-t) \frac{F(u_1(s)) - F(u_2(s))}{\cosh^2 s} ds \Big| \\ &\le \int_t^\infty (s-t) \frac{\Big|F(u_1(s)) - F(u_2(s))\Big|}{\cosh^2 s} ds \\ &\le \int_t^\infty (s-t) \frac{1}{\cosh^2 s} G\Big(|u_1(s) - u_2(s)|\Big) ds \\ &\le \int_t^\infty \frac{s}{\cosh^2 s} G\Big(|u_1(s) - u_2(s)|\Big) ds. \end{aligned}$$

Let $\phi(t) = |u_1(t) - u_2(t)|$. Then $\phi(t) \ge 0$ for all $t \ge T_0$ and

$$\phi(t) \le \int_t^\infty \frac{s}{\cosh^2 s} G\Big(\phi(s)\Big) ds.$$

We only need to show that $\phi(t) \equiv 0$ for all $t \ge T_0$. On the contrary, suppose that

$$\Phi(t) = \max_{s \ge t} \phi(s) > 0.$$

Then $\phi(t) \le \Phi(t)$ and it follows from (3.4) that

$$\lim_{t\to\infty} \Phi(t) = \lim_{t\to\infty} \phi(t) = 0,$$

which implies that for each $t \ge T_0$, there exists a $t^* \ge t$ such that $\Phi(t) = \phi(t^*)$. Therefore,

$$\Phi(t) = \phi(t^*) \le \int_{t^*}^\infty \frac{s}{\cosh^2 s} G\Big(\phi(s)\Big) ds \le \int_t^\infty \frac{s}{\cosh^2 s} G\Big(\phi(s)\Big) ds.$$

Set

$$\Psi(t) = \int_t^\infty \frac{s}{\cosh^2 s} G\Big(\phi(s)\Big) ds.$$

Then $\phi(t) \le \Psi(t)$ and $\lim_{t\to\infty} \Psi(t) = 0$. Moreover,

$$\Psi'(t) = -\frac{t}{\cosh^2 t} G(\phi(t)) \ge -\frac{t}{\cosh^2 t} G(\Psi(t)).$$

Therefore,

$$\frac{\Psi'(t)}{G(\Psi(t))} \ge -\frac{t}{\cosh^2 t}.$$

Integrating the above from t^* to ∞, we obtain

$$\int_{t^*}^\infty \frac{\Psi'(t)}{G(\Psi(t))} dt \ge -\int_{t^*}^\infty \frac{t}{\cosh^2 t} dt \ge -\frac{1}{\cosh t^*},$$

which is equivalent to

$$\int_0^{r^*} \frac{1}{G(r)} dr \le \frac{1}{\cosh t^*}, \tag{3.5}$$

where

$$r^* = \Psi(t^*) = \int_{t^*}^{\infty} \frac{s}{\cosh^2 s} G\Big(\phi(s)\Big) ds \in (0, \infty).$$

Now (3.5) contradicts the Osgood condition. This contradiction shows that $\Phi(t) \equiv 0$, and hence $\phi(t) \equiv 0$ for all $t \ge t_0$. □

REMARK 3.3. As a special case of Theorem 3.2, the equation (2.3) has a unique bounded continuous solution if F satisfies a local Lipschtiz condition. We thus recover the existence results proved in [4] for linear vorticity functions F and in [5] under the more restrictive hypothesis of a global Lipschitz condition on the nonlinear function F. For examples of non-Lipschitz functions F which satisfy an Osgood-type condition we refer to the discussion in [1] and [7].

Finally in this section, we construct an example having at least two nontrivial bounded solutions, in which the nonlinearity F is continuous but does not satisfy the Osgood condition.

EXAMPLE 3.4. Consider again the differential equation (2.1) with the asymptotic conditions (2.2). It is easy to see that the function

$$u_+(t) = \frac{1}{\cosh^3(t)}, \qquad t \ge 1, \tag{3.6}$$

satisfies (2.2) since

$$u_+'(t) = -\frac{3\sinh(t)}{\cosh^4(t)}, \qquad t \ge 1.$$

Note that

$$u_+''(t) = \frac{-3}{\cosh^3(t)} + \frac{12\sinh^2(t)}{\cosh^5(t)}.$$

Since

$$\tanh^2(t) = 1 - \frac{1}{\cosh^2(t)} = 1 - [u_+(t)]^{2/3},$$

we obtain that

$$\begin{aligned} u_+''(t) + \frac{2\omega\sinh(t)}{\cosh^3(t)} &= \frac{-3}{\cosh^3(t)} + \frac{12\sinh^2(t)}{\cosh^5(t)} + \frac{2\omega\sinh(t)}{\cosh^3(t)} \\ &= \frac{1}{\cosh^2(t)}\left\{\frac{-3}{\cosh(t)} + \frac{12\sinh^2(t)}{\cosh^3(t)} + \frac{2\omega\sinh(t)}{\cosh(t)}\right\} \\ &= \frac{1}{\cosh^2(t)}\Big\{ -3[u_+(t)]^{1/3} + 12\Big(1 - [u_+(t)]^{2/3}\Big)[u_+(t)]^{1/3} \\ &\qquad + 2\omega\sqrt{1 - [u_+(t)]^{2/3}}\Big\}. \end{aligned}$$

Consequently, $u_+(t)$ is a solution to (2.1)-(2.2) if F satisfies

$$F(u) = 9\,u^{1/3} - 12\,u + 2\omega\sqrt{1 - u^{2/3}}, \qquad 0 \le u \le 1.$$

Note that if F is given as above, then

$$F(0) = 2\omega\,, \qquad F(1) = -3.$$

Similarly we can verify that the function

$$u_-(t) = -\frac{1}{\cosh^3(t)}\,, \qquad t \geq 1\,, \tag{3.7}$$

is a solution of (2.1)-(2.2) if F satisfies

$$F(u) = 9\,u^{1/3} - 12\,u + 2\omega\sqrt{1-u^{2/3}}\,, \quad -1 \leq u \leq 0,$$

and in this case

$$F(0) = 2\omega\,, \qquad F(-1) = 3\,.$$

The above analysis shows that if the function $F : \mathbb{R} \to \mathbb{R}$ is given by

$$F(u) = \begin{cases} -3\,, & u \geq 1\,, \\ 9\,u^{1/3} - 12\,u + 2\omega\sqrt{1-u^{2/3}}\,, & -1 \leq u < 1\,, \\ 3\,, & u < -1\,, \end{cases} \tag{3.8}$$

then $u_+(t)$ and $u_-(t)$ given by (3.6) and (3.7), respectively, are two distinct solutions to (2.1)-(2.2). Obviously, F given by (3.8) is continuous, but does not satisfy the Osgood condition. $\square$

Remark 3.5. We can ensure that $T_0 = t_0$ in Theorem 3.1 and in Theorem 3.2 by a simple continuation argument for the differential equation (2.1), provided that its solutions do not blow up in finite time. This is the case, for example, if the function $F(u)$ has a linear growth at infinity. Indeed, in this case the fact that blow-up does not occur for linear equations ensures $T_0 = t_0$, by means of the comparison method (see the discussion in [14] or, alternatively, the considerations in [15]).

Acknowledgment. The author would like to express his great thanks to the referees for their valuable suggestions.

References

[1] R. P. Agarwal and V. Lakshmikantham, *Uniqueness and nonuniqueness criteria for ordinary differential equations*, Series in Real Analysis, vol. 6, World Scientific Publishing Co., Inc., River Edge, NJ, 1993. MR1336820

[2] J. Chu and P. J. Torres, *Applications of Schauder's fixed point theorem to singular differential equations*, Bull. Lond. Math. Soc. **39** (2007), no. 4, 653–660, DOI 10.1112/blms/bdm040. MR2346946

[3] J. Chu, P. J. Torres, and M. Zhang, *Periodic solutions of second order non-autonomous singular dynamical systems*, J. Differential Equations **239** (2007), no. 1, 196–212, DOI 10.1016/j.jde.2007.05.007. MR2341553

[4] J. Chu, *On a differential equation arising in geophysics*, Monatsh. Math. https://doi.org/10.1007/s00605-017-1087-1.

[5] J. Chu, *On a nonlinear model for arctic gyres*, Ann. Mat. Pura Appl. (4), https://doi.org/10.1007/s10231-017-0696-6.

[6] A. Constantin, *On the existence of positive solutions of second order differential equations*, Ann. Mat. Pura Appl. (4) **184** (2005), no. 2, 131–138, DOI 10.1007/s10231-004-0100-1. MR2149089

[7] A. Constantin, *A note on the uniqueness of solutions of ordinary differential equations*, Appl. Anal. **64** (1997), no. 3-4, 273–276, DOI 10.1080/00036819708840535. MR1460083

[8] A. Constantin and R. S. Johnson, *The dynamics of waves interacting with the Equatorial Undercurrent*, Geophys. Astrophys. Fluid Dyn. **109** (2015), no. 4, 311–358, DOI 10.1080/03091929.2015.1066785. MR3375654

[9] A. Constantin and R. S. Johnson, *An exact, steady, purely azimuthal equatorial flow with a free surface*, J. Phys. Oceanogr. **46** (2016), 1935–1945.

[10] A. Constantin and R. S. Johnson, *An exact, steady, purely azimuthal flow as a model for the Antarctic Circumpolar Current*, J. Phys. Oceanogr. **46** (2016), 3585–3594.

[11] A. Constantin and R. S. Johnson, *Large gyres as a shallow-water asymptotic solution of Euler's equation in spherical coordinates*, Proc. A. **473** (2017), no. 2200, 20170063, 17. MR3650591

[12] A. Constantin and R. S. Johnson, *A nonlinear, three-dimensional model for ocean flows, motivated by some observations of the Pacific Equatorial Undercurrent and thermocline*, Physics of Fluids, **29** (2017), 056604.

[13] A. Constantin and S. G. Monismith, *Gerstner waves in the presence of mean currents and rotation*, J. Fluid Mech. **820** (2017), 511–528, DOI 10.1017/jfm.2017.223. MR3659720

[14] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, D. C. Heath and Co., Boston, Mass., 1965. MR0190463

[15] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, Cambridge, 1991. MR1109491

[16] R. S. Johnson, *An ocean undercurrent, a thermocline, a free surface, with waves: a problem in classical fluid mechanics*, J. Nonlinear Math. Phys. **22** (2015), no. 4, 475–493, DOI 10.1080/14029251.2015.1113042. MR3434074

[17] E. Zeidler, *Nonlinear functional analysis and its applications. I*, Springer-Verlag, New York, 1986. Fixed-point theorems; Translated from the German by Peter R. Wadsack. MR816732