AN ESTIMATE FOR THE VOLUME ENTROPY OF NONPOSITIVELY CURVED GRAPH-MANIFOLDS

S. BUYALO

ABSTRACT. Let $M$ be a closed 3-dimensional graph-manifold. It is proved that $h(g) > 1$ for every geometrization $g$ of $M$, where $h(g)$ is the topological entropy of the geodesic flow of $g$.

§1. INTRODUCTION

Asymptotic geometry of a nonpositively curved (for brevity, NPC) graph-manifold is a complicated mixture of flat and hyperbolic parts, which both contribute nontrivially to the general picture. We recall that an NPC-metric on a closed 3-dimensional graph-manifold $M$ recovers the JSJ-decomposition of $M$ in the following sense. There is a unique (up to isotopy) minimal finite collection $E$ of pairwise disjoint flat geodesically embedded tori and Klein bottles such that the metric completion of each connected component of the complement of $E$ is a Seifert space called a block of $M$. Each block $M_v$ is fibered over a 2-orbifold $S_v$ with negative Euler characteristic, $\chi(S_v) < 0$. Furthermore, along the interior of each block the metric locally splits as $U \times (-\varepsilon, \varepsilon)$, where $U$ is an NPC-surface, this splitting is compatible with the fibration, the fibers are closed geodesics, and the regular fibers have one and the same length $l_v > 0$ depending only on the block.

Since we are interested in asymptotic properties, which are certainly the same for any finite covering of $M$, we may assume for simplicity that $M$ is orientable, the collection $E$ consists of tori, and each block is a trivial $S^1$-bundle over a compact surface $S_v$ with boundary, $M_v = S_v \times S^1$. We also assume that the graph-manifold structure of $M$ is nontrivial, i.e., that $M$ itself is not a Seifert fibered space (though it may consist of one block).

The flat part of the asymptotic geometry of $M$ was studied in [BS, CK]; see also [HS]. Roughly speaking, this part can be described by fairly special geodesic rays $[0, \infty) \to M$, which leave any block, passing through separating tori $e \in E$ almost tangentially and spending most of the time near tori; moreover, this time rapidly increases at each step. Though the set of such rays is a negligible part of all rays, it keeps an important information about the geometry of $M$: in [CK] it was shown how this information allows one to recover (up to scaling) the marked length spectrum of the closed geodesics on $S_v$, as well as the fiber length for each block $M_v$.

In this paper we study the hyperbolic part of the asymptotic geometry of $M$, assuming that the surface $U$ occurring in the local splitting $U \times (-\varepsilon, \varepsilon)$ as above has constant curvature $K = -1$. In other words, each block fibers over a hyperbolic orbifold (surface) $S_v$. An NPC-metric on $M$ satisfying this condition is called a geometrization of $M$.
(because the metric of each block is modeled on $H^2 \times \mathbb{R}$). We note that any geometrization of $M$ is only $C^{1,1}$-smooth, being analytic along the interior of each block. It is known [1] that $M$ admits an NPC-metric if and only if $M$ admits a geometrization. In topological terms, necessary and sufficient conditions for $M$ to carry an NPC-metric were found in [BK].

A relevant metric invariant that measures hyperbolicity of a space is the volume entropy $h$. Let $\pi : X \to M$ be the universal covering, and let $x_0 \in X$. We recall that $h = h(X)$ is defined by the formula

$$h = \lim_{R \to \infty} \frac{1}{R} \log \text{vol} B_R(x_0),$$

where $B_R(x)$ is the ball in $X$ of radius $R$ and centered at $x$. It is well known (see [M]) that the limit above exists, the quantity $h$ is independent of the choice of $x_0$, and that if $M$ is NPC, then $h$ coincides with the topological entropy of the geodesic flow of $M$. The volume entropy scales as $l^{-1}$, where $l$ is the length. Thus, the choice of a geometrization $g$ of $M$ also serves as a normalization. For the sectional curvatures of $g$ we have $-1 \leq K \leq 0$. Consequently, $h(g) \leq 2$, by comparison with $H^3$. Moreover, for the Ricci curvatures of $g$ we have $-1 \leq \text{Ric}_g \leq 0$ (whereas $\text{Ric}_{H^3} = -2$). The space $\sqrt{2} H^3$, where the distances are those of $H^3$ multiplied by $\sqrt{2}$, has the constant Ricci curvature $\text{Ric}_{\sqrt{2} H^3} = -1$ and the volume entropy $h(\sqrt{2} H^3) = \frac{1}{\sqrt{2}} h(H^3) = \sqrt{2}$. The Bishop comparison theorem yields

$$\text{vol} B_R^g \leq \text{vol} B_R^{\sqrt{2} H^3}$$

for the volumes of the balls of one and the same radius $R$ in the metric $g$ and in $\sqrt{2} H^3$. Thus, $h(g) \leq \sqrt{2}$ for every geometrization $g$.

Our main result is as follows.

**Theorem 1.1.** For any geometrization $g$ of a graph-manifold $M$ we have $h(g) > 1$.

**Remark 1.2.** Though the universal covering $X$ of $M$ looks much more complicated than the model space $H^2 \times \mathbb{R}$, even the estimate $h(g) \geq 1 = h(H^2 \times \mathbb{R})$ is not obvious and nontrivial: $X$ contains no isometrically and geodesically embedded $H^2$, which would lead to $h(g) \geq 1$; on the other hand, the attempt to compare $X$ and $H^2 \times \mathbb{R}$ via exponential maps identifying some tangent spaces fails, because the Jacobian of such a map is greater than 1 at some points. Finally, the estimate in [BW] for the measure-theoretic entropy of the geodesic flow, which never exceeds $h(g)$, gives only $\pi/4 < 1$ as a lower bound for any geometrization $g$ of $M$ (even if we ignore the fact that the $C^{1,1}$-smoothness of $g$ is not sufficient for the application of that estimate).

In the proof of Theorem 1.1 we use the well-known fact that $h(g)$ coincides with the critical exponent of the Poincaré series

$$\mathcal{P}(t) = \sum_{\gamma \in \Gamma} e^{-t|x_0 - \gamma x_0|},$$

where the fundamental group $\Gamma = \pi_1(M)$ acts on $X$ isometrically as the deck transformation group. Actually, instead of $\mathcal{P}$ we use a modified Poincaré series $\mathcal{P}_W$ in which summation is taken over some set $W$ of walls in $X$. Our proof involves three ingredients:

(i) A local estimate, which is technical and used in (ii): this estimate is obtained in §2.

(ii) An accumulating procedure, which consists in the inductive construction of appropriate broken geodesics in $X$ between the base point $x_0$ and the walls in $W$; the choice of these paths is the key point of the proof. The outcome of the accumulating procedure is the generating set for $\mathcal{P}_W$ to be used in (iii); the procedure is described in §3.
(iii) A self-similarity type argument. In this part of the proof we use a standard idea from self-similarity theory and the generating set obtained in (ii) to show that $\mathcal{P}_W(h)$ diverges for some $h > 1$. This is done in §4.

Acknowledgment. I am grateful to W. Ballmann for useful discussions of the topic treated in this paper. This work has been done during my stay at MPI of Mathematics (Bonn), and I thank MPI for the invitation and excellent working conditions.

§2. Local estimate

Let $F$ be the universal covering of a compact hyperbolic surface $S$ with geodesic boundary. We identify $F$ with a convex subset $F \subset \mathbb{H}^2$ bounded by countably many disjoint geodesic lines and fix a point $o \in \mathbb{H}^2 \setminus F$. Let $w_0$ be the boundary line of $F$ closest to $o$, and let $o_0 \in w_0$ be the point on $w_0$ closest to $o$, so that $|o - o_0| = \text{dist}(o, F) =: l > 0$.

We denote by $A$ the set of the boundary lines of $F$ different from $w_0$. For $w \in A$, let $o_w \in w$ be the point closest to $o$. Then the geodesic segment $oo_w$ intersects $w_0$ at some point $t_w$, and for $\tilde{l}_w = |o - o_w|$ we have

$$\tilde{l}_w = \tilde{l}_w' + l''_w,$$

where $\tilde{l}_w' = |o - t_w|$, $l''_w = |t_w - o_w|$ (all distances are taken in $\mathbb{H}^2$).

Next, we identify $\mathbb{H}^2$ with $\mathbb{H}^2 \setminus 0 \subset \mathbb{H}^2 \times \mathbb{R}$, so that $F$ becomes a subset of $\mathbb{H}^2 \times \mathbb{R}$; we keep the notation introduced above. Observe that the point $o_0$ is the closest to $o$ among the points of the wall $w_0 \times \mathbb{R}$. We take a nonhorizontal geodesic line $\sigma \subset w_0 \times \mathbb{R}$ through $o_0$, i.e., $\sigma \neq w_0 \times 0$, and take $s_w \in \sigma$ with $|s_w - o_0| = |t_w - o_0|$. Now, we put $l'_w := |o - s_w|$ (the distance is taken in $\mathbb{H}^2 \times \mathbb{R}$), and $\Delta_w := \tilde{l}_w' - l'_w$.

In other words, we replace the distance $\tilde{l}_w'$ between $o$ and $t_w$ in the hyperbolic plane $\mathbb{H}^2$ by the distance $l'_w$ in $\mathbb{H}^2 \times \mathbb{R}$, which is shorter by comparison: the triangles $oo_0t_w \subset \mathbb{H}^2 \setminus 0$, $oo_0s_w \subset \mathbb{H}^2 \times \mathbb{R}$ have right angles at $o_0$ ($\angle(oo_0t_w) = \pi/2 = \angle(oo_0s_w)$), have a common side $oo_0$, and have equal sides $|o_0 - t_w| = |o_0 - s_w|$. Since $oo_0t_w$ lies in the hyperbolic plane $\mathbb{H}^2 \times 0$, but $oo_0s_w$ does not, we have $\Delta_w > 0$ except in the case where $t_w = o_0 = s_w$. Now, we want to estimate the accumulation of the differences $\Delta_w$ from below. The precise statement is as follows.

Lemma 2.1. For any $l_0 > 0$, and any $\alpha_0 \in (0, \pi/2]$, there exists $\lambda_0 > 1$, which depends only on $l_0$, $\alpha_0$, and the compact surface $S$, so that

$$\lambda(F, l, \alpha) := e^l \sum_{w \in A} e^{\Delta_w} e^{-l_w} \geq \lambda_0$$

whenever $l = \text{dist}(o, w_0) \geq l_0$ and the angle $\alpha$ between the lines $w_0 \times 0$ and $\sigma$ is at least $\alpha_0$, i.e., $\alpha_0 \leq \alpha \leq \pi/2$.

Proof. By a well-known formula of hyperbolic geometry, we have $e^l = (\tan \frac{\psi}{4})^{-1}$, $e^{-l_w} = \tan \frac{\psi_w}{4}$, where $\psi$ and $\psi_w$ are the angles under which $w_0$ and $w \in A$ are observed in $\mathbb{H}^2$ from $o$, respectively. The boundary at infinity $\partial_{\infty} F \subset \partial_{\infty} \mathbb{H}^2 = S^1$ coincides with the limit set of $\pi_1(S)$ represented in $\text{Iso}(\mathbb{H}^2)$ as a Fuchsian group of the second kind. It is well known that the Hausdorff dimension of $\partial_{\infty} F$ (with respect to the angle metric) is less than 1; in particular, the Lebesgue measure of $\partial_{\infty} F$ is zero. Thus, $\psi = \sum_{w \in A} \psi_w$.

Therefore, $\tan \psi/4 \leq \sum_{w \in A} \tan \frac{\psi_w}{4}$, and

$$e^l \sum_{w \in A} e^{-l_w} = \sum_{w \in A} \tau_w \geq 1,$$
where \( \tau_w = e^{l_w} \). However, the sum \( \sum_{w \in A} \tau_w \) can be made as close to 1 as we like (e.g., we can let \( l \to \infty \)). As was mentioned above, \( \Delta_w > 0 \) unless \( s_w = t_w \). Thus, we always have \( \lambda(F, l, \alpha) > 1 \). The point is that \( \lambda(F, l, \alpha) \) is separated away from 1 uniformly over all \( l \geq l_0, \alpha \geq \alpha_0 \).

Consider the subset \( A_0 \subset A \) that consists of all \( w \in A \) with \( |t_w - o_0| \geq 1 \). Then \( \Delta_w \geq \delta_0 > 0 \) for all \( w \in A_0 \), where \( \delta_0 \) depends only on \( l_0 \) and \( \alpha_0 \). We claim that

\[
(1) \quad \sum_{w \in A_0} \tau_w \geq m_0 > 0,
\]

where \( m_0 = m_0(\delta, l_0) \) is independent of \( l \).

Assuming (1), we obtain

\[
\lambda(F, l, \alpha) = \sum_{w \in A} e^{\Delta_w} \tau_w \geq \sum_{w \in A_0} e^{\Delta_w} \tau_w + \sum_{w \in A \setminus A_0} \tau_w \geq e^{\delta_0} \sum_{w \in A_0} \tau_w + \sum_{w \in A \setminus A_0} \tau_w = (e^{\delta_0} - 1) \sum_{w \in A_0} \tau_w + \sum_{w \in A \setminus A_0} \tau_w \geq (e^{\delta_0} - 1)m_0 + 1 =: \lambda_0 > 1.
\]

It remains to prove (1). Let \( \delta'_w \in w \) be the point closest to \( o_0 \in w_0 \). Then \( \delta'_w - \delta_0 \leq |o_0 - o'_w| \leq l + |o_0 - o'_w| \), whence \( \delta'_w - l \leq \text{dist}(o_0, w) \). Consequently, \( \tau_w \geq e^{-\text{dist}(o_0, w)} \geq \delta'_w / 4 \), where \( w \) is observed from \( o_0 \) under the angle \( \delta'_w \). Since the Lebesgue measure class on \( \partial_{\infty} H^2 \) is independent of the choice of a marked point, we have \( \sum_{w \in A} \psi_w = \pi \).

Since \( l \geq l_0 \), for a sufficiently small \( m_0 = m_0(S, l_0) > 0 \) the sectors \( S^+(m_0) \) and \( S^-(m_0) \) to be defined below intersect no line \( w \in A \setminus A_0 \). The definition of the \( S^+(m_0) \) is as follows. The common vertex \( o_0 \) of \( S^+(m_0) \) divides the line \( w_0 \) into two opposite rays \( w_0^\pm \). The sectors \( S^+(m_0) \subset H^2 \) are bounded by the rays \( w_0^\pm \), \( s^+(m_0) \), where \( \angle_{\alpha_0}(s^+(m_0), w_0^+) = 2m_0 \), and \( s^+(m_0) \cap \partial_{\infty} F = \emptyset \).

Therefore, from the relation \( \sum_{w \in A} \psi_w = \pi \) it follows that \( \sum_{w \in A_0} \tau_w \geq m_0 \), which completes the proof. \( \square \)

\section*{3. Accumulating procedure}

In order to describe the accumulating procedure, we need some information about the metric structure of the universal covering \( X \) of \( (M, g) \), where \( g \) is a geoinetrisation.

\subsection*{3.1. Metric structure of the universal covering.}

We recall (see, e.g., [BS], [CK]) that \( X \) can be represented as a countable union \( X = \bigcup_v X_v \) of blocks, where each \( X_v \) is a closed convex subset in \( X \) isometric to the metric product \( F_v \times \mathbb{R} \), and \( F_v \) is the universal covering of a compact hyperbolic surface \( S_v \) with geodesic boundary. Any two blocks are either disjoint, or intersect each other along a boundary component that is 2-flat in \( X \) and separates them; consequently, no three blocks have a point in common. The 2-flats in \( X \) that separate blocks are called \emph{walls}. A common wall \( w \) of two blocks \( X_v \) and \( X_{v'} \) covers a 2-torus \( e \subset M \), which separates (possibly, only locally) the blocks \( M_v = \pi(X_v) \), \( M_{v'} = \pi(X_{v'}) \) of \( M \). The metric decompositions \( X_v = F_v \times \mathbb{R} \) and \( X_{v'} = F_{v'} \times \mathbb{R} \) do not agree on \( w \), and their \( \mathbb{R} \)-factors induce two fibrations of \( w \) by parallel geodesics. We denote by \( \alpha_w \) the angle between these fibrations, \( 0 < \alpha_w \leq \pi / 2 \). Since \( M \) is compact and the set \( E \) of separating tori in \( M \) is finite, we have \( \alpha_0 := \inf w \alpha_w > 0 \), where the infimum is taken over all walls in \( X \).
Theorem 1.1 will be proved if we show that

and take

where

all segments of $F$ as to ensure that

and is orthogonal to the wall

broken geodesic

agree with the metric splitting $X$ component of $W$.

We use induction on $n$. For each $n \geq 1$, we construct a broken geodesic $\xi_w$ in $X$ between $x_0$ and $w$ as follows. For $w \in W_0$, we put $\xi_w = x_0x_w$, where $x_w \in w$ is the point closest to $x_0$. The segment $x_0x_w$ lies in a horizontal slice $F_{x_w} \times \{r_0\}$ of the block $X_{x_w} = F_{x_w} \times \mathbb{R}$.

Suppose that the broken geodesic $\xi_w$ is already defined for all $k$ with $0 \leq k \leq n-1$ and all $w \in W_k$, and that $\xi_w$ is a juxtaposition $\eta_{w_0}\eta_{w_1} \cdots \eta_{w_k}$ of geodesic segments, where the sequence $w_i \in W_i$ of walls leads to $w = w_k$. Moreover, we assume that each segment of $\xi_w$ lies in a block and connects its different boundary components, the last segment $\eta_{w_k}$ lies in a horizontal slice of the block that contains it, and $\eta_{w_k}'$ is orthogonal to the wall $w$.

Let $w \in W_n$. Then there is a unique $\overline{w} \in W_{n-1}$ that precedes $w$. By assumption, the last edge $\eta_{\overline{w}}'$ of $\xi_{\overline{w}}$ lies in a horizontal slice $F_{x_{\overline{w}}} \times \{r_{\overline{w}}\}$ of the block $X_{x_{\overline{w}}} = F_{x_{\overline{w}}} \times \mathbb{R}$ and is orthogonal to the wall $\overline{w}$ (at the endpoint $x_{\overline{w}}$). It is important that $\xi_w$ contains all segments of $\xi_{\overline{w}}$ except for the last segment $\eta_{\overline{w}}'$.

Let $X_v \subset X$ be the other block adjacent to $\overline{w}$; in particular, the walls $w$ and $\overline{w}$ are boundary components of $X_v$. We recall that the metric splitting $X_v = F_v \times \mathbb{R}$ does not agree with the metric splitting $X_{x_v}$ along $\overline{w}$. Let $F_v \times \{r_v\}$ be the horizontal slice of $X_v$ that contains the point $x_{\overline{w}}$ on the corresponding boundary component. The boundary lines of the slices $F_v \times \{r_v\}$ and $F_v \times \{r_{\overline{w}}\}$, which lie in $\overline{w}$, contain $x_{\overline{w}}$ and form an angle of $\alpha_{\overline{w}} \in (0, \pi/2]$.

To construct $\xi_w$, we act as follows. We take an isometric copy $F_v \subset H^2 \times \{r_v\}$ of $F_v \times \{r_v\}$ (by rotating the latter through the angle $\alpha_{\overline{w}}$), where $F_v \times \{r_v\} \subset H^2 \times \{r_v\}$, so as to ensure that $F_v \times \{r_v\}$ and $F_v$ are sitting in the hyperbolic plane $H^2 \times \{r_v\}$ and are adjacent along their common boundary component, for which we use the same notation $\overline{w}$. Now, we connect the initial point $x_{\overline{w}}$ of the last edge $\eta_{\overline{w}}' \subset \xi_{\overline{w}}$ with the boundary component of $F_v$ corresponding to $w$ by the shortest geodesic segment $s_{\overline{w}w'} \subset H^2 \times \{r_v\}$ and take $t_w = s_{\overline{w}w'} \cap \overline{w}$. The segment $t_w' = s_{\overline{w}w} \cap \overline{w}$. The segment $t_w'$ turned back to $F_v \times \{r_v\}$ gives the last segment $\eta_{w}' = s_wx_w$ of $\xi_w$. Thus, $s_w \in \overline{w} \cap F_v \times \{r_v\}$, $|s_w - x_{\overline{w}}| = |t_w - x_{\overline{w}}|$, and the
segment \( \eta_w' \subset F_w \times \{ r_w \} \) is orthogonal to \( w \) at \( x_w \). To complete the construction of \( \xi_w \), we delete the last segment \( \eta_w'' \subset \xi_w \), replacing it with \( \eta_w'' \), where \( \eta_w'' = s_w \eta_w' \subset X_{r_w} \).

Clearly, the resulting \( \xi_w \) has all properties advertised above.

Let \( l = |s_w - x_w| = L(\eta_w') \) be the length of the last segment of \( \xi_w \), and let \( l'_w = L(\eta_w), l''_w = L(\eta_w') \). Then

\[
L(\xi_w) = L(\xi_w') - l + l'_w + l''_w = L(\xi_w') - l + \bar{t}_w - \Delta_w,
\]

where \( \bar{t}_w = L(s_w x_w') \) and \( t'_w = L(s_w x_w') \), so that \( \bar{t}_w = \tilde{t}_w + l''_w \), and \( \Delta_w = \bar{t}_w - l'_w \).

We arrive at the same configuration that was studied in §2, and we are going to apply Lemma 2.1 to estimate \( P_n(1) \) from below. Since \( l \) is the length of a segment in \( X \) that connects different boundary components of a block, \( l \) is separated away from 0 by some positive constant \( l_0 \), depending only on \( M, l \geq l_0 > 0 \).

For \( n = 0 \) we have

\[
P_0(1) = \sum_{w \in W_0} e^{-|x_0 - x_w|} = \sum_{w \in W_0} \tan \frac{\psi_w}{4} \geq \frac{\pi}{4},
\]

where \( \psi_w \) is the angle under which the boundary component \( w \) of \( X_{r_w} \) is observed from \( x_0 \) (in the horizontal direction).

By the induction assumption, we have

\[
P_{n-1}(1) \geq \sum_{w \in W_{n-1}} e^{-L(\xi_w)} \geq \frac{\pi}{4} \lambda_0^{n-1}.
\]

We write \( W_n = \bigcup_{w} W_{n, w} \), where the union is taken over all \( w \in W_{n-1} \) and each \( w \in W_{n, w} \) follows \( w \). Then

\[
P_n(1) = \sum_{w \in W_n} e^{-L(\xi_w)} = \sum_{w \in W_{n-1}} e^{-L(\xi_w)} e^l \sum_{w \in W_{n, w}} e^{\Delta_w} e^{-\bar{t}_w}.
\]

Applying Lemma 2.1 with \( A = W_{n, w} \), we obtain

\[
P_n(1) \geq \lambda_0 \sum_{w \in W_{n-1}} e^{-L(\xi_w)} \geq \frac{\pi}{4} \lambda_0^n,
\]

which completes the proof. \( \square \)

§4. Self-similarity argument

The constant \( \lambda_0 > 1 \) occurring in Proposition 3.1 depends only on some metric data of \( M \). Therefore, for some \( n \in \mathbb{N}, n = n(M) \), we have \( \frac{\pi}{4} \lambda_0^n > 1 \). Proposition 3.1 implies that

\[
P_n(\bar{h}) = \sum_{w \in W_n} e^{-\bar{h} \text{dist}(x_0, w)} \geq 1
\]

for some \( \bar{h} > 1 \). Furthermore, taking \( n \) sufficiently large, we can find \( \bar{h} > 1 \) such that \( P_n(\bar{h}) \geq 1 \) for any choice of the initial block \( X_{r_w} \), of its wall \( w^* \), and of the base point \( x_0 \in w^* \), because the set of such choices (up to isometries of \( X \)) is compact.

We fix \( n \in \mathbb{N} \) with the above property and select a subset \( W^* = \bigcup_{k \geq 1} W^*_k \subset W \), where \( W^*_k = W_{k, x_0} \). The set \( W^*_k \) serves as the generating set for \( W^* \). Connecting \( x_0 \) with each wall \( w \in W^*_1 \) by the shortest geodesic segment \( x_0 x_w \), we obtain new base points \( x_w \in w \) (these \( x_w \) may differ from the \( x_w \) constructed in the proof of Proposition 3.1). By induction, we find a base point \( x_w \in w \) for each \( w \in W^*_k, k \geq 1 \), with the property
that $\text{dist}(x_{\mathbf{m}}, w) = |x_{\mathbf{m}} - x_w|$, where $\mathbf{m} \in W^*_k$ precedes $w$. Furthermore, by the choice of $n$ and $\overline{h}$, for each $\mathbf{m} \in W^*_k$ we have

$$\sum_w e^{-\overline{h}|x_{\mathbf{m}} - x_w|} \geq 1,$$

where summation is over all $w \in W^*_k$ that follow $\mathbf{m}$. Since $\text{dist}(x_0, w) \leq \text{dist}(x_0, x_{\mathbf{m}}) \leq \text{dist}(x_0, x_{\mathbf{m}}) + |x_{\mathbf{m}} - x_w|$, we see that

$$P_{W^*_k}(\overline{h}) = \sum_{w \in W^*_k} e^{-\overline{h}\text{dist}(x_0, w)}$$

$$\geq \sum_{w \in W^*_{k-1}} e^{-\overline{h}\text{dist}(x_0, x_w)} \geq \sum_{w \in W^*_n} e^{-\overline{h}\text{dist}(x_0, x_w)} \geq 1$$

for all $k \geq 1$. Therefore, the modified Poincaré series

$$P_{W^*_k}(\overline{h}) \geq P_{W^*}(\overline{h}) = \sum_{k \geq 1} P_{W^*_k}(\overline{h})$$

diverges at $\overline{h}$, whence $h(g) \geq \overline{h} > 1$. This completes the proof of Theorem 1.1.

REFERENCES


St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191011, Russia

E-mail address: buyalo@pdmi.ras.ru

Received 2/SEP/2002

Translated by THE AUTHOR