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DUALITY RELATIONS IN THE THEORY OF ANALYTIC CAPACITY

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Abstract. This is a survey of duality relations arising in the theory of analytic capacity and its modifications, namely, the Cauchy capacities with various types of measures. Principal attention is paid to the material not published earlier. Also, new modifications of the capacities mentioned above are treated. Linear extremal problems (such as the problem of calculating the analytic capacity of a set) are dual to approximation processes with size constraints (the size of the approximants and the rate of approximation are measured in terms of different metrics). For the new versions of capacity introduced in this paper, approximation with size constraints is extended to the case where the approximants are taken from a fixed conical wedge (rather than from a linear subspace, as it has always been before). Extremely peculiar approximation processes arise via duality from the modifications of the analytic capacity the definition of which involves positive measures (a typical example is the so-called "positive" analytic capacity γ^+). More precisely, in such situations it is not even required to find an approximant within a small distance from a given element. Instead, in a fixed subspace or conical wedge we seek an addition that brings the above element to a fixed cone. Moreover, this addition must be as small as possible in a certain sense.

Extension of the collection of relations for the analytic capacity and its modifications, and comparison of various relations of this sort make it possible to better understand exceptional sets arising in various branches of holomorphic function theory. In particular, some information is obtained about possible approximation processes on null-sets in the sense of a particular capacity.

Contents

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§0. Introduction

Notation. Throughout the paper, we use the following notation: $\mathbb{C}$ is the complex plane; $\overline{\mathbb{C}}$ is the extended complex plane; G is a domain in $\overline{\mathbb{C}}$ with $\infty \in G$; $F = \overline{\mathbb{C}} \setminus G$; $B(G)$ is the class of single-valued holomorphic functions in G; $B^1(G)$ is the subclass of $B(G)$ consisting of functions $f(z)$ with

$$|f(z)| \le 1, \ \ z \in G; \quad f(\infty) = 0; \tag{0.1}$$

μ or $d\mu$ is a measure (in general, complex-valued); $|\mu|$, $|d\mu|$, or $d|\mu|$ is the total variation measure for μ; S_μ is the closed support of μ; $\int_{S_\mu} |d\mu| = \|\mu\|$; $\widehat{\mu}(z)$ is the Cauchy transform of μ (Cauchy potential for μ):

$$\widehat{\mu}(z) = \int_{S_\mu} \frac{d\mu}{\zeta - z}, \quad z \in \overline{\mathbb{C}} \setminus S_\mu. \tag{0.2}$$

We only consider finite regular Borel measures μ such that S_μ is compact. The symbol $\overline{A}$ stands for the closure of A in the space where A lies. Further notation is explained as the need arises.

Analytic capacity. The analytic capacity of a compact set F is the quantity

$$\gamma(F) = \sup |f'(\infty)|_{f \in B^1(G)} = \sup_{f \in B^1(G)} \lim_{|z| \to \infty} |zf(z)|. \tag{0.3}$$

This quantity was introduced by Ahlfors in the paper [1], which suggested for the first time the use of similar set functions to describe removable singularities of analytic functions in various classes (and in the first place, in $B(G)$). The problem of describing the sets of removable singularities for $B(G)$ was posed as early as at the end of the 19th century by P. Painlevé. In accordance with V. D. Erokhin's proposal (1958), the quantity $\gamma(F)$ has been called the *analytic capacity* or the *Ahlfors capacity* since that time. The extremal function in (0.3) (it exists because $B^1(G)$ is compact) is called the *Ahlfors function*. This function is unique up to a constant multiple of the form $e^{i\alpha}$. In [1], problem (0.3) was studied for a finitely connected domain G. The further study of problem (0.3) has been the subject of many papers. The corresponding results and references can be found in [2]–[8].

Analytic capacity and approximation questions. The notion of the analytic capacity became quite popular after the work of A. G. Vitushkin, who used it to solve the main problems of uniform approximation by rational functions on subsets of the complex plane. These results and their applications were based on certain important properties (established by A. G. Vitushkin and M. S. Mel′nikov) of the analytic capacity and of its analog for holomorphic functions continuous up to the boundary. See, e.g., [2], [4]–[6], [9]–[11] for expositions (detailed to a variable extent) of these studies.

Cauchy capacity. Along with the analytic capacity (0.3), the quantities $\gamma^c(F)$, $\gamma^R(F)$, and $\gamma^+(F)$ are considered. The first is the *Cauchy capacity* defined as follows:

$$\gamma^c(F) = \sup_\mu \left| \int_F d\mu \right| = \sup_\mu |\widehat{\mu}'(\infty)| : S_\mu \subset F, \quad \widehat{\mu}(z) \in B^1(G). \tag{0.4}$$

Clearly, $\gamma^c(F) \le \gamma(F)$. It is still an open question whether $\gamma^c(F) = \gamma(F)$. This identity has been proved only for the compact sets F of quite a special structure. In (0.4), the supremum is taken over the complex-valued measures μ. The quantities $\gamma^R(F)$ and $\gamma^+(F)$ are also defined by (0.4) with the difference that the supremum is taken over the real measures in the case of γ^R, and over the positive measures in the case of γ^+. It should be noted that, apparently, for the first time the term "the Cauchy capacity" was used in my lecture [12].

New developments. In recent years, remarkably intense investigations into the analytic capacity problems have been witnessed. I mention the innovative paper [13] by Mel′nikov, in which the notion of the curvature of a measure was introduced. This notion has turned out to be most fruitful. The paper [13] served as the origin for the subsequent work of many authors. Fusion of these ideas with Calderón's earlier results [14] on singular integrals (in particular, the Cauchy integral), and further development of this approach have led to the solution of several important problems about the analytic capacity. In particular, G. David described the sets F with $\gamma(F) = 0$ in geometric terms (yet, this was done under the additional assumption that the Hausdorff 1-measure of F is finite). This yielded a "near-solution" of the Painlevé problem on the removable singularities for $B(G)$. The geometric description mentioned above corroborated an old conjecture by Vitushkin. A survey of this material, and an extensive bibliography can be found in the paper [15] by David. Another problem, posed and reemphasized repeatedly by Vitushkin, is about the semiadditivity of the analytic capacity: is it true that for any compact sets F_1 and F_2 we have

$$\gamma(F_1 \cup F_2) \le \gamma(F_1) + \gamma(F_2)$$

or at least

$$\gamma(F_1 \cup F_2) \le c[\gamma(F_1) + \gamma(F_2)] \tag{0.5}$$

with some universal constant c? In the thesis [16], Mel′nikov's student X. Tolsa proved among other things that

$$\gamma^+(F_1 \cup F_2) \le \gamma^+(F_1) + \gamma^+(F_2), \tag{0.6}$$

and in [17] he established the remarkable relation

$$\gamma^+(F) \ge d \cdot \gamma(F), \tag{0.7}$$

where $d > 0$ is a universal constant. Therefore, γ, γ^c, γ^R, and γ^+ are equivalent characteristics of a set F, and (0.5) is a consequence of (0.7) and (0.6). (However, the question as to whether $\gamma(F) = \gamma^c(F)$ remains open.)

The subject of the paper. Duality relations. The present paper pertains to another direction in the study of capacities. We intend to obtain alternative, dual formulas for the analytic capacity and its modifications. In [18], [19], Garabedian started the exploration of the problem dual to (0.3). For a finitely connected domain with analytic boundary, he proved the formula

$$\gamma(F) = \min \frac{1}{2\pi} \int_{\partial G} |1 + \varphi(\zeta)|\,|d\zeta|, \tag{0.8}$$

where the minimum (it is attained indeed) is taken over all functions $\varphi(z)$ analytic in the closed domain $\overline{G}$ and satisfying $\varphi(\infty) = 0$. If $\varphi^*(z)$ is an extremal function for (0.8), then the function $L(z) = 1 + \varphi^*(z)$ is called the Garabedian function. The latter is closely related to the Szegő kernel, which is linked with systems of functions analytic in G and orthogonal on ∂G. The properties of the Garabedian function in arbitrary domains and various applications of it were studied in [20]–[30]. Naturally, formula (0.8) may fail for an arbitrary domain; however, the function $L(z)$ still plays an important part in the analytic capacity theory. For an arbitrary domain G, we shall need only the following properties of this function:

$$L(z) \neq 0 \text{ (and even } \ln L(z) \text{ is single-valued)}, \quad L(\infty) = 1. \tag{0.9}$$

For G arbitrary, the author obtained a different duality relation in [20]–[23], [30]. Let $\{a_i\}$ be a countable dense subset of G. Then

$$\gamma(F) = \inf \varliminf_n \sum_1^n |\nu_i||L(a_i)|, \tag{0.10}$$

where the infimum is taken over all functions $R_n(\zeta)$ of the form

$$R_n(\zeta) = \sum_1^n \nu_i (a_i - \zeta)^{-1} \tag{0.11}$$

with

$$R_n(\zeta) \to 1 \quad \text{uniformly on } F. \tag{0.12}$$

Khavin [24], [25] found yet another dual description for $\gamma(F)$:

$$\gamma(F) = \inf_\nu \int_{S_\nu} |d\nu| = \inf_\nu \{\|\nu\| : S_\nu \subset G\}, \quad \widehat{\nu}(\zeta) \equiv 1, \ \zeta \in F. \tag{0.13}$$

In (0.11)–(0.12) we approximate the constant function equal to 1, while in (0.13) we have the equality. In (0.10) we consider a weighted sum of the variations of some point masses at the points a_i (the weights are determined by the function $L(z)$), whereas there is no weights in (0.13). It has also been known that if the weights $|L(a_i)|$ are discarded in (0.10), then we arrive at the formula

$$\gamma^c(F) = \inf \varliminf_n \sum_1^n |\nu_i|, \tag{0.14}$$

where again the infimum is taken over all functions $R_n(\zeta)$ as in (0.11)–(0.12). However, it has already been mentioned that it is still unknown whether $\gamma(F) = \gamma^c(F)$. The reasons of the above "dissimilarity" between (0.10) and (0.13) have remained unclear for a long time.

Approximation with size constraints. The basis for the proofs of relations like (0.10)–(0.12) is provided by the theory of approximation with size constraints. In this theory, the size of an approximant and the deviation of it from the element to be approximated are measured in different metrics. The study of such problems started in the paper [31] by Ky Fan and Davis, where some special cases of such approximation were treated. In the author's papers [32]–[35], [30], a general theory of such approximation processes was developed, and various applications were presented, in particular, to the analytic capacity problems; in [35] there is also an extensive bibliography. The starting point for the treatment of such problems is duality between best approximations and the extrema of some linear functionals on the elements to be approximated. This duality was established in the papers [36] by Kreĭn and [37] by Nikol′skiĭ with the use of the Hahn–Banach theorem. In [38], Garkavi extended the duality relations to the case of approximation by elements of a convex set in a normed space. Duality questions are a major part of the material presented in the monographs [39]–[41], and the references [42]–[43] are devoted entirely to this topic. In [8, §2] new aspects of the theory were treated, namely, the problem of "placing" a vector to a certain cone: this must be done using shifts by elements of the approximating subspace; these elements should be as small as possible in a certain metric. Processes of this sort have turned out to be dual to linear extremal problems in which positive measures are sorted out. In particular, this refers to the study of the "positive" analytic capacity $\gamma^+(F)$. In all the papers mentioned, the role of the approximating set was played by a linear subspace. New modifications of the analytic capacity introduced in Chapter II of the present paper have required a generalization of approximation with size constraints to the case where the approximants are taken from a conical wedge.

The content of the paper. In §1, we present duality relations for approximation by elements of a conical wedge in a linear space. The deviation from the vector to be approximated is measured with the help of a convex functional r, whereas the size of approximants is controlled by another convex functional p (r and p may fail to be symmetric). In particular, we treat a specific way of "placing" a vector in a cone with the help of the shift by an element of a conical wedge. The formulas of §1 are given without proofs, which will appear elsewhere; here we concentrate on applications to the analytic capacity. In the other sections of Chapter 1, we consider the special case where the approximating set is a subspace rather than a conical wedge. Full-scale applications of the formulas of §1 are given in Chapter II.

§2 is devoted to new relations among γ, γ^+, and γ^R. It should be noted that in the paper [8] (where the problems dual to linear extremal problems, like that of calculating the Cauchy capacity with real or positive measures, were considered for the first time) the conditions

$$|\operatorname{Re} f(z)| \leq 1, \quad |\operatorname{Im} f(z)| \leq 1 \tag{0.15}$$

were imposed instead of the condition $f(z) \in B^1(G)$, as in (0.3) and (0.4). In our case of the Cauchy potentials (in [8] more general representations of functions were employed, namely, the Golubev sums), this leads to the quantities

$$\begin{aligned} \beta^R(F) &= \sup\left\{\left|\int_F d\mu\right| : S_\mu \subset F\right\}, \quad \mu \text{ is real;} \\ \beta^+(F) &= \sup\left\{\int_F d\mu : S_\mu \subset F\right\}, \quad \mu \geq 0, \end{aligned} \tag{0.16}$$

where the supremum is taken over all f such that $f(z) - \widehat{\mu}(z)$ satisfies (0.15). Clearly,

$$\frac{1}{\sqrt{2}}\beta^R(F) \leq \gamma^R(F) \leq \beta^R(F), \quad \frac{1}{\sqrt{2}}\beta^+(F) \leq \gamma^+(F) \leq \beta^+(F). \tag{0.17}$$

The passage to β^R and β^+ was caused by the purely real nature of the method. In the present paper, this difficulty is bypassed: we deal directly with γ^R and γ^+. In §3 it is shown that, in the Khavin formula (0.13), the variations of the measures $d\nu$ can be replaced by the weighted variations $|\Phi(z)d\nu|$, where $\Phi(z)$ is an arbitrary analytic function in G that satisfies

$$\Phi(z) \neq 0, \ \ z \in G; \quad \Phi(\infty) = 1. \tag{0.18}$$

In particular, the Garabedian function $L(z)$ fits, because it satisfies (0.18) (see (0.9)). This eliminates the dissimilarity between the duality relation (0.10) and (0.13). At the end of §3, we present a table of duality relations for γ, γ^c, γ^+, and γ^R.

In §4, the extremal problem of calculating $\gamma^+(F)$ is studied thoroughly in the interplay with the dual problem. In §5 we present several new facts about the structure of some linear functionals whose extremal values describe the properties to be empty or nonempty for certain classes of analytic functions, in the same way as the corresponding capacities do. The study of such functionals was started by Khavin [25] and continued by Val′skiĭ [26].

In Chapter II we study new modifications of the notion of analytic capacity. It would have been natural to call them "angular capacities" because some angle (in the plane where the functions in question have their values) is involved in the definition. Here the content of §1 is used at the full-scale level. The capacities under consideration are also "weighted": some weight $h(z)$, i.e., an arbitrary positive continuous function on the domain G, is involved. Let Δ be a convex angle in the w-plane with vertex at $w = 0$ and with the positive real semiaxis as a bisector. (By an *angle* we mean the open domain between the angle sides.) In the space $C(F)$ of continuous complex-valued functions on F, we consider two cones E_Δ and E^1_Δ:

$$(0.19)\qquad \begin{aligned} E_\Delta &= \{y(\zeta) \text{ is analytic on } F, \text{ and } y(F) \subset \Delta\}; \\ E^1_\Delta &= \Big\{ y(\zeta) = \sum_1^n \frac{\nu_i}{\zeta - a_i} : n \geq 1, a_i \in G, \nu_i \in \mathbb{C}, y(F) \subset \Delta \Big\}. \end{aligned}$$

On E_Δ and E^1_Δ (respectively), we define seminorms $P_h(y)$ and $p_h(y)$ by putting

$$(0.20)\qquad \begin{aligned} P_h(y) &= \inf_\nu \Big\{ \int_{S_\nu} h(z)\,|d\nu| : S_\nu \subset G \Big\}, \quad \widehat{\nu}(\zeta) \equiv y(\zeta), \ \ \zeta \in F; \\ p_h(y) &= \inf \sum_1^n h(a_i)|\nu_i|. \end{aligned}$$

In the latter formula, the infimum is taken over all representations of $y(\zeta) \in E^1_\Delta$ as a sum of elementary fractions, as in (0.19). In §6 we study approximation of functions in $C(F)$ by elements of the cones E_Δ and E^1_Δ, with constraints on the size of approximants (the size is measured via the seminorms (0.20)). The duality relations that we obtain lead to new modifications of the analytic capacity. Let us view $C(F)$ as a linear space over the reals. Then an arbitrary linear functional on $C(F)$ is represented by a pair (μ_1, μ_2) of real measures. On the function $u(\zeta) + iv(\zeta)$, this functional takes the value

$$(0.21)\qquad \int u\,d\mu_1 - v\,d\mu_2.$$

In §7 (see Theorem 7.3) it is proved that the functional (0.21) takes positive values on the cone E_Δ if and only if

$$(0.22)\qquad \text{the measure } \mu_1 - \tan\frac{|\Delta|}{2}|\mu_2| \text{ is positive,}$$

where $|\Delta|$ is the opening of Δ. Also, a necessary and sufficient condition is established for an analytic function G to be equal to the Cauchy potential $\widehat{\mu}(z)$ of the measure $\mu_1 + i\mu_2$ for some pair (μ_1, μ_2) of measures satisfying (0.22). In §8, the following capacities are introduced:

$$(0.23)\qquad \gamma^+_{h,\Delta}(F) = \sup \int_F d\mu_1,$$

where the supremum is taken over the measures $\mu = \mu_1 + i\mu_2$ satisfying (0.22) and such that

$$(0.24)\qquad |\widehat{\mu}(z)| \leq h(z), \quad z \in G.$$

If $|\Delta| = \pi$ and $h(z) \equiv 1$, we obtain $\gamma^+_{1,\pi}(F) = \gamma^+(F)$ and $\gamma^+_{|L|,\pi}(F) = \gamma(F)$. We prove the duality theorem

$$(0.25)\qquad \gamma^+_{h,\Delta}(F) = \inf_\nu \Big\{ \int_{S_\nu} |h|\,|d\nu| : S_\nu \subset G \Big\}, \quad \widehat{\nu}(\zeta) \in E_\Delta, \ \operatorname{Re}\widehat{\nu}(\zeta) \geq 1, \ \zeta \in F.$$

This formula generalizes the relations for $\gamma^+(F)$ and $\gamma(F)$ proved in §2. In §9 an "equimeasurability" theorem is presented: for a totally disconnected compact set F, the existence of a measure $\mu \neq 0$ satisfying (0.24) implies that $\gamma^+_{h,\Delta}(F) > 0$ for $0 < |\Delta| < \pi$ (however, some restrictions must be imposed on $h(z)$ and the structure of μ).

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Chapter 1. New relations for the analytic capacity and Cauchy capacities

§1. Approximation with size constraints

In this section, we present some results on approximation by elements of a conical wedge with size constraints imposed on the approximants. The detailed proofs of these and related results will appear elsewhere.

Let X be a linear space over the reals. We denote by X' the space of linear functionals on X. Consider some nonnegative convex functional $r(x)$ on X:

$$r(x) \geq 0, \quad r(x_1 + x_2) \leq r(x_1) + r(x_2), \quad r(\alpha x) = \alpha r(x) \text{ for all } \alpha \geq 0. \tag{1.1}$$

In general, it may happen that $r(-x) \neq r(x)$, i.e., r may fail to be symmetric. We introduce the following notation:

$$\begin{cases} R(r, M) = \{l \in X' : l(x) \leq M r(x) \text{ for all } x \in X\}, \quad M \geq 0 \text{ is a constant;} \\ R(r) = \bigcup_{M \geq 0} R(r, M); \\ \|l\| = \inf\{M : l \in R(r, M)\} \quad (\text{for } l \in R(r)). \end{cases} \tag{1.2}$$

Let Y be a linear subspace in X; suppose that some nonnegative convex functional $p(y)$ is defined on X. (Again, p may fail to be symmetric.) We put

$$R(p, M) = \{l \in X' : l(y) \leq M p(y) \text{ for all } y \in Y\}, \quad R(p) = \bigcup_{M \geq 0} R(p, M)$$

(these formulas are similar to (1.2); however, the functionals are still taken from X' rather than Y'). Suppose some conical wedge E is given in Y. We recall that a convex set E is said to be a *conical wedge* if $\alpha E \subset E$ for $\alpha \geq 0$. A conical wedge E is called a *cone* if $E \cap (-E) = \{0\}$ (see [44]). If E is a conical wedge, we consider the following sets of linear functionals:

$$\begin{cases} E^+ = \{l \in X',\, l(x) \geq 0 \text{ for } x \in E\}, \\ E^- = \{l \in X',\, l(x) \leq 0 \text{ for } x \in E\}. \end{cases} \tag{1.3}$$

If E is a linear subspace, we have

$$E^+ = E^- = E^\perp, \tag{1.4}$$

where $E^\perp$ is the annihilator of E, i.e., the set of all linear functionals vanishing on E. Consider the following approximation problem with size constraints imposed on the approximants:

$$d = \inf_{y \in E} [r(\omega - y) + p(y)], \quad \text{where } \omega \in X. \tag{1.5}$$

Theorem 1.1 (Duality for problem (1.5))**.** *We have*

$$d = \max l(\omega), \tag{1.6}$$

where the maximum is taken over all linear functionals $l \in R(r,1)$ for which there exists a functional $\lambda \in R(p,1)$ with the property

$$l - \lambda \in E^{-}. \tag{1.7}$$

If $E = Y$ is a linear subspace of X, then

$$d = \max\{l(\omega) : l \in R(r,1) \cap R(p,1)\}. \tag{1.8}$$

In (1.6) and (1.8) we write max in place of sup, because the extremal value is attained. For $\omega \in X$ we put

$$\tilde{p}(\omega) = \inf \lim_{k\to\infty} p(y_k), \tag{1.9}$$

where the infimum is taken over all sequences $\{y_k\} \subset E$ such that

$$r(\omega - y_k) \to 0, \quad k \to \infty. \tag{1.10}$$

If there are no such sequences, we put

$$\tilde{p}(\omega) = \infty. \tag{1.11}$$

Surely, it may happen that $\tilde{p}(\omega) = \infty$ even if there exist sequences $\{y_k\} \subset E$ that approximate ω as in (1.10). Also, we introduce the following sets of linear functionals $l \in X'$:

$$\begin{cases} R(M); M \geq 0 : \\ l \in R(M) \iff l \in R(r,M) \text{ and } \exists \lambda \in R(p,1) : l - \lambda \in E^{-}; \\ R = \bigcup_{M \geq 0} R(M). \end{cases} \tag{1.12}$$

Thus

$$l \in R \iff l \in R(r) \text{ and } \exists \lambda \in R(p,1) : l - \lambda \in E^{-}. \tag{1.12$'$}$$

If $E = Y$ is a linear subspace of X, the definitions of $R(M)$ and R simplify to

$$R(M) = R(r,M) \cap R(p,1), \quad R = R(r) \cap R(p,1). \tag{1.13}$$

Theorem 1.2 (The main duality theorem)**.** *We have*

$$\tilde{p}(\omega) = \sup_{l \in R} l(\omega). \tag{1.14}$$

If $\tilde{p}(\omega) = 0$, we sometimes say that *the element ω is $(r, o(p))$-approximable by the conical wedge E (the subspace Y)*. If r is a norm in a normed space X, then we simply say that *ω is $o(p)$-approximable* if $\tilde{p}(\omega) = 0$. Now we consider convex functionals $r(x)$ related to cones. Concerning the notions of the support (or Minkowski) functional for a set and a C-interior point of a set, see [44] and [45, p. 445].

Lemma 1.3. *Let K be a cone in a vector space X over the reals. Suppose K contains a C-interior point x_0, and put*

$$K_0 = K - x_0. \tag{1.15}$$

Let $r_0(x)$ denote the Minkowski functional for K_0. Then $r_0(x)$ is a nonnegative functional on X, and

$$r_0(x) = 0 \text{ for } x \in K, \quad r_0(-x_0) = 1. \tag{1.16}$$

Suppose

$$r(x) = r_0(-x). \tag{1.17}$$

Then

$$K^+ = R(r), \quad R(r, M) = \{l : l \in K^+, l(x_0) \le M\}. \tag{1.18}$$

The first statements of the lemma are well known. In a slightly less general form, relations (1.18) were proved in [8, Lemma 2.17] (there it was assumed that X is locally convex and x_0 is an interior point for K). The definition of the support functional implies that

$$r(\omega - x) = r_0(-\omega + x) = \inf\{t : t > 0, -\omega + x + tx_0 \in K\}. \tag{1.19}$$

Again, we define $\tilde{p}(\omega)$ by (1.9)–(1.11). Now, by (1.19), the infimum in (1.9) is taken over the sequences $\{y_k\} \subset E$ such that there exists a nonnegative sequence $\{t_k\}$, $t_k \to 0$, with

$$-\omega + y_k + t_k x_0 \in K. \tag{1.20}$$

The approximation problem with the smallest possible p-size of approximants transforms to the problem of "placing" the element $-\omega$ into K by adding to it elements $\{y_k\} \subset E$ with possibly small $p(y_k)$. Theorem 1.2 reshapes as follows.

Theorem 1.4. *Let X be a linear space over the reals, Y a linear subspace of X, $p(y) \ge 0$ a convex functional on Y, $E \subset Y$ a conical wedge, and K a cone in X containing a C-interior point x_0. For $\omega \in X$, let $\tilde{p}(\omega)$ be defined by* (1.9) *and* (1.20) *or* (1.11)*. Then*

$$\tilde{p}(\omega) = \sup l(\omega); \tag{1.21}$$

here the supremum is taken over the linear functionals l satisfying

$$l \in K^+ \quad \text{and} \quad \exists \lambda \in R(p,1) : l - \lambda \in E^-. \tag{1.22}$$

If $E = Y$ is a linear space, (1.21) *takes the form*

$$\tilde{p}(\omega) = \sup\{l(\omega) : l \in K^+ \cap R(p,1)\}. \tag{1.23}$$

§2. New duality relations for the analytic capacity γ and its modifications γ^R and γ^+

We already mentioned in the Introduction that γ^R and γ^+ are defined by formulas similar to (0.4) for γ^c but with other kinds of measures:

$$\gamma^R(F) = \sup\left\{\left|\int d\mu\right| : S_\mu \subset F, \mu \text{ is real}, |\widehat{\mu}(z)| \le 1, z \in G\right\}, \tag{2.1}$$

$$\gamma^+(F) = \sup\left\{\int d\mu : S_\mu \subset F, \mu \ge 0, |\widehat{\mu}(z)| \le 1, z \in G\right\}. \tag{2.2}$$

Theorem 2.1. *The following formulas are true*:

$$\gamma^R(F) = \inf \varliminf_{k\to\infty} \sum_1^{n_k} |\nu_i|, \tag{2.3}$$

where the infimum is taken over all sums (0.11) *of elementary fractions such that*

$$\operatorname{Re} \sum_1^{n_k} \nu_i (a_i - \zeta)^{-1} \to 1 \quad \textit{in the metric of } C(F); \tag{2.4}$$

$$\gamma^+(F) = \inf \sum_1^{n_k} |\nu_i|, \tag{2.5}$$

where the infimum is taken over all sums of the form (0.11) *for which*

$$\operatorname{Re}\sum_{1}^{n}\nu_i(a_i-\zeta)^{-1}\geq 1,\quad \zeta\in F\text{ , } n \textit{ is arbitrary;} \tag{2.6}$$

$$\gamma^{+}(F)=\inf_{\nu}\int_{S_\nu}|d\nu|,\quad S_\nu\subset G, \tag{2.7}$$

where the infimum is taken over all complex measures μ *such that*

$$\operatorname{Re}[\widehat{\nu}(\zeta)]\geq 1,\quad \zeta\in F; \tag{2.8}$$

$$\gamma(F)=\inf\sum_{1}^{n}|L(a_i)\nu_i|, \tag{2.9}$$

where the infimum is taken over the same sums as in (2.6);

$$\gamma(F)=\inf\int_{S_\nu}|L(z)|\,|d\nu|,\quad S_\nu\subset G, \tag{2.10}$$

where the infimum is taken over the complex measures ν *that satisfy* (2.8).

Proof. 1) We prove (2.3) (under the condition (2.4))). For this, we employ (1.14). We consider the space $X=C_R(F)$ of continuous real functions on F with the max-norm, and take this norm for the role of the convex functional r. As is easily seen, the relation $l\in R(r)$ means now that l is simply a continuous linear functional, i.e., $l\in C_R(F)^*$. (In distinction from X', as usual, X^* denotes the space of continuous linear functionals on a normed or a linear topological space X.) In our case, such a continuous linear functional is determined by a real measure μ on E. In X, we distinguish the subspace Y formed by the real parts of the sums (0.11). Thus, $y\in Y$ if

$$y=\operatorname{Re}\sum_{1}^{n}\nu_i(a_i-\zeta)^{-1},\quad n\geq 1,\ \ a_i\in G,\ \ \nu_i\in\mathbb{C},\ \ i=1,\dots,n. \tag{2.11}$$

We introduce a convex functional $p(y)$ on Y by putting

$$p(y)=\inf\sum|\nu_i|, \tag{2.12}$$

where the infimum is taken over all representations of y in the form (2.11). (In principle, such a representation may fail to be unique.) The relation $l\in R(p,1)$ means that

$$|l(y)|\leq p(y),\quad y\in Y. \tag{2.13}$$

However, it is easily seen that (2.13) is equivalent to the condition

$$|\widehat{\mu}(z)|\leq 1,\quad z\in G. \tag{2.14}$$

Thus, the set $R=R(r)\cap R(p,1)$ of linear functionals (formula (1.13)) consists of the real measures μ on F that satisfy (2.14). Now, we apply (1.14) to the function $\omega(\zeta)\equiv 1$ and recall the definition of $\tilde{p}(\omega)$. This shows that formulas (2.3)–(2.4) reproduce (1.14) in the setting in question.

2) We prove (2.5) under condition (2.6). In $X=C_R(F)$, we choose the cone K of nonpositive functions on F. The interior points of K are the strictly positive functions on F. An arbitrary functional $l\in K^+$ is determined by a positive measure $\mu\geq 0$ on F. The set $R(p,1)$ consists of the functionals satisfying (2.13), or, what is the same, of the measures μ on F satisfying (2.14). As an interior point x_0 of K, we take the function

$$x_0(\zeta)=\operatorname{Re}[(a-\zeta)^{-1}], \tag{2.15}$$

where a is chosen in such a way that $x_0(\zeta) > 0$ on F. For the function $\omega(\zeta) \equiv 1$, relation (1.20) means that

$$\operatorname{Re}\sum_1^{n_k} \nu_i(a_i-\zeta)^{-1} + t_k \operatorname{Re}(a-\zeta)^{-1} \geq 1. \tag{2.16}$$

Since $t_k > 0$ and $t_k \to 0$, it is easily seen that the infimum in (2.5) under condition (2.6) is the same as under condition (2.16). Now, (2.5) turns into (1.23) with $\omega(\zeta) \equiv 1$.

3) Observe that the infimum in (2.7) is less than or equal to the infimum in (2.5), because (2.5) is a special case of (2.7) for a discrete measure ν with atoms of size ν_i at the points a_i. On the other hand, for any positive measure μ on F and any measure ν satisfying (2.8) and such that $S_\nu \subset G$, we have

$$\begin{aligned}\int_F d\mu &\leq \int_F d\mu \cdot \operatorname{Re}[\widehat{\nu}(\xi)] = \operatorname{Re}\int_F d\mu_\zeta \int_{S_\nu} \frac{d\nu_z}{z-\zeta} = \operatorname{Re}\int_{S_\nu} [-\widehat{\mu}(z)]\, d\nu_z \\ &\leq \int_{S_\nu} |\widehat{\mu}(z)|\, |d\nu_z| \\ &\leq \int_{S_\nu} |d\nu|.\end{aligned} \tag{2.17}$$

Therefore, $\gamma^+(F)$ does not exceed the infimum in (2.7). Together with (2.5), this yields equality in (2.7).

4) We prove (2.9). On Y, we introduce another convex functional:

$$p_1(y) = \inf \sum_1^n |\nu_i| \cdot |L(a_i)|, \tag{2.18}$$

which differs from (2.12) by the presence of the weights $|L(a_i)|$ at the points a_i; the infimum in (2.18) is taken over the same collection of the parameters n, a_i, v_i as in (2.12). (We recall that $L(z)$ is the Garabedian function mentioned in the Introduction.) Now the condition

$$|l(y)| \leq p_1(y), \tag{2.19}$$

required for $l \in R(p_1, 1)$, is equivalent to the condition

$$|\widehat{\mu}(z)| \leq |L(z)|. \tag{2.20}$$

We invoke the representation of analytic functions obtained by the author in [20]–[22]. In those papers it was proved that every function $f(z)$ in $B^1(G)$ can be represented in the form

$$f(z) = \widehat{\mu}(z)/L(z), \tag{2.21}$$

where $\widehat{\mu}(z)$ is the Cauchy potential of a measure. Moreover, if we normalize the Ahlfors function $f^*(z)$ by the condition $\lim_{z\to\infty} zf^*(z) < 0$, then

$$f^*(z) = \widehat{\mu}^*(z)/L(z) \tag{2.22}$$

with a positive measure μ^*, and for every measure μ that may occur in (2.21) we have

$$\|\mu\| \leq \int d\mu^* = \mu^*(F). \tag{2.23}$$

It follows that

$$\gamma(F) = \int_F d\mu^* = \sup \int_F d\mu, \tag{2.24}$$

where the supremum is taken over all measures $\mu \geq 0$ on F that satisfy (2.20). Now, we apply (1.23) to the function $\omega(\zeta) \equiv 1$. Observe that $l \in K^+$ if and only if $\mu \geq 0$, and $l \in R(p_1, 1)$ if and only if (2.20) is fulfilled. Consequently,

$$\gamma(F) = \tilde{p}_1(\mathbb{1}) \tag{2.25}$$

($\mathbb{1}$ is the function $\omega(\zeta) \equiv 1$ on F). But $\tilde{p}_1(\mathbb{1})$ coincides with (2.9) under condition (2.6) for the same reasons that, at an earlier stage of the proof, implied the coincidence of $\tilde{p}(\mathbb{1})$ with (2.5) under condition (2.6).

5) Relation (2.10) is deduced from (2.9) in the same way as (2.7) was deduced from (2.5) in item 3) of the proof. The proof of Theorem 2.1 is finished. All relations established in this theorem are published here for the first time. □

§3. Extension of the Khavin duality theorem

It was already mentioned in the Introduction that the absence of symmetry in the dual expressions (0.13) (see the Khavin theorem) and (0.10)–(0.12) for the analytic capacity γ had been a reason for bewilderment for many years. However, symmetry turns out to be accessible in (0.10) and (0.13).

Theorem 3.1. *Let $\Phi(z)$ be an analytic function in G that satisfies the condition*

$$\Phi(z) \neq 0 \quad \text{for } z \in G,\ \Phi(\infty) = 1. \tag{3.1}$$

Then

$$\gamma(F) = \inf \int_{S_\nu} |\Phi(z)|\, |d\nu|, \tag{3.2}$$

where the infimum is taken over all complex measures ν such that

$$S_\nu \subset G, \quad \widehat{\nu}(\zeta) = \int_{S_\nu} \frac{d\nu_z}{z - \zeta} \equiv 1, \quad \zeta \in F. \tag{3.3}$$

In particular, the Garabedian function $L(z)$ can be taken as the weight $\Phi(z)$ in (3.2).

Proof. The argument splits into several steps.

1. Let $D_1, \dots, D_m$ be simply connected domains bounded by closed simple analytic curves. Suppose that the closures $\overline{D}_1, \dots, \overline{D}_m$ are disjoint For $j = 1, \dots, m$, let F_j be an infinite closed set inside D_j. We put

$$D = \bigcup_{j=1}^{m} D_j, \quad Q = \overline{\mathbb{C}} \setminus \overline{D}, \quad F = \bigcup_{j=1}^{m} F_j \tag{3.4}$$

and consider the extremal problem

$$\alpha = \sup \left| \int_F d\mu \right|, \tag{3.5}$$

where the supremum is taken over all complex measures on F satisfying

$$|\widehat{\mu}(z)| \leq |\Phi(z)|, \quad z \in \partial Q. \tag{3.6}$$

We assume that $\Phi(z)$ is an analytic function on Q such that

$$\Phi(z) \text{ is continuous in } \overline{Q}, \quad \Phi(z) \neq 0 \text{ on } \overline{Q}, \quad \Phi(\infty) = 1. \tag{3.1$'$}$$

In what follows, we consider the case where $Q \subset G$ (G is the domain mentioned in Theorem 3.1). Then condition (3.1) on $\Phi(z)$ implies (3.1′). First, we prove that

$$\alpha = \gamma(\overline{D}). \tag{3.7}$$

We put $\varphi(z) = \widehat{\mu}(z)/\Phi(z)$. If a measure μ on F satisfies (3.6), then

$$\varphi(z) \in B^1(Q), \quad |\varphi'(\infty)| = \left|\int_F d\mu\right|. \tag{3.8}$$

Therefore,

$$\alpha \leq \gamma(\overline{D}). \tag{3.9}$$

We choose a sequence $\{b_i\}$ of different points of F in such a way that the portion of it in any particular F_j, $j = 1, \ldots, m$, be infinite. The system of functions

$$\{(b_i - z)^{-1}\}_{i=1,2,\ldots} \tag{3.10}$$

is complete in $C_A(\partial Q)$ (the space of restrictions to $\partial D = \partial Q$ of analytic functions in Q that are continuous in $\overline{Q}$ and vanish at infinity). Indeed, if λ is a measure on ∂Q orthogonal to the system (3.10), i.e.,

$$\int_{\partial Q} (b_i - z)^{-1}\, d\lambda_z = 0, \quad i = 1, \ldots, \tag{3.11}$$

then the fact that each F_j contains infinitely many b_i's and the usual uniqueness theorem for analytic functions imply that

$$\widehat{\lambda}(\zeta) \equiv 0, \quad \zeta \in D. \tag{3.12}$$

By the Riesz brothers general theorem (see, e.g., [46, Theorem 6.11]), λ is absolutely continuous; moreover,

$$d\lambda = q(z)dz, \tag{3.13}$$

where $q(z)$ is an analytic function in Q, $q(\infty) = 0$, and $q \in E_1(Q)$ (see [47], [46] concerning the Smirnov class E_1; in (3.13) we mean the boundary function for this q). Since for every function $\psi(z) \in C_A(\partial Q)$ the product $\psi(z)q(z)$ has a zero of order at least 2 at infinity, we have

$$\int_{\partial Q} \psi(z)q(z)\, dz = 0, \quad \psi \in C_A(\partial Q). \tag{3.14}$$

Thus, (3.11) implies that the linear functional determined by λ is identically zero on $C_A(\partial Q)$. This means that the system (3.10) is complete in $C_A(\partial Q)$. Let $\varphi^*(z)$ be the Ahlfors function for Q. It is well known that $\varphi^*(z)$ is continuous on $\overline{Q}$ and $|\varphi^*(z)| = 1$ on ∂Q. We put

$$\psi^*(z) = \varphi^*(z)\Phi(z). \tag{3.15}$$

Then $\psi^*(z) \in C_A(\partial Q)$ and $|\psi^*(z)| = |\Phi(z)|$, $z \in \partial Q$. Since the system (3.10) is complete, for every $\varepsilon > 0$ there exist complex numbers $\lambda_1, \ldots, \lambda_n$ such that the function

$$\beta(z) = \sum_1^n \lambda_i (b_i - z)^{-1} \tag{3.16}$$

satisfies the inequality

$$|\beta(z) - \psi^*(z)| < \varepsilon, \quad z \in \partial Q. \tag{3.17}$$

Since $|\psi^*(z)| = |\Phi(z)|$ for $z \in \partial Q$, we may assume that

$$|\beta(z)| \leq |\Phi(z)|, \quad z \in \partial Q. \tag{3.18}$$

We also observe that $\beta(z)$ is the Cauchy transform of a discrete measure concentrated at $b_1, \ldots, b_n$ and taking the value λ_j on $\{b_j\}$, $j = 1, \ldots, n$. Let $\varphi^{**}(z) = \beta(z)/\Phi(z)$. Clearly,

$$\varphi^{**} \in B^1(Q), \tag{3.19}$$

and we have

$$
\begin{aligned}
|\varphi^{**'}(\infty)| &= |\beta'(\infty)| = \left|\sum_1^n \lambda_i\right| = \left|\frac{1}{2\pi}\int_{\partial Q}\beta(z)\,dz\right| \\
&\ge \left|\frac{1}{2\pi}\int_{\partial Q}\psi^*(z)\,dz\right| - \frac{1}{2\pi}\varepsilon\sigma = |\psi^{*'}(\infty)| - \frac{1}{2\pi}\varepsilon\sigma \\
&= |\varphi^{*'}(\infty)| - \frac{1}{2\pi}\varepsilon\sigma = \gamma(\overline{D}) - \frac{1}{2\pi}\varepsilon\sigma,
\end{aligned} \tag{3.20}
$$

where σ is the length of ∂Q. Thus, the quantity α in (3.5) satisfies the inequality

$$
\alpha \ge \left|\sum_1^n \lambda_i\right| \ge \gamma(\overline{D}) - \frac{1}{2\pi}\varepsilon\sigma. \tag{3.21}
$$

Together with (3.9), this yields (3.7).

2. Under the assumptions of the preceding item, let $\{a_i\}$ be a dense subset of $\partial\overline{D}$. As usual, we denote by $C(F)$ the space of complex-valued continuous functions on F with the uniform norm. We put

$$
\tilde{p} = \inf \varliminf_{n\to\infty} \sum_1^n |\nu_i||\Phi(a_i)|, \tag{3.22}
$$

where the infimum is taken over all sequences with

$$
\sum_1^n \nu_i(\zeta - a_i)^{-1} \to 1 \quad \text{in } C(F). \tag{3.23}
$$

Then

$$
\tilde{p} = \gamma(\overline{D}). \tag{3.24}
$$

Indeed, by the duality theorem (Theorem 1.2; see formula (1.14)) we have

$$
\tilde{p} = \sup\left|\int_F d\mu\right|, \tag{3.25}
$$

where the supremum is taken over all measures μ on F such that

$$
|\widehat{\mu}(a_i)| \le |\Phi(a_i)|. \tag{3.26}
$$

But (3.26) is none other than (3.6), so the supremum in (3.25) coincides with that in (3.5), i.e., with $\alpha = \gamma(\overline{D})$.

3. We put

$$
\tilde{\tilde{p}} = \inf \int_{\partial Q} |\Phi(z)|\,|d\nu|, \tag{3.27}
$$

where the infimum is taken over all measures on ∂Q for which

$$
\widehat{\nu}(\zeta) \equiv 1, \quad \zeta \in F, \tag{3.28}
$$

and prove the relation

$$
\tilde{\tilde{p}} = \tilde{p} = \gamma(\overline{D}). \tag{3.29}
$$

Moreover, there exists a measure ν^* on which the infimum in (3.27) is attained. Indeed, the existence of ν^* is a consequence of the fact that $|\Phi(z)| \ge \delta > 0$ for $z \in \partial Q$, so, when looking for the infimum in (3.27), we may restrict ourselves to measures the variations of which are bounded by some fixed constant, and such measures form a compact set. Next, taking Riemann-type integral sums in place of integrals, we can construct a sequence

$\{\nu^N\}$, $N = 1, \dots$, of discrete measures with atoms at some of the points a_i that converges to ν^* in the weak* topology. The family of functions

$$t_\zeta(z) = \{(z-\zeta)^{-1}\}, \quad \zeta \in F \text{ is a parameter,} \tag{3.30}$$

is equicontinuous on ∂Q; therefore,

$$\int_{\partial Q} (z-\zeta)^{-1}\, d\nu_z^N = \widehat{\nu}^N(\zeta) \to \widehat{\nu}^*(\zeta) \quad \text{in } C(F). \tag{3.31}$$

Moreover, for an arbitrarily small $\nu > 0$ we can choose points among $\{a_i\}$ at which the measures ν^N have atoms in such a way that

$$\int_{\partial Q} |\Phi(z)|\, |d\nu^N| \le \int_{\partial Q} |\Phi(z)|\, |d\nu^*| + \eta. \tag{3.32}$$

We have

$$\begin{gathered} \widehat{\nu}^N(\zeta) = \sum_1^{n_N} \nu_i^N (a_i - \zeta)^{-1} = \sum_1^{n_N} (-\nu_i^N)(\zeta - a_i)^{-1}, \\ \int_{\partial Q} |\Phi(z)|\, |d\nu^N| = \sum_1^{n_N} |\Phi(a_i)||\nu_i^N| \end{gathered} \tag{3.33}$$

(ν_i^N is the value of ν^N on the singleton $\{a_i\}$). So, the $\widehat{\nu}^N(\zeta)$ are sums of the form (3.23). Therefore,

$$\tilde{p} \le \lim_{N\to\infty} \sum_1^{n_N} |\nu_i^N \Phi(a_i)| \le \int_{\partial Q} |\Phi(z)\, d\nu^*| + \eta = \tilde{\tilde{p}} + \eta,$$

whence

$$\tilde{p} \le \tilde{\tilde{p}}. \tag{3.34}$$

Now, we construct a sequence of sums with the properties

$$\begin{gathered} \sum_1^{n_N} \nu_i^N (\zeta - a_i)^{-1} \xrightarrow[N\to\infty]{} 1 \quad \text{in } C(F), \\ \sum_1^{n_N} |\Phi(a_i)\nu_i^N| \to \tilde{p}, \quad N \to \infty. \end{gathered} \tag{3.35}$$

Since $|\Phi(z)| \ge \delta > 0$ on ∂Q, the discrete measures $-\nu^N$ that take the values $-\nu_i^N$ on the singletons $\{a_i\}$ have uniformly bounded variations. Thus, we may assume that the measures $\{-\nu^N\}$ converge in the weak* topology to a measure ν^0, and

$$\widehat{\nu}^0(\zeta) = \lim_{N\to\infty} [(-\nu_i^N)(a_i - \zeta)^{-1}] = \lim_{N\to\infty} [-\widehat{\nu}^N(\zeta)] = 1. \tag{3.36}$$

Moreover, the weighted measures $\{\Phi(z) d(-\nu^N)\}$ converge in the weak* topology to $\Phi(z) d\nu^0$. Therefore, for the variations we have

$$\int_{\partial Q} |\Phi(z)\, d\nu_z^0| \le \lim_{N\to\infty} \int_{\partial Q} |\Phi(z)\, [-d\nu^N]| = \lim_{N\to\infty} \sum_1^{n_N} |\Phi(a_i)\nu_i^N| = \tilde{p}. \tag{3.37}$$

Combining (3.36), (3.37), and (3.27), we obtain

$$\tilde{\tilde{p}} \le \int_{\partial Q} |\Phi(z)\, d\nu^0| \le \tilde{p}. \tag{3.38}$$

Formula (3.38) is a consequence of (3.38) and (3.34).

4. Now, let $\{a_i\}$ be a subset inside Q (rather than on ∂Q) which is dense in Q. Problem (3.25) does not depend on whether the $\{a_i\}$ are points on ∂Q or inside Q. Repetition of

Table 1. Principal formulas for various capacities.

Capacity	Dual expression	Comments	Author, year of publication
$\gamma(F)$	$\inf_{\varphi\in E_1} \frac{1}{2\pi}\int_{\partial G}\lvert 1+\varphi(\zeta)\rvert\lvert d\zeta\rvert$ $= \frac{1}{2\pi}\int_{\partial G}\lvert L(\zeta)\rvert\lvert d\zeta\rvert$	G is a finitely connected domain with rectifiable boundary, E_1 is the Smirnov class, $\varphi(\infty)=0$, and L is the Garabedian function	Garabedian, 1949 [18], [19]
$\gamma(F)$	$\inf\int\lvert d\nu\rvert$, $S_\nu\subset G$ under the condition $(*)$: $\widehat{\nu}(\zeta)\equiv 1$, $\zeta\in F$	G is arbitrary	Khavin, 1960 [24], [25]
$\gamma(F)$	$\inf\int\lvert\Phi(z)d\nu\rvert$, $S_\nu\subset G$ under condition $(*)$	G is arbitrary, $\Phi(z)\neq 0$, $\Phi(\infty)=1$	The present paper
$\gamma(F)$	$\inf\varliminf\sum\lvert\nu_i L(a_i)\rvert$ under the condition $(**)$: $\sum\nu_i(\zeta-a_i)^{-1}\to 1$ in $C(F)$	G is arbitrary	Khavinson, 1960 [23]
$\gamma(F)$	$\inf\sum\lvert\nu_i L(a_i)\rvert$ under the condition $(***)$: $\operatorname{Re}\sum\nu_i(\zeta-a_i)^{-1}\geq 1, \quad \zeta\in F$	G is arbitrary	The present paper
$\gamma(F)$	$\inf\int\lvert L(z)\rvert\lvert d\nu\rvert$, $S_\nu\subset G$, under the condition $(****)$: $\operatorname{Re}\widehat{\nu}(\zeta)\geq 1, \quad \zeta\in F$	G is arbitrary	The present paper
$\gamma^c(F)$	$\inf\sum\lvert\nu_i\rvert$ under condition $(**)$	G is arbitrary	Khavinson, 1966 [12]
$\gamma^R(F)$	$\inf\sum\lvert\nu_i\rvert$ under the condition $\operatorname{Re}\sum\frac{\nu_i}{(\zeta-a_i)}\to 1$ in $C(F)$	G is arbitrary	The present paper
$\gamma^+(F)$	$\inf\sum\lvert\nu_i\rvert$ under condition $(***)$	G is arbitrary	The present paper
$\gamma^+(F)$	$\inf\int\lvert d\nu\rvert$, $S_\nu\subset G$ under condition $(****)$	G is arbitrary	The present paper

the arguments of item 3 shows that, in formula (3.27) for $\tilde{\tilde{p}}$, $d\nu$ may range over arbitrary measures on $\overline{Q}$ and not merely on ∂Q.

5. Now, we pass to the proof of Theorem 3.1 itself. We return to the notation described in that statement. Let Δ denote the infimum in (3.2), and let $\varepsilon>0$ be arbitrary. We

choose a measure μ satisfying (3.3) and such that

$$\Delta \leq \int_{S_\nu} |\Phi(z)\, d\nu| < \Delta + \varepsilon. \tag{3.39}$$

We construct a union D of simply connected domains D_i, $i = 1, \dots, m$, and a domain Q in the same way as in items 1–4; moreover, we require that

$$F \subset D, \quad \gamma(\overline{D}) \leq \gamma(F) + \varepsilon, \quad S_\nu \subset Q. \tag{3.40}$$

For Q, we find $\tilde{\tilde{p}}$ as described in item 4. Then

$$\Delta \leq \tilde{\tilde{p}} = \gamma(\overline{D}) \leq \int_{S_\nu} |\Phi(z)\, d\nu| \leq \Delta + \varepsilon. \tag{3.41}$$

From (3.40) and (3.41) it follows that $\Delta - \varepsilon \leq \gamma(F) \leq \gamma(\overline{D}) < \Delta + \varepsilon$, and, consequently, (3.2) is fulfilled. □

At the end of §6, we shall give another proof of Theorem 3.1. That proof will be considerably shorter. However, the origin of the duality (3.2)–(3.3) and its relationship with some other constructions are less clear from that argument. At the same time, somewhat lengthy procedures of the present section are quite simple in essence, and this gives hope for a refinement that might shed light on the relationship between γ and γ^c.

§4. Extremal problem related to γ^+

We treat the extremal problem (2.2) and the dual problem (2.7)–(2.8) in more detail. The measures ν with $S_\nu \subset G$ that satisfy (2.8) will be called *admissible*. A sequence $\{\nu_n\}$ of admissible measures will be called *extremal* (for the dual problem) if

$$\int_{S_{\nu_n}} |d\nu_n| \to \gamma^+(F) \quad (n \to \infty). \tag{4.1}$$

For a measure ν, we denote by ϕ_ν the Radon–Nikodym derivative of ν with respect to the total variation of ν:

$$\phi_\nu(z) = \frac{d\nu}{d|\nu|}. \tag{4.2}$$

The function $\phi_\nu(z)$ is defined a.e. on S_ν with respect to $|d\nu|$, and $|\phi_\nu(z)| = 1$ a.e. relative to $|d\nu|$; see [45], [48], [49]. Let $\mu \geq 0$ be a measure on F with $|\widehat{\mu}(z)| \leq 1$, and let ν be an admissible measure. We introduce the following notation ($\varepsilon > 0$):

$$\begin{cases} E_\nu(\varepsilon) = \{\zeta \in F : \operatorname{Re}\widehat{\nu}(\zeta) \geq 1 + \varepsilon\}, \\ E^1_\nu(\varepsilon) = F \setminus E_\nu(\varepsilon), \\ D_\nu(\varepsilon) = \{z \in S_\nu : |\widehat{\mu}(z)\phi_\nu(z) + 1| \geq \varepsilon\}, \\ D^1_\nu(\varepsilon) = S_\nu \setminus D_\nu(\varepsilon). \end{cases} \tag{4.3}$$

Theorem 4.1. 1. *In problem* (2.2), *there is an extremal measure* μ^*. 2. *If* μ^* *is an extremal measure in* (2.2) *and* $\{\nu_n\}$ *is an extremal sequence, then for any* $\varepsilon > 0$ *we have*

$$\int_{E_{\nu_n}(\varepsilon)} \operatorname{Re}\widehat{\nu}_n(\zeta)\, d\mu^* \xrightarrow[n\to\infty]{} 0 \quad \textit{and, moreover,} \quad \mu^*(E_{\nu_n}(\varepsilon)) \to 0; \tag{4.4}$$

$$|\nu_n| D_{\nu_n}(\varepsilon) \xrightarrow[n\to\infty]{} 0, \tag{4.5}$$

where $D_{\nu_n}(\varepsilon)$ *is given by one of the formulas* (4.3) *with* $\mu = \mu^*$. *Conversely, if a measure* $\mu^* \geq 0$ *on* F *with* $|\widehat{\mu}^*(z)| \leq 1$ *and a sequence* $\{\nu_n\}$ *of admissible measures with uniformly bounded total variations* $\{\|\nu_n\|\}$ *satisfy* (4.4) *and* (4.5), *then* μ^* *is an extremal measure in* (2.2), *and* $\{\nu_n\}$ *is an extremal sequence for the dual problem* (4.1).

Proof. 1. Let Γ be a rectifiable contour enclosing F, and let σ be the length of Γ. If $\mu \geq 0$ is a measure with $S_\mu \subset F$ and $|\widehat{\mu}(z)| \leq 1$, then

$$\int_F d\mu = \left| \frac{1}{2\pi i} \int_\Gamma \widehat{\mu}(z)\, dz \right| \leq \frac{1}{2\pi} \int_\Gamma |dz| = \frac{\sigma}{2\pi}.$$

Therefore, all measures occurring in (2.2) are uniformly bounded; the compactness of such a family of measures implies the existence of an extremal measure.

2. Let μ^* be an extremal measure, and let $\{\nu_n\}$ be the extremal sequence (4.1). For these measures, we analyze the chain of inequalities (2.17). In order to have

$$\int_F d\mu^* = \lim_{n\to\infty} \int_{S\nu_n} |d\nu_n|, \tag{4.6}$$

we must have

$$\int_F d\mu^* = \lim_{n\to\infty} \int_F \operatorname{Re}[\widehat{\nu}_n(\zeta)]\, d\mu^*, \tag{4.7}$$

i.e.,

$$\int_F [\operatorname{Re} \widehat{\nu}_n(\zeta) - 1]\, d\mu^* \to 0 \tag{4.8}$$

and, *a fortiori*,

$$\int_{E_{\nu_n(\varepsilon)}} [\operatorname{Re} \widehat{\nu}_n(\zeta) - 1]\, d\mu^* \to 0, \quad \varepsilon > 0. \tag{4.9}$$

Suppose that $\mu^*(E_{\nu_n}(\varepsilon_0)) \nrightarrow 0$ as $n \to \infty$ for some $\varepsilon_0 > 0$. Passing (if necessary) to a subsequence of the sequence $\{\nu_n\}$, we may assume that $\mu^*(E_{\nu_n}(\varepsilon_0)) \geq \delta > 0$ for some δ, whence

$$\int_{E_{\nu_n}(\varepsilon_0)} d\mu^* \leq \int_{E_{\nu_n}(\varepsilon_0)} \operatorname{Re} \widehat{\nu}_n(\zeta)\, d\mu^* - \varepsilon_0\delta, \tag{4.10}$$

which is incompatible with (4.7). So,

$$\mu^*(E_{\nu_n}(\varepsilon)) \to 0, \quad \varepsilon > 0. \tag{4.11}$$

But (4.9) and (4.11) imply the relation

$$\int_{E_{\nu_n(\varepsilon)}} \operatorname{Re}[\widehat{\nu}_n(\zeta)]\, d\mu^* \to 0, \quad \varepsilon > 0.$$

We have proved the necessity of (4.4); now, we prove that of (4.5). If for some $\varepsilon_0 > 0$ condition (4.5) fails for the measure μ^*, then we may assume that there exists $\delta > 0$ with

$$|\nu_n|(D_{\nu_n}(\varepsilon_0)) \geq \delta, \quad n = 1, 2, \dots . \tag{4.12}$$

By the definition of $D_\nu(\varepsilon)$, on $D_{\nu_n}(\varepsilon_0)$ we have

$$|1 - (-\widehat{\mu}^*(z)\phi_{\nu_n}(z))| \geq \varepsilon_0,$$

and a geometric argument shows that

$$\operatorname{Re}[-\widehat{\mu}^*(z)\phi_{\nu_n}(z)] \leq q < 1 \tag{4.13}$$

on $D_{\nu_n}(\varepsilon_0)$, where q depends on ε_0. Therefore,

$$\begin{aligned} \operatorname{Re} \int_{S_{\nu_n}} [-\widehat{\mu}^*(z)]\, d\nu_n &= \operatorname{Re} \int_{S_{\nu_n}} [-\widehat{\mu}^*(z)\phi_{\nu_n}(z)]\, |d\nu_n| \\ &\leq \int_{D^1_{\nu_n}(\varepsilon_0)} |d\nu_n| + q \int_{D_{\nu_n}(\varepsilon_0)} |d\nu_n| \leq \int_{S_{\nu_n}} |d\nu_n| - (1-q)\delta. \end{aligned} \tag{4.14}$$

However, by (2.17), if $\{\nu_n\}$ is extremal, then

$$\lim_{n\to\infty} \operatorname{Re}\left[\int_{S_{\nu_n}} -\hat{\mu}^*(z)\, d\nu_n\right] = \lim_{n\to\infty}\int_{S_{\nu_n}} |d\nu_n|. \tag{4.15}$$

So, relation (4.14) (implied by (4.12)) contradicts (4.15). We see that (4.12) must fail, and (4.5) is necessary. Conversely, suppose that (4.4) and (4.5) are fulfilled for a measure $\mu^* \geq 0$ on F and a sequence $\{\nu_n\}$ of admissible measures. Then

$$\begin{aligned}
\int_F d\mu^* &= \lim_{n\to\infty}\int_{E^1_{\nu_n(\varepsilon)}} d\mu^* \geq \lim_{n\to\infty}\int_{E^1_{\nu_n}(\varepsilon)} \operatorname{Re}[\widehat{\nu}_n(\zeta)]\, d\mu^* - \varepsilon\mu^*(F)\\
&= \lim_{n\to\infty}\int_F [\operatorname{Re}\widehat{\nu}_n(\zeta)]\, d\mu^* - \varepsilon\mu^*(F) \quad \text{(by (4.4))}\\
&= \lim_{n\to\infty} \operatorname{Re}\int_{S_{\nu_n}} [-\widehat{\mu}^*(z)]\, d\nu_n - \varepsilon\mu^*(F)\\
&= \lim_{n\to\infty} \operatorname{Re}\int_{S_{\nu_n}} [-\widehat{\mu}^*(z), \phi_{\nu_n}(z)]\, |d\nu_n| - \varepsilon\mu^*(F)\\
&= \lim_{n\to\infty} \operatorname{Re}\int_{D^1_{\nu_n}(\varepsilon)} [-\widehat{\mu}^*(z)\phi_{\nu_n}(z)]\, |d\nu_n| - \varepsilon\mu^*(F)
\end{aligned} \tag{4.16}$$

(we have used (4.5) and the fact that $|\widehat{\mu}^*(z)\phi_{\nu_n}(z)| \leq 1$). On $D^1_{\nu_n}(\varepsilon)$, we have

$$|1 - [-\widehat{\mu}^*(z)\phi_{\nu_n}(z)]| < \varepsilon,$$

which implies the inequality

$$\operatorname{Re}[-\widehat{\mu}^*(z)\phi_{\nu_n}(z)] \geq 1 - \varepsilon \tag{4.17}$$

on the same set. Let A denote an upper bound for the total variations $\|\nu_n\|$ of the measures ν_n (recall that we have assumed the uniform boundedness of the sequence $\{\|\nu_n\|\}$). Then, with the help of (4.17), we finish (4.16) as follows:

$$\begin{aligned}
\int_F d\mu^* &\geq \lim_{n\to\infty}\int_{D^1_{\nu_n}(\varepsilon)} |d\nu_n| - \varepsilon|\nu_n(D^1_{\nu_n})| - \varepsilon\mu^*(F)\\
&\geq \lim_{n\to\infty}\int_{S_{\nu_n}} |d\nu_n| - \varepsilon(\mu^*(F) + A)
\end{aligned} \tag{4.18}$$

(we have used (4.5) once again). Since ε is arbitrary, comparison of (4.18) and (2.17) shows that the measure μ^* and the sequence $\{\nu_n\}$ are extremal.

This completes the proof of Theorem 4.1. □

The following fact was established in the course of the proof.

Proposition 4.2. *If μ^* is an extremal measure for the problem concerning γ^+, and $\{\nu_n\}$ is an extremal sequence for the dual problem, then the functions $\{\nu_n(\zeta)\}$ tend to* 1 *on F in the measure μ^*.*

Proposition 4.3. *In the relation*

$$\inf \sum_1^n |\nu_i| \leq \inf \sum_1^n |L(a_i)\nu_i|, \tag{4.19}$$

where the infimum is taken under condition (2.6), *as well as in the relation*

$$\inf \int_{S_\nu} |d\nu| \leq \inf \int_{S_\nu} |L(z)|\, |d\nu|, \tag{4.20}$$

where the infimum is taken under condition (2.8), *strict inequality may occur.*

Proof. By Theorem 2.1, both (4.19) and (4.20) express the fact that $\gamma^+(F) \leq \gamma(F)$. There are domains (see [27], [50], [51]) for which the extremal function for the problem of calculating $\gamma(F)$ (the Ahlfors function) is not representable as a Cauchy potential. Since there always exists an extremal measure for the problem of calculating $\gamma^+(F)$, we see that necessarily $\gamma^+(F) < \gamma(F)$ for F as above. □

In connection with Proposition 4.3, we recall that it is still not clear if strict inequality may occur in the relation $\gamma^c(F) \leq \gamma(F)$. In the dual form, this can be stated as follows. Consider the inequality

$$\inf \sum_1^n |\nu_i| \leq \inf \sum_1^n |L(a_i)\nu_i|,$$

where the infima are taken under the condition that $\sum_1^n \nu_i(\zeta - a_i)^{-1} \to 1$ uniformly on F. Do we always have equality in it, or may strict inequality occur?

It is natural to ask about the properties of the subclasses of $B^1(G)$ that correspond, via the Cauchy transformation, to some particular classes of measures. In the usual manner, we introduce a metric in $B^1(G)$ that corresponds to the topology of pointwise convergence on G. Let $B_+(G)$ denote the class of functions $f(z) \in B^1(G)$ representable by Cauchy potentials with a positive measure, and let $B_c(G)$ denote the class of functions $f(z) \in B^1(G)$ representable by Cauchy potentials with an arbitrary complex measure. Finally, let $B_M(G)$ be the class of functions $f(z) \in B^1(G)$ representable by the Cauchy potential of a complex measure μ with $\|\mu\| \leq M$, where $M > 0$ is a fixed number.

Proposition 4.4. *The sets $B_+(G)$ and $B_M(G)$ are closed in $B^1(G)$, and $B_c(G)$ is of type F_σ in $B^1(G)$.*

Proof. Since the set of measures with variations not exceeding M is weak*-compact, we see that $B_M(G)$ is closed for any $M > 0$. In the proof of statement 1 in Theorem 4.1 it was shown that $B_+(G) \subset B_M(G)$ for some $M > 0$, so $B_+(G)$ is also closed. Finally, $B_c(G) = \bigcup_{N=1}^\infty B_N(G)$ is of type F_σ. □

Proposition 4.5. *If $B^1(G)$ contains a function $f(z)$ not representable as a Cauchy potential, then $B_M(G)$ and $B_+(G)$ are nowhere dense in $B^1(G)$, and $B_c(G)$ is of the first category.*

Proof. The sets $B_M(G)$ and $B_+(G)$ are also norm-closed in the space of bounded functions. Suppose that $B_M(G)$ fails to be nowhere dense in $B^1(G)$. Then there exists a ball $S(f_0, r) = \{f : \|f - f_0\| \leq r\}$ with $S(f_0, r) \subset B_M(G)$. Take an arbitrary function $\varphi \in B^1(G)$. Then $f = f_0 + r\varphi \in S(f_0, r)$. Since f and f_0 are representable by the Cauchy potentials of measures the variations of which do not exceed M, we see that φ is representable by the Cauchy potential of a measure μ with $\|\mu\| \leq 2M/r$. This contradicts the existence of a function in $B^1(G)$ not representable by a Cauchy potential. Now, the statements about B_+ and B_c follow immediately. □

In [51], M. V. Samokhin even constructed a simply connected domain G_0 in which not only does the Ahlfors function fail to be a Cauchy potential, but such functions exist in abundance, forming a dense subset of $B^1(G_0)$. In the same paper, a complete description was given of the domains G with the property that the functions belonging to $B^1(G)$ and continuous in $\overline{G}$ are representable by certain specific Cauchy potentials that are quite similar to those occurring in the usual Cauchy integral formula. In this connection, the following problem arises: characterize the domains G with $B^1(G) = B_c(G)$. For such domains, the situation simplifies as described below.

Proposition 4.6. *If $B_c(G) = B^1(G)$, then $B^1(G) = B_M(G)$ for some $M > 0$.*

Proof. By the Baire category theorem, the complete metric space $B^1(G)$ cannot be of the first category; consequently, $B_M(G)$ fails to be nowhere dense for some M. As in the proof of Proposition 4.5, we then show that $B_{M_0}(G) = B^1(G)$ for some M_0. □

Remark. Theorem 4.1 leaves the question of uniqueness of an extremal measure μ^* in the problem of calculating γ^+ open. It would be desirable to clarify the situation. It was already mentioned in the Introduction that, in the problem of calculating $\gamma(F)$, an extremal function is unique (this is the Ahlfors function). For a finitely connected domain, this was proved in the Ahlfors fundamental paper [1]; the case of an arbitrary domain G was settled by the author in [22]. Another uniqueness proof was given by Carleson [52]. The question of uniqueness of the measure μ^* in (2.22) (i.e., the question of an extremal measure for problem (2.24)) has also been resolved: it was proved in [22] that μ^* is unique under some additional assumptions, and Samokhin lifted them in [53].

§5. The structure of some linear functionals

Let $\omega(\zeta)$ be a function belonging to $C(F)$. Consider the linear extremal problem

$$\gamma^c(F,\omega) = \sup_{\mu} \left| \int_F \omega(\zeta)\, d\mu \right|, \tag{5.1}$$

where the supremum is taken over the complex measures μ satisfying

$$S_\mu \subset F, \quad |\widehat{\mu}(z)| \le 1 \text{ for } z \in G. \tag{5.2}$$

If $\omega(\zeta) \equiv 1$, then $\gamma^c(F,\omega)$ is the Cauchy capacity $\gamma^c(F)$ of F. It was already mentioned in the Introduction (see (0.7)) that the quantities γ, γ^c, γ^R, and γ^+ are all equivalent; in particular,

$$\gamma(F) = 0 \iff \gamma^c(F) = 0 \iff \gamma^R(F) = 0 \iff \gamma^+(F) = 0. \tag{5.3}$$

In fact, (5.3) is weaker than (0.7). It should be noted that the equivalence

$$\gamma^c(F) = 0 \iff \gamma^+(F) = 0 \tag{5.3'}$$

was proved in [54], [55] somewhat prior to [17].

We want to know the conditions on $\omega(\zeta)$ that ensure the equivalence

$$\gamma^c(F,\omega) = 0 \iff \gamma(F) = 0 \tag{5.4}$$

(in other words, we want to know when the relation $\gamma^c(F,\omega) = 0$ plays the same "fatal" role for $B(G)$ as the relation $\gamma(F) = 0$, i.e., implies that $B(G)$ is trivial, consisting of constant functions only). The study of this question was initiated by Khavin in [24], [25] (see Theorems 3 and 4 in those papers). Val′skiĭ refined Khavin's result somewhat; see [26, Theorem 1]. The further development is described below.

Theorem 5.1. *Let F be a totally disconnected compact set, and let $\omega(\zeta) \in C(F)$. Suppose that the set E of zeros of $\omega(\zeta)$ satisfies $\gamma(E) = 0$. If $\gamma^c(F,\omega) = 0$, then $\gamma(F) = 0$, i.e.,* (5.4) *is true.*

Proof. Let μ be an arbitrary measure satisfying (5.2). The condition $\gamma^c(F,\omega) = 0$ means that

$$\int_F \omega(\zeta)\, d\mu = 0. \tag{5.5}$$

Let Δ be an open set that includes a portion F_Δ of F and does not intersect $F \setminus F_\Delta$. Denote by μ_Δ the restriction of μ to F_Δ. The potential $\widehat{\mu}(z)$ is bounded on $\Delta \setminus F_\Delta$, and so is the potential $(\widehat{\mu - \mu_\Delta})(z)$. Thus, the potential $\widehat{\mu}_\Delta(z)$ is bounded on $\Delta \setminus F_\Delta$ and, consequently, on G. Therefore, (5.5) must be fulfilled for μ_Δ:

$$\int_F \omega(\zeta)\, d\mu_\Delta = 0. \tag{5.6}$$

Putting

$$\omega_\Delta(\zeta) = \begin{cases} \omega(\zeta), & \zeta \in F_\Delta, \\ 0, & \zeta \in F \setminus F_\Delta, \end{cases} \tag{5.7}$$

we rewrite (5.5) as follows:

$$\int_F \omega_\Delta(\zeta)\, d\mu = 0. \tag{5.8}$$

Let H be the subspace of $C(F)$ consisting of all functions that vanish on F, and let $f \in H$. Fixing $\varepsilon > 0$, we cover F by mutually disjoint sets $\Delta_1, \dots, \Delta_N$ in such a way that the oscillation of $f(\zeta)$ on each Δ_j, $j = 1, \dots, N$, be smaller than ε. Among these sets, we take those intersecting E, denote them by $\Delta_1, \dots, \Delta_n$ $(n \le N)$, and put $\Delta = \bigcup_1^n \Delta_j$. On $F_\Delta = F \cap \Delta$, we have

$$|f(\zeta)| < \varepsilon. \tag{5.9}$$

The set $F_1 = F \setminus F_\Delta$ is totally disconnected. We cover it by open sets $\delta_1, \dots, \delta_m$ that are disjoint and do not intersect Δ. Moreover, we make the diameters of these sets so small that the oscillation of $f(\zeta)/\omega(\zeta)$ on each of the sets $F \cap \delta_j$ be smaller than ε/M, where $M = \max_{\zeta \in F} |\omega(\zeta)|$. For every $j = 1, \dots, m$, we can find a constant λ_j such that

$$\left| \frac{f(\zeta)}{\omega(\zeta)} - \lambda_j \right| < \frac{\varepsilon}{M}, \quad \zeta \in F \cap \delta_j. \tag{5.10}$$

(For instance, we may put $\lambda_j = \frac{f(\zeta_j)}{\omega(\zeta_j)}$ for some point $\zeta_j \in F \cap \delta_j$.) In terms of the functions

$$\omega_j(\zeta) = \begin{cases} \omega(\zeta), & \zeta \in F \cap \delta_j, \\ 0, & \zeta \in F \setminus \delta_j, \end{cases} \tag{5.11}$$

relation (5.10) can be rewritten as follows:

$$|f(\zeta) - \lambda_j \omega_j(\zeta)| < \varepsilon, \quad \zeta \in F \cup \delta_j. \tag{5.12}$$

This implies that

$$\left| f(\zeta) - \sum_{j=1}^m \lambda_j \omega_j(\zeta) \right| < \varepsilon, \quad \zeta \in F_1. \tag{5.13}$$

Applying (5.8) to ω_j, we obtain

$$\int_F \omega_j\, d\mu = \int_{F \cap \delta_j} \omega(\zeta)\, d\mu = 0. \tag{5.14}$$

Finally, from (5.9), (5.13), (5.14) we deduce that

$$\begin{aligned} \left| \int_F f(\zeta)\, d\mu \right| &\le \int_{F_\Delta} |f(\zeta)|\, |d\mu| + \left| \int_{F_1} f(\zeta)\, d\mu \right| \\ &\le \int_{F_\Delta} |f(\zeta)|\, |d\mu| + \left| \int_{F_1} \left[\sum_1^m \lambda_j \omega_j(\zeta) \right] d\mu \right| \\ &\quad + \int_{F_1} \left| f(\zeta) - \sum_1^m \lambda_j \omega_j(\zeta) \right| |d\mu| \le 2\varepsilon \|\mu\|, \end{aligned}$$

where $\|\mu\|$ is the total variation of μ on F. Thus, for every measure μ with bounded Cauchy potential and every $f(\zeta) \in H$, we have

$$\int_F f(\zeta)\, d\mu = 0. \tag{5.15}$$

But a function $f(\zeta) \in H$ may take arbitrary values of E; therefore,

$$S_\mu \subset E. \tag{5.16}$$

Since the potential $\widehat{\mu}(z)$ is bounded on G, it is also bounded off E. Since $\gamma(E) = 0$, we see that $\mu \equiv 0$. Consequently, $\gamma^c(F) = 0$, which is the same as $\gamma(F) = 0$. □

If E is finite, Theorem 5.1 implies the result of Val′skiĭ [26]. If F has finite Painlevé length (finite girth), any bounded analytic function in G is representable by a Cauchy potential, and the Cauchy capacity coincides with the analytic capacity. Then, if E is finite and $\omega(\zeta)$ is analytic on F, we obtain the result of Khavin [24], [25].

The topological condition for F to be totally disconnected can be replaced by a condition of metric nature.

Theorem 5.2. *Suppose a compact set F has zero area, and a function $\omega(\zeta)$ has the same properties as in Theorem* 5.1. *If $\gamma^c(F,\omega) = 0$, then $\gamma(F) = 0$.*

Proof. Condition (5.2) is equivalent to the conditions

$$S_\mu \subset F, \quad |\widehat{\mu}(a_k)| \le 1, \;\; k = 1, 2, \ldots, \tag{5.17}$$

where $\{a_k\}$ is a countable dense subset of a disk $|z| < r$ including F. Consider the subspace $Y \subset C(F)$ consisting of the fractions

$$y(\zeta) = \sum_1^n \nu_i (\zeta - a_i)^{-1},$$

where the ν_i are complex numbers and n is a positive integer. We endow Y with the norm

$$p(y) = \sum_1^n |\nu_i|$$

(cf. the proof of formula (2.3) in Theorem 2.1). By Theorem 1.2 (see (1.13) and (1.14)), the condition $\gamma^c(F,\omega) = 0$ means that $\omega(\zeta)$ is $o(p)$-approximable by Y. In other words, for any $\varepsilon > 0$ there exist $\nu_1, \ldots, \nu_n$ such that

$$\max_{\zeta \in F} \Big| \omega(\zeta) - \sum_1^n \nu_i (\zeta - a_i)^{-1} \Big| < \varepsilon, \quad \sum_1^n |\nu_i| < \varepsilon. \tag{5.18}$$

We use the identity

$$\frac{1}{(\zeta - z)(\zeta - a_i)} = \frac{1}{(a_i - z)(\zeta - a_i)} - \frac{1}{(\zeta - z)(a_i - z)}$$

and (5.17) and (5.18) to show that if $|z| > 2r$, then

$$\begin{aligned} \left| \int_F \frac{\omega(\zeta)\, d\mu}{\zeta - z} \right| &\le \int_F \left| \frac{\omega(\zeta)}{\zeta - z} - \frac{1}{\zeta - z} \sum_1^n \frac{\nu_i}{(\zeta - a_i)} \right| |d\mu| \\ &\quad + \sum_1^n \left| \frac{\widehat{\mu}(a_i)\nu_i}{a_i - z} \right| + \sum_1^n \left| \frac{\widehat{\mu}(z)\nu_i}{a_i - z} \right| \\ &< \varepsilon \|\mu| + \frac{2\varepsilon}{r} \end{aligned} \tag{5.19}$$

(we recall that the a_i lie in the disk $|z| < r$). Thus, the Cauchy potential of the measure $d\lambda = \omega d\mu$ is equal to zero at the points z lying "far away" from F, whence it follows that it vanishes identically on G. Since the area of F is equal to zero, we see that the potential of $d\lambda$ vanishes a.e. on the plane. Consequently (see [2, Chapter I, Corollary 8.3]), $d\lambda = \omega d\mu \equiv 0$. But then $d\mu$ may be nonzero only on the zero set of $\omega(\zeta)$. Since the potential $\widehat{\mu}(z)$ belongs to $B^1(G)$, it is easily seen that it also belongs to $B^1(\overline{\mathbb{C}} \setminus E)$. The condition $\gamma^c(E) = 0$ implies that $\mu \equiv 0$, and the theorem is proved. □

Now, we define $\gamma^R(F, \omega)$ and $\gamma^+(F, \omega)$ again by (5.1) under the restriction (5.2), but with the difference that the supremum is taken over the real measures for $\gamma^R(F, \omega)$ and over the positive measures for $\gamma^+(F, \omega)$. We recall that $C_R(F)$ is the space of real-valued continuous functions on F.

Theorem 5.3. *Let F be a totally disconnected compact set, and let $\omega(\zeta) \in C_R(F)$. Suppose that for the zero set E of $\omega(\zeta)$ we have $\gamma(E) = 0$. If $\gamma^R(F, \omega) = 0$, then $\gamma(F) = 0$.*

Theorem 5.4. *Suppose $\omega(\zeta) \in C_R(F)$ and $\omega(\zeta) \geq 0$ on F. Suppose also that $\gamma(E) = 0$ for the zero set E of $\omega(\zeta)$. If $\gamma^+(F, \omega) = 0$, then $\gamma(F) = 0$.*

Theorem 5.3 is proved in the same way as Theorem 5.1, whereas Theorem 5.4 is obvious, and no additional conditions on the structure of F are required for it. We note that, in order to verify the condition $\gamma^R(F, \omega) = 0$, we need to examine a smaller collection of measures than for the verification of the condition $\gamma^c(F, \omega) = 0$. The condition $\gamma^+(F, \omega) = 0$ involves an even smaller collection.

Chapter II. New modifications of the analytic capacity

§6. Approximation by cones of analytic functions on a compact set

In this chapter we continue to study approximation with size constraints. In contrast to Chapter I, we shall use the content of §1 entirely, because we shall approximate by elements of some cones that consist of analytic or rational functions. In accordance with §1, such approximation processes admit a dual characterization in terms of extremal values of certain linear functionals. These extremal values can be viewed as further modifications of the notion of analytic capacity. Often, the spaces we deal with will consist of complex-valued functions. Since the machinery of §1 applies to spaces over the reals, we shall often need to view a complex linear space X as a real space to be denoted by X_R. Thus, X and X_R consist of the same elements, but the distinction is that multiplication by imaginary scalars is not permitted in X_R. So, $C(F)_R$ is the space of complex-valued continuous functions on F regarded as a space over the reals, whereas the space of real-valued continuous functions on F is denoted by $C_R(F)$. The topological dual to a linear topological space X will be denoted by X^* (in distinction from X', which is the algebraic dual of X). The standard proof of the Hahn–Banach theorem for complex scalars shows that if X is a complex linear topological space, then

$$l \in X_R^* \iff \exists L \in X^* : l(x) = \operatorname{Re} L(x) \text{ for all } x \in X. \tag{6.1}$$

In $C(F)$, we distinguish the subspace Y of analytic functions on F; in order to be able to identify such functions with their restrictions to F, we always assume that F has no isolated points (otherwise some unessential explanations would have been needed frequently). Let Q be a domain with $Q \supset F$, and let $h(z) > 0$ be a function defined and continuous on $Q \cap G$. In Y, we introduce a pre-norm P_h as follows:

$$P_h(y) = \inf \|h(z)\nu\| = \int_{S_\nu} h(z)\,|d\nu|, \quad y \in Y, \tag{6.2}$$

where the infimum is taken over all complex measures ν satisfying

$$S_\nu \subset Q \cap G; \quad \widehat{\nu}(\zeta) = y(\zeta), \;\; \zeta \in F. \tag{6.3}$$

The pre-norm (6.2) turns Y into a linear topological space (Y, P_h). If $f(z)$ is an analytic function on G, it determines a linear functional

$$\Lambda_f(y) = \frac{1}{2\pi i}\int_{\partial D} f(z)y(z)\,dz \tag{6.4}$$

on Y, where D is a neighborhood of F that has rectifiable boundary and possesses the property that $f(z)$ is analytic in $\overline{D}$. Without loss of generality, we may also assume that $\overline{D} \subset Q$. Let B_h denote the set of analytic functions $f(z)$ in G such that

$$\|f\|_h \stackrel{\text{def}}{=} \sup_{z\in Q\cap G}\left|\frac{f(z)}{h(z)}\right| < +\infty. \tag{6.5}$$

An important result by Khavin says that the space $(Y, P_h)^*$ consists of functionals of the form (6.4) such that the analytic function $f(z)$ is in B_h; moreover,

$$\|\Lambda_f\| = \|f\|_h. \tag{6.6}$$

We denote by B_h^1 the collection of functions $f(z)$ in B_h with $\|f\|_h \le 1$ (i.e., the unit ball of B_h). It should be noted that the case where $h(z) \equiv 1$ in G was considered in Theorem 1 in [24], [25], but Remark 4 in [25] extends the result to an arbitrary continuous weight $h(z) > 0$ defined on G. The fact that in our case the weight is defined only on $Q \cap G$ affects neither the result itself, nor its proof. We note that the case of $h(z) \equiv 1$ was also treated in [56, Theorem 4.5].

Along with Y, we consider the subspace Y^1 of $C(F)$ that consists of rational functions of the following form:

(6.7)

$$y(\zeta) \in Y^1 \iff y(\zeta) = \sum_1^n \nu_j(\zeta - a_j)^{-1},\; n = 1, \dots,\;\; a_j \in Q\cap G,\;\; a_j \neq a_k \text{ for } j \neq k.$$

In Y^1, we introduce the norm

$$p_h(y) = \sum_1^n h(a_j)|\nu_j|. \tag{6.8}$$

We have already encountered some special cases of Y^1 and p_h: in §2 we took $Q = \overline{\mathbb{C}}$ and, consequently, $Q \cap G = G$, and the weight $h(z)$ was identically equal either to 1 or to $|L(z)|$; the weight $h(z) \equiv 1$ arose also in the proof of Theorem 5.2. In §3 we considered the above subspace Y and took the absolute value of an arbitrary analytic function $\phi(z)$ with $\phi(z) \neq 0$, $\phi(\infty) = 1$ for the role of a weight.

We agree that the functions in question take values in the complex w-plane. In this plane, we distinguish an angle Δ with vertex at $w = 0$. The opening of Δ will be denoted by $|\Delta|$. We always assume that Δ is a convex set. Consequently,

$$\text{either } \;|\Delta| \le \pi, \;\text{ or } \Delta \text{ is the entire } w\text{-plane.} \tag{6.9}$$

Finally, we can define the main objects to be treated in this section:

$$\begin{aligned} E_\Delta &= \{y(\zeta) \in Y : y(F) \subset \Delta\} \cup \{\mathbb{O}\}, \\ E_\Delta^1 &= \{y(\zeta) \in Y^1 : y(F) \subset \Delta\} \cup \{\mathbb{O}\}. \end{aligned} \tag{6.10}$$

Here $\mathbb{O}$ denotes the function on F identically equal to zero. The following statement is obvious.

Lemma 6.1. *If Δ is not the entire w-plane, then E_Δ and E^1_Δ are cones. Otherwise, $E_\Delta = Y$ and $E^1_\Delta = Y^1$.*

Let X be a Banach space of complex-valued functions on F such that

$$\begin{cases} X \supset C(F) \text{ (in the set-theoretic sense), and} \\ \text{the embedding } C(F) \to X \text{ is continuous.} \end{cases} \tag{6.11}$$

As X, we can take, e.g., the Lebesgue spaces $L^\delta(F, \beta)$, where $\beta \geq 0$ is a fixed measure on F and $\delta \geq 1$. Let $l \in X^*$. The function

$$\hat{l}(z) = l_\zeta\left(\frac{1}{\zeta - z}\right), \quad z \in G, \tag{6.12}$$

will be called the Cauchy transform of l. The subscript ζ in (6.12) means that $(\zeta - z)^{-1}$ is viewed as a function of ζ, and z plays the role of a parameter. (If l is determined by a measure μ, the function $\hat{l}(z) = \widehat{\mu}(z)$ is the Cauchy transform of μ.) The following lemma is easy.

Lemma 6.2. *Let $l \in X^*$, and let X satisfy* (6.11)*. Then $\hat{l}(z)$ is an analytic function in G, and for $y = y(\zeta) \in Y$ we have*

$$l(y) = \frac{1}{2\pi i} \int_{\partial D\downarrow} \hat{l}(z) y(z)\, dz, \tag{6.13}$$

where D is a domain as in (6.4)*.*

Finally, suppose that X is a space satisfying (6.11), and that a nonnegative convex continuous functional $r(x)$ is defined on X. We shall approximate an arbitrary element $\omega \in X$ by elements of the cones E_Δ and E^1_Δ with respect to the distance determined by r and with size constraints expressed in terms the pre-norms $P_h(y)$ on Y and $p_h(y)$ on Y^1.

Lemma 6.3. *Let $\omega \in X$. Suppose there exists a sequence $\{y_k\} \subset E_\Delta$ such that*

$$r(\omega - y_k) \to 0, \quad r \to \infty. \tag{6.14}$$

Then there exists a sequence $\{y^1_k\} \subset E^1_\Delta$ with

$$r(\omega - y^1_k) \to 0, \quad k \to \infty. \tag{6.15}$$

Proof. For every y_k, $k = 1, 2, \dots$, there exists a sequence $\{z^k_n\}$ that tends in $C(F)$ (as $n \to \infty$) to some element y_k. To see this, it suffices to represent $y_k(\zeta)$ by the Cauchy integral formula over the boundary ∂D of a "good" domain $D \supset F$, $\overline{D} \subset Q$, and to take a sequence $\{z^k_n\} \to y_k$ of integral sums that tend to y_k as $n \to \infty$ in $C(F)$ and, consequently, in X. In the w-plane, the image $y_k(F)$ is included in the angle Δ. Since Δ is open, for $n \geq n_0$ the images $z^k_n(F)$ are also included in Δ. Therefore, $\{z^k_n\} \subset F^1_\Delta$ for $n \geq n_0$. Since r is a continuous functional on X, we have

$$r(y_k - z^k_n) \to 0, \quad n \to \infty. \tag{6.16}$$

Now, we fix $\varepsilon_k > 0$ with $\varepsilon_k \to 0$. For fixed k, among the terms of the sequence $\{z^k_n\}$ we find an element $y^1_k \in E^1_\Delta$ such that

$$r(y_k - y^1_k) < \varepsilon_k. \tag{6.17}$$

Since

$$r(\omega - y^1_k) \leq r(\omega - y_k) + r(y_k - y^1_k),$$

from (6.14) and (6.17) we deduce (6.15).

In accordance with formulas (1.9)–(1.11) in §1, for $\omega \in X$ we define the quantity

$$\widetilde{P}_h(\omega) = \inf \varliminf_{k\to\infty} P_h(y_k), \tag{6.18}$$

where the infimum is taken over all sequences $\{y_k\} \subset E_\Delta$ that satisfy (6.14); we put

$$\widetilde{P}_h(\omega) = \infty \tag{6.19}$$

if there are no such sequences. Similarly, we define

$$\tilde{p}_h(\omega) = \inf \varliminf_{k} p_h(y_k^1), \tag{6.20}$$

where the infimum is taken over all sequences $\{y_k^1\} \subset E_\Delta^1$ that satisfy (6.15), and we put

$$\tilde{p}_h(\omega) = \infty \tag{6.21}$$

if there are no such sequences.

Clearly, $\widetilde{P}_h(y) \le P_h(y)$ for $y \in Y$ and $\tilde{p}_h(y) \le p_h(y)$ for $y \in Y^1$. Since P_h and p_h are not related directly either to r or to the norm in X, these inequalities may be strict. (The case of (6.19) is impossible for $y \in Y$.) When various h, r, and Δ occur, we shall use more detailed notation for $\widetilde{P}_h$ and $\tilde{p}_h$, namely

$$\widetilde{P}_{h,r,\Delta} \quad \text{and} \quad \tilde{p}_{h,r,\Delta}. \tag{6.22}$$

□

Proposition 6.4. *If $\omega \in X$, then*

$$\widetilde{P}_h(\omega) = \tilde{p}_h(\omega). \tag{6.23}$$

Proof. If there are no sequences $\{y_k^1\} \subset E_\Delta^1$ that converge to ω in the sense of (6.15), then, by Lemma 6.3, there are also no sequences $\{y_k\} \subset E_\Delta$ satisfying (6.14). Thus, in this case $\widetilde{P}_h(\omega) = \tilde{p}_h(\omega) = \infty$. Suppose a sequence $\{y_k^1\} \subset E_\Delta^1 \subset E_\Delta$ converges to ω in the sense of (6.15). Since $p_h(y_k^1) \ge P_h(y_k^1)$, we have $\varliminf_{k\to\infty} p_h(y_k^1) \ge \varliminf_{k\to\infty} P_h(y_k^1)$, and, therefore,

$$\tilde{p}_h(\omega) \ge \widetilde{P}_h(\omega). \tag{6.24}$$

Now, let $\varepsilon > 0$ be arbitrarily small, and let a sequence $\{y_k\} \subset E_\Delta$ satisfy (6.14) and also the condition

$$\varliminf_{k\to\infty} P_h(y_k) < \widetilde{P}_h(\omega) + \varepsilon. \tag{6.25}$$

Also, we take positive numbers ε_k with $\varepsilon_k \to 0$ as $k \to \infty$. Recalling the definition of the $P_h(y_k)$ (see formulas (6.2)–(6.3)), we find measures ν_k with $S_{\nu_k} \subset G \cap Q$ such that

$$y_k(\zeta) = \widehat{\nu}_k(\zeta), \quad \|h\nu_k\| < P_h(y_k) + \varepsilon_k. \tag{6.26}$$

We replace the integral in the formula for $\widehat{\nu}_k(\zeta)$ by an integral sum $y_k^1(\zeta) \in E_\Delta^1$ (see the proof of Lemma 6.3) so as to have

$$p_h(y_k^1) = \|h\nu_k\| < P_h(y_k) + \varepsilon_k, \quad r(y_k - y_k^1) < \varepsilon_k.$$

Then the sequence $\{y_k^1\} \subset E_\Delta^1$ satisfies (6.15), and

$$\widetilde{p}_h(\omega) \le \varliminf_{k\to\infty} p_h(y_k^1) \le \varliminf_{k\to\infty} P_h(y_k) < \widetilde{P}_h(\omega) + \varepsilon. \tag{6.27}$$

Comparison of (6.24) and (6.27) yields (6.23). □

Theorem 6.5. *If $\omega \in X$, then*

$$\tilde{p}_h(\omega) = \widetilde{P}_h(\omega) = \max \operatorname{Re} l(\omega), \tag{6.28}$$

where the maximum is taken over all linear functionals $l \in X^*$ such that $\operatorname{Re} l \in R(r)$ and there is an analytic function $f(z)$ in G with $\|f\|_h \le 1$ and

$$\operatorname{Re}\left[\frac{1}{2\pi i}\int_{\partial D\uparrow}(\hat{l}(z) + f(z))y(z)\,dz\right] \ge 0 \quad \textit{for all } y \in E_\Delta. \tag{6.29}$$

Here D is a neighborhood of F with rectifiable boundary ∂D and such that $y(z)$ is analytic in $\overline{D}$. (*Generally speaking, D depends on the function $y \in E_\Delta$.*)

Proof. We treat X and Y as the real spaces X_R and Y_R, and apply Theorem 1.2. Since the convex functional $r(x)$ is continuous, the sets $R(r, M)$ and $R(r)$ consist of continuous linear functionals on X_R. Every such functional has the form $\operatorname{Re} l$, where $l \in X^*$. Consider the set $R(P_h, 1)$ of linear functionals λ on the space Y_R with the pre-norm P_h. Every such functional has the form $\lambda = \operatorname{Re} L$, where $L \in (Y, P_h)^*$. Since P_h is a symmetric functional (a pre-norm), the relation $\lambda \in R(P_h, 1)$ is equivalent to the inequality $\|\lambda\| \le 1$; moreover, $\|\lambda\| = \|\operatorname{Re} L\| = \|L\|$. The functional l acts on $y \in Y^*$ in accordance with (6.13), and the functional L acts in accordance with (6.4), where, by the Khavin theorem cited above, the function $f(z)$ is analytic in G and satisfies $\|f\|_h = \|L\| < +\infty$. By Theorem 1.2, $\operatorname{Re} L \in R(P_h, 1)$, whence we obtain $\|f\|_h \le 1$. Condition (6.29) expresses the requirement

$$\operatorname{Re} l - \lambda = \operatorname{Re}[l - L] \in E_\Delta^-$$

in Theorem 1.2 (see (1.12)–(1.14)). In a similar way, on the basis of Theorem 1.1 (see formulas (1.6)–(1.7)), we deduce the following statement. □

Theorem 6.6. *We have*

$$\inf_{y \in E_\Delta}[r(\omega - y) + P_h(y)] = \inf_{y \in E_\Delta^1}[r(\omega - y) + p_h(y)] = \max \operatorname{Re} l(\omega), \tag{6.30}$$

where the maximum is taken over all $l \in X^*$ such that $\operatorname{Re} l \in R(r, 1)$ and there is an analytic function $f(z)$ in G with $\|f\|_h \le 1$ and satisfying (6.29).

Theorem 6.7. *If the class B_h of analytic functions $f(z)$ in G with finite norm $\|f\|_h$ is trivial* (*contains only constant functions*), *then for every element $\omega \in X$ admitting a sequence $\{y_k\} \subset E_\Delta$ that approximates ω as in* (6.14) *we have $\widetilde{P}_h(\omega) = \tilde{p}_h(\omega) = 0$, and for the other elements ω of X we have $\widetilde{P}_h(\omega) = \tilde{p}_h(\omega) = \infty$. So the first statement of the theorem says that the functions $\omega \in X$ admitting "usual" approximation as in* (6.14) *admit also $o(r, P_h)$- and $o(r, p_h)$-approximation.*

Proof. Since B_h is trivial, the maximum in (6.28) is taken over all $l \in X^*$ satisfying $\operatorname{Re} l \in R(r) \cap E_\Delta^-$. Since r is continuous, for ω satisfying (6.14) we have

$$\operatorname{Re} l(\omega) = \lim_{k\to\infty} \operatorname{Re} l(y_k) \le 0.$$

Therefore, the maximum in (6.28) is attained on the functional $l \equiv 0$, and $\widetilde{P}_h(\omega) = \tilde{p}_h(\omega) = 0$. But there are no sequences $\{y_k\} \subset E_\Delta$ convergent to ω in the sense of (6.14), then $\widetilde{P}_h(\omega) = \infty$ by definition. □

Consider the case where $h(z) \equiv 1$. Then B_h is simply the class $B(G)$ of bounded functions analytic in G, and this class is trivial if and only if $\gamma(F) = 0$. In this case we denote P_h and p_h simply by P and p. Now for $y \in E_\Delta$ we have

$$P(y) = \inf\{\|\nu\| : S_\nu \subset G,\ \widehat{\nu}(\zeta) \equiv y(\zeta),\ \zeta \in F\},$$

and for $y \in E^1_\Delta$ we have

$$p(y) = \inf\Big\{\sum |\nu_k| : \sum \nu_k(\zeta - a_k)^{-1} \equiv y(\zeta),\ a_k \in G\Big\}.$$

(We encountered these functionals earlier.) This yields the following statement.

Corollary 6.8. *If $\gamma(F) = 0$, then for the elements $\omega \in X$ admitting a sequence $\{y_k\} \subset E_\Delta$ that converges to ω in the sense of* (6.14) *we have $\widetilde{P}(\omega) = \tilde{p}(\omega) = 0$, and for the other $\omega \in X$ we have $\widetilde{P}(\omega) = \tilde{p}(\omega) = \infty$.*

Let C_Δ be the set of functions $\varphi(\zeta)$ in $C(F)$ with $\varphi(F) \subset \Delta$, and let $\overline{C}_\Delta$ be the closure of C_Δ in X relative to the metric generated by r. It is easily seen that, if $\gamma(F) = 0$, then the condition $\widetilde{P}(\omega) = \tilde{p}(\omega) = 0$ is fulfilled if and only if $\omega \in \overline{C}_\Delta$. If $X = C(F)$, r is the norm in $C(F)$, and Δ is the entire plane, then the condition $\gamma(F) = 0$ implies that $\overline{C}_\Delta$ coincides with $C(F)$, and Corollary 6.8 yields the following:

$$\text{for every } \omega \in C(F) \text{ we have } \widetilde{P}(\omega) = \tilde{p}(\omega) = 0. \tag{6.31}$$

This fact was established by Khavin [24], [25] and Khavinson [23].

In conclusion of this section, we present a new and shorter proof of Theorem 3.1. We have already cited and applied Khavin's result saying that the space B_h is isometrically isomorphic to $(Y, P_h)^*$. By a well-known consequence of the Hahn–Banach theorem, for every $y \in Y$ we have

$$P_h(y) = \sup_{\substack{f \in B_h \\ \|f\|_h \le 1}} \left| \frac{1}{2\pi} \int_{\partial D} f(z) y(z)\, dz \right|, \tag{6.32}$$

where D is a neighborhood of F such that the function $y(z)$ is analytic in $\overline{D}$. In particular, for the function $y(\zeta) \equiv 1$, which will be denoted simply by $\mathbb{1}$, we have

$$P_h(\mathbb{1}) = \sup_{f, \|f\|_h \le 1} \left| \frac{1}{2\pi} \int_{\partial D} f(z)\, dz \right|. \tag{6.33}$$

Moreover, the suprema in (6.32) and (6.33) are always attained. Let $h(z) = |\phi(z)|$, where $\phi(z)$ is an analytic function in G satisfying (3.1). Then the condition $\|f\|_h \le 1$ is equivalent to

$$f(z) = \varphi(z)\phi(z), \tag{6.34}$$

where $\varphi(z)$ is analytic in G and $|\varphi(z)| \le 1$ for $z \in G$. By (6.33),

$$P_h(\mathbb{1}) = \max_{|\varphi(z)| \le 1} \frac{1}{2\pi} \left| \int_{\partial D} \varphi(z)\phi(z)\, dz \right| = \max_{|\varphi(z)| \le 1} \frac{1}{2\pi} \left| \int_{\partial D} \varphi(z)\, dz \right|. \tag{6.35}$$

By the definition of $P_h(\mathbb{1})$ (see (6.2)), identity (6.35) is the claim of Theorem 3.1.

§7. Representing measures for positive linear functionals on the cone E_Δ. Cauchy potentials with such measures

Let $X = C(F)$. Then X^* consists of regular Borel measures $d\mu = d\mu_1 + i d\mu_2$, where μ_1 and μ_2 are real measures,

$$\operatorname{Re} \int_F (u + iv)\, d\mu = \int_F u\, d\mu_1 - v\, d\mu_2. \tag{7.1}$$

So, an arbitrary linear functional in $C(F)^*_R$ is represented by a pair (μ_1, μ_2) of real measures; the action of this pair on a function $u + iv \in C(F)_R$ is described by (7.1). In X, we distinguish the cone X_Δ consisting of the function $\mathbb{O}$ and all functions $\varphi(\zeta) \in X$ the values of which lie in the open angle Δ. We restrict ourselves to the angles Δ of opening smaller than π whose bisector coincides with the positive real axis. Let $y = \pm kx$ be the equations of the angle sides, and let $k = \tan\frac{|\Delta|}{2}$.

Theorem 7.1. *A linear functional in $C(F)^*_R$ is positive on X_Δ if and only if the measures (μ_1, μ_2) occurring in its representation* (7.1) *satisfy the following condition*:

$$\mu_1 - k|\mu_2| = \mu_1 - \tan\frac{|\Delta|}{2}|\mu_2| \quad \textit{is a positive measure.} \tag{7.2}$$

Here $|\mu_2|$ is the measure coinciding with the total variation of μ.

(If should be noted that, by (7.2), μ_1 is a positive measure *a fortiori*.)

Proof. The "if" part. If $u + iv \in X_\Delta$, then $u(\zeta) > 0$ and $|v(\zeta)| < ku(\zeta)$. We have

$$\int_F u\,d\mu_1 - v\,d\mu_2 \geq \int_F u\,d\mu_1 - |v|\,|d\mu_2| \geq \int_F u[d\mu_1 - k\,|d\mu_2|] > 0.$$

The "only if" part. Let u be a positive function in $X = C(F)$. Then $(1 + \tau i)u \in X_\Delta$ for all $\tau \in (-k, k)$. Consequently,

$$\int u\,d\mu_1 - \tau \int u\,d\mu_2 \geq 0$$

for every positive function $u \in X$ and every $\tau \in (-k, k)$. Therefore, $\mu_1 - \tau\mu_2 \geq 0$ for all $\tau \in (-k, k)$, whence we see that $\mu_1 \geq k\mu_2$ and $\mu_1 \geq -k\mu_2$. Now the definition of the total variation implies that $\mu_1 \geq k|\mu_2|$. □

Corollary 7.2. *If Δ_0 is the right half-plane, then $X^+_{\Delta_0}$ consists of the positive measures on F.*

Indeed, an arbitrary functional $(\mu_1, \mu_2) \in X^+_{\Delta_0}$ must belong to every set X^+_Δ, where Δ is an angle of opening strictly smaller than π. By (7.2), the measure $\mu_1 - k|\mu_2|$ must be positive for arbitrarily large $k > 0$. But this is possible only if $\mu_2 \equiv 0$. Surely, this corollary is a well-known fact, which can easily be checked directly.

Theorem 7.3. *The functionals belonging to E^+_Δ and E^{1+}_Δ are described by pairs (μ_1, μ_2) of measures as in Theorem* 7.1 (*i.e., by pairs satisfying* (7.2)).

Proof. Clearly, $E_\Delta = Y \cap X_\Delta$ and $E^1_\Delta = Y^1 \cap X_\Delta$, where Y and Y^1 are the subspaces of $C(F)$ described in §6 in the definition of E_Δ and E^1_Δ. Let $u_0 > 0$ be a constant. The function $\varphi(\zeta) \equiv u_0$, $\zeta \in F$, belongs to $E_\Delta \subset X_\Delta$ and is an interior point of X_Δ (note that $X_\Delta \setminus \mathbb{O}$ is open in $C(F)$). By the M. G. Kreĭn theorem (see, for instance, [58, p. 63]), in this case a positive linear functional on E_Δ can be extended to a positive linear functional on X_Δ; therefore, the description in Theorem 7.1 applies. Now we approximate the analytic function $\varphi(\zeta) \equiv u_0 > 0$ on F by sums of elementary fractions with poles in $G \cap Q$ (i.e., by elements of Y^1) to see that E^1_Δ contains some interior points of X_Δ. Consequently, a positive functional on E^1_Δ extends to a positive functional on X_Δ and, again, admits a description as in (7.2). □

Theorem 7.4. *An analytic function $f(z)$ in a domain G with $f(\infty) = 0$ is representable as the Cauchy potential of a measure $\mu = \mu_1 + i\mu_2$ with $S_\mu \subset F$ such that μ_1 and μ_2 are real measures satisfying* (7.2) *if and only if the functional Λ_f given by* (6.4) *satisfies the condition*

$$\operatorname{Re}\Lambda_f(y) = \operatorname{Re}\left[\frac{1}{2\pi i}\int_{\partial D\downarrow} f(z)y(z)\,dz\right] \geq 0, \quad y \in E_\Delta. \tag{7.3}$$

Proof. The "only if" part. Let $f(z) = \widehat{\mu}(z)$, where $S_\mu \subset F$ and μ_1, μ_2 satisfy (7.2). By Theorem 7.3, we have

$$\operatorname{Re} \int_F y(\zeta)\, d\mu \geq 0, \quad y \in E_\Delta. \tag{7.4}$$

However, for $y \in E_\Delta$ we have

$$\operatorname{Re} \int_F y(\zeta)\, d\mu = \operatorname{Re} \left[\frac{1}{2\pi i} \int_{\partial D \downarrow} \widehat{\mu}(z) y(z)\, dz \right]. \tag{7.5}$$

Thus, (7.4) is the same as (7.3).

The "if" part. Suppose $f(z)$ satisfies (7.5). On the subspace $Y \subset C(F)_R$ (we remind the reader that Y consists of all analytic functions on F), consider the functional

$$\operatorname{Re} \Lambda_f(y) = \operatorname{Re} \left[\frac{1}{2\pi i} \int_{\partial D \downarrow} f(z) y(z)\, dz \right]. \tag{7.6}$$

Condition (7.3) means that this functional is positive on E_Δ. By Theorem 7.3, there is a measure $\mu = \mu_1 + i\mu_2$ such that μ_1 and μ_2 are real and satisfy (7.2) and the linear functional

$$\operatorname{Re} \int_F y\, d\mu$$

represented by μ on $C(F)_R$ coincides with $\operatorname{Re} \Lambda_f(y)$ on Y. But (7.5) implies that

$$\begin{aligned} \operatorname{Re} \int_F y\, d\mu &= \operatorname{Re} \left[\frac{1}{2\pi i} \int_{\partial D \downarrow} \widehat{\mu}(z) y(z)\, dz \right] \\ &= \operatorname{Re} \left[\frac{1}{2\pi i} \int_{\partial D \downarrow} f(z) y(z)\, dz \right], \quad y \in Y. \end{aligned} \tag{7.7}$$

Let $z_0 \in G$ be an arbitrary point. The function $(\zeta - z_0)^{-1}$ belongs to Y. Taking $D \supset F$ in such a way that $z_0 \notin \overline{D}$, from (7.7) we deduce the formula

$$\operatorname{Re} \left[\frac{1}{2\pi i} \int_{\partial D \downarrow} \frac{\widehat{\mu}(z)\, dz}{z - z_0} \right] = \operatorname{Re} \widehat{\mu}(z_0) = \operatorname{Re} \left[\frac{1}{2\pi i} \int_{\partial D \downarrow} \frac{f(z)\, dz}{z - z_0} \right] = \operatorname{Re} f(z_0).$$

Thus, we have two analytic functions $\widehat{\mu}(z)$ and $f(z)$ in G satisfying

$$\operatorname{Re} \widehat{\mu}(z) \equiv \operatorname{Re} f(z), \quad \widehat{\mu}(\infty) = 0 = f(\infty).$$

Therefore,

$$f(z) \equiv \widehat{\mu}(z), \quad z \in G,$$

and the theorem is proved. □

Corollary 7.5. *An analytic function $f(z)$ in G satisfying $f(\infty) = 0$ is representable by the Cauchy potential $\widehat{\mu}(z)$ of a positive measure μ on F if and only if* (7.3) *is fulfilled for all $y \in Y$ such that* $\operatorname{Re} y(\zeta) \geq 0$ *for $\zeta \in F$.*

Proof. Here Δ is the right half-plane, and, together with Theorem 7.4, we must use Corollary 7.2. □

Corollary 7.5 is a special case of [8, Proposition 1.3], where a more general approximation device than the Cauchy integral was applied, namely, the Golubev sums. In §1 of that paper detailed references can be found to papers on criteria of representability by Cauchy potentials (the literature devoted to this question is fairly extensive).

§8. Further results on approximation by cones of analytic functions, and new modifications of analytic capacity

As in §7, here we assume that the positive real semiaxis is the bisector of an angle Δ.

Proposition 8.1. *In formula* (6.29) *for approximation problems treated in Theorems* 6.4 *and* 6.5, *the function* $\hat{l}(z) + f(z)$ *can be represented as* $[-\hat{\lambda}(z)]$, *where* $\lambda = \lambda_1 + i\lambda_2$ *and the real measures* λ_1 *and* λ_2 *satisfy* (7.2).

This is an immediate consequence of Theorem 7.4 and inequality (6.29).

Let M_Δ denote the class of complex measures $\mu = \mu_1 + i\mu_2$ such that the real measures μ_1 and μ_2 satisfy (7.2). Thus, in Proposition 8.1 a measure $\lambda \in M_\Delta$ was discussed.

Proposition 8.2. *Suppose that, in the approximation problems treated in Theorems* 6.4 *and* 6.5, *the space* X *has the property that every linear functional* $l \in X^*$ *can be represented by a measure* μ *with* $S_\mu \subset F$:

$$l(x) = \int_F x\, d\mu, \quad x \in X. \tag{8.1}$$

Then the function $f(z) \in B_h^1$ *in* (6.29) *is representable by the Cauchy potential of a measure* σ *with* $S_\sigma \subset F$.

Proof. In the case in question we have $\hat{l}(z) = \widehat{\mu}(z)$. The preceding proposition implies that

$$\hat{l}(z) + f(z) = [-\hat{\lambda}(z)],$$

whence

$$f(z) = \hat{\sigma}(z), \quad \sigma = -[\mu + \lambda]. \tag{8.2}$$

The condition of representability of functionals by measures is fulfilled, for instance, if X is the Lebesgue space $L^\delta(F, \beta)$, where β is a positive measure on F and $\delta \geq 1$. □

Proposition 8.3. *If* X *is as in Proposition* 8.2, *and the only function in* $\mathcal{B}_h$ *representable by a Cauchy potential is identically zero*, *then the conclusion of Theorem* 6.7 *is fulfilled.*

This is an immediate consequence of Proposition 8.2 and Theorem 6.7. From now on and till the end of this section, we take $X = C(F)$.

Let $y_0(\zeta)$ be a function belonging to E_Δ^1 and to the interior of the cone X_Δ. The existence of such a function was mentioned in Theorem 7.3. Next, let $r(x)$ be the convex functional on $C(F)$ constructed in accordance with (1.17) by the Minkowski functional r_0 of the shifted cone $X_\Delta - y_0$. For $\omega \in C(F)$, we define $\tilde{p}_{h,r,\Delta}(\omega)$ and $\widetilde{P}_{h,r,\Delta}(\omega)$ by (6.18)–(6.21). Now, conditions (6.14) or (6.15) mean that there exists a sequence $\{t_k\}$ of positive numbers with $t_k \to 0$ and

$$-\omega + t_k y_0 + y_k \in X_\Delta. \tag{8.3}$$

Clearly, now the quantities $\tilde{p}_{h,r,\Delta}(\omega)$ and $\widetilde{P}_{h,r,\Delta}(\omega)$ are finite for every $\omega \in C(F)$.

Theorem 8.4. *Let* $X = C(F)$, *and let the functional* r *be defined by* (1.17) *starting with the cone* $X_\Delta - y_0$. *Then*

$$\widetilde{P}_{h,r,\Delta}(\omega) = \tilde{p}_{h,r,\Delta}(\omega) = \max \operatorname{Re} \int_F \omega\, d\mu \tag{8.4}$$

for every $\omega \in X$, *where the maximum is taken over the measures* $\mu \in M_\Delta$ *for which there exists a measure* $\lambda \in M_\Delta$ *with*

$$\|\hat{\mu}(z) + \hat{\lambda}(z)\|_h \leq 1. \tag{8.5}$$

Proof. We employ Theorems 1.1 and 6.5. Since the functional r is continuous, the set $R(r)$ consists of functionals of the form $\operatorname{Re} l$, $l \in X^*$, which are continuous on X_R. Such an l is represented by a measure μ. But in Lemma 1.3 it was shown that $R(r) = X^+_\Delta$ (see (2.18)); therefore, $\mu \in M_\Delta$. By Proposition 8.1, the function $f(z)$ in (6.29) can be represented in the form $f(z) = -[\widehat{\mu}(z) + \hat{\lambda}(z)]$ with $\lambda \in M_\Delta$ (see (8.2)). The condition $\|f\|_h \le 1$ in Theorem 6.5 coincides with (8.5), and the theorem is proved. □

Theorem 8.5. *Under the assumptions of the preceding theorem, let $\omega(\zeta) \ge 0$ on F. Then*

$$\widetilde{P}_{h,r,\Delta}(\omega) = \tilde{p}_{h,r,\Delta}(\omega) = \max \operatorname{Re} \int_F \omega \, d\mu, \tag{8.6}$$

where the maximum is taken over the measures $\mu \in M_\Delta$ satisfying

$$\|\widehat{\mu}(z)\|_h \le 1. \tag{8.7}$$

Proof. Let $\mu = \mu_1 + i\mu_2 \in M_\Delta$ be a measure as in Theorem 8.4. For it there is a measure $\lambda = \lambda_1 + i\lambda_2 \in M_\Delta$ such that (8.5) is fulfilled. The measure $\widetilde{\mu} = \mu + \lambda$ belongs to M_Δ, and

$$\operatorname{Re} \int_F \omega \, d\mu = \int_F \omega \, d\mu_1 \le \int_F \omega \, [d\mu_1 + d\lambda_1] = \operatorname{Re} \int_F \omega \, d\widetilde{\mu}. \tag{8.8}$$

Since (8.5) is fulfilled for $\widetilde{\mu}$ with $\tilde{\lambda} \equiv 0$, we see that $\widetilde{\mu}$ is among the measures occurring in (8.4), and (8.8) shows that in (8.4) we can restrict ourselves to measures satisfying (8.7).

The quantities

$$\gamma^+_{h,\Delta}(F) = \max \operatorname{Re} \int_F d\mu, \tag{8.9}$$

where the maximum is taken over the measures $\mu \in M_\Delta$ satisfying (8.7), are natural analogs (and generalizations) of the capacity γ^+. It is natural to call them "angular capacities" and to apply them in connection with classes of analytic functions possessing a majorant h. By (8.6), we have

$$\gamma^+_{h,\Delta}(F) = \tilde{p}_{h,r,\Delta}(\mathbb{1}) = \widetilde{P}_{h,r,\Delta}(\mathbb{1}), \tag{8.10}$$

where $\mathbb{1}$ is the function $\omega(\zeta) \equiv 1$, $\zeta \in F$. If Δ is the right half-plane, the corresponding quantity in (8.10) is denoted by $\gamma^+_h(F)$. In this case $M_\Delta = M^+$ is the collection of all positive measures on F. If $h(z) \equiv 1$, then $\gamma^+_h(F) = \gamma^+(F)$. If $h(z) = |L(z)|$, where $L(z)$ is the Garabedian function, then the explanations before formulas (2.21)–(2.24) show that $\gamma^+_{|L|}(F) = \gamma(F)$ is the usual analytic capacity of F. We introduce also the quantity

$$\gamma^c_h(F) = \sup \left| \int_F d\mu \right|, \tag{8.11}$$

where the supremum is taken over all complex measures μ on F satisfying (8.7). The following inequalities are obvious:

$$\gamma^+_h(F) \le \gamma^+_{h,\Delta_1}(F) \le \gamma^+_{h,\Delta_2}(F) \le \gamma^c_h(F), \quad 0 < |\Delta_2| < |\Delta_1| < \pi. \tag{8.12}$$

Therefore,

$$\text{if } \gamma^c_h(F) = 0, \text{ then } \gamma^+_{h,\Delta}(F) = 0 \text{ for } 0 < |\Delta| \le \pi. \tag{8.13}$$

In the case where the weight $h(z)$ is identically equal to 1, the remarkable results of [17], [54], [55] cited in the Introduction and in §5 yield the converse statement: $\gamma^c(F) > 0 \implies \gamma^+(F) > 0$. In the next section we shall prove a sort of converse to (8.13) in the weighted case. □

§9. Equimeasurability theorem for angular capacities

A complex measure μ on a compact set F is said to be *strongly regular* if for every closed subset $E \subset F$ there exists an open set T such that $E \subset T \subset F$ and $|\mu|(E) = |\mu|(T)$. In the case of usual regularity, we can only assert that there is a G_δ-set T with $|\mu|(E) = |\mu|(T)$. The Lebesgue measure on a segment is not strongly regular, but for measures on totally disconnected sets the notion of strong regularity may be useful.

Theorem 9.1 (on the equimeasurability of angular capacities)**.** *Let F be a totally disconnected compact set, and let the weight $h(z)$ be a positive superharmonic function in some domain Q_1 such that $\overline{Q} \subset Q_1$. If there exists a strongly regular measure $\mu \not\equiv 0$ on F satisfying* (8.7), *then*

$$\gamma_h^c(F) > 0 \quad \text{and} \quad \gamma_{h,\Delta}^+(F) > 0 \text{ for all } \Delta. \tag{9.1}$$

Remarks. 1) First, we comment on the assumptions about the weight $h(z)$. In §6 by a weight we understood a positive continuous function on the domain $Q \cap G$, where Q was a domain containing F. In Theorem 9.1 it is required that $h(z)$ be defined on F as well, but $h(z)$ may fail to be continuous and may be infinite at some points of F. Next, since $h(z)$ is superharmonic in a domain $Q_1 \supset \overline{Q}$, we see that there exists a number A such that

$$h(z) \geq A > 0, \quad z \in \overline{Q}. \tag{9.2}$$

2) Condition (9.2) guarantees that every bounded and analytic function $f(z)$ in G satisfying $f(\infty) = 0$ belongs to B_h. Therefore, if $\gamma^c(F) > 0$, then $\gamma_h^c(F) > 0$. Next, by the "equimeasurability" theorem in [17], [54], [55], some positive measure on F also has bounded potential, so that all capacities occurring in (8.12) are positive, and this potential belongs to B_h. However, since h may grow infinitely near F, the implication $\gamma_h^c(F) > 0 \implies \gamma^c(F) > 0$ may fail in general. It should be emphasized that, for the problem in question, it is precisely the behavior of $h(z)$ near F that really matters. The behavior of $h(z)$ on the "remote" boundary $\partial\overline{Q}$ is immaterial. Unfortunately, our method of proof of Theorem 9.1 required a stronger restriction on the measure μ whose potential $\hat{\mu}(z)$ belongs to B_h. Next, Theorem 9.1 does not guarantee that $\gamma_h^+(F)$ is positive (the case where $|\Delta| = \pi$) and involves an *a priori* assumption about the structure of F.

Proof of the theorem. I. We show that $\gamma_h^c(F) > 0$. Consider arbitrary closed polygonal domains $\overline{D}$ on the plane such that $\overline{D} \cap F \neq \varnothing$, but $\partial D \cap F = \varnothing$. The sets $\overline{D} \cap F$ are closed and open simultaneously in the relative topology of F, and constitute a base of this topology. Since $\mu \not\equiv 0$, there exists a set $\overline{D}$ such that for $F_1 = \overline{D} \cap F$ we have

$$\int_{F_1} d\mu \neq 0. \tag{9.3}$$

Let μ_1 and μ_2 denote the restrictions of μ to F_1 and $F_2 = F \setminus F_1$, and let ρ_1 and ρ_2 be the distances of F_1 and F_2 to $\partial\overline{D}$. We choose a positive number t such that $t < 1/2$ and $t/\rho_i < A/2$ for $i = 1, 2$ (A is the constant in (9.2)). By (8.7), we have

$$|t\widehat{\mu}_1(z) + t\widehat{\mu}_2(z)| \leq th(z), \quad z \in \overline{Q} \setminus F.$$

For $z \in \overline{D}$ we obtain

$$|t\widehat{\mu}_1(z)| \leq th(z) + \frac{t}{\rho_2} \leq \frac{1}{2}h(z) + \frac{1}{2}A \leq h(z). \tag{9.4}$$

For $z \in Q \setminus D$ we can write

$$|t\widehat{\mu}_1(z)| \leq \frac{t}{\rho_1} < \frac{A}{2} < \frac{1}{2}h(z) < h(z). \tag{9.5}$$

Thus, the potential $\widehat{\sigma}(z)$ of the measure $\sigma = t\mu_1$ satisfies (8.7), and (9.3) yields

$$\int_F d\sigma = t\int_F d\mu_1 = t\int_{F_1} d\mu \neq 0.$$

Consequently, $\gamma_h^c(F) > 0$.

II. Returning to the initial notation, by the reduction in item I we arrive at the following setting: on a totally disconnected compact set F, we have a measure μ with

$$\int_F d\mu = B \neq 0. \tag{9.6}$$

(This measure was called σ in item I, and F_1 played the role of F.) For the Radon–Nikodym derivative of μ with respect to $|\mu|$, we have $\frac{d\mu}{d|\mu|} = e^{i\alpha(\zeta)}$, where $\alpha(\zeta)$ is a Borel measurable real function. By the Luzin theorem (see, e.g., [48], [49]), there exists a closed set $F_1 \subset F$ on which $\alpha(\zeta)$ is continuous and

$$|\mu|(F \setminus F_1) < \frac{B}{2}. \tag{9.7}$$

Since μ is strongly regular, there is an open set $T \subset F$ such that

$$F_1 \subset T \subset F, \quad |\mu|(T) = |\mu|(F_1). \tag{9.8}$$

Let μ_1 be the restriction of μ to F_1. Then

$$\int_F d\mu_1 = \int_{F_1} d\mu = \int_F d\mu - \int_{F\setminus F_1} d\mu \geq B - \int_{F\setminus F_1} |d\mu| > \frac{B}{2}. \tag{9.9}$$

On F_1 we have the formula

$$\frac{d\mu_1}{d|\mu_1|} = \frac{d\mu}{d|\mu|} = e^{i\alpha(\zeta)}, \quad \alpha(\zeta) \text{ is continuous.} \tag{9.10}$$

Next, by (9.8), F_1 is separated from $F \setminus F_1$ by the open set T. This allows us to argue as in item I (with slight changes), obtaining the measure $\sigma = t\mu_1$ with $t > 0$ on F_1 such that

$$\hat{\sigma}(z) \text{ satisfies (8.7) }, \quad \text{the function } \frac{d\sigma}{d|\sigma|} \text{ is continuous,} \quad \text{and} \quad \int_{F_1} d\sigma \neq 0. \tag{9.11}$$

Returning again to the initial notation, we arrive at the following setting:

F is a totally disconnected compact set, $\frac{d\mu}{d|\mu|} = e^{i\alpha(\zeta)}$, the function $\alpha(\zeta)$ is continuous on F, and

$$|\widehat{\mu}(z)| \leq h(z), \quad z \in Q \cap G, \quad \int_F d\mu \neq 0. \tag{9.12}$$

Denote by M^+ the collection of positive measures λ on F and consider the extremal problem

$$\gamma = \sup_{\lambda \in M^+} \int_F d\lambda : \left\| \int_F \frac{e^{i\alpha(\zeta)}\, d\lambda}{\zeta - z} \right\|_h \leq 1, \quad z \in Q \cap G. \tag{9.13}$$

Since $|\mu|$ is among the measures λ over which the supremum is taken, formula (0.12) shows that $\gamma > 0$. By duality,

$$\gamma = \inf \sum_1^n h(a_j)|\nu_j|, \tag{9.14}$$

where the infimum is taken over all n, all $\{a_j\} \subset G \cap Q$, and all $\{\nu_j\}$ with

$$\operatorname{Re} \sum_1^n \frac{\nu_j e^{i\alpha(\zeta)}}{\zeta - a_j} \geq 1, \quad \zeta \in F. \tag{9.15}$$

Relation (9.14) is proved much as (8.10) (for $|\Delta| = \pi$). However, now in the arguments leading to (8.10) (see §§6–8) we must replace the Cauchy kernel $(\zeta - z)^{-1}$ by the functions $e^{i\alpha(\zeta)}(\zeta - z)^{-1}$. Since $h(z)$ is superharmonic, in order to ensure the relation $f(z) \in B_h$ it suffices to assume that $|f(z)| \leq h(z)$ for two kinds of points: first, for z in a fixed neighborhood of E (this neighborhood may be taken as small as we wish) and, second, for the points on ∂Q (that are far away from F). Therefore, the points $\{a_j\}$ occurring in (9.14) are also of two kinds: they belong either to a certain subset dense "near" F or to a certain subset dense on ∂Q. Similarly, in all expressions like (9.14)–(9.15) to be written below, the points $\{a_j\}$ will also be of two kinds: first, belonging to an arbitrarily small neighborhood of the boundary set in question, and second, lying on ∂Q. Now, suppose that, contrary to our claim, we have $\gamma^+_{h,\Delta}(F) = 0$ for some Δ with $0 < |\Delta| < \pi$. Fixing $\varepsilon > 0$, we split F into disjoint closed subsets F_j, $j = 1, \dots, m$, in such a way that the oscillation of the real function $\alpha(\zeta)$ on each F_j be smaller than ε. This can be done because we may choose the F_j with arbitrarily small diameters. Let ρ denote the smallest distance between F_j and F_k, $j \neq k$, $j, k = 1, \dots, m$; $\rho > 0$. There is no loss of generality in assuming that the distance between F and ∂Q is greater than ρ. Clearly, $\gamma^+_{h,\Delta}(F_j) = 0$, $j = 1, \dots, m$. However, this statement requires some additional comments. The point is that, before this section, the entire theory had been developed under the assumption that the weight h was a continuous function. But now the superharmonic function $h(z)$ may be discontinuous or equal to $+\infty$ at some points outside a given set F_j. We enclose $F \setminus F_j$ in a neighborhood $S \subset Q$ such that the closure $\overline{S}$ does not intersect F_j, and the boundary ∂S is a "good" curve. Next, on $\overline{S}$ we replace $h(z)$ with a harmonic function $H(z)$ equal to $h(z)$ on ∂S, and consider the weight

$$\tilde{h}(z) = h(z), \;\; z \notin \overline{S}; \quad \tilde{h}(z) = H(z), \;\; z \in \overline{S}. \tag{9.16}$$

The properties of subharmonic, harmonic, and superharmonic functions imply that for a function $f(z)$ analytic off F_j we have the equivalence

$$\|f\|_h \leq 1 \iff \|f\|_{\tilde{h}} \leq 1 \tag{9.17}$$

and the weight $\tilde{h}(z)$ is continuous. This construction allows us to freely use the weight $h(z)$ in what follows. Since $\gamma^+_{h,\Delta}(F_j) = 0$, by (8.10) there are points $a_1^j, \dots, a_{n_j}^j$ outside F, and there are numbers $\nu_1^j, \dots, \nu_{n_j}^j$ such that

$$\varphi_j(\zeta) = \sum_1^{n_j} \nu_k^j (\zeta - a_k^j)^{-1} \in E^1_\Delta(F_j), \quad \operatorname{Re} \varphi_j(\zeta) \geq 1, \;\; \zeta \in F_j,$$

$$\sum_1^{n_j} h(a_k^j)|\nu_k^j| < \frac{3}{4}\frac{\gamma\rho}{mt}, \tag{9.18}$$

where $t > 1$ is chosen in such a way that

$$\frac{\gamma}{tA} < \frac{1}{4}, \quad \frac{\rho}{t} < \frac{1}{2}. \tag{9.19}$$

By the above, some points among $\{a_k^j\}$ in (9.17) can be assumed to lie in the $\rho/4$-neighborhood of F_j, whereas the other lie on ∂Q. It is important that, at the same time, all these points are at least $\frac{3}{4}\rho$-distant from $F \setminus F_j$. Let β_j denote some value of $\alpha(\zeta)$ on F_j. Then

$$|\alpha(\zeta) - \beta_j| < \varepsilon, \quad \zeta \in F_j. \tag{9.20}$$

Consider the sums

$$\psi_j(\zeta) = \sum_1^{n_j} \mu_k^j (\zeta - a_k^j)^{-1}, \quad \mu_k^j = \nu_k^j e^{-i\beta_j}, \tag{9.21}$$

$$\phi_j(\zeta) = e^{i\alpha(\zeta)} \psi_j(\zeta), \quad \zeta \in F, \quad j = 1, \ldots, m. \tag{9.22}$$

By (9.18), we have

$$\operatorname{Re}[e^{i\beta_j} \psi_j(\zeta)] = \operatorname{Re} \varphi_j(\zeta) \geq 1, \quad \zeta \in F_j. \tag{9.23}$$

Next,

$$\begin{aligned} \operatorname{Re}[\phi_j(\zeta) - e^{i\beta_j}\psi_j(\zeta)] &= \operatorname{Re}\{\varphi_j(\zeta)[e^{i(\alpha(\zeta)-\beta_j)} - 1]\} \\ &= \operatorname{Re}[\varphi_j(\zeta)][\cos(\alpha(\zeta) - \beta_j) - 1] - \operatorname{Im}[\varphi_j(\zeta)] \sin(\alpha(\zeta) - \beta_j) \\ &\leq \operatorname{Im}[\varphi_j(\zeta)]| \sin\varepsilon \leq \tan\frac{|\Delta|}{2} \operatorname{Re}[\varphi_j(\zeta)] \sin\varepsilon. \end{aligned} \tag{9.24}$$

We have used (9.20) and the inequality $\cos(\alpha(\zeta) - \beta_j) - 1 \leq 0$, which shows that the values of $\varphi_j(\zeta)$ lie in the angle Δ and, therefore,

$$|\operatorname{Im}[\varphi_j(\zeta)]| \leq \tan\frac{|\Delta|}{2} \operatorname{Re}[\varphi_j(\zeta)].$$

If ε is sufficiently small, (9.24) implies the relations

$$\begin{aligned} \operatorname{Re}\phi_j(\zeta) &= \operatorname{Re}\sum_1^{n_j} \mu_k^j e^{i\alpha(\zeta)} (\zeta - a_k^j)^{-1} \\ &\geq \operatorname{Re}[\varphi_j(\zeta)] \cdot \left(1 - \tan\frac{|\Delta|}{2} \sin\varepsilon\right) \geq \frac{3}{4} \operatorname{Re}[\varphi_j(\zeta)] \geq \frac{3}{4}, \quad \zeta \in F_j. \end{aligned} \tag{9.25}$$

Furthermore, by (9.18) we have

$$\sum_1^{n_j} h(a_k^j)|\mu_k^j| = \sum_1^{n_j} h(a_k^j)|\nu_k^j| < \frac{\gamma}{m}\frac{\rho}{t} \cdot \frac{3}{4} < \frac{3}{8}\frac{\gamma}{m}. \tag{9.26}$$

We do this procedure for all $j = 1, \ldots, m$, and put

$$\Phi(\zeta) = \sum_{j=1}^m \phi_j(\zeta) = \sum_{j=1}^m \sum_1^{n_j} \mu_k^j e^{i\alpha(\zeta)} (\zeta - a_k^j)^{-1}. \tag{9.27}$$

For an arbitrary j and for $\zeta \in F_j$ we have the following estimate:

$$\begin{aligned} \left|\sum_{\substack{l=1 \\ l\neq j}}^m \phi_l(\zeta)\right| &\leq \sum_{l\neq j} \sum_1^{n_l} |\mu_k^l|\, |\zeta - a_k^l|^{-1} < \frac{1}{A} \sum_{l\neq j} \sum_1^{n_l} h(a_k^l) |\mu_k^l|\, |\zeta - a_k^l|^{-1} \\ &< \frac{1}{A} \cdot \frac{1}{3/4\rho} \cdot \frac{\gamma}{m} \cdot \frac{\rho}{t} \cdot \frac{3}{4}(m-1) < \frac{\gamma}{tA} < \frac{1}{4}. \end{aligned} \tag{9.28}$$

To deduce (9.28), we have used (9.24), (9.26), and (9.19). Also, we have used the fact that $|a_k^l - \zeta| \geq \frac{3}{4}\rho$ if $l \neq j$ and $\zeta \in F_j$. From (9.28), (9.27), and (9.25) with $\zeta \in F_j$ we deduce that

$$\operatorname{Re}\Phi(\zeta) = \operatorname{Re}\phi_j(\zeta) + \sum_{l\neq j}^m \operatorname{Re}\phi_l(\zeta) \geq \operatorname{Re}\phi_j(\zeta) - \left|\sum_{l\neq j}^m \phi_l(\zeta)\right| \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}. \tag{9.29}$$

Being true for all $j = 1, \dots, m$, this estimate applies to all $\zeta \in F$. At the same time,

$$\sum_{j=1}^{m} \sum_{1}^{n_j} h(a_k^j)|\mu_k^j| < \frac{\gamma}{m} \cdot \frac{\rho}{t} \cdot \frac{3}{4} m < \frac{3}{8}\gamma \tag{9.30}$$

(we have used (9.26) and the second inequality in (9.19)). Finally, for the function $\Psi(\zeta) = 2\Phi(\zeta)$, estimate (9.29) yields

$$\operatorname{Re}\Psi(\zeta) \geq 1, \quad \zeta \in F, \tag{9.31}$$

and (9.30) yields

$$\sum_{j=1}^{m} \sum_{1}^{n_j} h(a_k^j)|\sigma_k^j| < \frac{3}{4}\gamma \tag{9.32}$$

(here the $\sigma_k^j = 2\mu_k^j$ are the coefficients of the elementary fractions involved in $\Psi(\zeta)$). Conditions (9.31) and (9.32) do not agree with (9.14)–(9.15), and the proof is finished. □

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