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INTEGRAL RELATIONS ASSOCIATED WITH THE SEMI-INFINITE HILBERT TRANSFORM AND APPLICATIONS TO SINGULAR INTEGRAL EQUATIONS

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Abstract. Integral relations with the Cauchy kernel on a semi-axis for the Laguerre polynomials, the confluent hypergeometric function, and the cylindrical functions are derived. A part of these formulas is obtained by exploiting some properties of the Hermite polynomials, including their Hilbert and Fourier transforms and connections to the Laguerre polynomials. The relations discovered give rise to complete systems of new orthogonal functions. Free of singular integrals, exact and approximate solutions to the characteristic and complete singular integral equations in a semi-infinite interval are proposed. Another set of the Hilbert transforms in a semi-axis are deduced from integral relations with the Cauchy kernel in a finite segment for the Jacobi polynomials and the Jacobi functions of the second kind by letting some parameters involved go to infinity. These formulas lead to integral relations for the Bessel functions. Their application to a model problem of contact mechanics is given. A new quadrature formula for the Cauchy integral in a semi-axis based on an integral relation for the Laguerre polynomials and the confluent hypergeometric function is derived and tested numerically. Bounds for the remainder are found.

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1. Introduction. The Hilbert transform, a convolution of a function and the Cauchy kernel, in the real axis was introduced by Hilbert at the beginning of the twentieth century as a tool for boundary value problems of the theory of analytic functions. Since then its properties and related methods in miscellaneous areas of applied sciences have been of interest for pure and applied mathematicians. Properties of the Hilbert transforms in the real axis are presented in detail in [23]. The Hilbert transform in a finite segment has been extensively studied due to applications to singular integral equations [19], [7], [16], [20], [25], [8], and quadrature formulas [12], [21] for the Cauchy integral in the interval $(-1, 1)$. In particular, a formula for the Hilbert transform in the interval $(-1, 1)$ of the weighted Jacobi polynomials was discovered in [24]. This formula generates many integral relations for orthogonal polynomials crucial for the solution of singular integral equations. In [11], the Fourier-Plancherel transformation was employed to derive a spectral representation of the finite Hilbert transform and an expansion of an arbitrary $L^2(a, b)$-function in terms of the Hilbert operator eigenfunctions. In [14], a self-adjoint differential equation was considered and integral operators whose eigenfunctions are the solutions of that differential equation were determined. This made it possible to recover spectral relations for the operators found and solve the corresponding singular integral equations in a finite interval. Finite Hilbert transforms of the Chebyshev, Bernstein, and Lagrange interpolating polynomials were used for an approximate solution of the Prandtl integro-differential equation in [15].

The Hilbert transform in a semi-axis has received less attention than its infinite and finite analogues. At the same time, many model problems including those arising in fluid mechanics, fracture, and penetration mechanics are governed by singular integral equations or their systems in a semi-axis whose kernels can be represented as a sum of the Cauchy kernel and a regular kernel. Also, vector Riemann–Hilbert problems, when the associated matrix Wiener–Hopf factors are infeasible, do not admit a closed-form solution. Alternatively, they can be written as systems of singular integral equations. The kernels of these systems may often be split into the Cauchy kernel and bounded functions (see, for example, [1], [2], [3]). Replacing the semi-axis $(0, \infty)$ by a finite segment $(0, A)$ and applying the numerical methods for singular integral equations in a finite segment do not preserve the properties of the solution far away from the point 0 and may generate a significant error of approximation due to the change of the weight function. In this paper, we aim to derive a series of integral relations in a semi-axis for the Laguerre polynomials, the confluent hypergeometric function, the Tricomi function, and the Bessel functions, and study a new system of functions $\{G_m(x)\}_{m=0}^{\infty}$, the G-functions, generated by these integral relations. We also intend, by means of these integral relations, to solve some singular integral equations and obtain a quadrature formula for the Cauchy integral in a semi-axis.

In Section 2, we introduce the G-functions, the Hilbert transforms of the weighted Hermite polynomials,

$$G_n(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{H_n(t) e^{-t^2/2} dt}{t - x}, \quad n = 0, 1, \ldots, \tag{1.1}$$

show that they constitute a complete orthogonal system in the associated space, and derive their representation in terms of the confluent hypergeometric function Φ,

$$G_{2m+j}(x) = \frac{\sqrt{2}(2m+j)!x^{1-j}}{(-1)^{j+1}e^{x^2/2}} \sum_{k=0}^{m} \frac{2^k \Phi(1/2-k-j, 3/2-j; x^2/2)}{(-1)^k (m-k)! \Gamma(k+j+1/2)}, \tag{1.2}$$

where $m = 0, 1, \ldots$ and $j = 0, 1$. We discover the Hilbert transforms in a semi-axis of the Laguerre polynomials in two weighted spaces, $L^2_{w_\pm}(0,\infty)$ and $L^2_{v_\pm}(0,\infty)$, where $w_\pm(x) = x^{\pm 1/2}e^{-x/2}$ and $v_\pm(x) = x^{\pm 1/2}e^{-x}$. It is found that the first group of the relations generates an orthogonal basis whose elements are the G-functions, while the second one gives rise to a nonorthogonal basis. We study the G-functions in Section 3. It is shown that these functions of even and odd indices satisfy certain second-order inhomogeneous differential equations with variable coefficients. Also, the asymptotics of the G-functions at the points 0 and ∞ is derived.

In Section 4, based on the Tricomi integral relation [24] written for the Jacobi polynomials $P_n^{(\alpha,\beta)}(1-2x/\beta)$ and by passing to the limit $\beta \to \infty$, we discover integral relations for the Laguerre polynomials $L_n^\alpha(x)$ to be used in Section 6 for a quadrature formula for the Cauchy integral in a semi-axis. In addition, by employing some representations of the Jacobi function of the second kind $Q_n^{(\alpha,\beta)}(x)$ and passing to the limit $n \to \infty$ in the representations for $n^{-\alpha}Q_n^{(\alpha,\beta)}(1-\frac{1}{2}z^2/n^2)$, we obtain integral relations for the Bessel functions.

In Section 5, we apply the integral relations in a semi-axis derived in the previous sections to find a closed-form solution of the integral equation with the Cauchy kernel in the interval $(0,\infty)$ in the classes of bounded and unbounded at zero functions. In addition, we construct two bilinear expansions of the Cauchy kernel in terms of the Laguerre polynomials and the G-functions. We also solve a system of two complete integral equations with the Cauchy kernel in a semi-axis by reducing it to an infinite system of linear algebraic equations of the second kind. By using the integral relation for the Bessel function obtained in Section 4 and the Hankel transform, we obtain a closed-form solution, that is free of singular integrals, to a contact problem on a semi-infinite stamp and an elastic half-plane.

In Section 6, we obtain the following quadrature formula for the Cauchy integral in a semi-axis:

$$\frac{1}{\pi}\int_0^\infty \frac{f(t)t^\alpha e^{-t}dt}{t-x} = -\frac{1}{n+\alpha}\sum_{m=1}^{n} \frac{x_m f(x_m)}{L_{n-1}^\alpha(x_m)} \frac{Q_n^\alpha(x) - Q_n^\alpha(x_m)}{x-x_m}, \quad 0<x<\infty, \tag{1.3}$$

where

$$Q_n^\alpha(x) = \frac{1}{\pi}e^{-x}\Gamma(\alpha)\Phi(-n-\alpha, 1-\alpha; x) - \cot\pi\alpha x^\alpha e^{-x}L_n^\alpha(x). \tag{1.4}$$

It is exact for a polynomial of degree $n-1$ and requires n zeros, x_m, of the Laguerre polynomial $L_n^\alpha(x)$.

2. Hilbert transforms of the weighted Hermite and Laguerre polynomials and the G- and V-functions. In this section we introduce the Hilbert transforms $G_n(x)$ and $V_n(x)$ of the weighted Hermite polynomials, derive associated integral relations for the weighted Laguerre polynomials, and study the properties of the functions $G_n(x)$ and $V_n(x)$.

2.1. *Relations for the weighted Laguerre polynomials $e^{-\eta/2}\eta^{\pm 1/2}L_m(\eta)$.* For real functions $\varphi_k(x)$ such that $\varphi_k(x) \in L^2(-\infty,\infty)$, $k=1,2$, define their Hilbert transform $\Phi_k(x) = H[\varphi_k(t)](x)$ by

$$\Phi_k(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varphi_k(t)dt}{t-x}, \quad -\infty < x < \infty. \tag{2.1}$$

The integral is understood in the sense of the principal value at $t=x$ and the mean square sense at infinity. Then $\Phi_k(x) \in L^2(-\infty,\infty)$, $k=1,2$, and the generalized Parseval's relation holds [23]:

$$\int_{-\infty}^{\infty}\varphi_1(x)\varphi_2(x)dx = \int_{-\infty}^{\infty}\Phi_1(x)\Phi_2(x)dx. \tag{2.2}$$

The Hilbert transform H is a 1-1 map and a unitary operator in the space $L^2(-\infty,\infty)$. Since this operator preserves the inner product in the Hilbert space $L^2(-\infty,\infty)$, a complete orthogonal system in the space $L^2(-\infty,\infty)$ is transformed by the operator H into another complete orthogonal system in the same space.

THEOREM 2.1. Let $H_n(x)$ be the Hermite polynomials normalized by the condition

$$\lim_{x\to\infty} x^{-n}H_n(x) = 2^n. \tag{2.3}$$

For the orthogonal system of functions $H_n^{(1)}(x) = \exp(-x^2/2)H_n(x)$, denote their Hilbert transforms by

$$G_n(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_n^{(1)}(t)dt}{t-x}, \quad n=0,1,\ldots, \quad -\infty < x < \infty. \tag{2.4}$$

Then the functions $G_n(x)$ form a complete orthogonal system in the space $L_2(-\infty,\infty)$,

$$\int_{-\infty}^{\infty}G_n(x)G_m(x)dx = \sqrt{\pi}2^n n!\delta_{mn}, \tag{2.5}$$

and admit the following representations:

$$G_{2m}(x) = -\sqrt{2}(2m)!xe^{-x^2/2}\sum_{k=0}^{m}\frac{(-1)^k 2^k\Phi(1/2-k,3/2;x^2/2)}{(m-k)!\Gamma(k+1/2)},$$

$$G_{2m+1}(x) = \sqrt{2}(2m+1)!e^{-x^2/2}\sum_{k=0}^{m}\frac{(-1)^k 2^k\Phi(-1/2-k,1/2;x^2/2)}{(m-k)!\Gamma(k+3/2)}, \quad m=0,1,\ldots. \tag{2.6}$$

Here, δ_{mn} is the Kronecker symbol, $\Phi(a,c;x)$ is the confluent hypergeometric function,

$$\Phi(a,c;x) = \sum_{k=0}^{\infty}\frac{(a)_k x^k}{(c)_k k!}, \tag{2.7}$$

and $(a)_k$ is the factorial symbol, $(a)_k = a(a+1)\cdots(a+k-1)$.

Proof. The first statement of the theorem follows from the unitarity of the Hilbert operator, the Parseval's relation

$$\int_{-\infty}^{\infty}G_n(x)G_m(x)dx = \int_{-\infty}^{\infty}H_n^{(1)}(x)H_m^{(1)}(x)dx, \tag{2.8}$$

and the orthogonality of the functions $H_n^{(1)}(x)$. To prove formulas (2.6), we apply the Fourier transform to equation (2.4) and employ the convolution theorem and the spectral relation for the Fourier operator ([10, 7.376.1, p. 804])

$$F[H_n^{(1)}(x)](\lambda) = i^n \sqrt{2\pi} H_n^{(1)}(\lambda), \quad n = 0, 1, \dots, \tag{2.9}$$

to deduce

$$F[G_n(x)](\lambda) = -i^{n+1} \mathrm{sgn}\lambda \sqrt{2\pi} H_n^{(1)}(\lambda), \quad n = 0, 1, \dots. \tag{2.10}$$

Here,

$$F[\phi(x)](\lambda) = \int_{-\infty}^{\infty} \phi(x) e^{i\lambda x} dx. \tag{2.11}$$

The Fourier inversion applied for even and odd indices yields the following alternative integral representations for the G-functions:

$$G_{2m}(x) = (-1)^{m+1} \sqrt{\frac{2}{\pi}} \int_0^{\infty} H_{2m}^{(1)}(\lambda) \sin \lambda x d\lambda,$$

$$G_{2m+1}(x) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^{\infty} H_{2m+1}^{(1)}(\lambda) \cos \lambda x d\lambda, \quad m = 0, 1, \dots, \quad -\infty < x < \infty. \tag{2.12}$$

Next express the Hermite polynomials through the Laguerre polynomials ([5, 10.13 (2), (3), p. 193])

$$H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{-1/2}(x^2), \quad H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{1/2}(x^2), \tag{2.13}$$

where the Laguerre polynomials are given by ([10, 8.970.1, p. 1000])

$$L_m^{\alpha}(x) = \sum_{k=0}^{m} \left(\begin{array}{c} m+\alpha \\ m-k \end{array} \right) \frac{(-1)^k x^k}{k!}, \quad \left(\begin{array}{c} a \\ n \end{array} \right) = \frac{\Gamma(a+1)}{n! \Gamma(a-n+1)}. \tag{2.14}$$

Now substitute the expressions (2.13) into (2.12) and use formula (2.14) and the sine- and cosine-integral transforms of the function $x^{\beta} e^{-\alpha x^2}$ ([6, 2.4 (24), p. 74]). After a simple rearrangement we ultimately have the representations (2.6). □

Corollary 2.2. The semi-infinite Hilbert transforms of the weighted Laguerre polynomials $\eta^{\mp 1/2} e^{-\eta/2} L_m^{-1/2}(\eta)$ are given by

$$\frac{1}{\pi} \int_0^{\infty} \frac{e^{-\eta/2} L_m^{-1/2}(\eta) d\eta}{(\eta - \xi)\sqrt{\eta}} = \frac{(-1)^m G_{2m}(\sqrt{\xi})}{2^{2m} m! \sqrt{\xi}},$$

$$\frac{1}{\pi} \int_0^{\infty} \frac{e^{-\eta/2} L_m^{1/2}(\eta) \sqrt{\eta} d\eta}{\eta - \xi} = \frac{(-1)^m G_{2m+1}(\sqrt{\xi})}{2^{2m+1} m!}, \quad m = 0, 1, \dots, \quad 0 < \xi < \infty. \tag{2.15}$$

Proof. Write the Hilbert transforms (2.4) separately for even and odd indices, make the substitutions $\xi = x^2$ and $\eta = t^2$, and employ formulas (2.13). This brings us to the integral relations (2.15). □

The orthogonality relations (2.5), when written for the functions $G_n(\sqrt{\xi})$, imply

$$\int_0^\infty G_{2m}(\sqrt{\xi})G_{2n}(\sqrt{\xi})\frac{d\xi}{\sqrt{\xi}} = 2^{2n}(2n)!\sqrt{\pi}\delta_{mn},$$

$$\int_0^\infty G_{2m+1}(\sqrt{\xi})G_{2n+1}(\sqrt{\xi})\frac{d\xi}{\sqrt{\xi}} = 2^{2n+1}(2n+1)!\sqrt{\pi}\delta_{mn}, \quad m,n=0,1,\dots. \tag{2.16}$$

Notice that the orthogonality relations (2.5) can alternatively be derived by employing the orthogonality relation for the weighted Hermite polynomials $H_n^{(1)}(x)$ and formulas (2.12). Indeed, for even indices we have

$$\int_{-\infty}^\infty G_{2n}(x)G_{2m}(x)dx = 2(-1)^{m+1}\sqrt{\frac{2}{\pi}}\int_0^\infty G_{2n}(x)dx\int_0^\infty e^{-\lambda^2/2}H_{2m}(\lambda)\sin\lambda x d\lambda. \tag{2.17}$$

Upon changing the order of integration and applying the inverse sine-transform

$$\int_0^\infty G_{2n}(x)\sin\lambda x dx = (-1)^{n+1}\sqrt{\frac{\pi}{2}}e^{-\lambda^2/2}H_{2n}(\lambda), \quad \lambda>0, \tag{2.18}$$

we deduce formula (2.5) for even indices. In the same fashion, this formula is derived for odd indices.

2.2. *Relations for the weighted Laguerre polynomials* $e^{-\eta}\eta^{\pm1/2}L_m(\eta)$. Consider the Hilbert transform of the Hermite polynomials with the new weight e^{-x^2}, $H_n^{(2)}(x) = e^{-x^2}H_n(x)$.

Theorem 2.3. Denote the semi-infinite Hilbert transform of the weighted Hermite polynomials $H_n^{(2)}(x)$ by

$$V_n(x) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{H_n^{(2)}(t)dt}{t-x}, \quad n=0,1,\dots, \quad -\infty<x<\infty. \tag{2.19}$$

Then the functions $V_n(x)$ admit the following representations in terms of the confluent hypergeometric function:

$$V_{2m}(x) = \frac{(-1)^{m+1}}{\sqrt{\pi}}2^{2m+1}m!xe^{-x^2}\Phi\left(\frac{1}{2}-m,\frac{3}{2};x^2\right),$$

$$V_{2m+1}(x) = \frac{(-1)^m}{\sqrt{\pi}}2^{2m+1}m!e^{-x^2}\Phi\left(-\frac{1}{2}-m,\frac{1}{2};x^2\right), \ m=0,1,\dots, \ -\infty<x<\infty. \tag{2.20}$$

Proof. As before, apply the Fourier transform and use the convolution theorem to obtain

$$F[V_n(x)](\lambda) = -i\,\mathrm{sgn}\,\lambda F[H_n^{(2)}(x)](\lambda). \tag{2.21}$$

To compute the Fourier transform of the function $H_n^{(2)}(x)$, we consider the even and odd indices cases separately and employ the table integrals ([10, 7.388.1, p. 806])

$$\int_0^\infty H_{2m}^{(2)}(t)\cos\lambda t dt = \frac{1}{2}(-1)^m\sqrt{\pi}\lambda^{2m}e^{-\lambda^2/4},$$

$$\int_0^\infty H_{2m+1}^{(2)}(t)\sin\lambda t dt = \frac{1}{2}(-1)^m\sqrt{\pi}\lambda^{2m+1}e^{-\lambda^2/4}. \tag{2.22}$$

If combined, these give

$$F[V_{2m}(x)](\lambda) = -i\sqrt{\pi}(-1)^m \operatorname{sgn}\lambda\lambda^{2m}e^{-\lambda^2/4},$$
$$F[V_{2m+1}(x)](\lambda) = \sqrt{\pi}(-1)^m|\lambda|^{2m+1}e^{-\lambda^2/4}, \quad m = 0, 1, \ldots, \quad -\infty < \lambda < \infty. \tag{2.23}$$

By the Fourier inversion we deduce analogues of formulas (2.6) and express the functions $V_n(x)$, the Hilbert transforms of the functions $H_n^{(2)}(x)$, in terms of the confluent hypergeometric functions by (2.20). □

Upon transforming the interval $(-\infty, \infty)$ into the semi-infinite interval and making the substitutions $x^2 = \xi$ and $t^2 = \eta$, formulas (2.19) and (2.20) enable us to prove the following result.

Corollary 2.4. The semi-infinite Hilbert transforms of the weighted Laguerre polynomials $\eta^{\mp 1/2}e^{-\eta}L_m^{-1/2}(\eta)$ are expressed through the confluent hypergeometric functions by

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-\eta}L_m^{-1/2}(\eta)d\eta}{(\eta-\xi)\sqrt{\eta}} = -\frac{2e^{-\xi}}{\sqrt{\pi}}\Phi\left(\frac{1}{2}-m, \frac{3}{2}; \xi\right),$$
$$\frac{1}{\pi}\int_0^\infty \frac{e^{-\eta}L_m^{1/2}(\eta)\sqrt{\eta}d\eta}{\eta-\xi} = \frac{e^{-\xi}}{\sqrt{\pi}}\Phi\left(-\frac{1}{2}-m, \frac{1}{2}; \xi\right), \quad m = 0, 1, \ldots, \quad 0 < \xi < \infty. \tag{2.24}$$

Undoubtedly, the integral relations (2.24) are simpler than (2.15). However, if applied to singular integral equations with the Cauchy kernel in a semi-infinite interval, they have a disadvantageous feature: the right-hand sides of the relations (2.24), the functions

$$\phi_m^{(1)}(\xi) = -\frac{2e^{-\xi}}{\sqrt{\pi}}\Phi\left(\frac{1}{2}-m, \frac{3}{2}; \xi\right), \quad \phi_m^{(2)}(\xi) = \frac{e^{-\xi}}{\sqrt{\pi}}\Phi\left(-\frac{1}{2}-m, \frac{1}{2}; \xi\right), \tag{2.25}$$

do not form an orthogonal system in the space $L^2(0, \infty)$. At the same time, the systems $\{\phi_m^{(1)}(\xi)\}_{m=0}^\infty$ and $\{\phi_m^{(2)}(\xi)\}_{m=0}^\infty$ are linearly independent. To prove the linear independence of the first system, denote

$$\chi_m(\eta) = e^{-\eta}\eta^{-1/2}[c_0L_0^{-1/2}(\eta) + c_1L_1^{-1/2}(\eta) + \cdots + c_mL_m^{-1/2}(\eta)]. \tag{2.26}$$

From the first formula in (2.24) we deduce

$$\frac{1}{\pi}\int_0^\infty \frac{\chi_m(\eta)d\eta}{\eta-\xi} = c_0\phi_0^{(1)}(\xi) + c_1\phi_1^{(1)}(\xi) + \cdots + c_m\phi_m^{(1)}(\xi), \quad 0 < \xi < \infty. \tag{2.27}$$

Suppose

$$c_0\phi_0^{(1)}(\xi) + c_1\phi_1^{(1)}(\xi) + \cdots + c_m\phi_m^{(1)}(\xi) = 0, \quad 0 < \xi < \infty. \tag{2.28}$$

Since the homogeneous singular integral equation

$$\frac{1}{\pi}\int_0^\infty \frac{\chi_m(\eta)d\eta}{\eta-\xi} = 0, \quad 0 < \xi < \infty, \tag{2.29}$$

has only the trivial solution in the class of functions integrable in the interval $(0, \infty)$, we have

$$c_0L_0^{-1/2}(\eta) + c_1L_1^{-1/2}(\eta) + \cdots + c_mL_m^{-1/2}(\eta) = 0, \quad 0 < \xi < \infty. \tag{2.30}$$

Now, the system of the Laguerre polynomials $\{L_n^\alpha(\eta)\}_{n=0}^m$ is linearly independent in $(0, \infty)$. Therefore $c_0 = c_1 = \cdots = c_m = 0$, and the functions $\phi_0^{(1)}(\xi)$, $\phi_0^{(1)}(\xi)$, $\ldots$, $\phi_m^{(1)}(\xi)$

are linearly independent for any m. We can show that the second system $\{\phi_m^{(2)}(\xi)\}_{m=0}^{\infty}$ is linearly independent in a similar manner.

In what follows we orthogonalize the systems $\{\phi_m^{(1)}(\xi)\}_{m=0}^{\infty}$ and $\{\phi_m^{(2)}(\xi)\}_{m=0}^{\infty}$ and represent the elements of these orthogonal systems as linear combinations of the functions $\phi_m^{(1)}(\xi)$ and $\phi_m^{(1)}(\xi)$, respectively. Consider the integrals

$$J_m^{(1)}(\xi) = \frac{1}{\pi}\int_0^{\infty} \frac{e^{-\eta/2}L_m^{-1/2}(\eta)d\eta}{(\eta-\xi)\sqrt{\eta}},$$

$$J_m^{(2)}(\xi) = \frac{1}{\pi}\int_0^{\infty} \frac{e^{-\eta/2}L_m^{1/2}(\eta)\sqrt{\eta}d\eta}{\eta-\xi}, \quad m=0,1,\dots, \quad 0<\xi<\infty. \tag{2.31}$$

We aim to compute them by utilizing formulas (2.24). On making the substitutions $\eta = 2u$ and $\xi = 2t$ and employing the identity ([5, (40), p. 192]) that is

$$L_m^{-1/2}(2u) = \sum_{k=0}^{m} \binom{m-1/2}{m-k} (-1)^{m-k}2^k L_k^{-1/2}(u), \tag{2.32}$$

we have for the integral $J_m^{(1)}(\xi)$

$$J_m^{(1)}(2t) = \frac{1}{\sqrt{2}}\sum_{k=0}^{m} \binom{m-1/2}{m-k} \frac{(-1)^{m-k}2^k}{\pi}\int_0^{\infty} \frac{e^{-u}L_k^{-1/2}(u)du}{(u-t)\sqrt{u}}. \tag{2.33}$$

The integral in (2.33) is given by the first formula in (2.24). This brings us to the following relation:

$$J_m^{(1)}(2t) = -\sqrt{\frac{2}{\pi}}\sum_{k=0}^{m} \binom{m-1/2}{m-k} (-1)^{m-k}2^k e^{-t}\Phi\left(\frac{1}{2}-k,\frac{3}{2};t\right). \tag{2.34}$$

Analysis of this formula shows that the function $J_m^{(1)}(2t)$ is a linear combination of the functions $\phi_k^{(1)}(t)$. Simple transformations ultimately yield

$$J_m^{(1)}(\xi) = \frac{(-1)^m G_{2m}(\sqrt{\xi})}{2^{2m}m!\sqrt{\xi}}, \quad m=0,1,\dots, \quad 0<\xi<\infty. \tag{2.35}$$

In a similar fashion we obtain

$$J_m^{(2)}(\xi) = \frac{(-1)^m G_{2m+1}(\sqrt{\xi})}{2^{2m+1}m!}, \quad m=0,1,\dots, \quad 0<\xi<\infty. \tag{2.36}$$

Thus, we have deduced that the orthogonalization of the systems $\{\phi_m^{(1)}(\xi)\}_{m=0}^{\infty}$ and $\{\phi_m^{(2)}(\xi)\}_{m=0}^{\infty}$ leads to the integral relations (2.15) derived in Section 2.1.

To complete this section, we invert the relations (2.24) by representing them as the integral equation with the Cauchy kernel in a semi-infinite segment

$$\frac{1}{\pi}\int_0^{\infty} \frac{\phi(\eta)d\eta}{\eta-\xi} = f(\xi), \quad 0<\xi<\infty. \tag{2.37}$$

Its solution in the class of integrable functions in $(0,\infty)$ and unbounded at $\xi=0$ is

$$\phi(\xi) = -\frac{1}{\pi\sqrt{\xi}}\int_0^{\infty} \frac{\sqrt{\eta}f(\eta)d\eta}{\eta-\xi}, \tag{2.38}$$

and the solution bounded at the point $\xi = 0$ has the form

$$\phi(\xi) = -\frac{\sqrt{\xi}}{\pi}\int_0^\infty \frac{f(\eta)d\eta}{\sqrt{\eta}(\eta-\xi)}. \tag{2.39}$$

Note that one of the ways to obtain formulas (2.38) and (2.39) is to employ the solution of the singular integral equation in the segment (a, b) [8], put $a = 0$, and pass to the limit $b \to \infty$. Upon employing these expressions for the inverse operators and the relations (2.24), we obtain, respectively,

$$\frac{1}{\pi}\int_0^\infty \frac{\sqrt{\eta}e^{-\eta}\Phi(1/2-m,3/2;\eta)d\eta}{\eta-\xi} = \frac{\sqrt{\pi}}{2}e^{-\xi}L_m^{-1/2}(\xi),$$

$$\frac{1}{\pi}\int_0^\infty \frac{e^{-\eta}\Phi(-1/2-m,1/2;\eta)d\eta}{\sqrt{\eta}(\eta-\xi)} = -\sqrt{\pi}e^{-\xi}L_m^{1/2}(\xi), \quad m = 0, 1, \dots, \quad 0 < \xi < \infty. \tag{2.40}$$

3. Properties of the G-functions. We have proved that the Hilbert transforms of the functions $\exp(-x^2/2)H_n(x)$, the functions $G_n(x)$, form an orthogonal system in the space $L_2(-\infty,\infty)$ with the L^2-norm $||G_n(x)|| = \pi^{1/4}2^{n/2}\sqrt{n!}$, while the systems of the functions $\{G_{2n}(\sqrt{\xi})\}$ and $\{G_{2n+1}(\sqrt{\xi})\}$ $(n = 0, 1, \dots)$ are two orthogonal bases for the weighted space $L_w^2(0,\infty)$ with weight $w(\xi) = \xi^{-1/2}$. It was also deduced that, up to certain constant factors, the functions $\xi^{-1/2}G_{2m}(\xi)$ and $G_{2m+1}(\xi)$ are the semi-infinite Hilbert transforms of the weighted Laguerre polynomials $e^{-\eta/2}\eta^{-1/2}L_m^{-1/2}(\eta)$ and $e^{-\eta/2}\eta^{1/2}L_m^{1/2}(\eta)$, respectively.

In this section we aim to show that the functions $G_{2m}(\xi)$ and $G_{2m+1}(\xi)$ satisfy certain ordinary differential equations and also to study their asymptotics for small and large ξ. It is known ([5, (13), p. 193]) that the function $H_{2m}^{(1)}(\lambda) = e^{-\lambda^2/2}H_{2m}(\lambda)$ satisfies the differential equation

$$\left(\frac{d^2}{d\lambda^2} + 4m + 1 - \lambda^2\right)H_{2m}^{(1)}(\lambda) = 0. \tag{3.1}$$

Now, the function $G_{2m}(x)$, up to a factor, is the sine-transform of the function $H_{2m}^{(1)}(\lambda)$,

$$G_{2m}(x) = (-1)^{m+1}\sqrt{\frac{2}{\pi}}\int_0^\infty H_{2m}^{(1)}(\lambda)\sin\lambda x d\lambda. \tag{3.2}$$

By multiplying equation (3.1) by $(-1)^{m+1}\sqrt{2/\pi}\sin\lambda x$, integrating in $(0,\infty)$, and then integrating by parts we deduce

$$\left(\frac{d^2}{dx^2} + 4m + 1 - x^2\right)G_{2m}(x) = B_m^{(1)}x, \quad 0 < x < \infty, \quad G_{2m}(0) = 0, \tag{3.3}$$

where

$$B_m^{(1)} = \sqrt{\frac{2}{\pi}}\frac{(2m)!}{m!}. \tag{3.4}$$

In our derivations, we used the fact that $H_{2m}^{(1)}(0) = (-1)^m(2m)!/m!$.

Similar actions discover that the G-functions of odd indices satisfy the differential equation

$$\left(\frac{d^2}{dx^2}+4m+3-x^2\right)G_{2m+1}(x)=B_m^{(2)}x, \quad 0<x<\infty, \quad \frac{d}{dx}G_{2m+1}(0)=0, \tag{3.5}$$

where

$$B_m^{(2)}=\sqrt{\frac{2}{\pi}}\frac{2(2m+1)!}{m!}. \tag{3.6}$$

Now analyze the asymptotics of the functions $G_n(\xi)$ as $\xi\to 0$ and $\xi\to\infty$. The representations (2.6) of $G_{2m}(x)$ and $G_{2m+1}(x)$ and the series (2.7) imply

$$G_{2m}(x)\sim a_0x, \quad G_{2m+1}(x)\sim a_1, \quad x\to 0, \tag{3.7}$$

where

$$a_0=-\sqrt{2}(2m)!\sum_{k=0}^{m}\frac{(-1)^k2^k}{(m-k)!\Gamma(k+1/2)}, \; a_1=\sqrt{2}(2m+1)!\sum_{k=0}^{m}\frac{(-1)^k2^k}{(m-k)!\Gamma(k+3/2)}. \tag{3.8}$$

To derive the asymptotics of the G-functions for large x, we employ the formula ([4, 6.13.1(3), p. 278])

$$\Phi(a,c;x)=\frac{\Gamma(c)}{\Gamma(a)}e^xx^{a-c}[1+O(x^{-1})], \quad x\to\infty. \tag{3.9}$$

From the representations (2.6) we deduce

$$G_{2m}(x)\sim-\frac{2^{2m+1/2}\Gamma(m+1/2)}{\pi x}, \quad x\to\infty,$$
$$G_{2m+1}(x)\sim-\frac{2^{2m+5/2}\Gamma(m+3/2)}{\pi x^2}, \quad x\to\infty. \tag{3.10}$$

It is possible to obtain full asymptotic expansions for large x for these functions by expressing the function Φ in (2.6) through the Tricomi function Ψ and writing the asymptotic expansion of the function Ψ (see [4, 6.13.1(1) and 6.7(7)], respectively). Alternatively, we may use the relation (3.2) and the asymptotic formula ([13, (3), p. 56])

$$\int_0^\infty f(\lambda)\sin\lambda x d\lambda\sim\frac{f(0)}{x}-\frac{f''(0)}{x^3}+\frac{f^{IV}(0)}{x^5}-\ldots, \quad x\to\infty. \tag{3.11}$$

This asymptotic expansion holds for all functions $f(x)$ defined with all its derivatives for $x\geq 0$. We discover for the G-functions of even indices

$$G_{2m}(x)\sim(-1)^{m+1}\sqrt{\frac{2}{\pi}}\left[\frac{H_m^{(1)}(0)}{x}-\frac{d^2H_m^{(1)}(0)}{x^3dx^2}+\frac{d^4H_m^{(1)}(0)}{x^5dx^4}-\cdots\right], \quad x\to\infty, \tag{3.12}$$

where

$$\frac{d^{2n}H_n^{(1)}(0)}{dx^{2n}}=(-1)^{n+m}(2n)!(2m)!\sum_{j=0}^{n}\frac{2^{2j}}{(2j)!(m-j)!(2n-2j)!!}, \tag{3.13}$$

where $(2m)!!=2\cdot 4\cdot\ldots\cdot(2m)$. On computing the first several terms we have

$$G_{2m}(x)\sim-\sqrt{\frac{2}{\pi}}\frac{(2m)!}{m!}\left(\frac{c_1}{x}+\frac{c_3}{x^3}+\frac{c_5}{x^5}+\frac{c_7}{x^7}+\cdots\right), \quad x\to\infty, \tag{3.14}$$

where

$$c_1 = 1,\ c_3 = 4m + 1,\ c_5 = 16m^2 + 8m + 3,\ c_7 = 64m^3 + 48m^2 + 68m + 15. \tag{3.15}$$

A similar asymptotic expansion may be obtained for the functions $G_{2m+1}(x)$.

For applications to integral equations, it will be convenient to denote

$$G_m^{(1)}(\xi) = \frac{G_{2m}(\sqrt{\xi})}{2^m\sqrt{(2m)!}\pi^{1/4}},$$

$$G_m^{(2)}(\xi) = \frac{G_{2m+1}(\sqrt{\xi})}{2^{m+1/2}\sqrt{(2m+1)!}\pi^{1/4}}, \quad m = 0, 1, \ldots, \quad 0 < \xi < \infty. \tag{3.16}$$

These functions form two orthonormal bases for the weighted space $L_w^2(0, \infty)$ with weight $w(\xi) = \xi^{-1/2}$,

$$\int_0^\infty G_n^{(j)}(\xi)G_m^{(j)}(\xi)\frac{d\xi}{\sqrt{\xi}} = \delta_{mn}, \quad m, n = 0, 1, \ldots, \quad j = 1, 2. \tag{3.17}$$

In Figures 1 and 2, we plot the functions $G_m^{(1)}(\xi)$ and $G_m^{(2}(\xi)$ for $m = 0, 1, \ldots, 4$, respectively. It has been discovered that although the functions $G_m^{(1)}(\xi)$ and $G_m^{(2}(\xi)$ are not

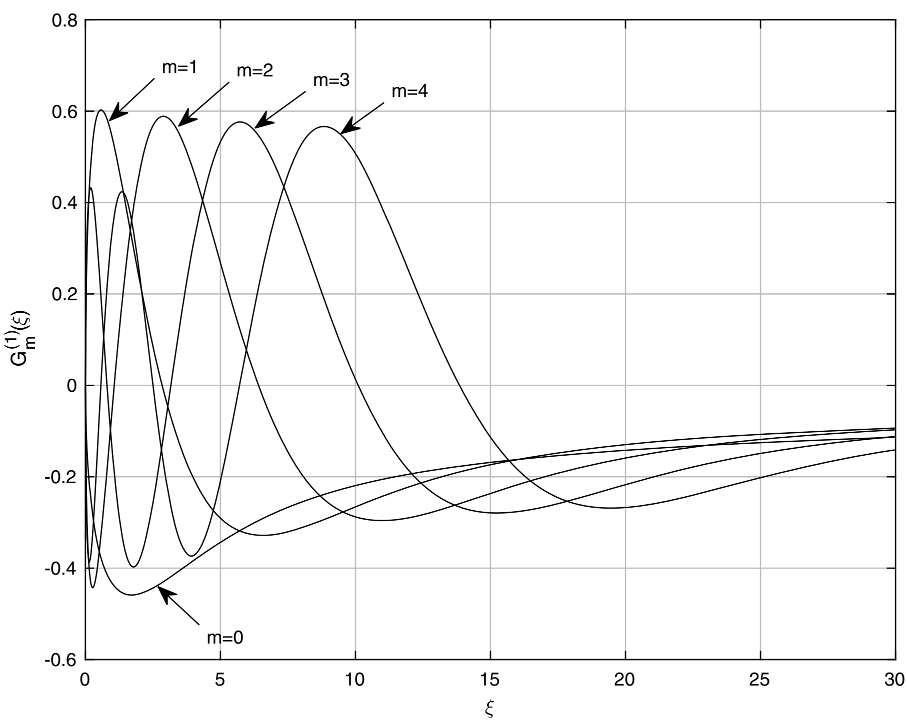

FIG. 1. The functions $G_m^{(1)}(\xi)$, $m = 0, 1, 2, 3, 4$

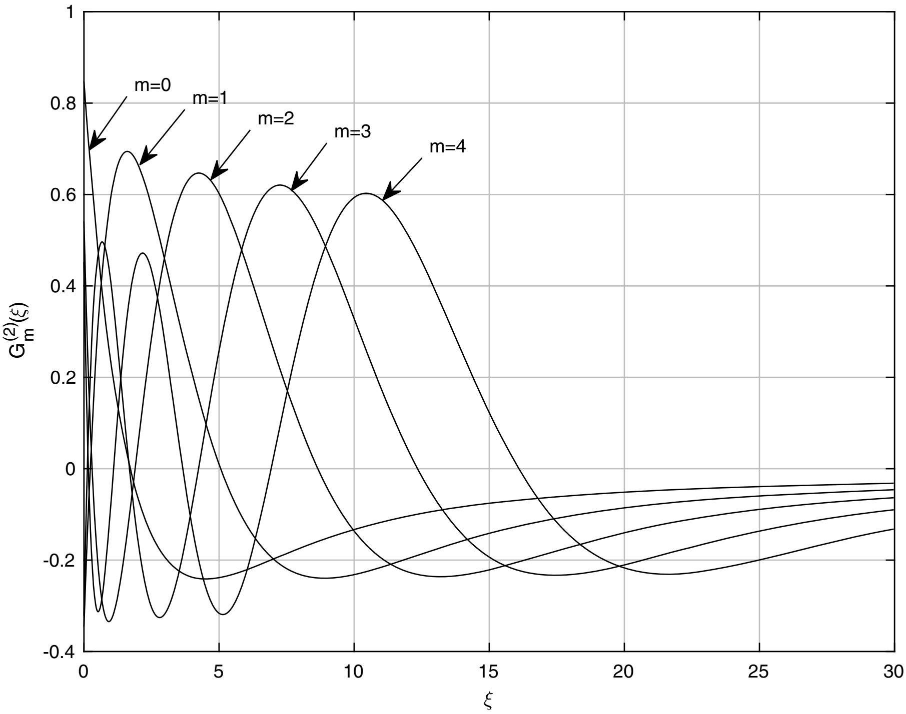

FIG. 2. The functions $G_m^{(2)}(\xi)$, $m=0,1,2,3,4$

polynomials, in addition to the orthogonality and being a basis of a certain weighted space, they share another property of classical orthogonal polynomials: in their interval of definition, $[0,\infty)$, the number of zeros correlates with the index and equals $m+1$ for both functions. Note that in the case of $G_m^{(1)}(\xi)$, $G_m^{(1)}(\xi)\sim \text{const}\sqrt{\xi}$, $\xi\to 0$, $m=0,1,\ldots$, and the point $\xi=0$ is counted as the first zero of the function $G_m^{(1)}(\xi)$. Due to the relations (3.10) and 3.16), the functions $G_m^{(1)}(\xi)$ and $G_m^{(2}(\xi)$ vanish at infinity,

$$G_m^{(1)}(\xi)\sim -\sqrt{\frac{2(2m-1)!!}{(2m)!!}}\frac{1}{\pi^{3/4}\sqrt{\xi}}, \quad G_m^{(2)}(\xi)\sim -\sqrt{\frac{(2m+1)!!}{(2m)!!}}\frac{2}{\pi^{3/4}\xi}, \quad \xi\to\infty,$$

where $(2m\pm 1)!!=1\cdot 3\cdot\ldots\cdot(2m\pm 1)$.

4. Hilbert transforms associated with limiting relations for the Jacobi polynomials and functions.

4.1. *Laguerre polynomials and the Hilbert transform of the Jacobi polynomials.* A number of Hilbert transforms for special functions including the weighted Jacobi and Laguerre polynomials and the confluent hypergeometric functions can be derived from integral relations in a finite segment by letting a parameter involved go to infinity. To pursue this goal, we analyze the integral relation for the Jacobi polynomials [24]

$$\pi \cot \pi\alpha\,(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x) - \int_{-1}^{1}\frac{(1-t)^{\alpha}(1+t)^{\beta}}{t-x}P_n^{(\alpha,\beta)}(t)dt$$
$$= \frac{2^{\alpha+\beta}\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}F\left(n+1,-n-\alpha-\beta;1-\alpha;\frac{1-x}{2}\right),$$
$$-1<x<1,\quad n=0,1,\dots. \tag{4.1}$$

Here, $\alpha > -1$, $\alpha \neq 0,1,\dots$, $\beta > -1$, $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials, and $F(a,b;c;x)$ is the Gauss hypergeometric function. On making the substitutions $x = 1-2\xi/\beta$ and $t = 1-2\eta/\beta$ we infer

$$\pi \cot \pi\alpha\;\xi^{\alpha}\left(1-\frac{\xi}{\beta}\right)^{\beta}P_n^{(\alpha,\beta)}\left(1-\frac{2\xi}{\beta}\right) + \int_0^{\beta}\frac{\eta^{\alpha}(1-\eta/\beta)^{\beta}}{\eta-\xi}P_n^{(\alpha,\beta)}\left(1-\frac{2\eta}{\beta}\right)d\eta$$
$$= \frac{\beta^{\alpha}\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}F\left(n+1,-n-\alpha-\beta;1-\alpha;\frac{\xi}{\beta}\right),$$
$$0<\xi<\beta,\quad n=0,1,\dots. \tag{4.2}$$

In what follows we use the connection between the Laguerre and Jacobi polynomials ([22, (5.3.4) p. 103])

$$L_n^{\alpha}(\xi) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2\xi}{\beta}\right) \tag{4.3}$$

and the asymptotic formula for the Γ-functions

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \sim z^{\alpha-\beta},\quad z\to\infty. \tag{4.4}$$

The last relation enables us to evaluate the limits

$$\lim_{\beta\to\infty}\frac{\beta^{\alpha}\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} = 1,$$
$$\lim_{\beta\to\infty} F\left(n+1,-n-\alpha-\beta;1-\alpha;\frac{\xi}{\beta}\right) = \Phi(n+1,1-\alpha;-\xi). \tag{4.5}$$

Now, on passing to the limit $\beta\to\infty$ in (4.2) and using formula (4.3) and Kummer's transformation ([4, (7), p. 253])

$$\Phi(n+1,1-\alpha;-\xi) = e^{-\xi}\Phi(-\alpha-n,1-\alpha;\xi), \tag{4.6}$$

we obtain the following result.

THEOREM 4.1. Let $\alpha > -1$, $\alpha \neq 0,1,\dots$. Then

$$\pi\cot\pi\alpha\;\xi^{\alpha}e^{-\xi}L_n^{\alpha}(\xi) + \int_0^{\infty}\frac{\eta^{\alpha}e^{-\eta}L_n^{\alpha}(\eta)d\eta}{\eta-\xi} = \Gamma(\alpha)e^{-\xi}\Phi(-n-\alpha,1-\alpha;\xi),$$
$$0<\xi<\infty,\quad n=0,1,\dots. \tag{4.7}$$

REMARK 4.2. This theorem generalizes Corollary 2.4: formulas (2.24) can be immediately deduced from (4.7) by putting $\alpha = \pm 1/2$ there.

4.2. *Laguerre polynomials and the Jacobi functions* $Q_n^{(\alpha,\beta)}(x)$. To derive an analogue of the integral relation (4.7) for the interval $(-\infty, 0)$, we analyze two representations of the Jacobi function $Q_n^{(\alpha,\beta)}(x)$ ([22, (4.61.4), (4.61.5), p. 74]),

$$Q_n^{(\alpha,\beta)}(x) = -\frac{(x-1)^{-\alpha}(x+1)^{-\beta}}{2}\int_{-1}^{1}(1-t)^{\alpha}(1+t)^{\beta}\frac{P_n^{(\alpha,\beta)}(t)dt}{t-x}, \quad n = 0, 1, \ldots, \tag{4.8}$$

and

$$Q_n^{(\alpha,\beta)}(x) = \frac{2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)}(x-1)^{-n-\alpha-1}(x+1)^{-\beta}$$
$$\times F\left(n+\alpha+1, n+1; 2n+\alpha+\beta+2; \frac{2}{1-x}\right), \quad n = 0, 1, \ldots, \tag{4.9}$$

where $\alpha > -1$, $\beta > -1$. The relations are valid in the whole complex plane cut along the segment $[-1, 1]$. We consider the case $x > 1$, employ the variables $x = 1 - 2\xi/\beta$ and $t = 1 - 2\eta/\beta$, and denote

$$f_n(\alpha, \beta; \xi) = 2(-\xi)^{\alpha}\left(1 - \frac{\xi}{\beta}\right)^{\beta} Q_n^{(\alpha,\beta)}\left(1 - \frac{2\xi}{\beta}\right). \tag{4.10}$$

This transforms the relations (4.8) and (4.9) to the following:

$$f_n(\alpha, \beta; \xi) = \int_0^{\beta}\frac{\eta^{\alpha}(1-\eta/\beta)^{\beta}P_n^{(\alpha,\beta)}(1-2\eta/\beta)d\eta}{\eta - \xi}$$
$$= \frac{\beta^{n+\alpha+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(2n+\alpha+\beta+2)(-\xi)^{n+1}}F\left(n+\alpha+1, n+1; 2n+\alpha+\beta+2; \frac{\beta}{\xi}\right). \tag{4.11}$$

Passing to the limit $\beta \to \infty$ in the expression for the function $f_n(\alpha, \beta; \xi)$ gives

$$\lim_{\beta\to\infty} f_n(\alpha, \beta; \xi) = \Gamma(n+\alpha+1)(-\xi)^{-n-1}{}_2F_0(n+1, n+\alpha+1; \xi^{-1}), \quad \xi < 0, \tag{4.12}$$

where ${}_2F_0(\alpha, \beta; z)$ is a generalized hypergeometric series that can also be expressed through the Tricomi function Ψ ([4, 6.6(3), p. 257])

$${}_2F_0(\alpha, \beta; -\xi^{-1}) = \xi^{\alpha}\Psi(\alpha, \alpha - \beta + 1; \xi). \tag{4.13}$$

Upon letting $\beta \to \infty$ in the first relation in (4.11) in view of formula (4.3), we have

$$\int_0^{\infty}\frac{\eta^{\alpha}e^{-\eta}L_n^{\alpha}(\eta)d\eta}{\eta - \xi} = \Gamma(n+\alpha+1)\Psi(n+1, 1-\alpha; -\xi), \quad n = 0, 1, \ldots; \quad -\infty < \xi < 0. \tag{4.14}$$

This formula can be rewritten in terms of the confluent hypergeometric function Φ if we employ the relation between the Ψ- and Φ-functions ([4, 6.5(7), p. 257]). We have the following result.

THEOREM 4.3. Let $\alpha > -1$, $\alpha \neq 0, 1, \ldots$. Then

$$\int_0^{\infty}\frac{\eta^{\alpha}e^{-\eta}L_n^{\alpha}(\eta)d\eta}{\eta - \xi} = \Gamma(\alpha)\Phi(n+1, 1-\alpha; -\xi)$$
$$+\frac{\Gamma(-\alpha)\Gamma(n+\alpha+1)}{n!}(-\xi)^{\alpha}\Phi(n+1+\alpha, 1+\alpha; -\xi), \quad n = 0, 1, \ldots, -\infty < \xi < 0. \tag{4.15}$$

REMARK 4.4. Alternatively, this relation can be derived by analytic continuation of the Gauss function in (4.11) ([4, 2.10(2), p. 108])

$$F\left(n+\alpha+1, n+1; 2n+\alpha+\beta+2; \frac{\beta}{\xi}\right) = \frac{\Gamma(2n+\alpha+\beta+2)\Gamma(\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}$$
$$\times\left(-\frac{\beta}{\xi}\right)^{-n-1} F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{\xi}{\beta}\right) + \frac{\Gamma(2n+\alpha+\beta+2)\Gamma(-\alpha)}{\Gamma(n+1)\Gamma(n+\beta+1)}$$
$$\times\left(-\frac{\beta}{\xi}\right)^{-n-\alpha-1} F\left(n+\alpha+1, -n-\beta; 1+\alpha; \frac{\xi}{\beta}\right), \quad \xi < 0, \tag{4.16}$$

and consequently passing to the limit $\beta \to \infty$ in (4.11).

4.3. *Integral relations for cylindrical functions.* By passing to the limit $n \to \infty$ in certain relations for the Jacobi functions of the second kind it is possible to discover some elegant formulas for cylindrical functions.

THEOREM 4.5. Let $\lambda > 0$, and let α be a complex number such that $-1 < \operatorname{Re}\alpha < 5/2$. Then

$$\frac{\pi}{\sin\pi\alpha}\left[z^{\alpha/2}J_{-\alpha}(\lambda\sqrt{z}) - (-z)^{\alpha}z^{-\alpha/2}J_{\alpha}(\lambda\sqrt{z})\right] = \int_0^{\infty}\frac{t^{\alpha/2}J_{\alpha}(\lambda\sqrt{t})dt}{t-z}, \tag{4.17}$$

where $J_\alpha(z)$ is the Bessel function and z^α is the single branch in the z-plane cut along the ray $(-\infty, 0]$ such that $\arg z \in [-\pi, \pi]$.

Proof. We start with the following representation of the Jacobi function ([5, (19), p. 171]):

$$Q_n^{(\alpha,\beta)}(\zeta) = -\frac{\pi}{2\sin\pi\alpha}P_n^{(\alpha,\beta)}(\zeta) + \frac{2^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}(\zeta-1)^{-\alpha}(\zeta+1)^{-\beta}$$
$$\times F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-\zeta}{2}\right), \tag{4.18}$$

valid in the whole ζ-plane cut along the segment $[-1, 1]$. On the cut sides, $\zeta = x \pm i0$, $\arg(\zeta-1) = \pm\pi$, and $\arg(\zeta+1) = 0$. Put $\zeta = 1 - w^2/(2n^2)$. Then the branch cut $[-1, 1]$ in the ζ-plane is transformed into the cut $[0, 2n]$ of the w-plane. Intending to pass to the limit $n \to \infty$ we multiply equation (4.18) by $n^{-\alpha}$ and use the limiting relation ([5, (41), p. 173])

$$\lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{w^2}{2n^2}\right) = \left(\frac{w}{2}\right)^{-\alpha}J_\alpha(w). \tag{4.19}$$

This relation holds for arbitrary α and β, uniformly in any bounded region of the complex plane. It is directly verified that

$$\lim_{n\to\infty} F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{w^2}{4n^2}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k w^{2k}}{k!(1-\alpha)_k 2^{2k}}$$
$$= \Gamma(1-\alpha)\left(\frac{w}{2}\right)^{\alpha}J_{-\alpha}(w). \tag{4.20}$$

Consequently, we deduce from (4.18) that

$$\lim_{n\to\infty} n^{-\alpha}Q_n^{(\alpha,\beta)}\left(1-\frac{w^2}{2n^2}\right) = \frac{\pi 2^{\alpha-1}}{\sin\pi\alpha}\left[(-w^2)^{-\alpha}w^{\alpha}J_{-\alpha}(w) - w^{-\alpha}J_\alpha(w)\right]. \tag{4.21}$$

Now, make the substitution $x = 1 - w^2/(2n^2)$ in the integral relation (4.8) to obtain

$$Q_n^{(\alpha,\beta)}\left(1-\frac{w^2}{2n^2}\right) = (-w^2)^{-\alpha}\left(2-\frac{w^2}{2n^2}\right)^{-\beta}$$

$$\times\int_0^{2n} u^{2\alpha+1}\left(2-\frac{u^2}{2n^2}\right)^{\beta}\frac{P_n^{(\alpha,\beta)}(1-\frac{1}{2}u^2/n^2)du}{u^2-w^2}. \tag{4.22}$$

We multiply this equation by $n^{-\alpha}$, pass to the limit $n \to \infty$, and use formulas (4.19) and (4.21). We have

$$\frac{\pi}{2\sin\pi\alpha}\left[w^\alpha J_{-\alpha}(w) - (-w^2)^\alpha w^{-\alpha} J_\alpha(w)\right] = \int_0^\infty \frac{u^{\alpha+1}J_\alpha(u)du}{u^2-w^2}. \tag{4.23}$$

Next we make the substitutions $u = \lambda\sqrt{t}$ and $w = \lambda\sqrt{z}$, where λ is a positive parameter, and ultimately deduce formula (4.17). □

Corollary 4.6. Let $\alpha = -1/2 - i\mu$, and let $-\infty < \mu < \infty$. Then the Bessel function $J_\alpha(\lambda\sqrt{x}$ satisfies the integral relation

$$\frac{1}{\pi}\int_0^\infty \frac{t^{\alpha/2}J_\alpha(\lambda\sqrt{t})dt}{t-x} + i\tanh\pi\mu x^{\alpha/2}J_\alpha(\lambda\sqrt{x}) = -\frac{x^{\alpha/2}}{\cosh\pi\mu}J_{-\alpha}(\lambda\sqrt{x}),\ 0<x<\infty. \tag{4.24}$$

Proof. Put $z = x \pm i0$, $0 < x < \infty$, in the last relation. Since $\arg(-z) = \mp\pi$, by the Sokhotski–Plemelj formulas

$$\frac{\pi x^{\alpha/2}}{\sin\pi\alpha}\left[J_{-\alpha}(\lambda\sqrt{x}) - e^{\mp i\pi\alpha}J_\alpha(\lambda\sqrt{x})\right] = \pm\pi i x^{\alpha/2}J_\alpha(\lambda\sqrt{x}) + \int_0^\infty \frac{t^{\alpha/2}J_\alpha(\lambda\sqrt{t})dt}{t-x}. \tag{4.25}$$

It is directly verified that both formulas may be put into the same form as (4.24). □

Corollary 4.7. Let $I_\alpha(x)$ be the modified Bessel function of the first kind, let $\alpha = -1/2 - i\mu$, and let $-\infty < \mu < \infty$. Then

$$\frac{1}{\pi}\int_0^\infty \frac{t^{\alpha/2}J_\alpha(\lambda\sqrt{t})dt}{t-x} = \frac{(-x)^{\alpha/2}}{\cosh\pi\mu}[I_\alpha(\lambda\sqrt{-x}) - I_{-\alpha}(\lambda\sqrt{-x})], \quad -\infty < x < 0. \tag{4.26}$$

Proof. Let $z \to x \pm i0$, $x < 0$. Since $\arg z = \pi$, $\arg(-z) = 0$, and $I_\alpha(x) = e^{-i\pi\alpha/2}J_\alpha(ix)$, we deduce from (4.17) the relation needed. □

Remark 4.8. On letting $\lambda \to 0^+$, we obtain from (4.25) the following spectral relation for the operator H in a semi-infinite interval:

$$\frac{1}{\pi i}\int_0^\infty \frac{t^{-1/2+i\mu}dt}{t-x} = \tanh\pi\mu\, x^{-1/2+i\mu}, \quad 0<x<\infty, \quad -\infty<\mu<\infty. \tag{4.27}$$

This means that the function $f(t) = t^{-1/2+i\mu}$ is a generalized eigenfunction of the Hilbert operator in the interval $(0, \infty)$, and $\tanh \pi\mu$ is its eigenvalue. We call $f(t)$ a generalized eigenfunction since it is not an $L_2(0, \infty)$-function. By virtue of the inequality $-\infty < \mu < \infty$, the interval $(-1, 1)$ is a continuous spectrum of the operator H.

REMARK 4.9. The particular case (4.24) of the general formula (4.17) can also be derived from the integral relation for the Jacobi polynomials ([24], [18])

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{P_n^{(\alpha,\bar{\alpha})}(t)(1-t)^{\alpha}(1+t)^{\bar{\alpha}}dt}{t-x} - i\tanh\pi\mu P_n^{(\alpha,\bar{\alpha})}(x)(1-x)^{\alpha}(1+x)^{\bar{\alpha}}$$

$$= \begin{cases} (2\cosh\pi\mu)^{-1}P_{n-1}^{(\bar{\alpha}+1,\alpha+1)}(x), & n = 1, 2, \dots, \\ 0, & n = 0, \end{cases} \quad -1 < x < 1, \tag{4.28}$$

by utilizing the substitutions $x = 1 - \xi^2/(2n^2)$ and $t = 1 - \eta^2/(2n^2)$ and passing to the limit $n \to \infty$.

Finally, we show that the classical Hilbert relation [23]

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos\lambda\eta d\eta}{\xi-\eta} = \sin\lambda\xi, \quad -\infty < \xi < \infty, \quad \lambda > 0, \tag{4.29}$$

can be deduced from (4.24) as a particular case. Put $\mu = 0$ in (4.24). Due to the relations

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z, \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z \tag{4.30}$$

we immediately get

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{\cos\lambda\sqrt{t}dt}{(x-t)\sqrt{t}} = \frac{\sin\lambda\sqrt{x}}{\sqrt{x}}, \quad 0 < x < \infty. \tag{4.31}$$

Equivalently, if the substitutions $\xi = \sqrt{x}$ and $\eta = \sqrt{t}$ are made, this may be written as the Hilbert relation (4.29).

5. Applications to singular integral equations.

5.1. *Integral equation with the Cauchy kernel in a semi-infinite axis.* Based on the integral relations (2.15) we derive an exact solution of the singular integral equation

$$\frac{1}{\pi}\int_{0}^{\infty}\frac{\chi(t)dt}{t-x} = f(x), \quad 0 < x < \infty, \tag{5.1}$$

in a series form free of singular integrals. In the class of functions unbounded at the point $x = 0$, we expand the solution through the Laguerre polynomials

$$\chi(x) = \frac{e^{-x/2}}{\sqrt{x}}\sum_{n=0}^{\infty} b_n\, L_n^{-1/2}(x). \tag{5.2}$$

By substituting the series (5.2) into equation (5.1) and using the first formula in (2.15) and the first orthogonality relation in (2.16), we obtain for the coefficients b_n

$$b_n = \frac{(-1)^n n!}{\sqrt{\pi}(2n)!} \int_0^\infty f(t) G_{2n}(\sqrt{t}) dt. \tag{5.3}$$

In terms of the elements of the orthonormal basis $\{G_n^{(1)}(t)\}_{n=0}^\infty$ given by (3.16) these coefficients have the form

$$b_n = \alpha_n \int_0^\infty f(t) G_n^{(1)}(t) dt, \tag{5.4}$$

where

$$\alpha_n = \frac{(-1)^n}{\pi^{1/4}} \sqrt{\frac{(2n)!!}{(2n-1)!!}}. \tag{5.5}$$

In the class of functions bounded at the point $x = 0$ we seek the solution of equation (5.1) in the form

$$\chi(x) = e^{-x/2}\sqrt{x} \sum_{n=0}^\infty b_n \, L_n^{1/2}(x). \tag{5.6}$$

Upon employing the second formulas in (2.15) and (2.16) we derive the coefficient b_n by quadratures possessing the G-functions of odd indices

$$b_n = \frac{(-1)^n n!}{\sqrt{\pi}(2n+1)!} \int_0^\infty f(t) G_{2n+1}(\sqrt{t}) \frac{dt}{\sqrt{t}} \tag{5.7}$$

or, in terms of the orthonormal basis functions $G_n^{(2)}(x)$,

$$b_n = \alpha_n \sqrt{\frac{2}{2n+1}} \int_0^\infty f(t) G_n^{(2)}(t) \frac{dt}{\sqrt{t}}. \tag{5.8}$$

Interesting representations of the Cauchy kernel are derived by comparing the series- and closed-form solutions of the singular integral equation (5.1).

Theorem 5.1. Let $0 < x < \infty$ and $0 < t < \infty$. Then the following two bilinear expansions of the Cauchy kernel in terms of the Laguerre polynomials and the G-functions are valid:

$$\frac{1}{t-x} = -\sqrt{\frac{\pi}{t}} e^{-x/2} \sum_{n=0}^\infty \frac{(-1)^n n!}{(2n)!} L_n^{-1/2}(x) G_{2n}(\sqrt{t}),$$

$$\frac{1}{t-x} = -\sqrt{\pi} e^{-x/2} \sum_{n=0}^\infty \frac{(-1)^n n!}{(2n+1)!} L_n^{1/2}(x) G_{2n+1}(\sqrt{t}). \tag{5.9}$$

Proof. The first representation of the Cauchy kernel is derived by comparing the series-form solution (5.2) and its integral form (2.38). Had we substituted the coefficients b_n given by (5.7) into the series (5.6) and compared the new series with the integral-form solution (2.39), we would have obtained the second formula. □

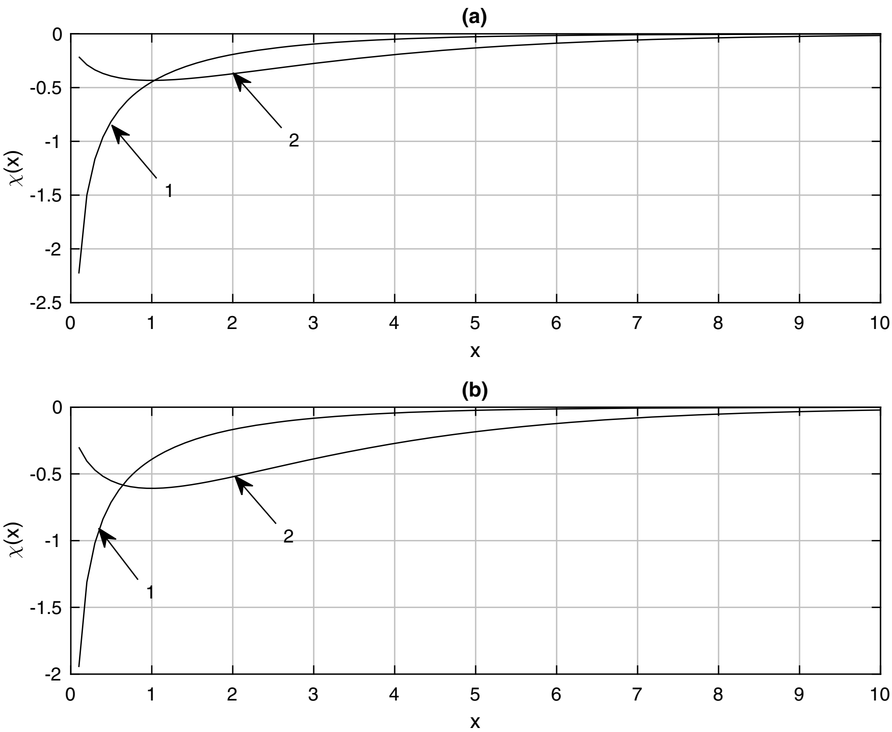

FIG. 3. The solution of the integral equation (5.1) in the class of functions unbounded at $x = 0$ (curve 1) and functions bounded at $x = 0$ (curve 2). (a): $f(x) = \sqrt{x}e^{-x/2}$. (b): $f(x) = 1$ if $0 < x < 1$, and $f(x) = 0$ if $x > 1$.

5.2. *System of two singular integral equations.* Consider the systems of complete singular integral equations of the first kind with the Cauchy kernel

$$\frac{1}{\pi}\int_0^\infty \left[\frac{1}{t-x} + K_{11}(t,x)\right]\chi_1(t)dt + \frac{1}{\pi}\int_0^\infty K_{12}(t,x)\chi_2(t)dt = f_1(x), \quad 0 < x < \infty,$$

$$\frac{1}{\pi}\int_0^\infty K_{21}(t,x)\chi_1(t)dt + \frac{1}{\pi}\int_0^\infty \left[\frac{1}{t-x} + K_{22}(t,x)\right]\chi_2(t)dt = f_2(x), \quad 0 < x < \infty, \tag{5.10}$$

where $K_{jl}(t,x)$ may have a weak singularity at the line $x = t$. Suppose that the functions $\chi_1(x)$ and $\chi_2(x)$ are integrable in the interval $(0,\infty)$ and not bounded at the point $x = 0$. Then, necessarily, they have the square root singularity at this point. We expand the unknown functions in terms of the Laguerre polynomials

$$\chi_j(x) = \frac{e^{-x/2}}{\sqrt{x}} \sum_{m=0}^{\infty} b_m^{(j)}\, L_m^{-1/2}(x), \quad j = 1, 2. \tag{5.11}$$

Because of the kernels k_{jl}, in general, the coefficients $b_m^{(1)}$ and $b_m^{(2)}$ cannot be found in explicit form. By applying the same argument as in the case of the characteristic equation

(5.1) we deduce an infinite system of linear algebraic equations

$$b_n^{(1)} + \sum_{m=1}^{\infty} (c_{nm}^{(1,1)} b_m^{(1)} + c_{nm}^{(1,2)} b_m^{(2)}) = f_n^{(1)},$$

$$b_n^{(2)} + \sum_{m=1}^{\infty} (c_{nm}^{(2,1)} b_m^{(1)} + c_{nm}^{(2,2)} b_m^{(2)}) = f_n^{(2)}, \quad n = 0, 1, \ldots . \tag{5.12}$$

Here,

$$c_{nm}^{(j,l)} = \frac{\alpha_n}{\pi} \int_0^\infty \int_0^\infty K_{jl}(t,x) \frac{e^{-t/2}}{\sqrt{t}} L_m^{-1/2}(t) G_n^{(1)}(x) dt dx, \quad j,l = 1,2,$$

$$f_n^{(j)} = \alpha_n \int_0^\infty f_j(x) G_n^{(1)}(x) dx, \quad j = 1,2, \tag{5.13}$$

where α_n are given by (5.5). Assume that the kernels of the system of integral equations are chosen such that the system (5.12) is regular. By solving the infinite system (5.12) by the reduction method we can approximately obtain the coefficients $b_m^{(j)}$ and therefore an approximate solution to the system of singular integral equations.

5.3. *Integral relation for the Bessel function: a contact problem for a semi-infinite stamp.* Suppose a semi-infinite rigid stamp of profile $y = g(x)$ is indented into an elastic half-plane $|x| < \infty$, $-\infty < y < 0$ such that the adhesion contact conditions hold everywhere in the contact zone, while the rest of the boundary of the half-plane is free of traction,

$$u(x,0) = c_1, \quad v(x,0) = g(x) + c_2, \quad 0 < x < \infty,$$

$$\sigma_y(x,0) = \tau_{xy}(x,0) = 0, \quad -\infty < x < 0. \tag{5.14}$$

Here, u and v are the x- and y-components of the displacement vector, σ_y and τ_{xy} are the stress tensor components, and c_1 and c_2 are constants. Denote $p(x) = -\sigma_y(x,0)$ and $\tau(x) = -\tau_{xy}(x,0)$. Then this model problem is equivalent [17], [9] to the system of integral equations

$$\frac{\kappa-1}{\kappa+1} p(x) + \frac{1}{\pi} \int_0^\infty \frac{\tau(t) dt}{t-x} = 0, \quad 0 < x < \infty,$$

$$\frac{\kappa-1}{\kappa+1} \tau(x) - \frac{1}{\pi} \int_0^\infty \frac{p(t) dt}{t-x} = \frac{4Gg'(x)}{\kappa+1}, \quad 0 < x < \infty, \tag{5.15}$$

where $\kappa = 3 - 4\nu$, ν is the Poisson ratio, and G is the shear modulus. In terms of the function $\varphi(x) = p(x) + i\tau(x)$, this system may be written as a single integral equation

$$\frac{1}{\pi} \int_0^\infty \frac{\varphi(t) dt}{t-x} + i \tanh \pi\mu \, \varphi(x) = f(x), \quad 0 < x < \infty, \tag{5.16}$$

where $f(x) = -4G(\kappa+1)^{-1} g'(x)$ and $\mu = (2\pi)^{-1} \ln(3 - 4\nu)$. For materials with the Poisson ratio $\nu \in (0, 1/2)$, $\mu \in (0, \mu_0)$, $\mu_0 \approx 0.17484958$. This equation can be solved exactly by the Mellin transform or by the method of the Riemann–Hillbert problem. In what follows, we propose an alternative technique based on the integral relation for the Bessel function (4.24) and the Hankel transform. Represent the unknown function $\varphi(x)$ in the integral form

$$\varphi(x) = x^{\alpha/2} \int_0^\infty \chi(\lambda) J_\alpha(\lambda\sqrt{x}) d\lambda, \quad \alpha = -\frac{1}{2} - i\mu. \tag{5.17}$$

Here, the density $\chi(\lambda)$ is to be determined. Upon substituting this integral into equation (5.16), changing the order of integration, and employing the relation (4.24) we deduce

$$\int_0^\infty \chi(\lambda) J_{-\alpha}(\lambda\sqrt{x})d\lambda = -x^{-\alpha/2}\cosh\pi\mu\, f(x), \quad 0 < x < \infty. \tag{5.18}$$

By applying Hankel inversion we find the function $\chi(\lambda)$,

$$\chi(\lambda) = -\frac{\lambda}{2}\cosh\pi\mu \int_0^\infty f(x)x^{-\alpha/2}J_{-\alpha}(\lambda\sqrt{x})dx. \tag{5.19}$$

We remark that the profile $g(x)$ of the stamp is assumed to be chosen such that the function $f(x) = -4G(\kappa+1)^{-1}g'(x)$ decays at infinity at the rate sufficient for the integral in (5.19) being convergent. It is directly verified that at the point $x = 0$ and at infinity, the solution (5.17) has the asymptotics required: it oscillates and $\varphi(x) = O(x^{-1/2})$, $x \to 0$, and $\varphi(x) = O(x^{-1/2})$, $x \to \infty$.

6. Quadrature formula for the Cauchy integral in a semi-infinite interval. In this section we obtain a quadrature formula for the Cauchy principal value of the singular integral

$$I^\alpha[f](x) = \frac{1}{\pi}\int_0^\infty \frac{f(t)w(t)dt}{t-x}, \quad w(t) = t^\alpha e^{-t},\ \alpha > -1,\ \alpha \neq 0, 1, \ldots,\ 0 < x < \infty. \tag{6.1}$$

THEOREM 6.1. Let $f(x)$ be Hölder-continuous in any finite interval $[0, a]$, $a > 0$, $|f(x)| \leq Ce^{\beta x}$, $x \to \infty$, $C = \text{const}$, $\beta < 1$, $w(t) = t^\alpha e^{-t}$, $\alpha > -1$, and $\alpha \neq 0, 1, \ldots$. Then

$$I^\alpha[f](x) = \sum_{m=1}^n \gamma_m f(x_m)\frac{Q_n^\alpha(x) - Q_n^\alpha(x_m)}{x - x_m} + R_n(x), \quad 0 < x < \infty, \quad x \neq x_m, \tag{6.2}$$

where

$$\gamma_m = -\frac{x_m}{(n+\alpha)L_{n-1}^\alpha(x_m)}, \tag{6.3}$$

x_m $(m = 1, 2, \ldots, n)$ are the zeros of the degree-n Laguerre polynomial $L_n^\alpha(x)$, and

$$Q_n^\alpha(x) = \frac{\Gamma(\alpha)}{\pi}e^{-x}\Phi(-n-\alpha, 1-\alpha; x) - \cot\pi\alpha\, x^\alpha e^{-x}L_n^\alpha(x). \tag{6.4}$$

For $x = x_j$,

$$I^\alpha[f](x_j) = \gamma_j f(x_j)\frac{dQ_n^\alpha(x_j)}{dx} + \sum_{m=1, m\neq j}^n \gamma_m f(x_m)\frac{Q_n^\alpha(x_j) - Q_n^\alpha(x_m)}{x_j - x_m} + R_n(x_j), \tag{6.5}$$

where

$$\frac{dQ_n^\alpha(x_j)}{dx} = \frac{1}{\pi}e^{-x_j}\Gamma(\alpha)\left[\frac{n+\alpha}{\alpha-1}\Phi(-n-\alpha+1, 2-\alpha; x_j) - \Phi(-n-\alpha, 1-\alpha; x_j)\right]$$
$$+ \cot\pi\alpha\, x_j^{\alpha-1}e^{-x_j}(n+\alpha)L_{n-1}^\alpha(x_j). \tag{6.6}$$

Formula (6.2) is exact, and $R_n(x) \equiv 0$ when $f(x)$ is a polynomial of degree not higher than $n-1$. Otherwise, if $f(x) = M_{n-1}(x) + r(x)$ and $M_{n-1}(x)$ is a polynomial of degree $n-1$, then the remainder $R_n(x)$ of the quadrature formula (6.2) is given by

$$R_n(x) = -\sum_{m=1}^{n} \frac{\gamma_m r(x_m)}{x - x_m}[Q_n^\alpha(x) - Q_n^\alpha(x_m)] + \frac{1}{\pi}\int_0^\infty \frac{r(t)w(t)dt}{t-x}. \tag{6.7}$$

Proof. For our derivations, we use the method [12] proposed for the Cauchy integral in the finite segment $(-1,1)$ and the integral relation (4.7) of Theorem 4.1. Introduce the system of orthonormal Laguerre polynomials,

$$p_j(t) = h_j^{-1/2} L_j^\alpha(t), \quad h_j = \frac{\Gamma(\alpha+j+1)}{j!}, \quad j = 0, 1, \dots, \tag{6.8}$$

and assume first that $f(x) = M_{n-1}(x)$ is a polynomial of degree $n-1$. It will be convenient to express it in terms of the polynomials $p_j(t)$,

$$f(t) = \sum_{j=0}^{n-1} f_j p_j(t), \tag{6.9}$$

where

$$f_j = \int_0^\infty f(t)p_j(t)w(t)dt, \quad j = 0, 1, \dots, n-1. \tag{6.10}$$

By using the Gauss quadrature formula exactly for polynomials of degree not higher than $2n-1$, we find

$$f_j = \sum_{m=1}^{n-1} A_m f(x_m) p_j(x_m) + \hat{R}_n(f), \quad j = 0, 1, \dots, n-1, \tag{6.11}$$

where x_m $(m = 1, 2, \dots, n)$ are the zeros of the Laguerre polynomial $L_n^\alpha(x)$, A_m are the Christoffel coefficients,

$$A_m = \frac{\Gamma(\alpha+n+1)}{n! x_m [\frac{d}{dx} L_n^\alpha(x_m)]^2}, \tag{6.12}$$

and $\hat{R}(f)$ is the reminder. Formula (6.11) is exact when $f(t) = M_{n-1}(t)$. Denote further that

$$q_j(x) = \int_0^\infty \frac{p_j(t)w(t)dt}{t-x}, \quad 0 < x < \infty. \tag{6.13}$$

Substitute the sum (6.9) into (6.1) and, in view of (6.11) and (6.12), obtain for the principal part of the integral (6.1)

$$I^\alpha[f](x) = \frac{1}{\pi}\sum_{m=1}^{n} A_m f(x_m) \sum_{j=0}^{n-1} p_j(x_m) q_j(x). \tag{6.14}$$

Next write the Christoffel–Daurboux formula ([22, (3.2.3), p. 43])

$$\sum_{j=0}^{n-1} p_j(x)p_j(t) = \frac{k_{n-1}}{k_n} \frac{p_n(x)p_{n-1}(t) - p_{n-1}(x)p_n(t)}{x-t}, \tag{6.15}$$

where $k_n = (-1)^n (n!\sqrt{h_n})^{-1}$. Integration of this identity with the weight $w(t)$ over the interval $(0, \infty)$ yields

$$p_{n-1}(x)q_n(x) - p_n(x)q_{n-1}(x) = \frac{k_n}{k_{n-1}}. \tag{6.16}$$

In particular,

$$q_n(x_m) = \frac{k_n}{k_{n-1}p_{n-1}(x_m)}. \tag{6.17}$$

Upon combining formulas (6.13), (6.15), and (6.16) it is possible to establish the following identity [12]:

$$\sum_{j=0}^{n-1} p_j(x)q_j(t) = \frac{k_{n-1}}{k_n(t-x)}[p_n(x)(q_{n-1}(x) - q_{n-1}(t)) - p_{n-1}(x)(q_n(x) - q_n(t))]. \tag{6.18}$$

Therefore, the internal sum in formula (6.14) transforms to

$$\sum_{j=0}^{n-1} p_j(x_m)q_j(x) = \frac{k_{n-1}p_{n-1}(x_m)}{k_n(x-x_m)}[q_n(x) - q_n(x_m)]. \tag{6.19}$$

Now, by substituting this expression into formula (6.14) we deduce

$$I^\alpha[f](x) = \frac{k_{n-1}}{\pi k_n} \sum_{m=1}^{n} A_m f(x_m) \frac{p_{n-1}(x_m)[q_n(x) - q_n(x_m)]}{x - x_m}. \tag{6.20}$$

Finally, since $Q_n^\alpha(x) = \pi^{-1}\sqrt{h_n}q_n(x)$ and

$$\frac{k_{n-1}A_m}{k_n\sqrt{h_n h_{n-1}}} = -\frac{x_m}{(n+\alpha)[L_{n-1}^\alpha(x_m)]^2}, \tag{6.21}$$

we derive the quadrature formula (6.2) for the singular integral (6.1). Here, we employed an alternative formula for the Christoffel coefficients (6.12)

$$A_m = \frac{\Gamma(\alpha+n)x_m}{n!(n+\alpha)[L_{n-1}^\alpha(x_m)]^2}. \tag{6.22}$$

For $x = x_m$, by the L'Hôpital's rule we transform formula (6.2) into the form (6.5) with $R_n(x) = 0$.

In the case $f(x) = M_{n-1}(x) + r(x)$ we derive the representation (6.7) from (6.1) and (6.2). $\square$

COROLLARY 6.2. Let $x = \xi_j$ be a zero of the function $Q_n^\alpha(x)$ given by (6.4). Then the quadrature formula (6.2) has the form

$$I^\alpha[f](\xi_j) = \frac{1}{\pi}\sum_{m=1}^{n} \frac{A_m f(x_m)}{x_m - \xi_j} + R_n(\xi_j), \tag{6.23}$$

where A_m are the Christoffel coefficients given by (6.22). This formula is exact for any polynomial $f(x)$ of degree $2n$.

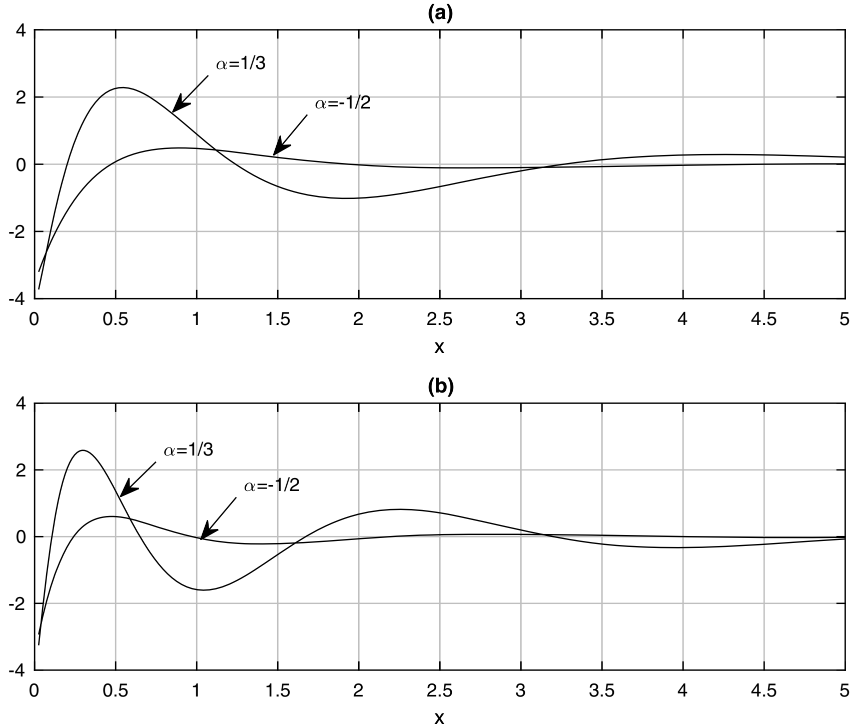

FIG. 4. The function $Q_n^\alpha(x)$ for $\alpha = 1/3$ and $\alpha = -1/2$. (a): $n = 5$. (b): $n = 10$.

Proof. As a consequence of the relation (6.17) we obtain

$$Q_n^\alpha(x_m) = -\frac{\Gamma(\alpha+n)}{\pi n! L_{n-1}^\alpha(x_m)}. \tag{6.24}$$

Putting $x = \xi_j$ in (6.2) and in view of $Q_n^\alpha(\xi_j) = 0$ we have formula (6.23). Since it is the Gauss quadrature formula in the interval $(0, \infty)$ associated with the Laguerre polynomials $L_n^\alpha(x)$, it is exact for any polynomial $f(x)/(x - \xi_j)$ of degree $2n - 1$. Therefore the last statement of Corollary 6.2 follows. $\square$

REMARK 6.3. Our numerical tests reveal that the function $Q_n^\alpha(x)$ has exactly $n + 1$ zeros in the interval $(0, \infty)$ for any values of the parameters n and $\alpha \in (-1, 0) \cup (0, 1)$ used. Sample curves of the function $Q_n^\alpha(x)$ for $n = 5$ and $n = 10$ when $\alpha = 1/3$ and $\alpha = -1/2$ are given in Figures 4(a) and 4(b). It is also found that the amplitude of the function $Q_n^\alpha(x)$ rapidly decreases as $x \to \infty$.

Considering that a continuous function cannot be uniformly approximated in the interval $[0, \infty)$ by a polynomial, it is infeasible to find an upper bound for the reminder in formula (6.5) in the form $|R_n(x)| < \delta_n$, $0 \le x < \infty$, $\delta_n \to 0$, $n \to \infty$, for the general class of functions employed in Theorem 6.1. However, it is possible to estimate $R_n(x)$ for functions decaying at infinity as x^{-m}, $m > 0$.

THEOREM 6.4. Let $f(x)$ be a continuously differentiable function in any finite segment $[0,a]$, $a>0$, and $f(x)\sim Cx^{-m}$, $x\to\infty$, $m>0$. For any $\varepsilon>0$ define a function

$$g_b(x)=\begin{cases} f'(x), & 0\le x\le b,\\ 0, & x>b,\end{cases}\qquad b>\sqrt[m+1]{\frac{m|C|}{\varepsilon}}. \tag{6.25}$$

Let $P'_{n-1}(x)$ be the polynomial of best approximation for the function $g_b(x)$ in the segment $[0,b]$ and

$$\max_{0\le x\le b}|g_b(x)-P'_{n-1}(x)|=e_n(b). \tag{6.26}$$

Then the remainder $R_n(x)$ of the quadrature formula (6.2) is estimated by

$$|R_n(x)|\le 2\left(\frac{2\Gamma(\alpha+1)}{\pi}+|Q_n^\alpha(x)|\sum_{m=1}^n|\gamma_m|\right)\tilde e_n(b)$$

$$\le 2\Gamma(\alpha+1)\left(\frac{2}{\pi}+\frac{n!|Q_n^\alpha(x)|}{\Gamma(n+\alpha)}\max_{1\le m\le n}|L_{n-1}^\alpha(x_m)|\right)\tilde e_n(b),\quad 0\le x<\infty, \tag{6.27}$$

where $\tilde e_n(b)=\max\{\varepsilon,e_n(b)\}$. If $x=\xi_j$, then the remainder has the bound

$$|R_n(\xi_j)|\le\frac{4\Gamma(\alpha+1)}{\pi}\tilde e_n(b). \tag{6.28}$$

Proof. Similar to [12], rewrite formula (6.7) for the remainder in the form

$$R_n(x)=\sum_{m=1}^n\frac{\gamma_m[r(x)-r(x_m)]}{x-x_m}[Q_n^\alpha(x)-Q_n^\alpha(x_m)]+\frac{1}{\pi}\int_0^\infty\frac{r(t)-r(x)}{t-x}w(t)dt. \tag{6.29}$$

By recalling that

$$\gamma_mQ_n^\alpha(x_m)=\frac{A_m}{\pi}, \tag{6.30}$$

we arrive at the formula

$$R_n(x)=\sum_{m=1}^n\left[\gamma_mQ_n^\alpha(x)-\frac{A_m}{\pi}\right]\frac{r(x)-r(x_m)}{x-x_m}+\frac{1}{\pi}\int_0^\infty\frac{r(t)-r(x)}{t-x}w(t)dt. \tag{6.31}$$

Now, if we take into account formula (6.26), the inequalities

$$|r'(x)|<|f'(x)-g_b(x)|+|g_b(x)-P'_{n-1}(x)|<\varepsilon+e_n(b),\quad 0\le x<\infty, \tag{6.32}$$

and also the integral

$$\int_0^\infty w(t)dt=\Gamma(\alpha+1), \tag{6.33}$$

we deduce the first bound in (6.27). To derive the second bound, we notice that $\gamma_mQ_n^\alpha(x_m)>0$,

$$\sum_{m=1}^n\gamma_mQ_n^\alpha(x_m)=\frac{1}{\pi}\int_0^\infty w(t)dt, \tag{6.34}$$

and that, due to formula (6.17),

$$\frac{1}{Q_n^\alpha(x_m)}=\frac{\pi n!}{\Gamma(n+\alpha)}L_{n-1}^\alpha(x_m). \tag{6.35}$$

Since $Q_n^\alpha(\xi_j)=0$, the bound (6.28) immediately follows from (6.27). □

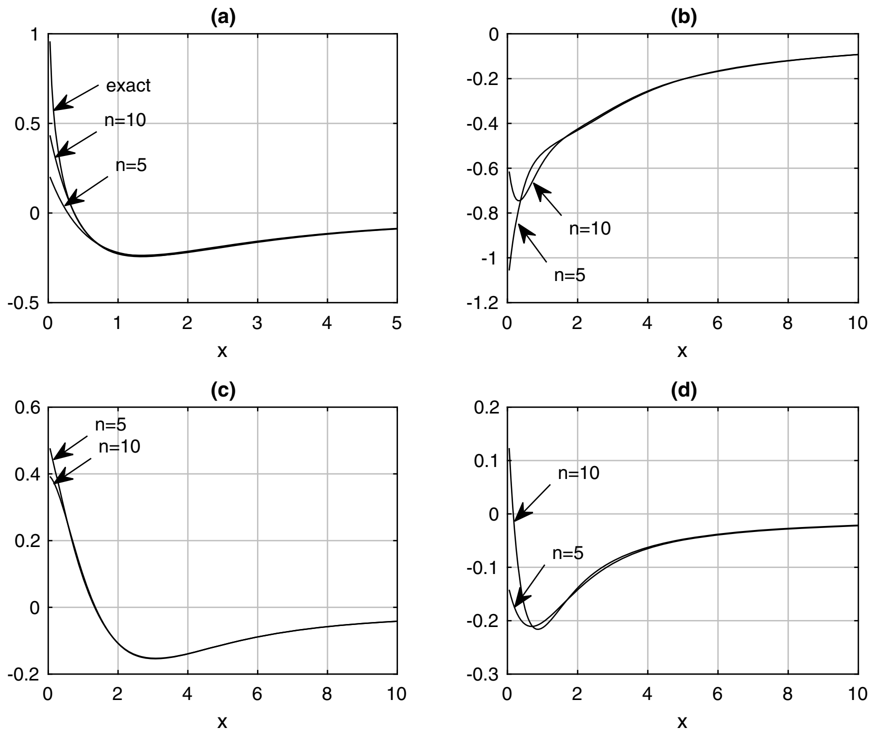

FIG. 5. The integral $I^\alpha[f](x)$ for $\alpha=-1/2$ when the number of zeros is $n=5$ and $n=10$. (a): $f(x)=\sqrt{x}$ (in this case the integral is expressed through the exponential integral $\mathrm{Ei}(x)$). (b): $f(x)=e^{x/2}$. (c): $f(x)=x\sqrt{x}$. (d): $f(x)=\sqrt{x}/(x+1)$.

The numerical tests implemented confirm numerical efficiency of the quadrature formula (6.2). Sample curves for the integral $I^\alpha[f](x)$ for $\alpha=-1/2$ when the degree of the Laguerre polynomial $L_n^{-1/2}$is $n=5$ and $n=10$ are shown in Figures 5(a)–5(d). In the case of Figure 5(a), the integral can be evaluated exactly,

$$I^{-1/2}[\sqrt{t}](x)=\frac{1}{\pi}\int_0^\infty\frac{e^{-t}dt}{t-x}=-\frac{e^{-x}}{\pi}\mathrm{Ei}(x),\quad 0<x<\infty, \tag{6.36}$$

where

$$\mathrm{Ei}(x)=\gamma+\ln x+\sum_{m=1}^\infty\frac{x^m}{mm!} \tag{6.37}$$

is the exponential integral and $\gamma\approx 0.57721566$ is the Euler constant.

7. Conclusions. We showed that the Hilbert transforms of the weighted Hermite polynomials $\exp(-x^2/2)H_n(x)$, the functions $G_n(x)$, form a complete orthogonal system of functions in the space $L_2(-\infty,\infty)$. We also discovered that the Hilbert transforms in a semi-axis of the weighted Laguerre polynomials $e^{-\eta/2}\eta^{-1/2}L_n^{-1/2}(x)$ and $e^{-\eta/2}\eta^{1/2}L_n^{-1/2}(x)$ up to constant factors equal the functions $\xi^{-1/2}G_{2m}(\sqrt{\xi})$ and

$G_{2m+1}(\sqrt{\xi})$, respectively. The system of functions $G_{2n}(x)$ and G_{2n+1} may be employed for solving singular integral equations or their systems of the form $S[\chi(t)](x) + K[\chi(t)](x) = f(x)$, $0 < x < \infty$, in the class of integrable functions unbounded and bounded at the point $x = 0$, respectively. Here, S is a singular operator with the Cauchy kernel and K is a regular operator. The method ultimately reduces the integral equations to systems of linear algebraic equations of the second kind. If these systems are regular or at least quasiregular, then they can be solved numerically by the reduction method. The sufficiency of this scheme needs to be verified by means of numerical tests. If $K = 0$, then the solution is exact and its representation is free of singular integrals. This method might also be employed for vector Riemann–Hilbert problems when the Wiener–Hopf factors are not available, and the associated system of integral equations has the structure $S[\chi(t)](x) + K[\chi(t)](x) = f(x)$, $0 < x < \infty$.

By employing an integral representation of the Jacobi function of the second kind $Q_n^{(\alpha,\beta)}(x)$, expressing it in terms of the hypergeometric Gauss function, and passing to the limit $n \to \infty$ in the representations for $n^{-\alpha}Q_n^{(\alpha,\beta)}(1 - \frac{1}{2}z^2/n^2)$, we obtained the semi-infinite Hilbert transforms of the Bessel function $J_\alpha(\lambda\sqrt{t})$ in terms of the functions $J_{\pm\alpha}(\lambda\sqrt{x})$ and $I_{\pm\alpha}(\lambda\sqrt{x})$ in the intervals $0 < x < \infty$ and $-\infty < x < 0$, respectively. Here, λ is a positive parameter. We applied this result to derive a closed-form solution to a model problem of contact mechanics. The solution is free of singular integrals, and the associated Riemann–Hilbert problem, as the standard way of dealing with such problems, was bypassed.

One of the most frequently applied methods for singular integral equations with the Cauchy kernel is the collocation method. To employ it for singular integral equations in a semi-axis, one needs to make the optimal choice of the collocation points and have at their disposal an efficient procedure for the Cauchy integral in a semi-axis. To find such a formula, we proposed to use the Hilbert transform of the weighted Laguerre polynomial $x^\alpha e^{-x}L_n^\alpha(x)$ derived in the paper, the Gauss quadrature formula for the integral $\int_0^\infty x^\alpha e^{-x}f(x)dx$ exact for polynomials of degree not higher than $2n - 1$, and the Christoffel–Daurboux formula for the Laguerre polynomials. The quadrature formula for the singular integral with the Cauchy kernel $1/(t - x)$ in a semi-axis with the density $t^\alpha e^{-t}f(t)$ is exact for any plynomial $f(t)$ of degree not higher than $n - 1$ and requires computing the Laguerre polynomials $L_n^\alpha(x)$ and $L_{n-1}^\alpha(x_m)$ and the confluent hypergeometric function $\Phi(-n - \alpha, 1 - \alpha; t)$ at the points $t = x$ and $t = x_m$, where x_m $(m = 1, 2, \dots, n)$ are the n zeros of the polynomial $L_n^\alpha(x)$. The numerical tests proved efficiency of the quadrature formula.

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