Алгебра и анализ
Том. 15 (2003), вып. 1

St. Petersburg Math. J.
Vol. 15 (2004), No. 1, Pages 81–102
S 1061-0022(03)00803-3
Article electronically published on December 31, 2003

LIMITING DISTRIBUTIONS OF THETA SERIES ON SIEGEL HALF-SPACES

F. GÖTZE AND M. GORDIN

ABSTRACT. Let $m \geq 1$ be an integer. For any Z in the Siegel upper half-space we consider the multivariate theta series

$$\Theta(Z) = \sum_{\overline{n} \in \mathbb{Z}^m} \exp(\pi i\, {}^t\overline{n} Z \overline{n}).$$

The function Θ is invariant with respect to every substitution $Z \longmapsto Z + P$, where P is a real symmetric matrix with integral entries and even diagonal. Therefore, for any real matrix $Y > 0$ the function $\Theta_Y(\cdot) = (\det Y)^{1/4}\Theta(\cdot + iY)$ can be viewed as a complex-valued random variable on the torus $\mathbb{T}^{m(m+1)/2}$ with the Haar probability measure. It is proved that the weak limit of the distribution of $\Theta_{\tau Y}$ as $\tau \to 0$ exists and does not depend on the choice of Y. This theorem is an extension of known results for $m = 1$ to higher dimension. Also, the rotational invariance of the limiting distribution is established. The proof of the main theorem makes use of Dani–Margulis' and Ratner's results on dynamics of unipotent flows.

§1. Introduction

Let $m \geq 1$ be an integer. For every rectangular matrix M we denote by ${}^tM$ the transpose of M. Set

$$\Theta(Z) = \sum_{\overline{n} \in \mathbb{Z}^m} \exp(\pi i\, {}^t\overline{n} Z \overline{n}), \tag{1.1}$$

where $Z = \operatorname{Re} Z + i \operatorname{Im} Z = X + iY$ is a complex symmetric $(m \times m)$-matrix such that the imaginary part Y of it is positive definite (this is denoted by $Y > 0$). We retain this notation throughout the paper. The set $\mathcal{H}_m$ of all such matrices Z is known as the *Siegel upper half-space.* Clearly, the function Θ is invariant under every substitution $X \longmapsto X + P$, where P is a symmetric matrix with integral entries and even diagonal. Such matrices P form a discrete uniform subgroup of the additive group of real symmetric $(m \times m)$-matrices with a quotient group isomorphic to the torus $\mathbb{T}^{m(m+1)/2}$. Therefore, for any $Y > 0$ the function $\Theta_Y(\cdot) = (\det Y)^{1/4}\Theta(\cdot + iY)$ may be viewed as a function on the torus, and, with respect to the Haar probability measure on $\mathbb{T}^{m(m+1)/2}$, Θ_Y can be regarded as a random variable taking values in $\mathbb{C}$. The main result in the present paper is the following (see Theorem 4.1).

2000 *Mathematics Subject Classification.* Primary 11Fxx, 37D30.

Key words and phrases. Theta series, Siegel's half-space, convergence in distribution, closed horospheres, unipotent flows.

Supported in part by the DFG-Forschergruppe FOR 399/1-1.

M. Gordin was also partially supported by RFBR (grant no. 02.01-00265) and by Sc. Schools grant no. 2258.2003.1. He was a guest of SFB-343 and the Department of Mathematics at the University of Bielefeld while the major part of this paper was prepared.

Theorem. *The weak limit of the distribution of* $\Theta_{\tau Y}$ *as* $\tau \to 0$ *exists and does not depend on the choice of* Y*. Convergence is uniform in* Y *running over every compact subset of the set of all* $(m \times m)$*-positive definite matrices.*

In fact, Theorem 4.1 gives also a description of the above limit as a distribution of some random variable on a certain probability space. From this description we deduce (see Corollary 4.1) that the limiting distribution is invariant with respect to the rotations of $\mathbb{C}$ around the origin.

Theorem 4.1 generalizes some earlier studies where the case of $m = 1$ was investigated. In [JVH2] (see also [JVH1], [JVH3]) the existence of a limiting distribution was established with the help of a careful analysis of the asymptotic behavior of $\Theta(X + iY)$ as $Y \to 0$. There the Diophantine nature of X played a crucial role. The corresponding mathematical technique goes back to Hardy and Littlewood [HL1], [HL2].

An alternative approach to the case of $m = 1$ can be found in the papers [Ma1], [Ma2] by J. Marklof, where the conclusion that the limiting distribution exists was based on some dynamical property of the geodesic flow on the unit tangent bundle of an appropriate noncompact surface of constant negative curvature and finite area. Namely, a closed horocycle (that is, a compact orbit of the horocycle flow on the same bundle) becomes uniformly distributed (relative to the natural probability measure on this bundle) under the action of the geodesic flow that carries this horocycle out of a cusp. The latter fact was proved in [Ma1] and [Ma2] in the spirit of P. Sarnak's paper [Sa], where this result was derived from some analytic properties of appropriate Eisenstein series. In some sense, this method makes it possible to retell the results about Eisenstein series in terms of the asymptotic behavior of the uniform measure on a moving closed horocycle. More precise asymptotics established in such a way can be found in [Ma1], [Ma2].

In our paper we deal with the case of an arbitrary $m \geq 1$, and restrict ourselves to the first term of the asymptotics only. The outline of the proof is similar to Marklof's approach: the main result is deduced from the fact that a compact orbit of some $m(m+1)/2$-dimensional unipotent subgroup of the metaplectic group $\mathbf{Mp}(2m, \mathbb{R})$ becomes uniformly distributed under the action of a flow (which is called *quasigeodesic* in the present paper) on $\widehat{\Gamma}_4 \backslash \mathbf{Mp}(2m, \mathbb{R})$, where $\widehat{\Gamma}_4$ is a certain lattice in $\mathbf{Mp}(2m, \mathbb{R})$. So, the first step in the proof is reduction to some uniform distribution problem. This is done by replacing the original theta function with another related function defined on the total space of a bundle over the original domain. We call this step "lifting" despite other meanings this term may have in the theta functions context. The second step consists in establishing the uniform distribution property.

However, the realization of both steps in our paper is quite different from the approach in [Ma1], [Ma2]. In the present paper lifting is done in a straightforward way and applies to any function with appropriate automorphic properties. We try to describe this reduction in rather general terms in §2, since we were unable to locate a suitable reference in the literature. Although a great deal is known about the multipliers in the "functional equation" (see Subsection 2.3), we avoid explicit expressions (for instance, in terms of the Gauss sums) because they are not important in this context.

The theta series represents an automorphic form of weight 1/2 related to a lattice in $\mathbf{Sp}(2m, \mathbb{R})$, which leads us to the consideration of the metaplectic group $\mathbf{Mp}(2m, \mathbb{R})$. One of our goals in §2 is to give a precise account of interconnections between the actions of $\mathbf{Mp}(2m, \mathbb{R})$ on two homogeneous spaces related to an appropriate lattice $\widehat{\Gamma}_4$ in $\mathbf{Mp}(2m, \mathbb{R})$. One of these spaces is $\widehat{\Gamma}_4 \backslash \mathbf{Mp}(2m, \mathbb{R})$, and the other is obtained from $\mathcal{H}_m$ by means of the group $\mathbf{Mp}(2m, \mathbb{R})$ acting via the standard homomorphism to $\mathbf{Sp}(2m, \mathbb{R})$. These interrelations are simple but not quite trivial because $\mathbf{Mp}(2m, \mathbb{R})$ acts on opposite sides in these two cases. However, there we start with a discussion of the analogous actions of $\mathbf{Sp}(2m, \mathbb{R})$, merely because the latter case can be handled in terms of matrices and is more intuitive. From these remarks it is clear that the main assertions of this paper apply also to the automorphic forms of integral weight related to some congruence subgroups of $\mathbf{Sp}(2m, \mathbb{Z})$.

The second step is the proof of the asymptotic uniform distribution property. Different approaches to this problem seem possible (apart from the Sarnak–Marklof's techniques for $m = 1$). One possibility is to deduce this property from rather general results about mixing for group actions on homogeneous spaces of finite volume. Since such mixing properties only concern nice functions on a homogeneous space, this method suggests some approximation of the uniform measure on a compact horosphere by smooth densities. Such an approach was sketched in [EMM, Theorem 7.1] for $m = 1$. The corresponding smoothening procedure is contained in the "wave front lemma" (Theorem 3.1 in [EMM]). As to results on mixing, the reader of [EMM] is addressed to some results known earlier and based substantially on group representation theory. The arguments used in [EMM] for $m = 1$ can be extended to cover the case where m is arbitrary.

In this paper, though, we choose to deduce the property in question from general results on unipotent flows. This is done in §3. The proof is based on the results by Dani and Margulis [DM], and also on M. Ratner's results concerning the Raghunathan conjecture and its measure-theoretic version.

The authors expect that, for the rather specific situation studied here, some relatively self-contained proof of this result might be achievable, together with certain explicit estimates of the rate of convergence.

The structure of the paper is as follows. In §2 the theta function is lifted to the principal bundle over its original domain. In the sequel we only need Propositions 2.1 and 2.2 of that section. In §3 the main result about uniform distribution is proved. Here the basic tool is the theory of unipotent flows along with a certain elementary construction. Finally, in §4 the results of the two preceding sections are combined to prove convergence in distribution.

Acknowledgments. The authors are grateful to G. Margulis and O. Schwarzman for useful discussions on the subject of the paper.

§2. Theta series and dynamics

Our goal in this section is to lift the function Θ from $\mathcal{H}_m$ to a new domain where there exists a dynamical system responsible for the chaotic behavior of Θ on $\mathcal{H}_m$ as the imaginary part of $Z \in \mathcal{H}_m$ goes to zero. As we shall see in Subsection 2.4, the function Θ plays a decisive role in this procedure when a discrete subgroup of $\mathbf{Sp}(2m, \mathbb{R})$ is lifted.

We need to settle some terminology to be used in the sequel. Let G_1 and G_2 be groups acting (for definiteness, from the left) on sets X_1 and X_2, respectively, and let $h : G_1 \to G_2$, $\varphi : X_1 \to X_2$ be a homomorphism and a map, respectively. We say that these actions are (h, φ)-*compatible* (also h-*compatible*, φ-*compatible*, etc.) if for every $x \in X_1$ and $g \in G_1$ we have $\varphi(gx) = h(g)\varphi(x)$. In particular, if $G_1 = G_2 = G$, $h = \mathrm{Id}_G$ (this is the identity isomorphism of G), and $\varphi : X_1 \to X_2$ is such that the actions of G on X_1 and X_2 are φ-compatible, then the map φ is said to be *equivariant*. The same terminology will be applied to right actions and even to actions of two groups on two sets from different sides (in the latter case h must be an antihomomorphism). A set with an action of a group G will be called a G-*space*.

2.1. The Siegel half-space and the group $\mathbf{Sp}(2m, \mathbb{R})$. By definition, the Siegel half-space $\mathcal{H}_m$ consists of all complex symmetric $(m \times m)$-matrices $Z = \operatorname{Re} Z + i \operatorname{Im} Z$ such that $\operatorname{Im} Z$, the imaginary part of Z, is positive definite. It is an $m(m+1)$-dimensional symmetric space of the noncompact Hermitian type for the action of the symplectic group $\mathbf{Sp}(2m,\mathbb{R})$ by fractional linear transformations. More precisely, every $g \in \mathbf{Sp}(2m,\mathbb{R})$ can be represented as a block matrix

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

with real entries of size $m \times m$ that satisfy

$$^t\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{2.2}$$

Relation (2.2) is equivalent to any of the following two conditions imposed on the entries A, B, C, D of the above block matrix (see [Kli, p. 1]):

$$^tCA = {}^tAC, \quad {}^tBD = {}^tDB, \quad {}^tAD - {}^tCB = 1, \tag{2.2$'$}$$

or

$$A\,{}^tB = B\,{}^tA, \quad C\,{}^tD = D\,{}^tC, \quad A\,{}^tD - B\,{}^tC = 1. \tag{2.2$''$}$$

The inverse of the matrix g in (2.1) is given by

$$g^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}. \tag{2.3}$$

The matrix g defined by (2.1) acts on $\mathcal{H}_m$ by the fractional linear transformation

$$Z \mapsto (AZ + B)(CZ + D)^{-1}. \tag{2.4}$$

2.2. 1-cocycles and the group $\mathbf{Mp}(2m,\mathbb{R})$. Let G be a group acting from the left on a set H. Recall that a function J defined on $G \times H$ and taking values in some group is called a cocycle (more precisely, a 1-cocycle) if for any $g_1, g_2 \in G$ and any $Z \in H$ we have

$$J(g_1 g_2, Z) = J(g_1, g_2 Z) J(g_2, Z).$$

For

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad Z \in \mathcal{H}_m,$$

we set

$$j(g, Z) = \det(CZ + D).$$

A straightforward calculation shows that j is a cocycle. Moreover, j is continuous in $g \in \mathbf{Sp}(2m,\mathbb{R})$ and holomorphic in $Z \in \mathcal{H}_m$. In the sequel an important role will be played by a cocycle ρ satisfying $\rho^2(g,Z) = j(g,Z)$. Such a cocycle that is continuous in g does not exist in a precise sense because the group $\mathbf{Sp}(2m,\mathbb{R})$ is not simply connected. We need to pass to a double cover of $\mathbf{Sp}(2m,\mathbb{R})$ in order to obtain an appropriate domain for ρ. The group $\mathbf{Sp}(2m,\mathbb{R})$ has a unique nonsplit central extension by the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The corresponding exact sequence is

$$0 \to \mathbb{Z}_2 \to \mathbf{Mp}(2m,\mathbb{R}) \to \mathbf{Sp}(2m,\mathbb{R}) \to 1,$$

with the metaplectic group $\mathbf{Mp}(2m,\mathbb{R})$ and the covering homomorphism $\pi : \mathbf{Mp}(2m,\mathbb{R}) \to \mathbf{Sp}(2m,\mathbb{R})$. There exists a unique continuous $\mathbf{Mp}(2m,\mathbb{R})$-cocycle ρ such that

$$\rho^2(g, Z) = j(\pi g, Z), \quad g \in \mathbf{Mp}(2m,\mathbb{R}), \;\; Z \in \mathcal{H}_m. \tag{2.5}$$

Now we describe a convenient realization of the group $\mathbf{Mp}(2m,\mathbb{R})$, the related extension of $\mathbf{Sp}(2m,\mathbb{R})$, and the cocycle ρ. The case where $m = 1$ was considered in detail in [LV, 1.8.25]; the general case exhibits no distinction.

$\mathbf{Mp}(2m,\mathbb{R})$ is defined as the set of all pairs (g, d) where $g \in \mathbf{Sp}(2m,\mathbb{R})$ and d is a holomorphic function on $\mathcal{H}_m$ such that $d^2(Z) = j(g, Z)$. The composition law in $\mathbf{Mp}(2m,\mathbb{R})$

is given by the relation $(g_1, d_1) \cdot (g_2, d_2) = (g_1 g_2, d)$, where $d(Z) = d_1(g_2 Z) d_2(Z)$. Also, we set $\pi(g, d) = g$. Thus, $\mathbf{Mp}(2m, \mathbb{R})$ is a connected Lie group locally isomorphic to $\mathbf{Sp}(2m, \mathbb{R})$ via π. The kernel of π is a cyclic group of order 2. It is contained in the center of $\mathbf{Mp}(2m, \mathbb{R})$, which is a cyclic group of order 4. The cocycle ρ is defined by

$$\rho((g, d), Z) = d(Z).$$

The definition of multiplication in $\mathbf{Mp}(2m, \mathbb{R})$ shows that ρ is a cocycle for $\mathbf{Mp}(2m, \mathbb{R})$. Relation (2.5) follows from the explicit construction of $\mathbf{Mp}(2m, \mathbb{R})$ described above.

2.3. Discrete subgroups related to Θ and the functional equation. Let (V_{2m}, Ω) be a standard symplectic space of dimension $2m$. This means that a symplectic basis $P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m$ is chosen in V so that for $i, j = 1, \ldots, m$ the skew-symmetric bilinear form Ω satisfies the relations $\Omega(P_i, P_j) = 0$, $\Omega(Q_i, Q_j) = 0$, and $\Omega(P_i, Q_j) = -\Omega(Q_i, P_j) = \delta_{i,j}$. Let $r = \mathbb{Z}P_1 \oplus \mathbb{Z}P_2 \oplus \cdots \oplus \mathbb{Z}P_m \oplus \mathbb{Z}Q_1 \oplus \mathbb{Z}Q_2 \oplus \cdots \oplus \mathbb{Z}Q_m$ be a lattice in V_{2m}, and let a function $\chi : r \to \{1, -1\}$ be defined by the relation

$$\chi\Big(\sum k_i P_i + \sum l_i Q_i\Big) = (-1)^{\sum k_i l_i}.$$

The group $\mathbf{Sp}(2m, \mathbb{R})$ acts on (V_{2m}, Ω) by linear symplectic transformations, and its discrete subgroup $\mathbf{Sp}(2m, \mathbb{Z})$ is exactly the group leaving the lattice r stable.

Consider a subgroup $\Gamma_{1,2}$ of $\mathbf{Sp}(2m, \mathbb{Z})$ leaving the function χ stable. The group $\Gamma_{1,2} \subset \mathbf{Sp}(2m, \mathbb{R})$ is a particular case of a subgroup $\Gamma(r, \chi) \in \mathbf{Sp}(2m, \mathbb{R})$ defined in [LV] for some class of pairs (r, χ). It is easy to verify (see [LV, 2.2.19] or [Mum, Chapter 2, Section 5]) that, whenever $g \in \mathbf{Sp}(2m, \mathbb{Z})$ is given by (2.1), g belongs to Γ if and only if the symmetric matrices ${}^tAB$ and ${}^tDC$ have even diagonal entries (or, equivalently, $A\,{}^tB$ and $D\,{}^tC$ have even diagonal entries). It is $\Gamma_{1,2}$ that is involved in the functional equation for Θ as defined by (1.1).

We denote by Γ_k (or by $\Gamma_k(m)$, to emphasize the dependence on m) the subgroup of $\mathbf{Sp}(2m, \mathbb{Z})$ consisting of all matrices g such that $g \equiv I_{2m} \pmod k$; this is the so-called *principal congruence subgroup*. Here I_{2m} is the identity matrix.

We are particularly interested in Γ_4, which has the following two properties:

i) $\Gamma_4 \subset \Gamma_{1,2} \subset \mathbf{Sp}(2m, \mathbb{Z})$, and Γ_4 is a normal subgroup of $\mathbf{Sp}(2m, \mathbb{Z})$ of finite index;

ii) for any $\gamma \in \Gamma_4, Z \in \mathcal{H}_m$ we have

$$\Theta^2(\gamma Z) = j(\gamma, Z)\Theta^2(Z). \tag{2.6}$$

Concerning ii) see [LV, 2.2.24] or [Mum, Chapter 2, (5.9)], and i) follows immediately from the definition of Γ_4. Property i) implies that $\Gamma_4 \backslash \mathbf{Sp}(2m, \mathbb{R})$ is a Galois cover of $\mathbf{Sp}(2m, \mathbb{Z}) \backslash \mathbf{Sp}(2m, \mathbb{R})$, and property ii) means that Θ^2 is an automorphic form of weight 1 with respect to Γ_4 (note that $\Gamma_{1,2}$ enjoys neither i) nor ii)).

For the convenience of the reader, we recall here some facts about the behavior of Θ under the action of $\Gamma_{1,2}$ on $\mathcal{H}_m$, including the role of "a rather tricky 8th root of 1" [Mum, p. 189].

Identity (2.6) is a consequence of the "functional equation" for the function Θ (see [LV, 2.2.37] or [Mum, Chapter 2, (5.1)]). In its full version it says that for every $Z \in \mathcal{H}_m$ and $\gamma \in \Gamma_{1,2}$ we have

$$\Theta(\gamma Z) = \xi(\gamma) j(\gamma, Z)^{1/2} \Theta(Z), \tag{2.7}$$

where $\xi(\gamma)$ is a certain 8th root of 1 (compare with (2.10) in the next section). The above formula determines $\xi(\gamma)$ up to a sign, so that there exists a character k of the group $\Gamma_{1,2}$ taking values in the 4th roots of 1 and satisfying

$$\Theta^2(\gamma Z) = k(\gamma) j(\gamma, Z)\Theta^2(Z). \tag{2.8}$$

On a certain subgroup of $\Gamma_{1,2}$, the character k can be calculated explicitly (see [LV, 2.2.24] and references therein). Furthermore, the character k is identically 1 on the group $\Gamma_4 \subset \Gamma_{1,2}$ (see [LV, 2.2.24] or [Mum, Chapter 2, (5.9)]). The latter fact is equivalent to (2.6).

2.4. Theta lifts Γ_4. The epimorphic homomorphism $\pi : \mathbf{Mp}(2m, \mathbb{R}) \to \mathbf{Sp}(2m, \mathbb{R})$ does not split. This means that there exists no continuous homomorphism $q : \mathbf{Sp}(2m, \mathbb{R}) \to \mathbf{Mp}(2m, \mathbb{R})$ such that $\pi q = \mathrm{Id}_{\mathbf{Sp}(2m,\mathbb{R})}$ (Id stands for the identity mapping). Nevertheless, for some subgroups of $\mathbf{Sp}(2m, \mathbb{R})$, in particular, for Γ_4, a homomorphism q as above can be constructed. Some examples of connected Lie subgroups sharing this feature will be discussed in Subsection 2.7.

For the discrete subgroup $\Gamma_4 \subset \mathbf{Sp}(2m, \mathbb{R})$, we shall describe a homomorphism $\lambda : \Gamma_4 \to \mathbf{Mp}(2m, \mathbb{R})$ such that $\pi\lambda = \mathrm{Id}_{\Gamma_4}$. Suppose $g \in \mathbf{Sp}(2m, \mathbb{R})$ and $Z \in \mathcal{H}_m$; we set

$$\theta_g(Z) = \Theta(gZ)/\Theta(Z).$$

It is clear that the map $(g, Z) \mapsto \theta_g(Z)$ is a (trivial) cocycle. Now we define the map $\lambda : \Gamma_4 \to \mathbf{Mp}(2m, \mathbb{R})$ by the formula

$$\lambda(\gamma) = (\gamma, \theta_\gamma), \quad \gamma \in \Gamma_4. \tag{2.9}$$

From (2.6) it follows that this definition really provides an element of $\mathbf{Mp}(2m, \mathbb{R})$. Furthermore, λ is a homomorphism because the map $(\gamma, Z) \mapsto \theta_\gamma(Z)$ is a Γ_4-cocycle (in fact, even a coboundary). Indeed, for every γ_1, $\gamma_2 \in \Gamma_4$ we have

$$\lambda(\gamma_1\gamma_2) = (\gamma_1\gamma_2, \theta_{\gamma_1}(\gamma_2\cdot)\theta_{\gamma_2}(\cdot)) = (\gamma_1, \theta_{\gamma_1})(\gamma_2, \theta_{\gamma_2}) = \lambda(\gamma_1)\lambda(\gamma_2).$$

The relation $\pi\lambda = \mathrm{Id}_{\Gamma_4}$ is an obvious consequence of (2.9); in particular, this implies that the kernel of λ is trivial. We denote $\lambda(\Gamma_4)$ by $\widehat{\Gamma}_4$. Thus, λ is an isomorphism of Γ_4 onto $\widehat{\Gamma}_4$. Again, formula (2.9) and the definition of the cocycle ρ in Subsection 2.2 show that

$$\rho(\lambda(\gamma), Z) = \theta_\gamma(Z)$$

for every $\gamma \in \Gamma_4$ and $Z \in \mathcal{H}_m$. The latter relation implies the following transformation formula for Θ (this formula refines (2.6)): for every $Z \in \mathcal{H}_m$ and $\gamma \in \widehat{\Gamma}_4$ we have

$$\Theta(\pi(\gamma)Z) = \rho(\gamma, Z)\Theta(Z). \tag{2.10}$$

Indeed,

$$\Theta(\pi(\gamma)Z)/\Theta(Z) = \theta_{\pi(\gamma)}(Z) = \rho((\pi(\gamma), \theta_{\pi(g)}), Z) = \rho(\lambda(\pi(\gamma)), Z) = \rho(\gamma, Z).$$

2.5. $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R})$ covers $\Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R})$. The symplectic group $\mathbf{Sp}(2m, \mathbb{R})$ acts transitively on the Siegel half-space $\mathcal{H}_m$ in accordance with formula (2.4). The stabilizer of the point $Z_0 = iI_m \in \mathcal{H}_m$ is a compact subgroup of $\mathbf{Sp}(2m, \mathbb{R})$. It is isomorphic to the unitary group $\mathbf{U}(m)$ and consists of all matrices of the form

$$\begin{pmatrix} \operatorname{Re} Q & \operatorname{Im} Q \\ -\operatorname{Im} Q & \operatorname{Re} Q \end{pmatrix}$$

with $Q \in \mathbf{U}(m)$. This embedding allows us to identify $\mathcal{H}_m$ and $\mathbf{Sp}(2m, \mathbb{R})/\mathbf{U}(m)$ as left $\mathbf{Sp}(2m, \mathbb{R})$-spaces.

The Siegel modular group $\mathbf{Sp}(2m, \mathbb{Z})$ is a discrete subgroup of $\mathbf{Sp}(2m, \mathbb{R})$. It is a nonuniform lattice, so that the homogeneous space $\mathbf{Sp}(2m, \mathbb{Z})\backslash\mathbf{Sp}(2m, \mathbb{R})$ is noncompact but has a finite volume induced by the Haar measure on $\mathbf{Sp}(2m, \mathbb{R})$. Indeed, $\mathbf{Sp}(2m, \mathbb{Z})\backslash\mathbf{Sp}(2m, \mathbb{R})$ is (the total space of) a principal fiber bundle with the fiber $\mathbf{U}(m)$ over $\mathbf{Sp}(2m, \mathbb{Z})\backslash\mathcal{H}_m = \mathbf{Sp}(2m, \mathbb{Z})\backslash\mathbf{Sp}(2m, \mathbb{R})/\mathbf{U}(m)$. The latter quotient space $\mathbf{Sp}(2m, \mathbb{Z})\backslash\mathcal{H}_m$ is noncompact and has finite volume, which follows from the well-known properties of the classical Siegel fundamental domain for the action of $\mathbf{Sp}(2m,\mathbb{Z})$ on $\mathcal{H}_m$ (see [Kli] for more detail). Since $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ is fibered over

$$\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathcal{H}_m = \mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})/\mathbf{U}(m)$$

with a compact fiber $\mathbf{U}(m)$, this implies that $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ is a homogeneous space (in fact, a manifold) of finite invariant measure.

In accordance with Subsection 2.2, $\pi : \mathbf{Mp}(2m,\mathbb{R}) \to \mathbf{Sp}(2m,\mathbb{R})$ is a two-to-one covering homomorphism, and the homogeneous space $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ is a double cover of the homogeneous space $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ via the map induced by π. More precisely, we define a covering map $\pi_4 : \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R}) \to \Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ as follows. Let $\widehat{\Gamma}_4 g \in \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ be a coset corresponding to some $g \in \mathbf{Mp}(2m,\mathbb{R})$. We put $\pi_4(\widehat{\Gamma}_4 g) = \Gamma_4\pi(g)$. It is clear that the $\mathbf{Mp}(2m,\mathbb{R})$-space $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ and the $\mathbf{Sp}(2m,\mathbb{R})$-space $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ are (π,π_4)-compatible.

Moreover, from Subsection 2.4 it follows that $\pi|_{\widehat{\Gamma}_4}$ is an isomorphism onto Γ_4 with inverse isomorphism λ defined by (2.7). Consequently, $\widehat{\Gamma}_4$ has only trivial intersection with the kernel of π. Recall also that the kernel of π is isomorphic to $\mathbb{Z}_2$ and belongs to the center of $\mathbf{Mp}(2m,\mathbb{R})$. Thus, every Γ_4-coset in $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ is the image under π_4 of exactly two $\widehat{\Gamma}_4$-cosets in $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$, which are interchanged by the action of a nontrivial element in the kernel of π.

Since Γ_4 is a normal subgroup of $\mathbf{Sp}(2m,\mathbb{Z})$, the homogeneous space $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$, in its turn, covers $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ with multiplicity equal to the index of Γ_4 in $\mathbf{Sp}(2m,\mathbb{Z})$.

2.6. Introducing $\widehat{\Theta}$ as a function on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$. We recall that $Z_0 = iI_m \in \mathcal{H}_m$. For $g \in \mathbf{Mp}(2m,\mathbb{R})$ we set

$$\widehat{\Theta}(g) = \Theta((\pi g)Z_0)/\rho(g,Z_0).$$

By (2.10), for any $\gamma \in \widehat{\Gamma}_4$ and $g \in \mathbf{Mp}(2m,\mathbb{R})$ we have

$$\begin{aligned}\widehat{\Theta}(\gamma g) &= \Theta(\pi(\gamma g)Z_0)/\rho(\gamma g,Z_0) = \Theta(\pi(\gamma)\pi(g)Z_0)/\rho(\gamma g,Z_0)\\ &= \big(\rho(\gamma,\pi(g)Z_0)\Theta(\pi(g)Z_0)\big)/\rho(\gamma g,Z_0) = \Theta(\pi(g)Z_0)/\rho(g,Z_0)\\ &= \widehat{\Theta}(g).\end{aligned}$$

Thus, $\widehat{\Theta}$ can be regarded as a function on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$.

2.7. Lifting subgroups $\mathbf{B}_+$ and $\mathbf{B}_-$. Let $\mathbb{GL}^+(m,\mathbb{R})$ be the multiplicative group of real $(m\times m)$-matrices with positive determinant, and let $\mathbf{S}_m(\mathbb{R})$ be the additive group of real symmetric $(m\times m)$-matrices. Let $\mathbf{B}_+$ and $\mathbf{B}_-$ be the subgroups of $\mathbf{Sp}(2m,\mathbb{R})$ defined by the relations

$$\mathbf{B}_+ = \left\{\begin{pmatrix} A & B\\ 0 & {}^tA^{-1}\end{pmatrix} : A \in \mathbb{GL}^+(m,\mathbb{R}), B\,{}^tA \in \mathbf{S}_m(\mathbb{R})\right\}, \tag{2.11}$$

$$\mathbf{B}_- = \left\{\begin{pmatrix} A & 0\\ C & {}^tA^{-1}\end{pmatrix} : A \in \mathbb{GL}^+(m,\mathbb{R}),\ {}^tAC \in \mathbf{S}_m(\mathbb{R})\right\}. \tag{2.12}$$

The groups $\mathbf{B}_+$ and $\mathbf{B}_-$ are connected Lie subgroups in $\mathbf{Sp}(2m,\mathbb{R})$. Here we shall describe the homomorphic embeddings of the subgroups $\mathbf{B}_+$ and $\mathbf{B}_-$ in $\mathbf{Mp}(2m,\mathbb{R})$ that are right inverse to π over these subgroups.

On $\mathbf{B}_+$, the cocycle j introduced in Subsection 2.2 reduces to the map

$$j_+ : (g,Z) = \left(\begin{pmatrix} A & B\\ 0 & {}^tA^{-1}\end{pmatrix}, Z\right) \mapsto (\det A)^{-1},$$

which is merely a positive character of $\mathbf{B}_+$. This character admits a positive square root $g \mapsto (\det A)^{-1/2}$. Let $j_+^{1/2}$ denote the corresponding cocycle. Thus, by Subsection 2.2, we can embed $\mathbf{B}_+$ in $\mathbf{Mp}(2m, \mathbb{R})$ via the homomorphism μ_+ defined by the relation

$$\mu_+(g) = (g, j_+^{1/2}(g)), \quad g \in \mathbf{B}_+.$$

We set $\widehat{\mathbf{B}}_+ = \mu_+(\mathbf{B}_+)$. Clearly, $\pi\mu_+ = \mathrm{Id}_{\mathbf{B}_+}$.

A similar embedding exists for $\mathbf{B}_-$ as well. The cocycle j introduced in Subsection 2.2 reduces on $\mathbf{B}_-$ to the function

$$j_- : (g, Z) = \left(\begin{pmatrix} A & 0 \\ C & {}^t A^{-1} \end{pmatrix}, Z \right) \mapsto \det(CZ + {}^t A^{-1}).$$

For every $g \in \mathbf{B}_-$ we have $\lim_{Z\to 0} j_-(g, Z) = \det({}^t A^{-1})$, where A is the upper left block of the matrix g. Since $\mathcal{H}_m$ is simply connected, for every $g \in \mathbf{B}_-$ there exists a unique holomorphic square root $j_-^{1/2}(g, \cdot)$ of $j_-(g, \cdot)$, with the property that $\lim_{Z\to 0} j_-^{1/2}(g, Z) = (\det A)^{-1/2}$, where $(\det A)^{-1/2}$ is the positive value of the square root of $(\det A)^{-1}$. It is clear that $j_-^{1/2} : (g, Z) \mapsto j_-^{1/2}(g, Z)$ satisfies the cocycle identity up to a sign. But this is a cocycle indeed for the following reasons: (a) $Z = 0$ is a fixed point for the group $\mathbf{B}_-$ acting by (2.4) on the closed upper Siegel half-space $\overline{\mathbf{H}}_m = \{Z = X + iY : X, Y$ are symmetric real $(m \times m)$-matrices, Y is nonnegative definite$\}$; (b) the map $g \mapsto \lim_{Z\to 0} j_-^{1/2}(g, Z)$ is a group homomorphism. Thus, we can embed $\mathbf{B}_-$ in $\mathbf{Mp}(2m, \mathbb{R})$ by means of a homomorphism μ_- defined as follows:

$$\mu_-(g) = (g, j_-^{1/2}(g, \cdot)), \quad g \in \mathbf{B}_+.$$

Again, setting $\widehat{\mathbf{B}}_- = \mu_-(\mathbf{B}_-)$, we see that $\pi\mu_- = I_{\mathbf{B}_-}$.

Now we consider the Lie subgroup $\mathbf{D}$ of $\mathbf{Sp}(2m, \mathbb{R})$ defined by the relation

$$\mathbf{D} = \mathbf{B}_+ \cap \mathbf{B}_-.$$

It consists of the matrices of the form

$$d(A) = \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix},$$

where $A \in \mathbb{GL}^+(m, \mathbb{R})$. Thus, $d : \mathbb{GL}^+(m, \mathbb{R}) \to \mathbf{Sp}(2m, \mathbb{R})$ is an embedding of Lie groups. There are two seemingly different embeddings of $\mathbf{D}$ into $\mathbf{Mp}(2m, \mathbb{R})$: by means of μ_+, and by means of μ_-. In fact, the restrictions of them to $\mathbf{D}$ agree because the cocycles $j_+^{1/2}$ and $j_-^{1/2}$ agree. Thus, we can set

$$\mu = \mu_+|_{\mathbf{D}} = \mu_-|_{\mathbf{D}}, \quad \widehat{d} = \mu d, \quad \widehat{\mathbf{D}} = \mu(\mathbf{D}) = \widehat{d}(\mathbb{GL}^+(m, \mathbb{R})).$$

2.8. Horospheric subgroups in $\mathbf{Sp}(2m, \mathbb{R})$ and $\mathbf{Mp}(2m, \mathbb{R})$. Let $\mathbf{H}_+$ be the Lie subgroup of $\mathbf{B}_+$ that consists of matrices of the form

$$h_+(B) = \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix},$$

where $B \in \mathbf{S}_m(\mathbb{R})$ (observe that $h_+ : \mathbf{S}_m(\mathbb{R}) \to \mathbf{Sp}(2m, \mathbb{R})$ is a Lie group embedding). The relation

$$\begin{pmatrix} A & B \\ 0 & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} I_m & F \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} I_m & AF\,{}^t A \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & B \\ 0 & {}^t A^{-1} \end{pmatrix} \tag{2.13}$$

(with $A \in \mathbb{GL}^+(m, \mathbb{R})$ and $B\,{}^t A, F \in \mathbf{S}_m(\mathbb{R})$) shows that $\mathbf{H}_+$ is a normal subgroup in $\mathbf{B}_+$. A special case of (2.13) is

$$\begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix} \begin{pmatrix} I_m & F \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} I_m & AF\,{}^t A \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{pmatrix}, \tag{2.14}$$

where $A \in \mathbb{GL}^+(m,\mathbb{R})$ and $F \in \mathbf{S}_m(\mathbb{R})$. Every element of $\mathbf{B}_+$ is a product of uniquely determined elements of $\mathbf{D}$ and $\mathbf{H}_+$, so that

$$\begin{pmatrix} A & B \\ 0 & {}^t\!A^{-1} \end{pmatrix} = \begin{pmatrix} I_m & B\,{}^t\!A \\ 0 & I_m \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix}\begin{pmatrix} I_m & A^{-1}B \\ 0 & I_m \end{pmatrix} \tag{2.15}$$

with some $A \in \mathbb{GL}^+(m,\mathbb{R})$ and $B\,{}^t\!A \in \mathbf{S}_m(\mathbb{R})$. Thus, $\mathbf{B}_+ = \mathbf{DH}_+ = \mathbf{H}_+\mathbf{D}$. Hence, $\mathbf{B}_+$ is a semidirect product of $\mathbf{H}_+$ and $\mathbf{D}$. In particular, $\mathbf{D}$ is a quotient of $\mathbf{B}_+$ by the normal subgroup $\mathbf{H}_+$. It is convenient to rewrite relations (2.14) and (2.15) in terms of the maps h_+ and d. We obtain

$$d(A)h_+(F) = h_+(AF\,{}^t\!A)d(A) \tag{2.16}$$

and

$$\begin{pmatrix} A & B \\ 0 & {}^t\!A^{-1} \end{pmatrix} = h_+(B\,{}^t\!A)d(A) = d(A)h_+(A^{-1}B), \tag{2.17}$$

where $A \in \mathbb{GL}^+(m,\mathbb{R})$, $B\,{}^t\!A \in \mathbf{S}_m(\mathbb{R})$. In particular, we see that

$$\mathbf{B}_+ = d(\mathbb{GL}^+(m,\mathbb{R}))h_+(\mathbf{S}_m(\mathbb{R})) = h_+(\mathbf{S}_m(\mathbb{R}))d(\mathbb{GL}^+(m,\mathbb{R})). \tag{2.18}$$

The images of $\mathbf{D}$ and $\mathbf{H}_+$ under μ_+ will be denoted by $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{H}}_+$, respectively. We fix the notation $\widehat{h}_+ = \mu_+h_+$ and $\widehat{d} = \mu d$ $(= \mu_+|_{\mathbf{D}}d)$ for the Lie embeddings of $\mathbf{S}_m(\mathbb{R})$ and $\mathbb{GL}^+(m,\mathbb{R})$ (respectively) into $\mathbf{Mp}(2m,\mathbb{R})$. Since μ_+ is a Lie group isomorphism of $\mathbf{B}_+$ onto $\widehat{\mathbf{B}}_+$, the subgroup $\widehat{\mathbf{H}}_+ = \mu_+(\mathbf{H}_+)$ is normal in $\widehat{\mathbf{B}}_+$. Also, any relation in the group $\mathbf{B}_+$ written in terms of h_+ and d transforms automatically into a similar relation in the group $\widehat{\mathbf{B}}_+$ written in terms of $\widehat{h}_+$ and $\widehat{d}$. In particular, from (2.16)–(2.18) it follows that for every $A \in \mathbb{GL}^+(m,\mathbb{R})$ and $B\,{}^t\!A \in \mathbf{S}_m(\mathbb{R})$ we have

$$\widehat{d}(A)\widehat{h}_+(F) = \widehat{h}_+(AF\,{}^t\!A)\widehat{d}(A) \tag{2.19}$$

and

$$\mu_+\left(\begin{pmatrix} A & B \\ 0 & {}^t\!A^{-1} \end{pmatrix}\right) = \widehat{h}_+(B\,{}^t\!A)\widehat{d}(A) = \widehat{d}(A)\widehat{h}_+(A^{-1}B), \tag{2.20}$$

and that

$$\widehat{\mathbf{B}}_+ = \widehat{d}(\mathbb{GL}^+(m,\mathbb{R}))\widehat{h}_+(\mathbf{S}_m(\mathbb{R})) = \widehat{h}_+(\mathbf{S}_m(\mathbb{R}))\widehat{d}(\mathbb{GL}^+(m,\mathbb{R})). \tag{2.21}$$

Thus, $\widehat{\mathbf{B}}_+ = \widehat{\mathbf{D}}\widehat{\mathbf{H}}_+ = \widehat{\mathbf{H}}_+\widehat{\mathbf{D}}$, and $\widehat{\mathbf{B}}_+$ is a semidirect product of $\widehat{\mathbf{H}}_+$ and $\widehat{\mathbf{D}}$. As above, $\widehat{\mathbf{D}}$ is a quotient of $\widehat{\mathbf{B}}_+$ by the normal subgroup $\widehat{\mathbf{H}}_+$.

The properties of $\mathbf{H}_+$, $\mathbf{D}$, and $\mathbf{B}_+$ deduced above are shared, with minor changes, by $\mathbf{H}_-$, $\mathbf{D}$, and $\mathbf{B}_-$. Let $\mathbf{H}_-$ be the Lie subgroup of $\mathbf{B}_-$ that consists of matrices of the form

$$h_-(C) = \begin{pmatrix} I_m & 0 \\ C & I_m \end{pmatrix}, \quad C \in \mathbf{S}_m(\mathbb{R}).$$

We set $\widehat{\mathbf{H}}_- = \mu_-(\mathbf{H}_-)$. Again, $\mathbf{H}_-$ $(\widehat{\mathbf{H}}_-)$ is a normal subgroup of $\mathbf{B}_-$ (respectively, of $(\widehat{\mathbf{B}}_-)$), $\mathbf{B}_- = \mathbf{DH}_- = \mathbf{H}_-\mathbf{D}$ (respectively, $\widehat{\mathbf{B}}_- = \widehat{\mathbf{D}}\widehat{\mathbf{H}}_- = \widehat{\mathbf{H}}_-\widehat{\mathbf{D}}$), and $\mathbf{B}_-$ $(\widehat{\mathbf{B}}_-)$ is a semidirect product of $\mathbf{H}_-$ and $\mathbf{D}$ (respectively, of $\widehat{\mathbf{H}}_-$ and $\widehat{\mathbf{D}}$). Let $\widehat{h}_- = \mu_-h_-$. Each of relations (2.13)–(2.18) has a natural analog in this situation. We shall mention part of them only. For every $A \in \mathbb{GL}^+(m,\mathbb{R})$ and $G \in \mathbf{S}_m(\mathbb{R})$, we have

$$\begin{pmatrix} I_m & 0 \\ G & I_m \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & {}^t\!A^{-1} \end{pmatrix}\begin{pmatrix} I_m & 0 \\ {}^t\!AGA & I_m \end{pmatrix}, \tag{2.14$'$}$$

$$h_-(G)d(A) = d(A)h_-(AG\,{}^t\!A), \tag{2.16$'$}$$

and

$$\widehat{h}_-(G)\widehat{d}(A) = \widehat{d}(A)\widehat{h}_-(AG\,{}^t\!A). \tag{2.19$'$}$$

We also need the one-parameter subgroup $(g(s))_{s\in\mathbb{R}}$ of $\mathbf{D}$ defined by

$$g(s) = \begin{pmatrix} I_m \exp(-s) & 0 \\ 0 & I_m \exp s \end{pmatrix}$$

and by the commutation relations

$$h_+(B)g(s) = g(s)h_+(B\exp(2s)), \quad h_-(B)g(s) = g(s)h_-(B\exp(-2s)), \tag{2.22}$$

which are fulfilled for every $B \in \mathbf{S}_m(\mathbb{R})$ and $s \in \mathbb{R}$. Relations (2.22) are lifted to $\mathbf{Mp}(2m,\mathbb{R})$ in the form

$$\widehat{h}_+(B)\widehat{g}(s) = \widehat{g}(s)\widehat{h}_+(B\exp(2s)), \quad \widehat{h}_-(B)\widehat{g}(s) = \widehat{g}(s)\widehat{h}_-(B\exp(-2s)), \tag{2.23}$$

where $\widehat{g}(s) = \mu(g(s))$, $s \in \mathbb{R}$. From (2.22) it follows that for every $f_+ \in \mathbf{H}_+$ and $f_- \in \mathbf{H}_-$ we have $g_s f_+ g_{-s} \to I_{2m}$ and $g_{-s} f_- g(s) \to I_{2m}$ as $s \to \infty$ (here I_{2m} denotes the identity $(2m \times 2m)$-matrix). The groups $\mathbf{H}_+$ and $\mathbf{H}_-$ are called the *expanding* and the *contracting horospheric subgroups*, respectively, for the subgroup $(g(s))_{s\in\mathbb{R}}$. Similarly, (2.16) implies that $\widehat{\mathbf{H}}_+, \widehat{\mathbf{H}}_- \subset \mathbf{Mp}(2m,\mathbb{R})$ are, the expanding and the contracting horospheric subgroups, respectively, for $(\widehat{g}(s))_{s\in\mathbb{R}}$. Under conjugation by $\widehat{g}(s)$, every element of $\widehat{\mathbf{H}}_+$ ($\widehat{\mathbf{H}}_-$) converges to $\widehat{I}_{2m}$, the identity of $\mathbf{Mp}(2m,\mathbb{R})$, as s tends to $+\infty$ (respectively, to $-\infty$).

2.9. The quasigeodesic flow on the groups $\mathbf{Sp}(2m,\mathbb{R})$ and $\mathbf{Mp}(2m,\mathbb{R})$ and on their homogeneous spaces. By mapping $(g,s) \in \mathbf{Sp}(2m,\mathbb{R}) \times \mathbb{R}$ to $g/g(s)$, we define a flow on $\mathbf{Sp}(2m,\mathbb{R})$. Similarly, the relation $(g,s) \mapsto g/\widehat{g}(s)$ for $(g,s) \in \mathbf{Mp}(2m,\mathbb{R}) \times \mathbb{R}$ determines a flow on $\mathbf{Mp}(2m,\mathbb{R})$. These flows agree in the sense that for every $g \in \mathbf{Mp}(2m,\mathbb{R})$ and $s \in \mathbb{R}$ we have $\pi(g\widehat{g}(s)) = \pi(g)g(s)$.

Furthermore, these two flows induce flows on the right homogeneous spaces of the groups $\mathbf{Sp}(2m,\mathbb{R})$ and $\mathbf{Mp}(2m,\mathbb{R})$. The action of the corresponding flows on every homogeneous space of the form $\Gamma\backslash\mathbf{Sp}(2m,\mathbb{R})$ or $\widehat{\Gamma}\backslash\mathbf{Mp}(2m,\mathbb{R})$ (where $\Gamma \subset \mathbf{Sp}(2m,\mathbb{R})$ and $\widehat{\Gamma} \subset \mathbf{Mp}(2m,\mathbb{R})$ are some discrete subgroups) is denoted by right multiplication by the corresponding element of $(g(s))_{s\in\mathbb{R}}$ and $(\widehat{g}(s))_{s\in\mathbb{R}}$. For instance, for $x \in \Gamma\backslash\mathbf{Sp}(2m,\mathbb{R})$, $xg(s)$ denotes the coset $\Gamma g g(s)$ if $x = \Gamma g$, with obvious changes for $(\widehat{g}(s))_{s\in\mathbb{R}}$. We shall call any of these actions a *quasigeodesic flow.*

As was outlined in Subsection 2.5, the isomorphisms $\lambda : \Gamma_4 \to \widehat{\Gamma}_4$ along with the map π allow us to construct a covering map $\pi_4 : \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R}) \to \Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ of multiplicity 2. Since the actions by left and right multiplications commute, for every $s \in \mathbb{R}$ and $x \in \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ we have

$$\pi_4(x\widehat{g}(s)) = \pi_4(x)g(s).$$

Precisely as in the case of $(g(s))_{s\in\mathbb{R}}$ and $(\widehat{g}(s))_{s\in\mathbb{R}}$, we introduce the actions of the group $\mathbf{S}_m(\mathbb{R})$ on $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ by the relation $(x,F) \to xh_+(F)$ and on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ by the relation $(x,F) \to x\widehat{h}_+(F)$, where $(x,F) \in \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R}) \times \mathbf{S}_m(\mathbb{R})$. Each of these actions will be called an *expanding horospheric* action. These actions are related by the formula

$$\pi_4(x\widehat{h_+}(F)) = \pi_4(x)h_+(F), \quad x \in \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R}), \ \ F \in \mathbf{S}_m(\mathbb{R}). \tag{2.24}$$

We skip the entirely similar notation and properties for the *contracting horospheric* actions. From now on, we set $\mathcal{M}_4 = \Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ and $\widehat{\mathcal{M}_4} = \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$. We also supply the homogeneous spaces $\mathcal{M}_4$ and $\widehat{\mathcal{M}_4}$ with the probability measures P_4 and $\widehat{P}_4$

induced by the Haar measures on $\mathbf{Sp}(2m,\mathbb{R})$ and $\mathbf{Mp}(2m,\mathbb{R})$. The measures P_4 and $\widehat{P}_4$ are invariant under the action of $\mathbf{Sp}(2m,\mathbb{R})$ and $\mathbf{Mp}(2m,\mathbb{R})$, respectively, and π_4 sends $\widehat{P}_4$ to P_4.

Remark 2.1. The flow induced on $\mathbf{Sp}(2m,\mathbb{R})$ by the one-parameter subgroup $(g(s))_{s\in\mathbb{R}}$ can be pushed down to the factor of $\mathbf{Sp}(2m,\mathbb{R})$ by its center $\{I_{2m}, -I_{2m}\}$. The latter flow can be identified with the flow of holomorphic orthonormal frames over the Siegel half-space $\mathcal{H}_m$; this is an extension of the geodesic flow on the unit tangent bundle of $\mathcal{H}_m$. From the viewpoint of differentiable dynamics, the quasigeodesic flows belong (for $m > 1$) to the class of *partially hyperbolic dynamical systems.*

2.10. The action of $\mathbf{B}_+$ on $\mathcal{H}_m$, and compact horospheres in Siegel cylinders ("left horospheres"). The action of $\mathbf{B}_+$ on $\mathcal{H}_m$ by formula (2.4) is transitive. To establish this, we consider the group $\mathbf{H}_+ \subset \mathbf{B}_+$ operating on $\mathcal{H}_m$ by the same formula. This action is free and reduces to translations of $\mathcal{H}_m$ parallel to the "real direction". The $\mathbf{H}_+$-orbit of a point $X + iY \in \mathcal{H}_m$ consists of all matrices $X' + iY$ with X' real symmetric. Next, we observe that the subgroup $\mathbf{H}_+$ is normal in $\mathbf{B}_+$ with the quotient group $\mathbf{D}$. This implies that every element of $\mathbf{D}$ moves every $\mathbf{H}_+$-orbit onto another $\mathbf{H}_+$-orbit. Thus, to check the transitivity of $\mathbf{B}_+$ on $\mathcal{H}_m$ it suffices to prove that $\mathbf{D}$ acts transitively on the set of $\mathbf{H}_+$-orbits in $\mathcal{H}_m$, or even to prove that there exists a $\mathbf{D}$-orbit that intersects every $\mathbf{H}_+$-orbit. The latter fact follows from considering the $\mathbf{D}$-orbit of the point $Z_0 = iI_m$. Actually, for every $A \in \mathbb{GL}^+(m,\mathbb{R})$ we have $d(A)Z_0 = iA\,{}^tA$, which implies that the $\mathbf{D}$-orbit of Z_0 consists of all $(m \times m)$-matrices iY with $Y > 0$, and we are done.

Remark 2.2. The action of the group $\mathbf{B}_+$ on $\mathcal{H}_m$ is not free for $m > 1$. Indeed, the stabilizer of the point Z_0 for this action consists of all matrices of the form $d(A)$ with $A \in \mathbf{SO}(m,\mathbb{R})$, the group of real orthogonal $(m \times m)$-matrices with determinant 1.

If we are given an action of $\mathbf{Sp}(2m,\mathbb{R})$ or $\mathbf{Mp}(2m,\mathbb{R})$ on some space, then, by definition, a *horosphere* is an orbit of the group $\mathbf{H}_+$ with respect to this action. More precisely, in this case we should talk about an *expanding horosphere*, calling an orbit of $\mathbf{H}_-$ a *contracting horosphere*. Since from now on we consider orbits of $\mathbf{H}_+$ only, we shall merely call these orbits horospheres.

Instead of the actions of $\mathbf{H}_+$ and $\widehat{\mathbf{H}}_+$, it is somewhat more convenient for us to consider the equivalent actions of the group $\mathbf{S}_m(\mathbb{R})$, as was defined in Subsection 2.9, via the isomorphisms h_+ and $\widehat{h}_+$. These actions of $\mathbf{S}_m(\mathbb{R})$ will be called $\mathbf{S}_m(\mathbb{R})$-*actions* in this section. In particular, an $\mathbf{S}_m(\mathbb{R})$-orbit in $\mathcal{M}_4$ (in $\widehat{\mathcal{M}}_4$) is the same as an $\mathbf{H}_+$-orbit (respectively, an $\widehat{\mathbf{H}}_+$-orbit). For every $A \in \mathbb{GL}^+(m,\mathbb{R})$ we define an automorphism i_A of the group $\mathbf{S}_m(\mathbb{R})$ by the relation $i_A(F) = A^{-1}F\,{}^tA^{-1}$, $F \in \mathbf{S}_m(\mathbb{R})$.

Compact left horospheres can be associated with an arbitrary *uniform subgroup* N of the group $\mathbf{H}_+$. Since $\mathbf{H}_+$ is Abelian and $N \subset \mathbf{H}_+$, the action of $\mathbf{H}_+$ on $\mathcal{H}_m$ gives rise to a left action of $\mathbf{H}_+$ on the space $N\backslash\mathcal{H}_m$, the *Siegel cylinder* of N. The stabilizer of every point in $N\backslash\mathcal{H}_m$ is N. Therefore, as a homogeneous $\mathbf{H}_+$-space, any $\mathbf{H}_+$-orbit in $N\backslash\mathcal{H}_m$ is isomorphic to $N\backslash\mathbf{H}_+$, or, equivalently, to $N'\backslash\mathbf{S}_m(\mathbb{R})$ with $N' = h_+^{-1}(N)$, which, in turn, is equivalent to the torus $\mathbb{T}^{m(m+1)/2}$ with the standard action of $\mathbb{R}^{m(m+1)/2}$. In particular, via this isomorphism a unique $\mathbf{H}_+$-invariant probability measure on an $\mathbf{H}_+$-orbit in $N\backslash\mathcal{H}_m$ corresponds to the Haar probability measure on the torus $\mathbb{T}^{m(m+1)/2}$. Thus, the Siegel cylinder $N\backslash\mathcal{H}_m$ is a union of disjoint compact left horospheres, and each of them is supplied with a unique $\mathbf{H}_+$-invariant probability measure.

Since $N \subset \mathbf{H}_+$, the space of $\mathbf{H}_+$-orbits in the Siegel cylinder $N\backslash\mathcal{H}_m$ is the same as the space of $\mathbf{H}_+$-orbits in $\mathcal{H}_m$, and the group $\mathbf{D}$ acts transitively on it. Thus, we have a transitive action of $\mathbf{D}$ on the space of $\mathbf{H}_+$-orbits in $N\backslash\mathcal{H}_m$, though there is no natural projection of the action of $\mathbf{D}$ on $\mathcal{H}_m$ by formula (2.4) to a similar action on $N\backslash\mathcal{H}_m$. Nevertheless, the action of $\mathbf{D}$ on the set of $\mathbf{H}_+$-orbits in $N\backslash\mathcal{H}_m$ can be represented as a quotient action of an action of $\mathbf{D}$ on $N\backslash\mathcal{H}_m$ itself, so that the latter action respects the $\mathbf{H}_+$-orbit structure. For every $A \in \mathbb{GL}^+(m,\mathbb{R})$ and $Z = X + iY \in \mathcal{H}_m$, we define

$$\varkappa_{d(A)}(Z) = X + iAY\,{}^tA. \tag{2.25}$$

In accordance with this formula, the group $\mathbf{D}$ acts on $N\backslash\mathcal{H}_m$ (or on $\mathcal{H}_m$), sending the $\mathbf{H}_+$-orbits onto $\mathbf{H}_+$-orbits. Observe that the action $\varkappa$ has no relationship with formula (2.4) and does not preserve Riemannian and complex structures of the manifolds under consideration. However, the map $\varkappa_{d(A)}$ sends the $\mathbf{H}_+$-orbit of $Z = X + iY \in \mathcal{H}_m$ onto the $\mathbf{H}_+$-orbit of $\varkappa_{d(A)}(Z)$, so that the corresponding two $\mathbf{S}_m(\mathbb{R})$-spaces are $(i_A, \varkappa_{d(A)})$-compatible, where i_A is the automorphism of the group $\mathbf{S}_m(\mathbb{R})$ defined above. Moreover, the map $\varkappa_{d(A)}$ transports the $\mathbf{H}_+$-invariant probability measure on the first orbit to that on the second. The latter conclusion follows from the uniqueness of such measures and the fact that, for φ-compatible spaces, φ takes an invariant measure to an invariant one.

As N we are going to take one of the following two discrete subgroups of $\mathbf{H}_+$ related to a pair of discrete subgroups of $\mathbf{Sp}(2m,\mathbb{R})$:

$$N_4 = \Gamma_4 \cap \mathbf{H}_+, \quad N_{1,2} = \Gamma_{1,2} \cap \mathbf{H}_+.$$

Both N_4 and $N_{1,2}$ are uniform subgroups of $\mathbf{H}_+$. These groups and their position in $\mathbf{H}_+$ are completely determined, up to isomorphism, by describing the groups $N_4' = h_+^{-1}(N_4)$ and $N_{1,2}' = h_+^{-1}(N_{1,2})$ as subgroups of $\mathbf{S}_m(\mathbb{R})$. Clearly, N_4' is the additive group of integral symmetric $(m \times m)$-matrices with entries congruent to 0 mod 4, and $N_{1,2}'$ is the additive group of integral symmetric matrices with even diagonal entries.

Comparing the $\mathbf{H}_+$-spaces $N_4\backslash\mathcal{H}_m$ and $N_{1,2}\backslash\mathcal{H}_m$, we see that the latter space is covered by the former one with multiplicity equal to the index of N_4 as a subgroup of $N_{1,2}$. This index is equal to that of N_4' as a subgroup of $N_{1,2}'$, which is $2m^2$. This covering map is compatible with the action of $\mathbf{H}_+$, it sends every $\mathbf{H}_+$-orbit in $N_4\backslash\mathcal{H}_m$ to an $\mathbf{H}_+$-orbit in $N_{1,2}\backslash\mathcal{H}_m$, and, by the above remark on invariant measures on compact $\mathbf{H}_+$-orbits in the Siegel cylinder, it also maps the invariant probability measure on every $\mathbf{H}_+$-orbit in $N_4\backslash\mathcal{H}_m$ to the similar measure on the image of that orbit. Moreover, the actions of the group $\mathbf{D}$ by formula (2.25) on two Siegel cylinders $N_4\backslash\mathcal{H}_m$ and $N_{1,2}\backslash\mathcal{H}_m$ are compatible with this covering map.

2.11. Compact horospheres in $\mathcal{M}_4$ and $\widehat{\mathcal{M}}_4$ ("right horospheres"). Every point of the space $\mathcal{M}_4$ is a coset of the form $\Gamma_4 g$ for some $g \in \mathbf{Sp}(2m,\mathbb{R})$. The set $x_0\mathbf{B}_+ \subset \mathcal{M}_4$ (the $\mathbf{B}_+$-orbit of the point $x_0 = \Gamma_4$) is a union of compact orbits of the group $\mathbf{H}_+ \subset \mathbf{B}_+$. Similarly, in the space $\widehat{\mathcal{M}}_4$ we set $\widehat{x}_0 = \widehat{\Gamma}_4$ and consider the $\widehat{\mathbf{B}}_+$-orbit of the point $\widehat{x}_0$; this is also a union of compact orbits of $\widehat{\mathbf{H}}_+ \subset \widehat{\mathbf{B}}_+$. Observe that $\pi_4(\widehat{x}_0) = x_0$ and that π_4 maps $\widehat{x}_0\widehat{\mathbf{B}}_+$ onto $x_0\mathbf{B}_+$ so that the action of $\widehat{\mathbf{B}}_+$ on $\widehat{x}_0\widehat{\mathbf{B}}_+$ and the action of $\mathbf{B}_+$ on $x_0\mathbf{B}_+$ are $(\mu_+^{-1}, \pi_4|_{\widehat{\mathbf{B}}_+})$-compatible.

For every $A \in \mathbb{GL}^+(m,\mathbb{R})$, we define maps $\varphi_A : x_0\mathbf{H}_+ \to x_0 d(A)\mathbf{H}_+$ and $\widehat{\varphi}_A \to \widehat{x}_0\widehat{d}(A)\widehat{\mathbf{H}}_+$ by the relations $\varphi_A(x_0 h(F)) = x_0 h(F) d(A)$ and $\widehat{\varphi}_A(\widehat{x}_0\widehat{h}(F)) = \widehat{x}_0\widehat{h}(F)\widehat{d}(A)$, respectively. As in Subsection 2.10, for every $A \in \mathbb{GL}^+(m,\mathbb{R})$ we define an automorphism i_A of the group $\mathbf{S}_m(\mathbb{R})$ by the relation $i_A(F) = A^{-1}F\,{}^tA^{-1}$, $F \in \mathbf{S}_m(\mathbb{R})$.

Proposition 2.1. i) *The set $x_0\mathbf{B}_+$ is a union of $\mathbf{S}_m(\mathbb{R})$-orbits; the action of the group $\mathbf{D}$ on $x_0\mathbf{B}_+$ preserves the partition of $x_0\mathbf{B}_+$ into $\mathbf{S}_m(\mathbb{R})$-orbits; $\mathbf{D}$ acts transitively on the set of $\mathbf{S}_m(\mathbb{R})$-orbits in $x_0\mathbf{B}_+$.*

ii) *Every* $\mathbf{S}_m(\mathbb{R})$*-orbit in* $x_0\mathbf{B}_+$ *is of the form* $x_0 d(A)\mathbf{H}_+ (= x_0\mathbf{H}_+ d(A))$ *with some* $A \in \mathbb{GL}^+(m,\mathbb{R})$; *the stabilizer of every point of this orbit is the group* $i_A(N_4') = A^{-1}N_4'\,{}^tA^{-1} \subset \mathbf{S}_m(\mathbb{R})$; *as an* $\mathbf{S}_m(\mathbb{R})$*-space, this orbit is isomorphic to the* $\mathbf{S}_m(\mathbb{R})$*-space* $A^{-1}N_4'\,{}^tA^{-1}\backslash \mathbf{S}_m(\mathbb{R})$; *both of these spaces are compact.*

iii) *For every* $A \in \mathbb{GL}^+(m,\mathbb{R})$, *the* $\mathbf{S}_m(\mathbb{R})$*-spaces* $x_0\mathbf{H}_+$ *and* $x_0 d(A)\mathbf{H}_+$ *are* (i_A, φ_A)*-compatible.*

iv) *Every* $\mathbf{S}_m(\mathbb{R})$*-orbit in* $x_0\mathbf{B}_+$ *carries a unique* $\mathbf{S}_m(\mathbb{R})$*-invariant probability measure; the map* φ_A *transports such measure from the* $\mathbf{S}_m(\mathbb{R})$*-orbit* $x_0\mathbf{H}_+$ *to the similar measure on the* $\mathbf{S}_m(\mathbb{R})$*-orbit* $x_0 d(A)\mathbf{H}_+$.

Moreover, for the homogeneous space $\widehat{\mathcal{M}}_4$ *we have assertions* $\widehat{\text{i}})$–$\widehat{\text{iv}})$ *that are obtained from assertions* i)–iv) *if* $\mathbf{B}_+$, $\mathbf{H}_+$, $\mathbf{D}$, x_0, d, φ_A, *and the* $\mathbf{S}_m(\mathbb{R})$*-action via* h_+ *are replaced with* $\widehat{\mathbf{B}}_+$, $\widehat{\mathbf{H}}_+$, $\widehat{\mathbf{D}}$, $\widehat{x}_0$, $\widehat{d}$, $\widehat{\varphi}_A$, *and the* $\mathbf{S}_m(\mathbb{R})$*-action via* $\widehat{h}_+$, *respectively.*

Finally, the $\widehat{\mathbf{B}}_+$*-space* $\widehat{x}_0\widehat{\mathbf{B}}_+$ *and the* $\mathbf{B}_+$*-space* $x_0\mathbf{B}_+$ *are* $(\mu_+^{-1}, \pi_4|_{\widehat{x}_0\widehat{\mathbf{B}}_+})$*-compatible, and the map* $\pi_4|_{\widehat{x}_0\widehat{\mathbf{B}}_+}$ *sends every* $\mathbf{S}_m(\mathbb{R})$*-orbit in* $\widehat{x}_0\widehat{\mathbf{B}}_+$ *to an* $\mathbf{S}_m(\mathbb{R})$*-orbit in* $x_0\mathbf{B}_+$ *in an equivariant and measure preserving way.*

Proof. We start with the proof of $\widehat{\text{i}})$–$\widehat{\text{iv}})$. We introduce a discrete subgroup of $\mathbf{Mp}(2m,\mathbb{R})$ by

$$\widehat{N}_4 = \widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+.$$

Recall that $N_4 = \Gamma_4 \cap \mathbf{H}_+$ and $h_+^{-1}(N_4) = N_4'$.

Lemma 2.1. *For* μ_+, λ, *and the subgroups* N_4, $\widehat{N}_4$, *and* N_4' *we have*

$$\mu_+|_{N_4} = \lambda|_{N_4} \tag{2.26}$$

and

$$\widehat{N}_4 = \mu_+(N_4), \quad \widehat{h}_+^{-1}(\widehat{N}_4) = N_4'. \tag{2.27}$$

Proof of the lemma. After restriction to $\widehat{\Gamma}_4$ or to $\widehat{\mathbf{H}}_+$, the map π becomes one-to-one, with $\pi(\widehat{\Gamma}_4) = \Gamma_4$ and $\pi(\widehat{\mathbf{H}}_+) = \mathbf{H}_+$, and with the inverse maps λ and μ_+, respectively. This implies that $\widehat{N}_4 \subset \mu_+(N_4)$, because $\widehat{N}_4 = \widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+ = \mu_+(\pi(\widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+)) \subset \mu_+(\pi(\widehat{\Gamma}_4) \cap \pi(\widehat{\mathbf{H}}_+)) = \mu_+(\Gamma_4 \cap \mathbf{H}_+) = \mu_+(N_4)$.

Next we verify that λ restricted to $\Gamma_4 \cap \mathbf{H}_+$ agrees with μ_+. This is equivalent to saying that the cocycles involved in the definitions of the maps λ and μ_+ coincide after restriction to $\Gamma_4 \cap \mathbf{H}_+$. In fact, both of them are equal to 1. This is clearly seen for the map μ_+ from the definition of it in Subsection 2.7. As to λ, the corresponding cocycle was defined in Subsection 2.4 as the map $(g, Z) \mapsto \Theta(gZ)/\Theta(Z)$. Observe that $h_+(B)Z = Z + B$, $B \in \mathbf{S}_m(\mathbb{R})$, and that $\Theta(Z+P) = \Theta(Z)$ if P is a symmetric integral matrix with even diagonal entries, which is the case if $P \in h_+^{-1}(\Gamma_4 \cap \mathbf{H}_+) = N_4'$. Thus, (2.26) is proved.

Now we see that $\mu_+(\Gamma_4 \cap \mathbf{H}_+) \subset \widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+$, because the maps $\mu_+ : \mathbf{H}_+ \to \widehat{\mathbf{H}}_+$ and $\lambda : \Gamma_4 \to \widehat{\Gamma}_4$ are one-to-one and agree on $\Gamma_4 \cap \mathbf{H}_+$. This establishes the first relation in (2.27). We also have $\widehat{h}_+^{-1}(\widehat{N}_4) = h_+^{-1}(\mu_+^{-1}(\widehat{N}_4)) = h_+^{-1}(N_4) = N_4'$. □

Since an $\mathbf{S}_m(\mathbb{R})$-orbit in $\widehat{x}_0\widehat{\mathbf{B}}_+$ is the same as an $\widehat{\mathbf{H}}_+$-orbit, $\widehat{\text{i}})$ follows from the fact that $\widehat{\mathbf{B}}_+$ is a semidirect product of $\widehat{\mathbf{H}}_+$ and $\widehat{\mathbf{D}}$ and from the transitivity of the action of $\widehat{\mathbf{B}}_+$ on $\widehat{x}_0\widehat{\mathbf{B}}_+$.

The stabilizer is one and the same for every point within an orbit of the group $\mathbf{S}_m(\mathbb{R})$ because $\mathbf{S}_m(\mathbb{R})$ is Abelian. By the first assertion in $\widehat{\text{ii}})$, every $\mathbf{S}_m(\mathbb{R})$-orbit must contain a point of the form $\widehat{x}_0\widehat{d}(A)$, and a matrix $F \in \mathbf{S}_m(\mathbb{R})$ belongs to the stabilizer of $\widehat{x}_0\widehat{d}(A)$ if and only if there exists $k \in \widehat{\Gamma}_4$ such that

$$\widehat{d}(A)\widehat{h}_+(F) = k\widehat{d}(A).$$

By (2.19), we have

$$k = \widehat{d}(A)\widehat{h}_+(F)(\widehat{d}(A))^{-1} = \widehat{h}_+(AF\,{}^tA),$$

which implies that $k \in \widehat{h}_+(\mathbf{S}_m(\mathbb{R})) = \widehat{\mathbf{H}}_+$. Thus, F stabilizes $\widehat{x}_0\widehat{d}(A)$ if and only if $\widehat{h}_+(AF\,{}^tA) \in \widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+ = \widehat{N}_4$, or, equivalently, if and only if $AF\,{}^tA \in \widehat{h}_+^{-1}(\widehat{N}_4) = N_4'$, or, finally, if and only if $F \in A^{-1}\widehat{h}_+^{-1}(\widehat{N}_4)\,{}^tA^{-1} = A^{-1}N_4'\,{}^tA^{-1}$, and the description of the stationary subgroup in $\widehat{\text{ii}})$ is established. Since the transitive actions of every Abelian group are classified by their stationary subgroups, isomorphism of the two $\mathbf{S}_m(\mathbb{R})$-spaces in $\widehat{\text{ii}})$ is proved. Compactness follows from the facts that N_4' is a uniform subgroup in $\mathbf{S}_m(\mathbb{R})$ and that the latter property is preserved by the automorphism i_A. This completes the proof of $\widehat{\text{ii}})$.

To prove $\widehat{\text{iii}})$, it suffices to note that for every F_0, $F \in \mathbf{S}_m(\mathbb{R})$ we have

$$\begin{aligned}\widehat{\varphi}_A(\widehat{x}_0\widehat{h}_+(F_0)\widehat{h}_+(F)) &= \widehat{x}_0\widehat{h}_+(F_0)\widehat{h}_+(F)\widehat{d}(A)\\ &= \widehat{x}_0\widehat{h}_+(F_0)\widehat{d}(A)\widehat{h}_+(A^{-1}F\,{}^tA^{-1}) = \widehat{\varphi}_A(\widehat{x}_0\widehat{h}_+(F_0))\widehat{h}_+(i_A(F))\end{aligned}$$

by (2.19).

The uniqueness of the invariant probability measure on every $\widehat{\mathbf{H}}_+$-orbit in $\widehat{x}_0\widehat{\mathbf{B}}_+$ is a consequence of $\widehat{\text{ii}})$ and the uniqueness of Haar measure on the torus $A^{-1}N_4'\,{}^tA^{-1}\backslash\mathbf{S}_m(\mathbb{R})$.

The last assertion of the proposition is a direct consequence of the (π, π_4)-compatibility of the $\mathbf{Mp}(2m, \mathbb{R})$-space $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R})$ and the $\mathbf{Sp}(2m, \mathbb{R})$-space $\Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R})$ established in Subsection 2.5. As a consequence, we get the compatibility of the actions of $\mathbf{S}_m(\mathbb{R})$ and $\mathbb{GL}^+(m, \mathbb{R})$ on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R})$ and on $\Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R})$. Combined with assertions $\widehat{\text{i}})$–$\widehat{\text{iv}})$, this implies i)–iv). ☐

2.12. Correspondence between the left and the right horospheres. In Subsection 2.11 we saw that the orbit $x_0\mathbf{B}_+ \subset \mathcal{M}_4$ is a union of compact $\mathbf{S}_m(\mathbb{R})$-orbits; similarly, the $\widehat{\mathbf{B}}_+$-orbit of the point $\widehat{x}_0$ in the space $\widehat{\mathcal{M}}_4$ is a union of compact $\mathbf{S}_m(\mathbb{R})$-orbits; the group $\mathbb{GL}^+(m, \mathbb{R})$ acts transitively on the set of $\mathbf{S}_m(\mathbb{R})$-orbits of both of these $\mathbf{S}_m(\mathbb{R})$-actions; and the actions of $\widehat{\mathbf{B}}_+$ and $\mathbf{B}_+$ on the orbits of $\widehat{x_0}$ and x_0, respectively, are $(\mu_+^{-1}, \pi_4|_{\widehat{x}_0\widehat{\mathbf{B}}_+})$-compatible. Let $Z = X + iY \in \mathcal{H}_m$, and let $Y^{1/2}$ be a unique positive definite square root of Y. We define a map $\psi : \mathcal{H}_m \to \mathbf{B}_+$ by the formula

$$\psi(Z) = h(X)d(Y^{1/2}).$$

The map ψ satisfies the relation

$$\psi(Z)Z_0 = Z, \quad Z \in \mathcal{H}_m$$

(recall that $Z_0 = iI_m \in \mathcal{H}_m$). Another important property of ψ is that it is $\mathbf{S}_m(\mathbb{R})$-equivariant with respect to the left actions of $\mathbf{S}_m(\mathbb{R})$ on $\mathcal{H}_m$ and on $\mathbf{B}_+$. In particular, if $Z' = X' + iY$ is an element of $\mathcal{H}_m$ such that $Z' - Z \in N_4'$, then $\psi(Z') = h(X')d(Y^{1/2}) = h(X' - X)h(X)d(Y^{1/2}) \in \Gamma_4\psi(Z)$. Thus, ψ determines a map (to be denoted by ψ_4) of the Siegel cylinder $N_4\backslash\mathcal{H}_m$ to the $\mathbf{B}_+$-orbit of $x_0 \in \mathcal{M}_4$. In a completely similar way, we define maps

$$\widehat{\psi}(\cdot) = \mu_+(\psi(\cdot))$$

and

$$\widehat{\psi}_4(\cdot) = \widehat{x_0}\widehat{\psi}(\cdot)$$

from $\mathcal{H}_m$ to $\widehat{\mathbf{B}}_+$ and from $N_4\backslash\mathcal{H}_m$ to $\widehat{x}_0\widehat{\mathbf{B}}_+ \subset \widehat{\mathcal{M}}_4$, respectively. For one-parameter groups $(g(s))_{s\in\mathbb{R}}$ and $(\widehat{g}(s))_{s\in\mathbb{R}}$, by $\varkappa$-actions we mean the actions on $N_4\backslash\mathcal{H}_m$ defined by $(s,Z)\mapsto \varkappa_{g(s)}(Z)$ and $(s,Z)\mapsto \varkappa_{\pi(\widehat{g}(s))}(Z)$, respectively.

Proposition 2.2. i) *Every $\mathbf{S}_m(\mathbb{R})$-orbit in $N_4\backslash\mathcal{H}_m$ is compatible with its ψ_4-image in $\mathcal{M}_4$. More precisely, as an $\mathbf{S}_m(\mathbb{R})$-space, the $\mathbf{S}_m(\mathbb{R})$-orbit of every point of the form $X+iY+N_4'\in N_4\backslash\mathcal{H}_m$ is $(i_{Y^{1/2}},\psi_4)$-compatible with the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $x_0g(Y^{1/2})\in\mathcal{M}_4$. The invariant probability measures on these $\mathbf{S}_m(\mathbb{R})$-orbits are transported by ψ_4 to one another.*

ii) *The map ψ_4 is equivariant relative to the $\varkappa$-action of the one-parameter group $(g(s))_{s\in\mathbb{R}}$ on $N_4\backslash\mathcal{H}_m$ and the action of $(g(s))_{s\in\mathbb{R}}$ on $\mathcal{M}_4$.*

iii) *For every $s\in R$ and every $Z=X+iY+N_4'\in N_4\backslash\mathcal{H}_m$, the map ψ_4 sends the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $\varkappa_{g(s)}(Z)\in N_4\backslash\mathcal{H}_m$ to the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $x_0d(Y^{1/2})g(s)$.*

The assertions similar to i)–iii) *hold true for* $\mathbf{Mp}(2m,\mathbb{R})$ *and the related groups, actions, and homogeneous spaces.*

Proof. First we verify the compatibility statement in i). For any $Z=X+iY+N_4'\in N_4\backslash\mathcal{H}_m$ and any $F\in\mathbf{S}_m(\mathbb{R})$, we have

$$\begin{aligned}\psi_4(h(F)(Z)) &= \psi_4(X+F+iY+N_4')\\ &= x_0h(X+F)d(Y^{1/2})\\ &= x_0h(X)d(Y^{1/2})h(Y^{-1/2}F\,{}^tY^{-1/2})\\ &= \psi_4(Z)h(i_{Y^{1/2}}(F)).\end{aligned}$$

Every $\mathbf{S}_m(\mathbb{R})$-orbit is a compact transitive space with a unique invariant probability measure, and ψ_4 must transport any such measure to another one because it preserves invariance.

Now we establish ii). If $Z=X+iY+N_4'\in N_4\backslash\mathcal{H}_m$, then

$$\begin{aligned}\psi_4(\varkappa_{g(s)}(Z)) &= \psi_4(X+i\exp(-2s)Y+N_4')\\ &= x_0h(X)d(\exp(-s)Y^{1/2})\\ &= x_0h(X)d(Y^{1/2})g(s)\\ &= \psi_4(Z)g(s).\end{aligned}$$

Thus, we have proved that

$$\psi_4(\varkappa_{g(s)}(Z)) = \psi_4(Z)g(s)$$

for every $s\in\mathbb{R}$ and $Z\in N_4\backslash\mathcal{H}_m$, which shows that ψ_4 is an equivariant map relative to the actions under consideration.

Finally, the $\mathbf{S}_m(\mathbb{R})$-orbit of $Z=X+iY+N_4'$ contains a point $iY+N_4'$ the image of which under ψ_4 is $x_0d(Y^{1/2})$. Thus, iii) is a direct consequence of i) and ii). We omit the entirely similar proof of assertions $\widehat{\text{i}})$–$\widehat{\text{iii}})$. □

2.13. Comparing Θ and $\widehat{\Theta}$. For any $(m\times m)$-matrix $Y>0$, we define a function Θ_Y on $\mathbf{S}_m(\mathbb{R})$ by the relation

$$\Theta_Y(X) = (\det Y)^{1/4}\Theta(X+iY), \quad X\in\mathbf{S}_m(\mathbb{R}).$$

As was mentioned in the Introduction, the function Θ_Y is invariant with respect to the group $N_{1,2}'\subset\mathbf{S}_m(\mathbb{R})$ and can be viewed as a function on the torus $N_{1,2}'\backslash\mathbf{S}_m(\mathbb{R})$. We recall that the function $\widehat{\Theta}$ was defined in Subsection 2.6.

Proposition 2.3. *Let $Z = X + iY \in \mathcal{H}_m$. Then the distribution of the function $\Theta_Y(\cdot)$ with respect to the Haar probability measure on $N'_{1,2}\backslash\mathbf{S}_m(\mathbb{R})$ is the same as the distribution of the function $\widehat{\Theta}$ on the right horosphere in $\widehat{\mathcal{M}}_4$ containing $\widehat{\psi}_4(Z + N'_4)$ and supplied with the $\mathbf{S}_m(\mathbb{R})$-invariant probability measure.*

Proof. Since $N'_4 \subset N'_{1,2} \subset \mathbf{S}_m(\mathbb{R})$, the function Θ_Y can be also regarded as a function on the torus $N'_4\backslash\mathbf{S}_m(\mathbb{R})$, which covers $N'_{1,2}\backslash\mathbf{S}_m(\mathbb{R})$. It is clear that the distributions of Θ_Y with respect to the Haar probability measures on these tori are the same, because the covering map is measure preserving. Thus, in the proof of the proposition we may replace $N'_{1,2}\backslash\mathbf{S}_m(\mathbb{R})$ (occurring in the statement) with $N'_4\backslash\mathbf{S}_m(\mathbb{R})$.

The function Θ_Y on $N'_4\backslash\mathbf{S}_m(\mathbb{R})$ is the restriction to the left horosphere containing $iY + N'_4$ of the function

$$Z = X + iY + N'_4 \mapsto (\det Y)^{1/4}\Theta(X + iY)$$

defined on the Siegel cylinder $N_4\backslash\mathcal{H}_m$. To complete the proof, we shall show that this function can be identified with $\widehat{\Theta}(\widehat{\psi}_4(\cdot))$. In accordance with Subsection 2.6, for $g \in \mathbf{Mp}(2m, \mathbb{R})$ we have

$$\widehat{\Theta}(\widehat{x}_0 g) = \Theta((\pi g)Z_0)/\rho(g, Z_0).$$

Then, from the definition of the functions $\widehat{\psi}_4$ and $\widehat{\psi}$, we conclude that if $Z = X+iY+N'_4$, then

$$\begin{aligned}\widehat{\Theta}(\widehat{\psi}_4(Z)) &= \Theta(\pi(\widehat{\psi}(Z))Z_0)/\rho(\widehat{\psi}(Z), Z_0)\\ &= \Theta(\psi(Z)Z_0)/\rho(\widehat{\psi}(Z), Z_0) = \Theta(Z)/\rho(\widehat{\psi}(Z), Z_0).\end{aligned}$$

The relations $\widehat{\psi}(Z) = \mu_+(\psi(Z))$ and $\psi(Z) = h(X)d(Y^{1/2})$, together with the definition of the map μ_+ in Subsection 2.7, imply that

$$\rho(\widehat{\psi}(Z), Z_0) = (\det Y)^{-1/4}.$$

Thus,

$$\widehat{\Theta}(\widehat{\psi}_4(X + iY + N'_4)) = (\det Y)^{1/4}\Theta(X + iY) = \Theta_Y(X),$$

which proves the proposition. □

Remark 2.3. It should be noted that, instead of Γ_4 and $\widehat{\Gamma}_4$, we can take any pair of lattices $\Gamma \subset \mathbf{Sp}(2m, \mathbb{R})$ and $\widehat{\Gamma} \subset \mathbf{Mp}(2m, \mathbb{R})$ such that $\pi|_{\widehat{\Gamma}}$ is a one-to-one map of $\widehat{\Gamma}$ onto Γ and Γ is a normal subgroup of finite index in $\mathbf{Sp}(2m, \mathbb{Z})$. Then $\widehat{\Gamma}\backslash\mathbf{Mp}(2m, \mathbb{R})$ is a double cover of $\Gamma\backslash\mathbf{Sp}(2m, \mathbb{R})$ via the map induced by π (and denoted by $\pi_{\widehat{\Gamma}}$). The arguments in Subsections 2.5, 2.9, 2.10 (except for the last part involving the subgroup $\Gamma_{1,2}$), 2.11, and 2.12 apply to this more general case without any changes.

§3. Uniform distribution of closed horospheres

In the following theorem we deal with the family of closed orbits for the action of the group $\widehat{\mathbf{H}}_+$ on the space $\widehat{\mathcal{M}}_4$ considered in Proposition 2.1. For every $A \in \mathbb{GL}^+(m, \mathbb{R})$, we denote by $\mathcal{O}(A)$ the compact $\widehat{\mathbf{H}}_+$-orbit $\widehat{x}_0\widehat{d}(A)\widehat{\mathbf{H}}_+ \subset \widehat{\mathcal{M}}_4$.

Theorem 3.1. *Let $\widehat{P}_4$ denote the $\mathbf{Mp}(2m, \mathbb{R})$-invariant probability measure on the homogeneous space $\widehat{\mathcal{M}}_4$, and, for every $A \in \mathbb{GL}^+(m, \mathbb{R})$, let $\widehat{P}_4(A)$ be the $\widehat{\mathbf{H}}_+$-invariant probability measure on the compact $\widehat{\mathbf{H}}_+$-orbit $\mathcal{O}(A)$. Then $\widehat{P}_4(\exp(-s)A)$ converges to $\widehat{P}_4$ weakly as $s \to +\infty$. This convergence is uniform in A running over an arbitrary compact subset of $\mathbb{GL}^+(m, \mathbb{R})$.*

In the proof of Theorem 3.1 we shall need the following definitions, which can be found in [DM] along with further references. Since we prefer that the Lie subgroups act from the right and lattices from the left, some relations look differently in the present paper. Let G be a connected Lie group, and let Γ be a discrete subgroup in G. For any closed subgroup W of G, let $\mathcal{S}(W)$ denote the set of all $x \in \Gamma\backslash G$ for which there exists a proper closed subgroup H of G such that $W \subset H$ and xH admits a finite invariant measure (under this condition xH is automatically a proper closed subset of $\Gamma\backslash G$). We set $\mathcal{G}(W) = \Gamma\backslash G - \mathcal{S}(W)$.

Furthermore, for any two closed subgroups H and W of G we put $X(H,W) = \{g \in G : gW \subset Hg\}$. Note that if H is a proper closed subgroup such that $H \cap \Gamma$ is a lattice in H, then $\Gamma\backslash\Gamma X(H,W)$ is contained in $\mathcal{S}(W)$. We denote by $\mathcal{H} = \mathcal{H}(G,\Gamma)$ the class of all proper closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H and $\operatorname{Ad}(H \cap \Gamma)$ is Zariski-dense in $\operatorname{Ad} H$. Here and below Ad denotes the adjoint representation of G.

One of the main tools in the proof of Theorem 3.1 is the following result by Dani and Margulis (see [DM, Theorem 2]).

Proposition 3.1. *Let G be a connected Lie group, Γ a lattice in G, and μ the G-invariant probability measure on $\Gamma\backslash G$. Suppose that a sequence $\left(u^{(n)}(\cdot)\right)_{n\ge 1}$ of* Ad*-unipotent one-parameter subgroups converges to an* Ad*-unipotent one-parameter subgroup $u(\cdot)$, i.e., $u^{(n)}(t) \to u(t)$ for all $t \in \mathbb{R}$. Let $\{x_n\}$ be a sequence in $\Gamma\backslash G$ converging to a point in $\mathcal{G}(\{u(t)\})$, and let $\{T_n\}$ be a sequence in $\mathbb{R}^+$ tending to infinity. Then for any bounded continuous function φ on $\Gamma\backslash G$ we have*

$$\frac{1}{T_n}\int_0^{T_n} \varphi(u^{(n)}(t)x_n)\,dt \to \int_{\Gamma\backslash G} \varphi\,d\mu.$$

Lemma 3.1. *There exists a real number $p > 0$ such that for every $A \in \mathbb{GL}^+(m,\mathbb{R})$ there are points x^A, $z^A \in \widehat{\mathcal{M}}_4$ that satisfy the following conditions*:

1) $x^A \in \mathcal{O}(A\widehat{g}(0))$;
2) *z^A is a p-periodic point for the quasigeodesic flow* (*i.e.*, $z^A\widehat{g}(p) = z^A$);
3) $x^A\widehat{g}(np) \to z^A$ *as* $n \to +\infty$.

Proof of Lemma 3.1. We choose a matrix

$$\gamma_0 = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in \Gamma_4(1)$$

with eigenvalues $0 < l^{-1} < 1 < l$ (for instance, we may set $\gamma_{11} = 13,\ \gamma_{12} = 8,\ \gamma_{21} = 8,\ \gamma_{22} = 5$). Then the matrix

$$\gamma = \gamma_0 \otimes I_m = \begin{pmatrix} \gamma_{11}I_m & \gamma_{12}I_m \\ \gamma_{21}I_m & \gamma_{22}I_m \end{pmatrix}$$

is an element of $\Gamma_4 = G_4(m)$. Let $r \in \mathbf{SL}(2,\mathbb{R})$ be a matrix that diagonalizes γ_0, so that

$$\gamma_0 r = rd,$$

where

$$d = \begin{pmatrix} l^{-1} & 0 \\ 0 & l \end{pmatrix}.$$

The matrix r can be represented as follows:

$$r = \begin{pmatrix} r_{11} & r_{12} \\ r_{13} & r_{22} \end{pmatrix} = r^{(+)}r^{(0)}r^{(-)} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

In the sequel we assume that $k > 0$ (otherwise we can diagonalize γ_0 conjugating it by $-r$ instead of r).

Setting

$$R = r \otimes I_m, \quad D = d \otimes I_m,$$
$$R^{(+)} = r^{(+)} \otimes I_m, \quad R^{(0)} = r^{(0)} \otimes I_m, \quad R^{(-)} = r^{(-)} \otimes I_m,$$

we obtain

$$RD = \gamma R$$

and

$$R = R^{(+)} R^{(0)} R^{(-)}.$$

These relations can be rewritten as

$$Rd(l^{-1} I_m) = \gamma R$$

and

$$R = h_+(uI_m) d(kI_m) h_-(vI_m)$$

(we have used the notation of §2). By Subsections 2.4 and 2.7, the first of the above relations implies that

$$\widehat{R}\widehat{d}(l^{-1} I_m) = q\lambda(\gamma)\widehat{R},$$

where $\widehat{R}$ is defined by

$$\widehat{R} = \widehat{h}_+(uI_m)\widehat{d}(kI_m)\widehat{h}_-(vI_m)$$

and q belongs to the center of $\mathbf{Mp}(2m, \mathbb{R})$ and satisfies $q^2 = 1$. Then

$$\widehat{R}\widehat{d}(l^{-2} I_m) = \lambda^2(\gamma)\widehat{R}.$$

It follows that

$$\begin{aligned}
\widehat{h}_+(uI_m)\widehat{g}(2n \log l) &= \widehat{h}_+(uI_m)\widehat{d}(l^{-2n} I_m) \\
&= \widehat{R}\widehat{h}_-(-vI_m)\widehat{d}(k^{-1} I_m)\widehat{d}(l^{-2n} I_m) \\
&= \widehat{R}\widehat{d}(l^{-2n} I_m)\widehat{d}(k^{-1} I_m)\widehat{h}_-(-vl^{-2n} k^{-1} I_m) \\
&= \lambda^{2n}(\gamma)\widehat{R}\widehat{d}(k^{-1} I_m)\widehat{h}_-(-vl^{-2n} k^{-1} I_m).
\end{aligned}$$

Multiplying by $\widehat{d}(A)$ from the right and using (2.16′), we obtain

$$\widehat{h}_+(uI_m)\widehat{d}(A)\widehat{g}(2n \log l) = \lambda^{2n}(\gamma)\widehat{R}\widehat{d}(k^{-1} A)\widehat{h}_-(-vl^{-2n} k^{-1} A\, {}^t\! A).$$

Passing to $\widehat{\Gamma}_4$-cosets, we conclude that

$$\begin{aligned}
&\widehat{\Gamma}_4 \widehat{h}_+(uI_m)\widehat{d}(A)\widehat{g}(2n \log l) \\
&\quad = \widehat{\Gamma}_4 \widehat{R}\widehat{d}(k^{-1} A)\widehat{h}_-(-vl^{-2n} k^{-1} A\, {}^t\! A) \to \widehat{\Gamma}_4 \widehat{R} d(k^{-1} A)
\end{aligned}$$

as $n \to \infty$.

Thus, we may set $p = 2 \log l$, $z^A = \widehat{\Gamma}_4 \widehat{R} d(k^{-1} A)$ and $x^A = \widehat{h}_+(uI_m)\widehat{d}(A) \in \mathcal{O}(A)$. □

Lemma 3.2. *Let $z \in \widehat{\mathcal{M}}_4$ be a periodic point for the quasigeodesic flow on $\widehat{\mathcal{M}}_4$. Then the set $z\widehat{\mathbf{H}}_+$ is dense in $\widehat{\mathcal{M}}_4$.*

Proof of Lemma 3.2. The subgroup $\widehat{\mathbf{H}}_+$ of $\mathbf{Mp}(2m, \mathbb{R})$ is *horospheric* relative to $\widehat{g}(\cdot)$, i.e.,

$$\widehat{\mathbf{H}}_+ = \{g \in \mathbf{Mp}(2m, \mathbb{R}) : \widehat{g}(t) g \widehat{g}(-t) \to I_{\mathbf{Mp}(2m,\mathbb{R})} \text{ as } t \to +\infty\}.$$

Since z is periodic, the semiorbit $\{z\widehat{g}(t), t \geq 0\}$ is bounded. The group $\widehat{\mathbf{H}}_+$ acts on $\widehat{\mathcal{M}}_4$ ergodically (this follows, e.g., from Theorem 7 in [Mo1]), and the assertion of the lemma follows from [Da, Theorem 7]. □

In the statement and proof of the following lemma, we shall use the notation introduced in this section after the statement of Theorem 3.1 with $\mathbf{Mp}(2m,\mathbb{R})$ and $\widehat{\Gamma}_4$ in place of G and Γ, respectively.

Lemma 3.3. *Let $z \in \widehat{\mathcal{M}}_4$ be a periodic point for the quasigeodesic flow on $\widehat{\mathcal{M}}_4$. Then there exists a one-parameter unipotent subgroup $u_0(\cdot)$ in $\widehat{\mathbf{H}}_+$ such that $z \in \mathcal{G}(u_0(\cdot))$.*

Proof of Lemma 3.3. We choose $g \in \mathbf{Mp}(2m,\mathbb{R})$ such that $z = \widehat{\Gamma}_4 g$. Suppose that a one-parameter unipotent group $u(\cdot) \subset \widehat{\mathcal{M}}_4$ is such that $z \in \mathcal{S}(u(\cdot))$. By Proposition 2.3 in [DM], for some $H \in \mathcal{H}(\mathbf{Mp}(2m,\mathbb{R}),\widehat{\Gamma}_4)$ we have $z \in \widehat{\Gamma}_4\backslash\widehat{\Gamma}_4 X(H,u)$ or, equivalently, $u(\cdot) \subset g^{-1}Hg$.

Thus, $u(\cdot) \subset g^{-1}Hg \cap \widehat{\mathbf{H}}_+$. Then $g^{-1}Hg \cap \widehat{\mathbf{H}}_+$ is a proper subgroup of $\widehat{\mathbf{H}}_+$. Otherwise, the orbit $zg^{-1}Hg$ is dense in $\widehat{\mathcal{M}}_4$ by Lemma 3.2. On the other hand, $zg^{-1}Hg = \Gamma_4 Hg$ is a proper closed subset of $\widehat{\mathcal{M}}_4$ because, up to right translations, this subset is a closed orbit $\Gamma_4 H$ of a proper subgroup H of $\mathbf{Mp}(2m,\mathbb{R})$ such that $\Gamma_4 \cap H$ is a lattice in H, and we arrive at a contradiction.

By [DM, Proposition 2.1], $\mathcal{H}(\mathbf{Mp}(2m,\mathbb{R}),\widehat{\Gamma}_4)$ is at most countable (see also [R]). Thus, $\{u(\cdot) : x \in \mathcal{S}(u(\cdot))\}$ is contained in a countable union of proper closed connected subgroups of $\widehat{\mathbf{H}}_+$, and off that union we can choose any one-parameter subgroup. This completes the proof. □

Proof of Theorem 3.1. We fix a compact subset $K \subset \mathbb{GL}^+(m,\mathbb{R})$. We must prove that for every bounded continuous function φ on $\widehat{\mathcal{M}}_4$ we have

$$\int_{\mathcal{O}(\exp(-s)A)} \varphi \, d\widehat{P}_4(A) \to \int_{\widehat{\mathcal{M}}_4} \varphi \, d\widehat{P}_4$$

uniformly in $A \in K$ as $s \to +\infty$. If this is not so, then there exists a bounded continuous function φ, a sequence $(A_n)_{n\geq 1}$ of matrices in K, and an increasing sequence $(s_n)_{n\geq 1}$ of positive numbers tending to infinity such that for some $\delta > 0$ the relation

$$\left| \int_{\mathcal{O}(\exp(-s_n)A_n)} \varphi \, d\widehat{P}_4(\exp(-s_n)A_n) - \int_{\widehat{\mathcal{M}}_4} \varphi \, d\widehat{P}_4 \right| \geq \delta$$

is true for all $n \geq 1$. We may and shall assume (passing to a subsequence if necessary) that $A_n \to A$ for some $A \in K$.

By Lemmas 3.1 and 3.3, there exists a sequence $(x_n)_{n\geq 1}$, a point $z \in \widehat{\mathcal{M}}_4$, and a one-parameter subgroup $u(\cdot)$ of the group $\widehat{\mathbf{H}}_+$ such that $x_n \in \mathcal{O}(\exp(-s_n)A)$ for every $n \geq 1$, $x_n \to z$, and $z \in \mathcal{G}(u(\cdot))$. Each x_n is of the form $x_n = \widehat{\Gamma}\widehat{g}(s_n)h_n A$ with $h_n \in \widehat{\mathbf{H}}_+$. For every $n \geq 1$ we set $x'_n = \widehat{\Gamma}\widehat{g}(s_n)h_n A_n$. Then

$$\begin{aligned} x'_n &= \widehat{\Gamma}\widehat{g}(s_n)h_n A_n = \widehat{\Gamma}\widehat{g}(s_n)h_n A(\widehat{g}(s_n)h_n A)^{-1}\widehat{g}(s_n)h_n A_n \\ &= x_n A^{-1}A_n \to z. \end{aligned}$$

From statements $\widehat{\text{ii}}$) and $\widehat{\text{iv}}$) of Proposition 2.1 it follows that the action of $(u(t))_{t\in\mathbb{R}}$ on $\mathcal{O}(\exp(-s_n)A_n)$ with invariant measure $\widehat{P}_4(\exp(-s)A_n)$ is merely a flow of translations on the torus with Haar measure, on which the entire group $\widehat{\mathbf{H}}_+$ acts transitively by translations. A generic one-parameter subgroup of $\widehat{\mathbf{H}}_+$ acts on $\mathcal{O}(\exp(-s_n)A_n)$ ergodically, and we can find a sequence $(u_n(\cdot))_{n\geq 1}$ of one-parameter subgroups of $\widehat{\mathbf{H}}_+$ such that for every $n \geq 1$ the subgroup u_n acts on $\mathcal{O}(\exp(-s_n)A_n)$ ergodically and $u_n \to u$ as $n \to \infty$. We choose a positive real sequence $(\epsilon_n)_{n\geq 1}$ tending to 0. Applying the ergodic theorem to every dynamical system $(\mathcal{O}(\exp(-s_n)A_n),\, u_n(\cdot),\, \widehat{P}_4(\exp(-s)A_n))$, we can construct a sequence $(T_n)_{n\geq 1}$ such that $T_n \to \infty$ and we have

$$\left| \frac{1}{T_n} \int_0^{T_n} \varphi(u^{(n)}(t)x'_n)\, dt - \int_{\mathcal{O}(\exp(-s_n)A_n)} \varphi\, d\widehat{P}_4(\exp(-s_n)A_n) \right| < \epsilon_n$$

for every $n \geq 1$. On the other hand, Proposition 3.1 (with (x'_n) in place of (x_n)) shows that

$$\frac{1}{T_n} \int_0^{T_n} \varphi(u^{(n)}(t)x'_n)\, dt \to \int_{\widehat{\mathcal{M}}_4} \varphi\, d\widehat{P}_4$$

as $n \to \infty$. The above two relations imply that

$$\int_{\mathcal{O}(\exp(-s_n)A_n)} \varphi\, d\widehat{P}_4(\exp(-s_n)A_n) \to \int_{\widehat{\mathcal{M}}_4} \varphi\, d\widehat{P}_4$$

as $n \to \infty$, a contradiction. □

§4. Existence of the limiting distribution

The main result of this section and of the entire paper is the following theorem.

Theorem 4.1. *Let the function Θ_Y be defined (as in Subsection 2.12) by the relation $\Theta_Y(X) = (\det Y)^{1/4}\Theta(X + iY)$, $X \in \mathbf{S}_m(\mathbb{R})$. Then the distribution of $\Theta_{\tau Y}(\cdot)$ with respect to the Haar probability measure on $N'_{1,2}\backslash\mathbf{S}_m(\mathbb{R})$ converges weakly as $\tau \to 0$ to the distribution of the function $\widehat{\Theta}$ with respect to the $\mathbf{Mp}(2m,\mathbb{R})$-invariant probability measure $\widehat{P}$ on the homogeneous space $\widehat{\mathcal{M}}_4$. This convergence is uniform in Y on an arbitrary compact subset of the set of positive definite matrices of size $m \times m$.*

The precise form of the limiting distribution is unknown. The following statement reduces the study of the limiting distribution to the study of its radial part.

Corollary 4.1. *The limiting distribution in Theorem 3.1 is invariant under the rotations of $\mathbb{C}$ around the origin.*

Proof. By Subsection 2.6, we have

$$\widehat{\Theta}(g) = \Theta((\pi g)Z_0)/\rho(g, Z_0)$$

for any $g \in \mathbf{Mp}(2m,\mathbb{R})$. Now we take $u \in \pi^{-1}(\mathbf{U}(m))$; here the unitary group $\mathbf{U}(m)$ is assumed to be embedded in $\mathbf{Sp}(2m,\mathbb{R})$ as the stabilizer of the point $Z_0 = iI_m \in \mathcal{H}_m$ (see Subsection 2.5). Observe that for all $u \in \pi^{-1}(\mathbf{U}(m))$ the functions $\widehat{\Theta}(\cdot u)$ have the same distribution as $\widehat{\Theta}$ with respect to the invariant probability measure $\widehat{P}$. On the other hand, the cocycle property of ρ implies that

$$\rho(gu, Z_0) = \rho(g, \pi(u)Z_0)\rho(u, Z_0) = \rho(g, Z_0)\rho(u, Z_0)$$

and

$$\widehat{\Theta}(gu) = \widehat{\Theta}(g)/\rho(u, Z_0).$$

By (2.5) and the definition of the cocycle j, we have

$$\rho^2(u, Z_0) = j(\pi u, Z_0) = \det(i \operatorname{Im} \pi(u) + \operatorname{Re} \pi(u)) = \det(\pi(u)),$$

which completes the proof, because the determinant of a unitary matrix may take arbitrary complex values of modulus 1, and so does the cocycle $\rho(\cdot, Z_0)$. □

Remark 4.1. It is possible to show that, for any positive definite $(m \times m)$-matrix Y, the integrals of $|\Theta_{\tau Y}|^2$ over the left compact horosphere containing $i\tau Y$ are bounded in τ if $0 < \tau < \tau_0$. This implies that the distributions of the family of functions $(\Theta_{\tau Y})_{0<\tau<\tau_0}$ is tight. In view of this fact, the main content of Theorem 4.1 is that all partial limits of the distribution of $\Theta_{\tau Y}$ as $\tau \to 0$ are in fact the same. Thus, the limiting distribution should have a finite second moment. From the results of [Ma2] it follows that for $m = 1$ the moment of order α is finite if $\alpha < 4$ and infinite if $\alpha > 4$.

Proof of Theorem 4.1. We set $\tau = \exp(-2s)$ and consider the limiting behavior of $\Theta_{\exp(-2s)Y}$ as $s \to \infty$. By Proposition 2.3 with $Z = Z_s = i\exp(-2s)Y + N'_{1,2}$, the distribution of the function $\Theta_{\exp(-2s)Y}$ with respect to the Haar probability measure on $N'_{1,2}\backslash\mathbf{S}_m(\mathbb{R})$ agrees with the distribution of the function $\widehat{\Theta}$ on $\mathcal{O}(\exp(-s)Y^{1/2})$ (which is the closed $\mathbf{S}_m(\mathbb{R})$-orbit of the point $\widehat{\psi}_4(Z_s) \in \widehat{\mathcal{M}}_4$) supplied with $\widehat{P}_4(\exp(-s)Y^{1/2})$ (which is the $\mathbf{S}_m(\mathbb{R})$-invariant probability measure on $\mathcal{O}(\exp(-s)Y^{1/2})$).

Now, from Theorem 3.1 it follows that the measure $\widehat{P}_4(\exp(-s)Y^{1/2})$ tends weakly to the $\mathbf{Mp}(2m,\mathbb{R})$-invariant probability measure $\widehat{P}$ on $\widehat{\mathcal{M}}_4$, and this convergence is uniform in Y running over any compact set. This implies that for any continuous function on $\widehat{\mathcal{M}}_4$ its distribution relative to $\widehat{P}_4(\exp(-s)Y^{1/2})$ converges weakly to its distribution relative to $\widehat{P}$. Applying this to $\widehat{\Theta}$, we complete the proof. □

References

[Ar] V. I. Arnol′d, *Some remarks on flows of line elements and frames*, Dokl. Akad. Nauk SSSR **138** (1961), no. 2, 255–257; English transl. in Soviet Math. Dokl. **2** (1961). MR **28:**1555

[Da] S. G. Dani, *Orbits of horospherical flows*, Duke Math. J. **53** (1986), no. 1, 177–188. MR **87i:**22026

[DM] S. G. Dani and G. A. Margulis, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Part 1, Amer. Math. Soc., Providence, RI, 1993, pp. 91–137. MR **95b:**22024

[EMM] A. Eskin and C. McMullen, *Mixing, counting, and equidistribution in Lie groups*, Duke Math. J. **71** (1993), no. 1, 181–209. MR **95b:**22025

[FJK] H. Fiedler, W. Jurkat, and O. Körner, *Asymptotic expansions of finite theta series*, Acta Arith. **32** (1977), no. 2, 129–146. MR **58:**27832

[Ig] J. Igusa, *On the graded ring of theta-constants.* II, Amer. J. Math. **88** (1966), 221–236. MR **34:**375

[HL1] G. H. Hardy and J. E. Littlewood, *Some problems of Diophantine approximation.* II, Acta Math. **37** (1914), 193–239.

[HL2] ———, *A further note on the trigonometrical series associated with the elliptic theta-functions*, Proc. Cambridge Philos. Soc. **21** (1922), 1–5.

[JVH1] W. B. Jurkat and J. W. Van Horne, *The proof of the central limit theorem for theta sums*, Duke Math. J. **48** (1981), no. 4, 873–885. MR **86m:**11059

[JVH2] ———, *On the central limit theorem for theta series*, Michigan Math. J. **29** (1982), no. 1, 65–77. MR **83h:**10075

[JVH3] ———, *The uniform central limit theorem for theta sums*, Duke Math. J. **50** (1983), no. 3, 649–666. MR **85g:**11071

[Kli] H. Klingen, *Introductory lectures on Siegel modular forms*, Cambridge Stud. Adv. Math., vol. 20, Cambridge Univ. Press, Cambridge, 1990. MR **91a:**11021

[LV] G. Lion and M. Vergne, *The Weil representation, Maslov index and theta series*, Progr. Math., vol. 6, Birkhäuser, Boston, MA, 1980. MR **81j:**58075

[Ma1] J. Marklof, *Theta sums, Eisenstein series, and the semiclassical dynamics of a precessing spin*, Emerging Applications of Number Theory (Minneapolis, MN, 1996), IMA Vol. Math. Appl., vol. 109, Springer, New York, 1999, pp. 405–450. MR **2000k:**81094

[Ma2] ———, *Limit theorems for theta sums*, Duke Math. J. **97** (1999), no. 1, 127–153. MR **2000h:**11088

[Mo1] C. C. Moore, *Ergodicity of flows on homogeneous spaces*, Amer. J. Math. **88** (1966), 154–178. MR **33:**1409

[Mo2] ———, *Exponential decay of correlation coefficients for geodesic flows*, Group Representations, Ergodic Theory, Operator Algebras, and Mathematical Physics (Berkeley, CA, 1984), Math. Sci. Res. Inst. Publ., vol. 6, Springer-Verlag, New York–Berlin, 1987, pp. 163–181. MR **89d:**58102

[Mum] D. Mumford, *Tata lectures on theta*. I, Progr. Math., vol. 28, Birkhäuser, Boston, MA, 1983. MR **85h:**14026

[R] M. Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. (2) **134** (1991), no. 3, 545–607. MR **93a:**22009

[Sa] P. Sarnak, *Asymptotic behavior of periodic orbits of the horocycle flow and Eisenstein series*, Comm. Pure Appl. Math. **34** (1981), no. 6, 719–739. MR **83m:**58060

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany

E-mail address: goetze@mathematik.uni-bielefeld.de

St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191011, Russia

E-mail address: gordin@pdmi.ras.ru

Received 2/SEP/2002

Originally published in English