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ON THE INVISCID LIMIT FOR THE COMPRESSIBLE NAVIER-STOKES SYSTEM WITH NO-SLIP BOUNDARY CONDITION

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Abstract. The proposal of this paper is to study the convergence of the compressible Navier-Stokes equations with no-slip boundary condition to the corresponding problem of the Euler equations in a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$. Motivated by Wang's work (2001), we obtain a sufficient condition for the convergence to take place in the energy space $L^2(\Omega)$ uniformly in time, by using Kato's idea (1984) of constructing an artificial boundary layer. This improves the result of Sueur in the sense that this sufficient condition contains the tangential or the normal component of velocity only.

Contents

1. Introduction. This paper is devoted to the issue of the inviscid limit for the compressible Navier-Stokes system with no-slip boundary condition. More specifically, for a smooth bounded domain $\Omega \subseteq \mathbb{R}^3$ we consider the question of convergence of the

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weak solution to the following problem for the compressible Naiver-Stokes system in $[0,T]\times\Omega$:

$$
\begin{cases}
\partial_t\rho + div(\rho u) = 0,\\
\partial_t(\rho u) + div(\rho u\otimes u) + \nabla p(\rho) = \varepsilon div\mathbb{S}(\nabla u),\\
u|_{x\in\partial\Omega} = 0,\\
\rho|_{t=0} = \rho_0, (\rho u)|_{t=0} = \rho_0 u_0,
\end{cases}
\tag{1.1}
$$

to the solution of the compressible Euler system:

$$
\begin{cases}
\partial_t\rho^E + div(\rho^E u^E) = 0,\\
\partial_t(\rho^E u^E) + div(\rho^E u^E\otimes u^E) + \nabla p(\rho^E) = 0,\\
(u^E\cdot\vec{n})|_{x\in\partial\Omega} = 0,\\
\rho^E|_{t=0} = \rho_0^E, (\rho^E u^E)|_{t=0} = \rho_0^E u_0^E,
\end{cases}
\tag{1.2}
$$

where

$$
\mathbb{S}(\nabla u) := \mu\left(\nabla u + (\nabla u)^T - \frac{2}{3}(div\ u)Id\right) + \eta(div\ u)Id,\ p(\rho) = \rho^\gamma\ with\ \gamma > \frac{3}{2},
$$

and $\vec{n}$ denotes the unit outward normal vector to the boundary of Ω, when the initial data of (1.1) goes to the initial data of (1.2) in energy space as ε vanishes. Since the Navier-Stokes equations and the Euler equations admit different boundary conditions in (1.1) and (1.2) respectively, this leads to the formulation of boundary layers in the small neighborhood of boundary $\partial\Omega$, in which the flow (1.1) changes very fast in the small viscosity limit.

The study of vanishing viscosity limit for solutions of the Navier-Stokes equations is a classical problem. Prandtl, in the pioneer work [20], studied the flow near the physical boundary and introduced the boundary layer concept. For the incompressible Navier-Stokes equations with no-slip boundary condition, Prandtl derived that the boundary layer is described by a degenerate parabolic equation coupled with the divergence free constrain, which is now called the Prandtl equations. Till now, there have been many interesting mathematical results on the well-posedness of the Prandtl equations; cf. see [11, 19, 21]. The rigorous justification of Prandtl's boundary layer theory was known only for some special cases. Lopes Filho et al. [17] studied the small viscosity limit for solutions to an incompressible circularly symmetric viscous flow, in which the boundary layer is described by the heat equation. The problem of circularly symmetric viscous compressible flow was studied in [16]. Recently, Guo et al. [12] showed the validity of the Prandtl boundary layer theory for two-dimensional steady incompressible Navier-Stokes flows with a no-slip boundary condition over a moving plate. Sammartino and Caflisch [22] obtained a rigorous theory of the Prandtl boundary layer problem in the class of analytic solutions in two or three space variables. By using the vorticity formulation, Maekawa [18] proved the convergence of the two-dimensional Navier-Stokes flow to the Euler flow away from the boundary and to the Prandtl flow in the boundary layer in the small viscosity limit when the initial vorticity of the Euler flow is supported away from the boundary. There are also many works on vanishing viscosity limit of solutions to the Navier-Stokes equations with a slip boundary condition, see [26–28], in which the boundary layer is much weaker than the problem with non-slip boundary condition.

Another approach for proving the convergence from the solutions of the Navier-Stokes equations to the solutions of the Euler equations was introduced by Kato in [13], in which he studied the small viscosity limit of the incompressible viscous flow with the non-slip boundary condition, and concluded that the viscous flow can be approximated by the inviscid flow in the energy space under a dissipation condition of energy in a neighborhood of the physical boundary with width proportional to the viscosity, by constructing an artificial boundary layer. Since then, this result has been improved in a series of works. Wang [25] observed that one can relax Kato's dissipation condition of energy to the case only containing the tangential derivatives of the tangential or normal velocity at the expense of increasing the size of the neighborhood of the boundary slightly. In [14], Kelliher extends Kato's result in the way that the gradient of velocity of Kato's energy condition can be replaced by the vorticity only of the flow. In [4], Constantin et al. obtained that under the assumption of the Oleinik condition of no back-flow in the trace of the Euler flow, and of a lower bound for the Navier-Stokes vorticity in a Kato-like boundary layer, the inviscid limit from the Navier-Stokes equations to the Euler equations holds in energy space.

Recently, there are some progresses in studying the small viscosity limit of the compressible viscous flow. For the compressible Navier-Stokes system (1.1), Sueur [24] gave the following sufficient condition for the convergence holding from the compressible viscous flow to the inviscid flow:

$$\varepsilon \int_{(0,T)\times\Gamma_{c\varepsilon}} \left(\frac{\rho|u|^2}{d_\Omega^2} + \frac{\rho^2(u\cdot n)^2}{d_\Omega^2} + |\nabla u|^2 \right) dxdt \to 0 \text{ when } \varepsilon \to 0, \tag{1.3}$$

where $u \cdot n$ denotes the normal component of u, $d_\Omega(x)$ is the distance of $x \in \Omega$ to the boundary $\partial\Omega$, and $\Gamma_{c\varepsilon} = \{x \in \Omega |\ d_\Omega(x) \le c\varepsilon\}$ for a positive $c > 0$. By using the Hardy inequality, one can regard (1.3) as an extension of Kato's result in the compressible flow. Besides (1.3), there are some criteria for the validity of the small viscosity limit for the compressible viscous flow; see [2].

The main proposal of this work is to weaken the above condition (1.3) to include only the tangential or the normal component of velocity in the integrand to have the small viscosity limit for the problem (1.1) in the energy space $L^\infty(0,T;L^2(\Omega))$. By developing the idea of [25], we improve the result of [24] by requiring only the second term in (1.3), at the cost of increasing the width of the boundary layer. Moreover, we shall obtain another similar condition with the integrand only containing tangential component of the velocity.

This paper is organized as follows. In Section 2, we recall the definition of the weak solutions to the compressible Navier-Stokes system with no-slip condition, and some results on existence of solutions to problems (1.1) and (1.2), then state the main result of this paper. In Section 3, we introduce an important relative energy inequality similar to that given in [24]. Finally, we prove the main result in Section 4.

2. Preliminaries and the main result. In the following calculation, we shall use the notation $o(1)$ or $O(1)$ to denote a quantity converging to zero or being bounded respectively, as $\varepsilon \to 0$, and C to denote a generic constant that may change from line to line.

For the system given in (1.1), first assume the lower bounds on μ and η entail that the tensor product

$$\mathbb{S}(\nabla u) : \nabla u = \sum_{1 \le i,j \le 3} \frac{\mu}{2}(\partial_i u_j + \partial_j u_i)^2 + (\eta - \frac{2}{3}\mu)|divu|^2$$

is a positively definite quadratic form with respect to $(\partial_i u_j)_{1 \le i,j \le 3}$, and there exists a constant $C_0 > 0$ such that for any $u \in H^1(\Omega)$,

$$\int_\Omega \mathbb{S}(\nabla u) : \nabla u dx \ge C_0 \int_\Omega |\nabla u|^2 dx. \tag{2.1}$$

Let's recall the definition of weak solutions to the compressible Naiver-Stokes equations with no-slip condition, cf. [8]:

Definition 2.1. For a fixed T >0, we say that (ρ, u) is a finite energy weak solution of the problem (1.1) for the compressible Navier-Stokes system with no-slip boundary condition on $[0, T]$ associated to the initial data satisfying

$$\rho_0 \ge 0,\ \rho_0 \in L^\gamma(\Omega),\ \rho_0 |u_0|^2 \in L^1(\Omega), \tag{2.2}$$

if:

$$\rho \in C_w([0, T]; L^\gamma(\Omega)),\ \rho u \in C_w([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega)),$$
$$u \in L^2([0, T]; H_0^1(\Omega)),\ \rho u^2 \in C_w([0, T]; L^1(\Omega)),$$

satisfy the problem (1.1) in the sense of distributions, and the energy inequality:

$$\mathcal{E}(\rho(\sigma, \cdot), u(\sigma, \cdot)) + \varepsilon \int_0^\sigma \int_\Omega \mathbb{S}(\nabla u) : \nabla u dx dt \le \mathcal{E}(\rho_0, u_0) \tag{2.3}$$

holds for almost all $\sigma \in [0, T]$, where

$$\mathcal{E}(\rho, u) = \int_\Omega E(\rho, u) dx,\ \text{with } E(\rho, u) := \frac{1}{2}\rho|u|^2 + H(\rho) \text{ and } H(\rho) := \frac{\rho^\gamma}{\gamma - 1}.$$

The following existence of such a weak solution of (1.1) was given in [9, 15]:

Proposition 2.1. Let (ρ_0, u_0) satisfy the assumption (2.2), for any fixed $T > 0$, there exists a finite energy weak solution of the problem (1.1) for the compressible Navier-Stokes equations on $[0, T]$.

Also, the following existence of a strong solution to the problem (1.2) for the compressible Euler equations can be found in many works; cf. [1, 3]:

Proposition 2.2. Assume that $\rho_0^E, u_0^E \in H^3(\Omega)$ satisfy the compatibility conditions of the problem (1.2), and $0 < \inf_\Omega \rho_0^E \le \sup_\Omega \rho_0^E < \infty$, then there exist $T > 0$ and a unique solution (ρ^E, u^E) of (1.2) in $[0, T] \times \Omega$ satisfying

$$0 < \inf_{(0,T)\times\Omega} \rho^E \le \sup_{(0,T)\times\Omega} \rho^E < \infty,$$

and

$$\partial_t^j u^E,\ \partial_t^j \rho^E \in C(0,T;H^{3-j}(\Omega)),\ j=0,1,2.$$

Denote by

$$d_\Omega(x) := dist(x,\partial\Omega) \text{ and } \Gamma_\varepsilon := \{x\in\Omega |\ d_\Omega(x)<\varepsilon\}$$

for $\varepsilon > 0$ small enough.

The main result of this paper is as follows:

THEOREM 2.1. Let (ρ^E, u^E) be a strong solution of the Euler equations on $[0,T]$ corresponding to an initial data (ρ_0^E, u_0^E) as given in Proposition 2.2, and $(\rho^\varepsilon, u^\varepsilon)$ be a weak solution of the compressible Navier-Stokes equations (1.1) on $[0,T]$ with initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$ satisfying (2.2) for any $\varepsilon \in (0,1)$, as given in Proposition 2.1. Assume that

$$\| \rho_0^\varepsilon - \rho_0^E \|_{L^\gamma(\Omega)} + \int_\Omega \rho_0^\varepsilon |u_0^\varepsilon - u_0^E|^2 dx \to 0 \quad \text{when} \quad \varepsilon\to 0, \tag{2.4}$$

we have

$$\sup_{t\in(0,T)} \left(\| \rho^\varepsilon - \rho^E \|_{L^\gamma(\Omega)} + \int_\Omega \rho^\varepsilon |u^\varepsilon - u^E|^2 dx \right)(t) \to 0 \quad \text{when} \quad \varepsilon \to 0$$

if one of the following conditions holds:

$$\varepsilon \int_{(0,T)\times\Gamma_\delta} \frac{\rho^{\varepsilon 2}(u^\varepsilon\cdot n)^2}{d_\Omega^2} dxdt \to 0 \quad \text{when} \quad \varepsilon\to 0, \tag{2.5}$$

$$\varepsilon \int_{(0,T)\times\Gamma_\delta} \frac{\rho^{\varepsilon 2}(u^\varepsilon\cdot \tau)^2}{d_\Omega^2} dxdt \to 0 \quad \text{when} \quad \varepsilon\to 0, \tag{2.6}$$

where $u^\varepsilon \cdot n$ and $u^\varepsilon\cdot\tau$ denote the normal and the tangential components of u^ε respectively, and $\delta \to 0$ when $\varepsilon\to 0$, with $\varepsilon = o(\delta)$.

3. Relative energy inequality. As in [24], we introduce the following relative energy $\mathcal{E}([\rho,u]|[r,U])$ of (ρ,u) with respect to (r,U):

$$\mathcal{E}([\rho,u]|[r,U]) := \int_\Omega E([\rho,u]|[r,U])dx,$$

where

$$E([\rho,u]|[r,U]) = \frac{1}{2}\rho|u-U|^2 + H(\rho|r),$$

with

$$H(\rho|r) = \frac{\rho^\gamma}{\gamma-1} - \frac{\gamma(\rho-r)r^{\gamma-1}}{\gamma-1} - \frac{r^\gamma}{\gamma-1}.$$

We shall use the following inequality of relative energy frequently, which has been given in [10, 24].

LEMMA 3.1. For any compact set $K\subset(0,\infty)$, there exist two positive constants C_1 and C_2 such that for any $\rho\geq 0$ and $r\in K$,

$$C_1(|\rho-r|^2 1_{|\rho-r|<1} + |\rho-r|^\gamma 1_{|\rho-r|\geq 1}) \leq H(\rho|r) \leq C_2(|\rho-r|^2 1_{|\rho-r|<1} + |\rho-r|^\gamma 1_{|\rho-r|\geq 1}), \tag{3.1}$$

where $1_{|\rho-r|<1}$ or $1_{|\rho-r|\geq 1}$ denotes the classical characteristic functions.

For completeness, we give the proof of this inequality.

Proof. i) If $|\rho - r| \le 1$, we use the Taylor expansion to get

$$H(\rho|r) = \frac{\gamma}{2}((1-\theta)\rho + \theta r)^{\gamma-2}|\rho - r|^2$$

for some $\theta \in (0,1)$. Since $r \in K$, $((1-\theta)\rho + \theta r)^{\gamma-2}$ is bounded from above and has a positive lower bound. Thus we have the estimate (3.1) as $|\rho - r| < 1$.

ii) For $|\rho - r| \ge 1$, we will discuss in two cases. If $\rho - r \ge 1$, obviously we have

$$H(\rho|r) = \frac{(\rho - r)^\gamma}{\gamma - 1}\left(\frac{\rho^\gamma}{(\rho-r)^\gamma} - \frac{\gamma r^{\gamma-1}}{(\rho-r)^{\gamma-1}} - \frac{r^\gamma}{(\rho-r)^\gamma}\right).$$

Since $r \in K$, we know that

$$\frac{\rho^\gamma}{(\rho-r)^\gamma} - \frac{\gamma r^{\gamma-1}}{(\rho-r)^{\gamma-1}} - \frac{r^\gamma}{(\rho-r)^\gamma} \text{ is bounded from above.}$$

Let $x := \frac{r}{\rho - r}$; then $\frac{\rho}{\rho-r} = 1 + x$. Using the Taylor expansion we have

$$\begin{aligned}\tfrac{\rho^\gamma}{(\rho-r)^\gamma} - \tfrac{\gamma r^{\gamma-1}}{(\rho-r)^{\gamma-1}} - \tfrac{r^\gamma}{(\rho-r)^\gamma} &= (1+x)^\gamma - \gamma x^{\gamma-1} - x^\gamma \\ &= \textstyle\int_x^{1+x} t^{\gamma-2}(1+x-t)dt = \int_0^1 (z+x)^{\gamma-2}(1-z)dz \\ &\ge \textstyle\int_0^1 z^{\gamma-2}(1-z)dz.\end{aligned}$$

This implies that $\frac{\rho^\gamma}{(\rho-r)^\gamma} - \frac{\gamma r^{\gamma-1}}{(\rho-r)^{\gamma-1}} - \frac{r^\gamma}{(\rho-r)^\gamma}$ has a positive lower bound.

When $r - \rho \ge 1$, we have

$$H(\rho|r) = \frac{(r-\rho)^\gamma}{\gamma - 1}\left(\frac{\rho^\gamma}{(r-\rho)^\gamma} + \frac{\gamma r^{\gamma-1}}{(r-\rho)^{\gamma-1}} - \frac{r^\gamma}{(r-\rho)^\gamma}\right).$$

Since $r \in K$, we know that

$$\frac{\rho^\gamma}{(r-\rho)^\gamma} + \frac{\gamma r^{\gamma-1}}{(r-\rho)^{\gamma-1}} - \frac{r^\gamma}{(r-\rho)^\gamma} \text{ is bounded from above.}$$

Let $x := \frac{\rho}{r-\rho}$; then $\frac{r}{r-\rho} = 1 + x$. Using the Taylor expansion we have

$$\begin{aligned}\tfrac{\rho^\gamma}{(r-\rho)^\gamma} + \tfrac{\gamma r^{\gamma-1}}{(r-\rho)^{\gamma-1}} - \tfrac{r^\gamma}{(r-\rho)^\gamma} &= x^\gamma + \gamma(x+1)^{\gamma-1} - (x+1)^\gamma \\ &= \textstyle\int_{1+x}^x t^{\gamma-2}(x-t)dt = \int_0^1 (z+x)^{\gamma-2} z dz \\ &\ge \textstyle\int_0^1 z^{\gamma-1}dz.\end{aligned}$$

This implies that $\frac{\rho^\gamma}{(\rho-r)^\gamma} - \frac{\gamma r^{\gamma-1}}{(\rho-r)^{\gamma-1}} - \frac{r^\gamma}{(\rho-r)^\gamma}$ has also a positive lower bound. Therefore, the inequality (3.1) holds when $|\rho - r| \ge 1$.

□

From Lemma 3.1 we deduce that for a bounded domain Ω and any compact $K \subset (0,\infty)$, there exist constants $C_3, C_4 > 0$ such that for any functions $\rho : \Omega \to [0,\infty)$ and $r : \Omega \to K$, one has

$$\begin{aligned} C_3\|\rho - r\|^\gamma_{L^\gamma(\Omega)} &\le \int_\Omega H(\rho|r)dx + \left(\int_\Omega H(\rho|r)dx\right)^{\frac{\gamma}{2}}, \\ C_4\int_\Omega H(\rho|r)dx &\le \|\rho - r\|^\gamma_{L^\gamma(\Omega)} + \|\rho - r\|^2_{L^\gamma(\Omega)}. \end{aligned} \tag{3.2}$$

Before the end of this section, let us recall the following relative energy inequality given in [8]:

PROPOSITION 3.1. Let $T > 0$ and (ρ, u) be a finite energy weak solution of the compressible Navier-Stokes system on [0,T] associated to an initial data (ρ_0, u_0) as given in Proposition 2.1. Then, for any smooth pairs $(r, U) : [0, T] \times \bar{\Omega} \to (0, \infty) \times \mathbb{R}^3$ satisfying the no-slip condition $U|_{\partial\Omega} = 0$, we have the following relative energy inequality:

$$\mathcal{E}([\rho, u]|[r, U])(\sigma) + \int_0^\sigma \int_\Omega \varepsilon \mathbb{S}(\nabla u) : \nabla u dx dt \le \mathcal{E}_0 + \mathcal{R}(\rho, u, r, U)$$

for almost all $\sigma \in (0, T)$, where

$$\mathcal{E}_0 = \mathcal{E}([\rho_0, u_0]|[r_0, U_0]), \tag{3.3}$$

with $r_0(x) = r(0, x)$, $U_0(x) = U(0, x)$, and

$$\begin{aligned}\mathcal{R}(\rho, u, r, U) := &\int_0^\sigma \int_\Omega \rho\,(\partial_t U + (u \cdot \nabla)U) \cdot (U - u) dx dt + \int_0^\sigma \int_\Omega \varepsilon \mathbb{S}(\nabla u) : \nabla U dx dt \\ &+ \int_0^\sigma \int_\Omega ((r - \rho)\partial_t H'(r) + \nabla H'(r) \cdot (rU - \rho u))\, dx dt \\ &- \int_0^\sigma \int_\Omega (div\ U)\,(p(\rho) - p(r))\, dx dt.\end{aligned}$$

4. Proof of the main result. In this section we will prove our main result, Theorem 2.1.

At first we introduce a Kato type "fake" boundary layer: Let $u^E = (u_1^E, u_2^E, u_3^E)^T$ be a smooth solution of (1.2) as given in Proposition 2.2. Define

$$v := \xi \left(\frac{d_\Omega(x)}{\delta} \right) u^E|_{\partial\Omega},$$

with

$$\xi \in C^\infty[0, \infty), \quad \xi(0) = 1, \quad \| \xi \|_{L^\infty} < \infty, \quad \| \xi' \|_{L^\infty} < \infty, \quad \text{supp } \xi \subseteq [0, 1),$$

and $\delta = \delta(\varepsilon)$ tending to zero as $\varepsilon \to 0$, which will be determined later.

It is obvious to see that v has the following properties:

$$\begin{aligned} v_n = 0,\ \|v\|_{L^\infty([0,T]\times\Omega)} = O(1), \\ \|\partial_t v\|_{L^\infty([0,T]\times\Omega)} = O(1),\ \|div\ v\|_{L^\infty([0,T]\times\Omega)} = O(1), \\ \|\partial_\tau v_\tau\|_{L^\infty([0,T]\times\Omega)} = O(1),\ \|\partial_n v_\tau\|_{L^\infty([0,T]\times\Omega)} = O(\delta^{-1}), \end{aligned} \tag{4.1}$$

where v_n and v_τ denote the normal and the tangential components of v, ∂_n and ∂_τ denote the normal and the tangential derivatives respectively.

For simplicity of notation, we drop the index ε, and simply denote by (ρ, u) the weak solution of the problem (1.1) as given in Proposition 2.1. Using (2.1), (2.3) and the assumption (2.4) of the initial data, we can easily obtain

$$\| \rho \|_{L^\infty([0,T];L^\gamma(\Omega))} + \| \rho u^2 \|_{L^\infty([0,T];L^1(\Omega))} + \sqrt{\varepsilon} \| \nabla u \|_{L^2([0,T]\times\Omega)} \leq O(1). \tag{4.2}$$

Set $(r, U) = (\rho^E, u^E - v)$. Since $U|_{\partial\Omega} = 0$, by applying Proposition 3.1 we obtain

$$\int_\Omega \left(\frac{1}{2}\rho|u - U|^2 + H(\rho|\rho^E) \right) dx + \int_0^\sigma \int_\Omega \varepsilon \mathbb{S}(\nabla u) : \nabla u dxdt \leq \mathcal{E}_0 + \mathcal{R}(\rho, u, r, U) \tag{4.3}$$

with $\mathcal{E}_0$ being given in (3.3), and

$$\begin{aligned}
\mathcal{R}(\rho, u, r, U) :=& \int_0^\sigma \int_\Omega \rho \left(\partial_t u^E + (u^E \cdot \nabla) u^E \right) \cdot (U - u) dxdt \\
&+ \int_0^\sigma \int_\Omega \rho \left(((u - u^E) \cdot \nabla) u^E \right) \cdot (U - u) dxdt \\
&- \int_0^\sigma \int_\Omega \rho \left(\partial_t v + (u \cdot \nabla) v \right) \cdot (U - u) dxdt + \int_0^\sigma \int_\Omega \varepsilon \mathbb{S}(\nabla u) : \nabla U dxdt \\
&+ \int_0^\sigma \int_\Omega \left((\rho^E - \rho) \partial_t H'(\rho^E) + \nabla H'(\rho^E) \cdot (\rho^E U - \rho u) \right) dxdt \\
&- \int_0^\sigma \int_\Omega (div\ U)(p(\rho) - p(\rho^E)) dxdt.
\end{aligned}$$

From (1.2), we deduce

$$\partial_t u^E + (u^E \cdot \nabla) u^E = -\nabla H'(\rho^E),$$

$$\partial_t H'(\rho^E) + u^E \cdot \nabla H'(\rho^E) = -(div u^E) p'(\rho^E).$$

As in [24], this gives

$$\begin{aligned}
&\mathcal{R}(\rho, u, \rho^E, U) \\
&= \int_0^\sigma \int_\Omega \rho \left(((u - u^E) \cdot \nabla) u^E \right) \cdot (U - u) dxdt - \int_0^\sigma \int_\Omega \rho \left(\partial_t v + (u \cdot \nabla) v \right) \cdot (U - u) dxdt \\
&+ \int_0^\sigma \int_\Omega \varepsilon \mathbb{S}(\nabla u) : \nabla U dxdt - \int_0^\sigma \int_\Omega (div\ u^E) \left(p(\rho) - p(\rho^E) - p'(\rho^E)(\rho - \rho^E) \right) dxdt \\
&- \int_0^\sigma \int_\Omega (\rho^E - \rho) \left(v \cdot \nabla H'(\rho^E) \right) dxdt + \int_0^\sigma \int_\Omega (div\ v)(p(\rho) - p(\rho^E)) dxdt \\
&= \sum_{j=1}^6 \mathcal{R}_j,
\end{aligned}$$

with obvious notions $\mathcal{R}_j (1 \leq j \leq 6)$. Denoting by $w = u - U$, we will estimate each $\mathcal{R}_j (1 \leq j \leq 6)$.

i) For $\mathcal{R}_1$, by using (4.1), (4.2) and the Hölder inequality, we deduce

$$\begin{aligned}|\mathcal{R}_1| &= \left|-\int_0^\sigma\int_\Omega \rho((u-u^E)\otimes(u-u^E)) : \nabla u^E dxdt - \int_0^\sigma\int_\Omega \rho(((u-u^E)\cdot\nabla)u^E)\cdot vdxdt\right| \\ &\le C\int_0^\sigma\int_\Omega \rho|u-u^E|^2dxdt + C\int_0^\sigma\left(\int_\Omega \rho|u-u^E|^2dx\right)^{\frac12}\left(\int_\Omega \rho|v|^2dx\right)^{\frac12}dt \\ &\le C\int_0^\sigma\int_\Omega \rho|u-u^E|^2dxdt + C\int_0^\sigma\int_\Omega \rho|v|^2dxdt \\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2dxdt + C\int_0^\sigma\int_\Omega \rho|v|^2dxdt \\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2dxdt + C\int_0^\sigma\left(\int_\Omega \rho^\gamma dx\right)^{\frac1\gamma}\left(\int_{\Gamma_\delta} |v|^{\frac{2\gamma}{\gamma-1}}dx\right)^{\frac{\gamma-1}{\gamma}}dt \\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2dxdt + o(1).\end{aligned}$$

ii) Decompose $\mathcal{R}_2$ into

$$\mathcal{R}_2 = \int_0^\sigma\int_\Omega \rho\partial_t v\cdot wdxdt + \int_0^\sigma\int_\Omega \rho(u\cdot\nabla)v\cdot wdxdt. \tag{4.4}$$

By using (4.1), (4.2) and the Hölder inequality again, we estimate the above first term by

$$\begin{aligned}\left|\int_0^\sigma\int_\Omega \rho\partial_t v\cdot wdxdt\right| &\le C\int_0^\sigma\left(\int_\Omega \rho|w|^2dx\right)^{\frac12}\left(\int_\Omega \rho|\partial_t v|^2dx\right)^{\frac12}dt \\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2dxdt + o(1).\end{aligned}$$

Now we turn to the second term on the right hand side of (4.4). For simplicity of presentation, we consider the case of boundary being flat. As usual, one can treat the problem with a general smooth boundary, by using the technique of localization and transforming the curved boundary into a flat one. Without loss of generality we assume that the domain lies in the upper half plane, $\Omega = \{(x_1,x_2,x_3)|(x_1,x_2)\in\mathbb{R}^2, x_3>0\}$, with $\{x_3=0\}$ being the boundary.

With the construction of v, we can decompose the second term on the right hand side of (4.4) into

$$\int_0^\sigma\int_\Omega \rho(u\cdot\nabla)v\cdot wdxdt = \int_0^\sigma\int_\Omega \rho\sum_{i=1}^2\left[\left(\sum_{j=1}^3 u_j\partial_j\right)v_iw_i\right]dxdt. \tag{4.5}$$

By using (4.1), (4.2) and the Hölder inequality, we have

$$\begin{aligned}\left|\int_0^\sigma\int_\Omega \rho u_jw_i\partial_jv_idxdt\right| &\le \left|\int_0^\sigma\int_\Omega \rho w_jw_i\partial_jv_idxdt\right| + \left|\int_0^\sigma\int_\Omega \rho U_jw_i\partial_jv_idxdt\right| \\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2dxdt + \int_0^\sigma\left(\int_\Omega \rho|w_i|^2dx\right)^{\frac12}\left(\int_\Omega \rho|U_j\partial_jv_i|^2dx\right)^{\frac12}dt \\ &\le C\int_0^\sigma\int_\Omega \rho|w|^2dxdt + o(1),\end{aligned} \tag{4.6}$$

for $i,j\in\{1,2\}$, by noting $supp_{x_3}v\subseteq[0,\delta)$.

Now let us study another two terms given in (4.5). By using (4.1), the Young inequality and the Poincaré inequality we have

$$\begin{aligned}\left|\int_0^\sigma\int_\Omega \rho u_3 w_1\partial_3 v_1 dxdt\right| =& \left|\int_0^\sigma\int_\Omega \frac{\rho u_3}{d_\Omega}\cdot w_1 d_\Omega\partial_3 v_1 dxdt\right|\\ \leq& C\int_0^\sigma \delta\|\frac{\rho u_3}{d_\Omega}\|_{L^2(\Gamma_\delta)}\|\partial_3 w_1\|_{L^2(\Omega)}dt\\ \leq& C\frac{\delta^2}{\varepsilon}\|\frac{\rho u_3}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}+\frac{C_0\varepsilon}{4}\|\partial_3 w_1\|^2_{L^2((0,T)\times\Omega)}. \end{aligned}\tag{4.7}$$

On the other hand, we can decompose this term into

$$\int_0^\sigma\int_\Omega \rho u_3 w_1\partial_3 v_1 dxdt = \int_0^\sigma\int_\Omega \rho u_3 u_1\partial_3 v_1 dxdt - \int_0^\sigma\int_\Omega \rho u_3 U_1\partial_3 v_1 dxdt. \tag{4.8}$$

For the first term on the right hand of (4.8), by using (4.1), the Young inequality and the Poincaré inequality we have

$$\begin{aligned}\left|\int_0^\sigma\int_\Omega \rho u_3 u_1\partial_3 v_1 dxdt\right| =& \left|\int_0^\sigma\int_\Omega \frac{\rho u_1}{d_\Omega}\cdot u_3 d_\Omega\partial_3 v_1 dxdt\right|\\ \leq& C\int_0^\sigma \delta\|\frac{\rho u_1}{d_\Omega}\|_{L^2(\Gamma_\delta)}\|\partial_3 u_3\|_{L^2(\Omega)}dt\\ \leq& C\frac{\delta^2}{\varepsilon}\|\frac{\rho u_1}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}+\frac{C_0\varepsilon}{4}\|\partial_3 u_3\|^2_{L^2((0,T)\times\Omega)}. \end{aligned}\tag{4.9}$$

Next we study the second term on the right hand side of (4.8). First from the definition of weak solution of (1.1), we deduce that for any $\phi \in C^1([0,T]\times\bar\Omega;\mathbb{R})$,

$$\int_\Omega \rho(\sigma,\cdot)\phi(\sigma,\cdot)dx - \int_\Omega \rho_0\phi(0,\cdot)dx = \int_0^\sigma\int_\Omega(\rho\partial_t\phi+\rho u\cdot\nabla\phi)dxdt. \tag{4.10}$$

Notice that $v_1u_1^E\in C^1([0,T]\times\bar\Omega;\mathbb{R}^3)$, we take $\phi=v_1u_1^E$ in (4.10) and get

$$\int_\Omega \rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx - \int_\Omega \rho_0(v_1u_1^E)(0,\cdot)dx = \int_0^\sigma\int_\Omega(\rho\partial_t(v_1u_1^E)+\rho u\cdot\nabla(v_1u_1^E))dxdt,$$

which implies

$$\begin{aligned}&\textstyle\int_0^\sigma\int_\Omega \rho u_3 u_1^E\partial_3 v_1 dxdt\\ &=\textstyle\int_\Omega \rho(\sigma,\cdot)(v_1u_1^E)(\sigma,\cdot)dx - \int_\Omega\rho_0(v_1u_1^E)(0,\cdot)dx - \int_0^\sigma\int_\Omega\rho\partial_t(v_1u_1^E)dxdt\\ &\textstyle -\int_0^\sigma\int_\Omega\rho u_2\partial_2(v_1u_1^E)dxdt - \int_0^\sigma\int_\Omega\rho u_1\partial_1(v_1u_1^E)dxdt - \int_0^\sigma\int_\Omega\rho u_3 v_1\partial_3 u_1^E dxdt.\end{aligned}\tag{4.11}$$

Similarly, we have the identity

$$\begin{aligned}\int_0^\sigma\int_\Omega \rho u_3 v_1\partial_3 v_1 dxdt =& \frac12\int_\Omega\rho(\sigma,\cdot)v_1^2(\sigma,\cdot)dx - \frac12\int_\Omega\rho_0 v_1^2(0,\cdot)dx - \frac12\int_0^\sigma\int_\Omega\rho\partial_t(v_1^2)dxdt\\ &-\int_0^\sigma\int_\Omega\rho u_2 v_1\partial_2 v_1 dxdt - \int_0^\sigma\int_\Omega \rho u_1 v_1\partial_1 v_1 dxdt\end{aligned}\tag{4.12}$$

by choosing $\phi = v_1^2$ in (4.10). Thus, from (4.11) and (4.12) we have by noting $U_1 = u_1^E - v_1$,

$$\begin{aligned}
&\int_0^\sigma \int_\Omega \rho u_3 U_1 \partial_3 v_1 dxdt \\
&= \int_\Omega \rho(\sigma,\cdot)(v_1 u_1^E)(\sigma,\cdot)dx - \int_\Omega \rho_0 (v_1 u_1^E)(0,\cdot)dx - \int_0^\sigma \int_\Omega \rho \partial_t (v_1 u_1^E) dxdt \\
&\quad -\tfrac{1}{2}\int_\Omega \rho(\sigma,\cdot) v_1^2(\sigma,\cdot)dx + \tfrac{1}{2}\int_\Omega \rho_0 v_1^2(0,\cdot)dx + \tfrac{1}{2}\int_0^\sigma \int_\Omega \rho\partial_t(v_1^2)dxdt \\
&\quad -\int_0^\sigma \int_\Omega \rho u_2 \partial_2 (v_1 u_1^E)dxdt - \int_0^\sigma \int_\Omega \rho u_1 \partial_1 (v_1 u_1^E)dxdt - \int_0^\sigma \int_\Omega \rho u_3 v_1 \partial_3 u_1^E dxdt \\
&\quad + \int_0^\sigma \int_\Omega \rho u_2 v_1 \partial_2 v_1 dxdt + \int_0^\sigma \int_\Omega \rho u_1 v_1 \partial_1 v_1 dxdt.
\end{aligned} \tag{4.13}$$

By using the Hölder inequality, we can deduce from (4.13) that

$$\begin{aligned}
&\left|\int_0^\sigma \int_\Omega \rho u_3 U_1 \partial_3 v_1 dxdt\right| \\
\le& C\left(\int_\Omega |\rho(\sigma,\cdot)|^\gamma dx\right)^{\frac{1}{\gamma}} \left(\int_{\Gamma_\delta} \left|(v_1 u_1^E)(\sigma,\cdot) - \frac{1}{2}v_1^2(\sigma,\cdot)\right|^{\frac{\gamma}{\gamma-1}} dx\right)^{\frac{\gamma-1}{\gamma}} \\
&+ C\left(\int_\Omega |\rho(0,\cdot)|^\gamma dx\right)^{\frac{1}{\gamma}} \left(\int_{\Gamma_\delta} \left|(v_1 u_1^E)(0,\cdot) - \frac{1}{2}v_1^2(0,\cdot)\right|^{\frac{\gamma}{\gamma-1}} dx\right)^{\frac{\gamma-1}{\gamma}} \\
&+ C\int_0^\sigma \left(\int_\Omega |\rho|^\gamma dx\right)^{\frac{1}{\gamma}} \left(\int_{\Gamma_\delta} \left|\partial_t(v_1 u_1^E) - \frac{1}{2}\partial_t(v_1^2)\right|^{\frac{\gamma}{\gamma-1}} dxdt\right)^{\frac{\gamma-1}{\gamma}} dt \\
&+ C\int_0^\sigma \int_{\Gamma_\delta} \rho\left(|\partial_2(v_1u_1^E)|^2 + |\partial_1(v_1u_1^E)|^2 + |v_1\partial_3 u_1^E|^2 + |v_1\partial_1 v_1|^2| + |v_1\partial_2 v_1|^2\right) dxdt \\
&+ C\int_0^\sigma \int_{\Gamma_\delta} \rho|u|^2 dxdt,
\end{aligned}$$

which implies that

$$\left|\int_0^\sigma \int_\Omega \rho u_3 U_1 \partial_3 v_1 dxdt\right| \le C\int_0^\sigma \int_\Omega \frac{1}{2}\rho|w|^2 dxdt + o(1). \tag{4.14}$$

By using (4.1) and (4.2), and plugging (4.9) and (4.14) into (4.8), we get

$$\begin{aligned}
&|\int_0^\sigma \int_\Omega \rho u_3 w_1 \partial_3 v_1 dxdt| \\
&\le C\frac{\delta^2}{\varepsilon}\|\frac{\rho u_1}{d_\Omega}\|_{L^2(\Gamma_\delta)}^2 + \frac{C_0\varepsilon}{4}\|\partial_3 u_3\|_{L^2(\Omega)}^2 + C\int_0^\sigma \int_\Omega \frac{1}{2}\rho|w|^2 dxdt + o(1).
\end{aligned} \tag{4.15}$$

One can get a similar estimate for the other term

$$\int_0^\sigma \int_\Omega \rho u_3 w_2 \partial_3 v_2 dxdt$$

given on the right hand of (4.5). Obviously we have

$$\begin{aligned}\varepsilon\int_0^\sigma\int_\Omega|\nabla w|^2dxdt &\le \varepsilon\int_0^\sigma\int_\Omega|\nabla u|^2dxdt+\varepsilon\int_0^\sigma\int_\Omega|\nabla v|^2dxdt+\varepsilon\int_0^\sigma\int_\Omega|\nabla u^E|^2dxdt\\ &\le \varepsilon\int_0^\sigma\int_\Omega|\nabla u|^2dxdt+C\frac{\varepsilon}{\delta}+o(1),\end{aligned}$$

thus combining (4.5), (4.6), (4.7), (4.15) with (4.4) it follows

$$|\mathcal{R}_2|\le C\int_0^\sigma\int_\Omega\frac{1}{2}\rho|w|^2dx+\frac{C_0\varepsilon}{2}\|\nabla u\|^2_{L^2((0,T)\times\Omega)}+C\frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}+C\frac{\varepsilon}{\delta}+o(1), \tag{4.16}$$

or

$$|\mathcal{R}_2|\le C\int_0^\sigma\int_\Omega\frac{1}{2}\rho|w|^2dx+\frac{C_0\varepsilon}{2}\|\nabla u\|^2_{L^2((0,T)\times\Omega)}+C\frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot \tau}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}+C\frac{\varepsilon}{\delta}+o(1). \tag{4.16'}$$

iii) For $\mathcal{R}_3$, we have

$$\begin{aligned}|\mathcal{R}_3|&\le\left|\varepsilon\int_0^\sigma\int_\Omega\mathbb{S}(\nabla u):\nabla u^Edxdt\right|+\left|\varepsilon\int_0^\sigma\int_\Omega\mathbb{S}(\nabla u):\nabla vdxdt\right|\\ &\le\frac{C_0\varepsilon}{8}\int_0^\sigma\int_\Omega|\nabla u|^2dxdt+o(1)+\frac{C_0\varepsilon}{8}\int_0^\sigma\int_\Omega|\nabla u|^2dxdt+C\frac{\varepsilon}{\delta}\\ &=\frac{C_0\varepsilon}{4}\int_0^\sigma\int_\Omega|\nabla u|^2dxdt+C\frac{\varepsilon}{\delta}+o(1).\end{aligned}$$

The estimates of $\mathcal{R}_4,\mathcal{R}_5$ and $\mathcal{R}_6$ have been studied in [24]. For completeness we recall them briefly.

$$|\mathcal{R}_4|=\left|(\gamma-1)\int_0^\sigma\int_\Omega(div\ u^E)H(\rho|\rho^E)dxdt\right|\le C\int_0^\sigma\int_\Omega H(\rho|\rho^E)dxdt,$$

$$\begin{aligned}|\mathcal{R}_5|\le& C\int_0^\sigma\int_\Omega|\rho^E-\rho|^2 1_{|\rho-r|<1}dxdt+C\int_0^\sigma\int_\Omega|v\cdot\nabla H'(\rho^E)|^2dxdt\\ &+C\int_0^\sigma\int_\Omega|\rho^E-\rho|^\gamma 1_{|\rho-r|\ge1}dxdt+C\int_0^\sigma\int_\Omega|v\cdot\nabla H'(\rho^E)|^{\frac{\gamma}{\gamma-1}}dxdt\\ \le& C\int_0^\sigma\int_\Omega H(\rho|\rho^E)dxdt+o(1),\end{aligned}$$

and

$$\begin{aligned}|\mathcal{R}_6|=&\left|\int_0^\sigma\int_\Omega(div\ v)(p(\rho)-p(\rho^E)-p'(\rho^E)(\rho-\rho^E))dxdt+\int_0^\sigma\int_\Omega(div\ v)p'(\rho^E)(\rho^E-\rho)dxdt\right|\\ \le&\int_0^\sigma\|div\ v\|_{L^\infty(\Gamma_\delta)}\int_\Omega H(\rho|\rho^E)dxdt+C\int_0^\sigma\|div\ v\|_{L^2(\Gamma_\delta)}\left(\int_\Omega H(\rho|\rho^E)dx\right)^{\frac12}dt\\ &+C\int_0^\sigma\|div\ v\|_{L^{\frac{\gamma}{\gamma-1}}(\Gamma_\delta)}\left(\int_\Omega H(\rho|\rho^E)dx\right)^{\frac1\gamma}dt\\ \le& C\int_0^\sigma\int_\Omega H(\rho|\rho^E)dxdt+o(1).\end{aligned}$$

Summing up all the above estimates for $\mathcal{R}_1$ to $\mathcal{R}_6$ with estimate (4.16) or (4.16') of $\mathcal{R}_2$, from (4.3) we get

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx + \int_0^\sigma \int_\Omega \varepsilon\mathbb{S}(\nabla u) : \nabla u dx dt \\ \le & \mathcal{E}_0 + C\int_0^\sigma \int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx dt + \frac{3C_0\varepsilon}{4}\int_0^\sigma \int_\Omega |\nabla u|^2 dx \\ &+ C\frac{\varepsilon}{\delta} + C\frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)} + o(1),\end{aligned}$$

or

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx + \int_0^\sigma \int_\Omega \varepsilon\mathbb{S}(\nabla u) : \nabla u dx dt \\ \le & \mathcal{E}_0 + C\int_0^\sigma \int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx dt + \frac{3C_0\varepsilon}{4}\int_0^\sigma \int_\Omega |\nabla u|^2 dx \\ &+ C\frac{\varepsilon}{\delta} + C\frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot \tau}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)} + o(1).\end{aligned}$$

From the assumption (2.4) and (3.2) we have $\mathcal{E}_0 \to 0$ when $\varepsilon \to 0$, thus from the above two inequalities we get that by using (2.1),

$$\begin{aligned}\int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx \le & C\int_0^\sigma \int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx dt \\ &+ C\frac{\varepsilon}{\delta} + C\frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)} + o(1),\end{aligned}$$

or

$$\begin{aligned}\int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx \le & C\int_0^\sigma \int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx dt \\ &+ C\frac{\varepsilon}{\delta} + C\frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot \tau}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)} + o(1),\end{aligned}$$

which implies

$$\sup_{0\le t\le T}\int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx \le C\left(\frac{\varepsilon}{\delta} + \frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}\right) + o(1), \quad (4.17)$$

or

$$\sup_{0\le t\le T}\int_\Omega \left(\frac{1}{2}\rho|w|^2 + H(\rho|\rho^E)\right) dx \le C\left(\frac{\varepsilon}{\delta} + \frac{\delta^2}{\varepsilon}\|\frac{\rho u\cdot \tau}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}\right) + o(1), \quad (4.17')$$

by using the Gronwall inequality, as $\delta \to 0$ when $\varepsilon \to 0$.

The above results that we obtained are summarized in the following proposition.

PROPOSITION 4.1. Assume that $T > 0$, and (ρ^E, u^E) is the strong solution of the problem (1.2) of the Euler equations corresponding to an initial data (ρ_0^E, u_0^E) as given in Proposition 2.2. For any $\varepsilon \in (0, 1)$, let $(\rho_0^\varepsilon, u_0^\varepsilon)$ be an initial data satisfying (2.2), and $(\rho^\varepsilon, u^\varepsilon)$ be the associated weak solution of the compressible Navier-Stokes equations (1.1) on $[0, T]$ as given in Proposition 2.1. Set $w = u^\varepsilon - u^E + v$ with v being the artificial boundary layer given at the beginning of this section, and

$$H(\rho|r) = \frac{\rho^\gamma}{\gamma - 1} - \frac{\gamma(\rho - r)r^{\gamma-1}}{\gamma - 1} - \frac{r^\gamma}{\gamma - 1}.$$

When

$$\| \rho_0^\varepsilon - \rho_0^E \|_{L^\gamma(\Omega)} + \int_\Omega \rho_0^\varepsilon |u_0^\varepsilon - u_0^E|^2 dx \to 0 \quad \text{when} \quad \varepsilon \to 0, \tag{4.18}$$

then, we have

$$\sup_{0\le t\le T} \int_\Omega \left(\frac{1}{2}\rho^\varepsilon |w|^2 + H(\rho^\varepsilon|\rho^E) \right) dx \le C \left(\frac{\varepsilon}{\delta} + \frac{\delta^2}{\varepsilon} \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \right) + o(1), \tag{4.19}$$

and

$$\sup_{0\le t\le T} \int_\Omega \left(\frac{1}{2}\rho^\varepsilon |w|^2 + H(\rho^\varepsilon|\rho^E) \right) dx \le C \left(\frac{\varepsilon}{\delta} + \frac{\delta^2}{\varepsilon} \| \frac{\rho^\varepsilon u^\varepsilon \cdot \tau}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \right) + o(1) \tag{4.20}$$

for any $\delta = \delta(\varepsilon)$ satisfying $\delta \to 0$ when $\varepsilon \to 0$.

It remains to prove the results given in Theorem 2.1, which will be obtained by using Proposition 4.1 and developing an idea similar to that given in [25].

Proof of Theorem 2.1. We shall only prove the condition (2.5) by using the inequality (4.19), while (2.6) can be derived similarly from the inequality (4.20).

Denote by $\alpha = \frac{\varepsilon}{\delta}$; then $\frac{\delta^2}{\varepsilon} = \frac{\varepsilon}{\alpha^2}$. Obviously, as a function of $a > 0$, $a + \frac{\varepsilon}{a^2} \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)}$ attains its minimum at

$$a = \alpha_{ct} = \left(2\varepsilon \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \right)^{\frac{1}{3}}.$$

Now fix α, that means δ being fixed. If $\alpha_{ct} \ge \alpha$, then $\delta_{ct} = \frac{\varepsilon}{\alpha_{ct}} \le \frac{\varepsilon}{\alpha} = \delta$, and

$$\begin{aligned} \alpha_{ct} + \frac{\varepsilon}{\alpha_{ct}^2} \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_{\delta_{ct}})} &\le \alpha_{ct} + \frac{\varepsilon}{\alpha_{ct}^2} \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \\ &= \min_{a>0} \left(a + \frac{\varepsilon}{a^2} \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \right) \\ &= \frac{3}{2} \left(2\varepsilon \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \right)^{\frac{1}{3}}. \end{aligned} \tag{4.21}$$

Since

$$\alpha_{ct} = \left(2\varepsilon \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \right)^{\frac{1}{3}} \longrightarrow 0$$

as $\varepsilon \to 0$ under the assumption (2.5), we know $\varepsilon = \alpha_{ct}\delta_{ct} = o(\delta_{ct})$. Moreover, from $\delta \to$ as $\varepsilon \to 0$ we get that $\delta_{ct} \le \delta$ gives rise to $\delta_{ct} \to 0$ when $\varepsilon \to 0$. Thus, the inequality (4.19) holds for $\delta = \delta_{ct}$. Together with (4.21), we obtain

$$\sup_{0\le t\le T} \int_\Omega \left(\frac{1}{2}\rho^\varepsilon |w|^2 + H(\rho^\varepsilon|\rho^E) \right) dx \le C \left(\varepsilon \| \frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega} \|^2_{L^2((0,T)\times\Gamma_\delta)} \right)^{\frac{1}{3}} + o(1). \tag{4.22}$$

If $\alpha_{ct} < \alpha$, that is,

$$\left(2\varepsilon\|\frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}\right)^{\frac{1}{3}} < \alpha,$$

we know

$$\varepsilon\|\frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)} < \frac{\alpha^3}{2},$$

which implies

$$\alpha + \frac{\varepsilon}{\alpha^2}\|\frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)} < \frac{3\alpha}{2}. \tag{4.23}$$

Combining (4.19),(4.22) with (4.23), we deduce that

$$\int_\Omega \frac{1}{2}\rho^\varepsilon|w|^2 + H(\rho^\varepsilon|\rho^E)dx \le C\max\left\{\alpha, \left(\varepsilon\|\frac{\rho^\varepsilon u^\varepsilon \cdot n}{d_\Omega}\|^2_{L^2((0,T)\times\Gamma_\delta)}\right)^{\frac{1}{3}}\right\} + o(1). \tag{4.24}$$

Noticing from the definition of w that

$$\int_\Omega \frac{1}{2}\rho^\varepsilon|u^\varepsilon - u^E|^2dx \le \int_\Omega \frac{1}{2}\rho^\varepsilon|w|^2dx + \int_\Omega \frac{1}{2}\rho^\varepsilon|v|^2dx = \int_\Omega \frac{1}{2}\rho^\varepsilon|w|^2dx + o(1),$$

from (3.2) and (4.24) we get the conclusion given in Theorem 2.1 in the case of (2.5), as $\delta = \delta(\varepsilon)$ is chosen such that $\varepsilon = o(\delta)$, and $\delta \to 0$ as $\varepsilon \to 0$. ☐

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