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CLASSIFICATION OF SIMPLE MULTIGERMS OF CURVES IN A SPACE WITH SYMPLECTIC STRUCTURE

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Abstract. A classification of stably simple germs of curves (both reducible and irreducible) in the complex space equipped with a symplectic structure is obtained. This classification extends the result by V. I. Arnol′d of 1999, which described the A_{2k} singularities in the symplectic complex space. The proofs involve the homotopy method and the Darboux–Givental theorem.

§1. Introduction

A significant number of papers is devoted to classification of simple singularities of curves. Bruce and Gaffney [4] classified the irreducible plane curves. Gibson and Hobbs [6] classified the simple singularities of irreducible curves in the complex 3-space. Arnol′d [1] classified the (stably) simple singularities of irreducible curves in the complex linear space of arbitrary dimension. The stably simple singularities of germs of reducible curves (or *multigerms*) in the complex space of arbitrary dimension were classified in [7], [8].

Now suppose that the complex space is equipped with a certain additional structure, e.g., a symplectic one. In this case, we consider not all local diffeomorphisms of our space, but only those that preserve this additional structure. Suppose that the complex $2n$-space $\mathbb{C}^{2n}$ is equipped with a symplectic structure, and we have a germ of a curve that is equivalent to (t^2, t^{2m+1}) in the sense of the usual RL-equivalence (where *all* local diffeomorphisms are allowed). Arnol′d studied this case in the paper [2]. His result is stated below. We consider singularities of both reducible and irreducible curves in a space equipped with a symplectic structure, and give a list of stably simple ones.

The necessary definitions are as follows. A *singularity* of an irreducible curve at the origin of $\mathbb{C}^n$ is a germ $f : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ of a complex analytic mapping. Let L (respectively, R) be the group of coordinate changes in $(\mathbb{C}^n, 0)$ (respectively, in $(\mathbb{C}, 0)$), i.e., the group of germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ (respectively, $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$) of nondegenerate analytic mappings. The group L (respectively, R) is called the group of *left* (respectively, *right*) *coordinate changes*.

Definition 1. A *multigerm* $\mathcal{F} = (f_1, \dots, f_k)$ in $(\mathbb{C}^n, 0)$ is a collection of germs $f_i : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ of analytic mappings, where $\operatorname{Im} f_i \cap \operatorname{Im} f_j = \{0\}$ for $i \neq j$ ($f_1, \dots, f_k$ are the *components* of $\mathcal{F}$).

Let $G \subset L$ be a subgroup. Then we define $RG := G \times R_{(1)} \times \dots \times R_{(k)}$, where, for each i, $R_{(i)}$ is a copy of R. The group RG (of right-left coordinate changes) acts on the space of multigerms by the formula

$$(g, h_1, \dots, h_k) \cdot (f_1, \dots, f_k) = (g \circ f_1 \circ h_1^{-1}, \dots, g \circ f_k \circ h_k^{-1}).$$

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Definition 2. A multigerm $\mathcal{F}$ is *RG-simple* if $\mathcal{F}$ has a neighborhood that intersects at most finitely many RG-orbits, in the space of multigerms (with the usual Whitney topology: the base of this topology consists of the preimages of open sets in the space of m-jets for each m).

Definition 3. Two multigerms $\mathcal{F}$ and $\mathcal{F}'$ in $(\mathbb{C}^n, 0)$ are *RG-equivalent* if they lie in one RG-orbit.

In what follows, we assume that our space is even-dimensional and symplectic, i.e., it is equipped with a closed nondegenerate 2-form ω. By the familiar Darboux theorem, there exist coordinates $(q_1, \dots, q_n; p_1, \dots, p_n)$ (they are said to be *symplectic*) in which locally we have

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

Now we let G be the group of *symplectomorphisms* of $(\mathbb{C}^{2n}, 0)$, i.e., the group of local diffeomorphisms preserving ω. In this situation, an RG-equivalence will be called a *symplectic equivalence*.

We denote by $\omega_{(n)}$ the germ of the symplectic form in $(\mathbb{C}^{2n}, 0)$ and by $\omega_{(N)}$ the germ of the symplectic form in $(\mathbb{C}^{2N}, 0)$. Let $i : (\mathbb{C}^{2n}, 0) \hookrightarrow (\mathbb{C}^{2N}, 0)$ be an embedding such that $i^*\omega_{(N)} = \omega_{(n)}$. A multigerm $\mathcal{F}$ is said to be *stably simple* if the multigerm $i \circ \mathcal{F} := \{i \circ f_1, \dots, i \circ f_k\}$ is also simple for each N. Two multigerms $\mathcal{F}$ and $\mathcal{F}'$ such that the multigerms $i \circ \mathcal{F}$ and $i \circ \mathcal{F}'$ are equivalent are said to be *symplectically stably equivalent*.

Now we state Arnol′d's result presented in [2]. Here and below, the coordinates $(q_1, \dots, q_n; p_1, \dots, p_n)$ are assumed to be symplectic if not stipulated otherwise; we separate the q-coordinates from the p-coordinates by a semicolon and sometimes drop the coordinates equal to zero.

Theorem 1. *Suppose $\mathcal{C}$ is a germ of a curve RL-equivalent to (t^2, t^{2m+1}). Then $\mathcal{C}$ is symplectically equivalent to exactly one germ on the following list*:

$$A_{2m,0} : (t^2; t^{2m+1}),$$
$$A_{2m,r} : (t^2, t^{2m+1}; t^{2m+2r+1}, 0), \quad 0 < r \le 2m.$$

Remark. For $r = 2m$, we can replace the monomial $t^{2m+2r+1}$ in p_1 by zero.

In particular, Theorem 1 implies that each multigerm RL-equivalent to (t^2, t^{2m+1}) is also symplectically simple.

We classify the other symplectically simple germs of irreducible curves, as well as the symplectically simple germs of reducible curves (multigerms). Our main result is as follows.

Theorem 2. *Up to permutation of components, each symplectically stably simple multigerm $\mathcal{F}$ is symplectically stably equivalent to exactly one multigerm on the list presented below.*

1. Germs of irreducible curves. Here, we do not present Arnol′d's list given above.

1. $(t^3, t^5; t^4, 0)$;	2. $(t^3; t^4)$;	3. $(t^3, t^4; t^5, 0)$;
4. $(t^3, t^4; 0, t^5)$;	5. $(t^3, t^4, t^5; t^7, 0, 0)$;	6. $(t^3, t^4, t^5; t^8, 0, 0)$;
7. $(t^3, t^4, t^5; 0, 0, 0)$;	8. $(t^3, t^7; t^5, 0)$;	9. $(t^3; t^5)$.

2. Two-component multigerms.

2.1. Both components are nonsingular. Here, $m > 1$ and k are positive integers.

1. $((t;0),(0;t))$;
2. $((t,0;0,0),(0,t;0,0))$;
3. $((t;0),(t;t^m))$;
4. $((t,0;0,0),(t,t^m;0,0))$;
5. $((t,0;0,0),(t,t^m;t^k,0))$, $\quad m < k < 3m-1$.

2.2. One of the components is singular. Here, m and k are positive integers.

1. $((t^2,t^{2m+1};t^{2m+1},0),(0,0;t,0))$;
2. $((t^2,t^{2m+1};t^{2k+1},0),(0,0;t,0))$, $\quad m < k < 3m$;
3. $((t^2,t^{2m+1};0,0),(0,0;t,0))$;
4. $((t^2;t^{2m+1}),(0;t))$;
5. $((t^2,0;t^3,t^3),(0,t;0,0))$;
6. $((t^2,0;t^3,0),(0,t;0,0))$;
7. $((t^2,t^3;t^5,0),(0,0;0,t))$;
8. $((t^2,t^3;0,0),(0,0;0,t))$;
9. $((t^2,t^3,0;t^5,0,0),(0,0,t;0,0,0))$;
10. $((t^2,t^3,0;0,0,0),(0,0,t;0,0,0))$;
11. $((t^2,t^4;t^3,0),(t,0;0,0))$;
12. $((t^2;t^3),(t;0))$.

3. Three-component multigerms. Here, $m > 1$ and k are positive integers.

1. $((t,0;0,0),(0,0;t,0),(t,t;t,0))$;
2. $((t,0;0,0),(0,0;t,0),(t,t;0,0))$;
3. $((t,0;0,0),(0,0;t,0),(0,t;0,0))$;
4. $((t,0,0;0,0,0),(0,t,0;0,0,0),(0,0,t;0,0,0))$;
5. $((t;0),(0;t),(t;t))$;
6. $((t;0),(0;t),(t;t^m))$;
7. $((t,0;0,0),(0,0;t,0),(t,t^m;t^m,0))$;
8. $((t,0;0,0),(0,0;t,0),(t,t^m;0,0))$;
9. $((t,0;0,0),(0,0;t,0),(t,t^m;t^k,0))$, $\quad m < k < 3m-1$.

Before starting the proof, we make some preliminary remarks.

The *tangent vector* (more precisely, the *tangent direction*) of the germ $\mathcal{C}$ of an irreducible curve C at the origin is the limit position of the (complex) line through the origin and a point on C approaching the origin. (In the general case, where we deal with a germ of a singular submanifold, this object is usually called the *tangent cone*.)

The *tangent space* of a multigerm $\mathcal{F}$ at the origin is the tangent space of the germ of a submanifold of minimal dimension containing $\mathcal{F}$. The *tangent space* of the k-jet of a germ is the tangent space of the polynomial mapping of degree at most k representing the jet. We easily see that this definition does not depend on the choice of a coordinate system.

Part I. Simple singularities of irreducible curves

§2. Curves with 4-jet (t^3,t^4)

Consider the 4-jet of our germ $\mathcal{C}$. It is obvious that in some coordinate system in the space of 4-jets this 4-jet has the form $(t^3,t^4,0,\dots,0)$. Two cases are possible: the restriction of ω to the tangent space of the 4-jet at the origin is either a nonzero or a zero form. Consider the first case.

2.1. The tangent space of the 4-jet of $\mathcal{C}$ at the origin is nonisotropic. In this situation, we obviously bring the 4-jet of $\mathcal{C}$ to the form

$$(q_1 = t^3, p_1 = t^4, q_{>1} = p_{>1} = 0).$$

We consider the case where $\mathcal{C}$ is RL-equivalent to the plane curve

$$(t^3, t^4). \tag{2.1}$$

Then there exists a germ $\mathcal{M}$ of a 2-dimensional submanifold in $(\mathbb{C}^{2n}, 0)$ containing $\mathcal{C}$, and the restriction of ω to the tangent space of $\mathcal{M}$ at the origin is nonzero. We show that there exist *symplectic* coordinates on $\mathcal{M}$ in which $\mathcal{C}$ has the form (2.1), and then apply the Darboux–Givental theorem [3], which asserts that, up to a germ of a symplectomorphism, a germ M of a submanifold in a symplectic space is uniquely determined by the restriction of ω to M.

Lemma 1. *Suppose $\mathcal{M}$ is a germ of a 2-manifold with coordinates (x, y) in which $\mathcal{C}$ has the form (2.1). We also assume that $\mathcal{M}$ is equipped with a symplectic form $\omega = f(x, y)dx \wedge dy$, where $f(0, 0) \neq 0$. Then there exist coordinates (p, q) in which we have $\omega = dp \wedge dq$ and (possibly, after some right change) $\mathcal{C}$ has the form (2.1).*

Proof. Without loss of generality, we may assume that $f(0, 0) > 0$. In the coordinates (x, y), we have $\mathcal{C} = \{y^3 = x^4\}$. We construct the required diffeomorphism by the homotopy method. Consider the following family of 2-forms:

$$\omega_t = (tf(x, y) + 1 - t)dx \wedge dy = f_t(x, y)dx \wedge dy.$$

Obviously, there exists a neighborhood of the origin in which the form ω_t is nondegenerate for each t. We must find a family $\phi_t(x, y)$ of diffeomorphisms preserving $\mathcal{C}$ and such that

$$\phi_t^* \omega_0 = \omega_t. \tag{2.2}$$

Let $\mathbf{v}_t(x, y)$ be a family of vector fields in a neighborhood of the origin that satisfies the following condition:

$$\frac{d}{dt}\phi_t(x, y) = \mathbf{v}_t(\phi_t(x, y)). \tag{2.3}$$

Differentiating (2.2) with respect to t, we obtain

$$L_{\mathbf{v}_t}\omega_t = (f(x, y) - 1)dx \wedge dy,$$

where $L_{\mathbf{v}_t}\omega_t$ is the Lie derivative of ω_t with respect to the vector field $\mathbf{v}_t$. The familiar formula $L_{\mathbf{v}} = i_{\mathbf{v}}d + di_{\mathbf{v}}$ (where $i_{\mathbf{v}}$ is the 1-form $\omega_t(\mathbf{v}, \cdot)$) and the fact that the form ω_t is closed for each t imply that

$$di_{\mathbf{v}_t}\omega_t = (f(x, y) - 1)dx \wedge dy.$$

Writing this relation in more detail, we obtain $\operatorname{div}(f_t\mathbf{v}_t) = f(x, y) - 1$.

We easily see that the vector field $\mathbf{v} = 3x\,\partial_x + 4y\,\partial_y$ is tangent to $\mathcal{C}$. Furthermore, the vector field $g(x, y)\mathbf{v}$ is also tangent to $\mathcal{C}$ for any function g. The above arguments show that it suffices to solve the "homological" equation $\operatorname{div}(g(x, y)\mathbf{v}) = h(x, y)$ for a given function $h(x, y)$. We do this in the analytic case. Let $h(x, y) = \sum h_{kl}x^k y^l$ and $g(x, y) = \sum g_{kl}x^k y^l$. Then $g_{kl} = \frac{h_{kl}}{3(k+1)+4(l+1)}$. □

Remark. The idea of this proof was borrowed from [2].

Now suppose that $\mathcal{C} \underset{\mathrm{RL}}{\sim} (t^3, t^4, t^5)$.

Lemma 2. *If the restriction of ω to the tangent space of the 4-jet of $\mathcal{C}$ is nonzero, then*

$$\mathcal{C} \underset{\mathrm{Sp}}{\sim} (q_1 = t^3, q_2 = t^5, p_1 = t^4, q_{>2} = p_{>1} = 0).$$

Proof. It is obvious that a symplectic change brings the 5-jet of $\mathcal{C}$ to the form

$$(q_1 = t^3 + o(t^3), q_2 = t^5, p_1 = t^4 + o(t^4), q_{>2} = p_{>1} = 0).$$

Projecting $\mathcal{C}$ to the $(2n-2)$-dimensional subspace $(q_1, q_3, \dots, q_n; p_1, p_3, \dots, p_n)$, we obtain a plane curve C equivalent to (2.1). By Lemma 1 and the Darboux–Givental theorem, a symplectic change brings C to the normal form, as before. Thus, $\mathcal{C}$ lies in the 4-dimensional subspace $(q_1, q_2; p_1, p_2)$, and we have

$$\mathcal{C} = (t^3, t^5 + o(t^5); t^4, o(t^5)).$$

The 6-jet of $\mathcal{C}$ has the form $(t^3, t^5 + o(t^5); t^4, \alpha t^6)$. The symplectic change of coordinates

$$Q = q, \quad P = p - \frac{\partial S(q)}{\partial q} \tag{2.4}$$

with $S(q_1, q_2) = \alpha q_1^2 q_2$ yields

$$\mathcal{C} = (t^3, t^5 + o(t^5); t^4 + o(t^7), o(t^6)).$$

In this case, the 7-jet of $\mathcal{C}$ has the form

$$(t^3, t^5 + o(t^5); t^4, \beta t^7). \tag{2.5}$$

We must "kill" the monomial t^7 in the last coordinate. Once again we use the homotopy method, i.e., we seek the required family of diffeomorphisms in the form of a solution of equation (2.3). The field $\mathbf{v}_t$ is chosen to be Hamiltonian (from [5] we know that the phase flow of a Hamiltonian system preserves the symplectic form). In the space of 7-jets with coordinate β we consider a one-parameter family of the form (2.5). It suffices to construct a family of vector fields smoothly depending on β and such that for each β the image of the corresponding vector field under the action on the space of 7-jets gives the tangent vector of the family, i.e., the vector $(0, 0; 0, t^7)$. It is easily seen that the vector field $(-q_1 q_2, 0; q_2 p_1, q_1 p_1)$ with the Hamiltonian $H = q_1 q_2 p_1$ is the required one. Thus, the corresponding right change yields

$$\mathcal{C} = (t^3, t^5 + o(t^5); t^4 + o(t^4), o(t^7)).$$

On the whole, this argument is similar to the proof of the well-known Mather lemma in [9]. Each function $o(t^7)$ can be written as $f(q_1, q_2)$. After the change (2.4) with $S(q_1, q_2) = \int_0^{q_2} f(q_1, q) dq$, the last coordinate of $\mathcal{C}$ becomes zero. Next, the change described in Lemma 1 in the (p_1, q_1)-plane yields

$$\mathcal{C} = (t^3, t^5 + o(t^5); t^4, 0).$$

The germ of any analytic function of order $o(t^5)$ has the form $f(q_1, p_1)$. Finally, in the subspace $\{p_2 = 0\}$ (containing $\mathcal{C}$), the vector field with the Hamiltonian $H = -f(q_1, p_1) p_2$ has only one nonzero component, namely, q_2. Therefore, the integral trajectories lying in this subspace are of the following form (s is the parameter on the trajectory):

$$(Q_1 = q_1, Q_2 = q_2 + s f(q_1, p_1); P_1 = p_1; P_2 = 0).$$

This proves the existence of a change that "kills" $o(t^5)$. □

2.2. The tangent space of the 4-jet of $\mathcal{C}$ at the origin is isotropic. In this situation, we can bring the 4-jet to the form $(q_1 = t^3, q_2 = t^4, q_{>2} = p = 0)$. We assume that

$$\mathcal{C} \underset{\mathrm{RL}}{\sim} (t^3, t^4, t^5). \tag{2.6}$$

Lemma 3. *Suppose the restriction of ω to the tangent space of the 4-jet of the germ* (2.6) *is zero. If the tangent space of $\mathcal{C}$ contains a vector that is not skew-orthogonal to the tangent vector of $\mathcal{C}$ at the origin, then*

$$\mathcal{C} \underset{\mathrm{Sp}}{\sim} (q_1 = t^3, q_2 = t^4; p_1 = t^5, q_{>2} = p_{>1} = 0).$$

Proof. We easily bring the 5-jet of $\mathcal{C}$ to the form

$$(q_1 = t^3 + o(t^3), q_2 = t^4 + o(t^4); p_1 = t^5, q_{>2} = p_{>1} = 0).$$

Consider the projection of $\mathcal{C}$ to the Lagrangian surface $\mathcal{L} := \{p = 0\}$. From [1] it follows that a diffeomorphism Φ of $\mathcal{L}$ brings the projection to the form (2.1). Then the symplectomorphism induced by Φ on the ambient space $\mathbb{C}^n$ preserves $\mathcal{L}$, whence $p = o(t^4)$. However, each function $o(t^4)$ is of the form $f(q_1, q_2, p_1)$. Consequently, we may assume that $\mathcal{C}$ lies in the 4-dimensional submanifold

$$\mathcal{M} := \{q_{>2} = 0, p_{>2} = F(q_1, q_2, p_1)\}$$

($(q_1, q_2; p_1, p_2)$ are coordinates on $\mathcal{M}$), and that the restriction of ω to $\mathcal{M}$ is equal to $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. By the Darboux–Givental theorem, $\mathcal{C} \underset{\mathrm{Sp}}{\sim} (t^3, t^4; t^5 + o(t^5), o(t^4))$. In order to get rid of t^5 in p_2, we perform the change

$$P_2 = p_2 - \alpha p_1, \quad Q_1 = q_1 + \alpha q_2, \quad Q_2 = q_2, \quad P_1 = p_1.$$

Now $p_2 = o(t^5)$; hence, p_2 has the form $f(q_1, q_2)$. The change (2.4) with $S(q_1, q_2) = \int_0^{q_2} f(q_1, q)dq$ yields

$$\mathcal{C} = (t^3 + o(t^3), t^4 + o(t^4); t^5 + o(t^5), 0).$$

As in the proof of Lemma 1, a symplectic change in the (q_1, p_1)-plane (and, possibly, a right change) leads to $\mathcal{C} = (t^3, t^4 + o(t^4); t^5, 0)$. First, integrating the vector field with the Hamiltonian $H = q_1 q_2 p_2$, we delete the term t^7 in the second coordinate. Now we delete the remaining terms. Clearly, we have $q_2 = t^4 + f(q_1, p_1)$. Consequently, the Hamiltonian vector field with the Hamiltonian $H = f(q_1, p_1)p_2$ generates the required transformation. □

Lemma 4. *If the tangent space of the curve* (2.6) *contains a vector skew-orthogonal to the tangent vector of $\mathcal{C}$ at the origin, but not skew-orthogonal to some vector in the tangent space of the 4-jet of $\mathcal{C}$, then $\mathcal{C} \underset{\mathrm{Sp}}{\sim} (t^3, t^4; 0, t^5)$.*

Proof. As in the proof of Lemma 3, we see that

$$\mathcal{C} \underset{\mathrm{Sp}}{\sim} (t^3, t^4; o(t^5), t^5 + o(t^5)).$$

Each function $o(t^5)$ is of the form $f(q_1, q_2)$. We present p_1 in this form. Then the change (2.4) with $S(q_1, q_2) = \int_0^{q_1} f(q, q_2)dq$ yields $\mathcal{C} = (t^3, t^4; 0, t^5 + o(t^5))$. The projection $\mathcal{C}'$ of $\mathcal{C}$ to the (q_2, p_2)-plane has the 5-jet (t^4, t^5). From [1] we know that each plane curve C with such a 5-jet is RL-equivalent either to (t^4, t^5), or to $(t^4, t^5 + t^7)$, and if the 7-jet of C has the form (t^4, t^5), then C is RL-equivalent to the former curve. We bring the 7-jet of $\mathcal{C}'$ precisely to this form. First we delete the monomial t^6 in the second coordinate. This can be done by performing a certain right change and subtracting p_2 from q_2, which does not affect the symplectic form. Thus, we may assume that

$$\mathcal{C} = (t^3 + o(t^3), t^4; 0, t^5 + \alpha t^7 + o(t^7)).$$

The right change $t \mapsto t + kt^3$, with k to be chosen later, yields

$$\mathcal{C} = (t^3 + o(t^3), t^4 + 4kt^6 + o(t^6); 0, t^5 + (5k + \alpha)t^7 + o(t^7)).$$

Next, we perform the symplectic change

$$Q_2 = -p_2, \quad P_2 = q_2. \tag{2.7}$$

The change (2.4) with $S = \beta q_1^2 q_2$ and the change converse to (2.7) yield

$$\mathcal{C} = (t^3 + o(t^3), t^4 + (4k - \beta)t^6 + o(t^6); 2\beta t^8 + o(t^8), t^5 + (5k + \alpha)t^7 + o(t^7)).$$

The change (2.4) with $S = 2\beta q_1 q_2^2$ yields

$$\mathcal{C} = (t^3 + o(t^3), t^4 + (4k - \beta)t^6 + o(t^6); o(t^8), t^5 + (5k - 4\beta + \alpha)t^7 + o(t^7)).$$

Let β and k be the solution of the linear system

$$\begin{cases} 4k - \beta = 0, \\ 4\beta - 5k = \alpha. \end{cases}$$

Using the change (2.4), we make $p_1 = 0$, and the addition to p_2 arising under this change has the form $o(t^7)$. Thus, we obtain

$$\mathcal{C} = (t^3 + o(t^3), t^4 + o(t^6); 0, t^5 + o(t^7)).$$

The above arguments imply that the projection of $\mathcal{C}$ to the (q_2, p_2)-plane is RL-equivalent to (t^4, t^5); consequently, in some (not necessarily symplectic) system of coordinates we have $\mathcal{C} = \{x^5 - y^4 = 0\}$. Applying the same arguments as in the proof of Lemma 1, we obtain $\mathcal{C} = (t^3 + o(t^3), t^4; 0, t^5)$.

Now we wish to delete t^6, t^7, and t^{11} in q_1. We show how we delete, e.g., t^7 if t^6 has already been deleted. We use symplectic changes on the space of 7-jets. Integrating the vector field with the Hamiltonian $H = q_1 q_2 p_1$, we obtain a change Φ that obviously does not affect the components q_2, p_1, and p_2. At the same time, the image of the tangent vector under the action on the space of 7-jets is of the form $(t^7, 0; 0, 0)$. This shows that Φ is the required transformation. In a similar way, we delete t^6 and t^{11}. After this, we present the remaining term $o(t^3)$ in the form $f(q_2, p_2)$ and, integrating the vector field with the Hamiltonian $H = f(q_2, p_2)p_1$, we bring $\mathcal{C}$ to the required form. □

Lemma 5. *Suppose that* $\mathcal{C} \underset{\mathrm{RL}}{\sim} (2.1)$*, and the restriction of* ω *to the tangent space of* $\mathcal{C}$ *is zero. Then*

$$\mathcal{C} \underset{\mathrm{Sp}}{\sim} (q_1 = t^3, q_2 = t^4; p_1 = c_1 t^7 + c_2 t^8 + c_3 t^{10} + c_4 t^{11} + c_5 t^{14}, p_2 = 0).$$

Proof. Obviously, a symplectomorphism yields

$$\mathcal{C} = (q_1 = t^3 + o(t^3), q_2 = t^4 + o(t^4), q_{>2} = p = o(t^5)).$$

We project $\mathcal{C}$ to the Lagrangian submanifold $\mathcal{L} := \{p = 0\}$. A diffeomorphism Φ of $\mathcal{L}$ brings the projection to the form (2.1). Since the symplectomorphism of $\mathbb{C}^n$ induced by Φ preserves $\mathcal{L}$, we have $p = o(t^5)$, whence $p_{>2} = F(q_1, q_2)$. Thus, $\mathcal{C}$ lies in the 4-dimensional submanifold

$$\mathcal{M} := \{q_{>2} = 0, p_{>2} = F(q_1, q_2)\},$$

and the restriction of ω to $\mathcal{M}$ has the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_1$. By the Darboux–Givental theorem, we have

$$\mathcal{C} \underset{\mathrm{Sp}}{\sim} (q_1 = t^3, q_2 = t^4; p_1 = f(t), p_2 = g(t), q_{>2} = p_{>2} = 0),$$

where $f, g = o(t^5)$.

For simplicity, we let $q_1 = x$ and $q_2 = y$. Clearly,

$$f(t) = u(x) + yz(x) + y^2 r(x) \quad \text{and} \quad g(t) = v(x) + yw(x) + y^2 s(x),$$

where $u, v = o(x)$ and $z, w = o(1)$.

Using the change (2.4) with

$$S(x,y) = A(x) + yB(x) + y^2C(x) + y^3D(x) + y^4E(x) + y^5F(x),$$

we obtain

$$\begin{cases} \frac{\partial S}{\partial x} = A' + yB' + y^2C' + y^3D' + y^4E' + y^5F', \\ \frac{\partial S}{\partial y} = B + 2yC + 3y^2D + 4y^3E + 5y^4F. \end{cases}$$

Since on $\mathcal{C}$ we have $x^4 = y^3$, it follows that on $\mathcal{C}$ the above equations take the form

$$\begin{cases} \frac{\partial S}{\partial x} = (A' + x^4D') + y(B' + x^4E') + y^2(C' + x^4F'), \\ \frac{\partial S}{\partial y} = (B + 4x^4E) + y(2C + 5x^4F) + y^2(3D). \end{cases}$$

Now we choose $A(x)$ and $D(x)$ so that

$$A'(x) + x^4D'(x) = u(x) \quad \text{and} \quad 3D(x) = s(x).$$

After that, we take $B(x)$ and $C(x)$ so that

$$B(x) + 4x^4E(x) = v(x) \quad \text{and} \quad 2C(x) + 5x^4F(x) = w(x).$$

Then

$$B' + x^4E' = v' - 16x^3E - 3x^4E' \quad \text{and} \quad 2C' + 2x^4F' = w' - 20x^3F - 3x^4F'.$$

After this choice, we obtain $g - \partial_y S = 0$ and

$$\begin{aligned} f - \frac{\partial S}{\partial x} &= y(z - v' + 16x^3E + 3x^4E') + \frac{y^2}{2}(2r - w' + 20x^3F + 3x^4F') \\ &= y(z - v' + x^3(16E + 3xE')) + \frac{y^2}{2}(2r - w' + x^3(20F + 3xF')). \end{aligned}$$

Clearly, $z(x) - v'(x) = O(x)$ and $2r(x) - w'(x) = O(1)$. Consequently, $E(x)$ and $F(x)$ can be chosen so that the expression in the first (respectively, second) parentheses be equal to $a_1x + a_2x^2$ (respectively, $b_1 + b_2x + b_3x^2$). Thus, recalling that $x = t^3$ and $y = t^4$, on $\mathcal{C}$ we obtain

$$f - \frac{\partial S}{\partial x} = y(a_1x + a_2x^2) + y^2(b_1 + b_2x + b_3x^2) = c_1t^7 + c_2t^8 + c_3t^{10} + c_4t^{11} + c_5t^{14}.$$

□

Lemma 6. *Suppose $\mathcal{C} \underset{\mathrm{RL}}{\sim} (t^3, t^4, t^5)$. If the restriction of ω to the tangent space of $\mathcal{C}$ is zero, then $\mathcal{C}$ is symplectically equivalent to one of the following three curves*:

(1) $(t^3, t^4, t^5; t^7, 0, 0)$,
(2) $(t^3, t^4, t^5; t^8, 0, 0)$,
(3) $(t^3, t^4, t^5; 0, 0, 0)$.

Remark. Below it will be shown that these three curves are not symplectically equivalent.

Proof. Arguments similar to those used in the proof of Lemma 5 yield

$$\mathcal{C} = (t^3, t^4, t^5; c_1t^7 + c_2t^8 + c_3t^{10} + c_4t^{11} + c_5t^{14}, 0, 0). \tag{2.8}$$

A change of the form (2.4) allows us to assume that $c_3 = c_4 = c_5 = 0$.

Suppose $c_1 \neq 0$. Then $\mathcal{C} \underset{\mathrm{Sp}}{\sim} (t^3, t^4, t^5; t^7 + \alpha t^8, 0, 0)$. We show how to delete t^8. The tangent space of an R-orbit contains the vector $(3t^4, 4t^5, 5t^6; 7t^8, 0, 0)$. Using the vector fields with the Hamiltonians $H = p_2q_3$, $q_1^2p_3$, p_1q_2, q_2^3, $q_1q_2^2$, we easily obtain the vector $(0, 0, 0; 0, t^8, 0)$. Now, integrating the corresponding vector fields on the space of 8-jets of right and left changes, we obtain the required change. The higher terms arising in p_1 are deleted by a change of the form (2.4), as before. Thus, we come to the normal form (1).

If in (2.8) we have $c_1 = 0$ and $c_2 \neq 0$, then we obtain the normal form (2).
Finally, if $c_1 = c_2 = 0$, we obtain the normal form (3). □

§3. Curves with 5-jet (t^3, t^5)

From [1] we know that up to RL-equivalence there are only two curves $\mathcal{C}$ with such a 5-jet, namely, (t^3, t^5) and (t^3, t^5, t^7).

Lemma 7. *If the restriction of ω to the tangent space of the 5-jet of $\mathcal{C}$ is nonzero, then $\mathcal{C}$ is symplectically equivalent to one of the following two curves*:

(1) $(q_1 = t^3, q_2 = t^7, p_1 = t^5, q_{>2} = p_{>1} = 0)$,
(2) $(q_1 = t^3, p_1 = t^5, q_{>1} = p_{>1} = 0)$.

Proof. If $\mathcal{C} \underset{\mathrm{RL}}{\sim} (t^3, t^5)$, then the proof is similar to that of Lemma 1. (We set $\mathbf{v} := 3x\,\partial_x + 5y\,\partial_y$.) If $\mathcal{C} \underset{\mathrm{RL}}{\sim} (t^3, t^5, t^7)$, then the proof is similar to that of Lemma 2. □

§4. Nonsimple families

We prove that all germs except for those indicated in [2] and in Lemmas 1–4, 6, and 7 are nonsimple.

Lemma 8. *The curves with the 5-jet $(q_1 = t^3, q_2 = t^4; q_{>2} = p = 0)$ are not simple.*

Proof. By Lemma 5, the 8-jet of such a curve $\mathcal{C}$ adjoins the family $(t^3, t^4; t^7 + \alpha t^8, 0)$. This family will be regarded as a 1-dimensional submanifold in the space of 8-jets. We show that at no point the tangent vector $(0, 0; t^8, 0)$ of this submanifold is contained in the tangent space of an orbit of the action of the group. For this, we use the following assertion from [5]: the tangent vector field of a one-parameter family of symplectomorphisms is Hamiltonian.

We argue by contradiction. The monomial t^8 in p_1 can be obtained with the help of some of the following four vectors:

$$\begin{aligned}
&(3t^4, 4t^5; 7t^8, 0), \\
&(3t^3, 4t^4; 7t^7 + 8\alpha t^8, 0), \\
&(0, 0; t^8, 2t^7), && H = q_1 q_2^2, \\
&(-t^3, 0; t^7 + \alpha t^8, 0), && H = p_1 q_1.
\end{aligned}$$

We cannot use the first vector because t^5 in q_2 cannot be compensated by anything. Since t^3 in q_1 can be obtained only by the two ways indicated above (the second and fourth vectors), we must compensate t^7 in p_1 somehow. This can be done only with the help of the vector $(0, 0; 2t^7, t^6)$ with the Hamiltonian $H = q_1^2 q_2$. But t^6 in p_2 cannot be obtained in any other way, and so this monomial cannot be compensated by anything. If we try to use the third vector, then we must compensate t^7 in p_2. This can be done only with the help of the vector $(-t^4, 0; 0, t^7 + \alpha t^8)$ with the Hamiltonian $H = p_1 q_2$. However, t^4 in q_1 can be compensated only by the first vector; as was observed before, we cannot use it. A contradiction. □

Lemma 9. *If $\mathcal{C} \underset{\mathrm{RL}}{\sim} (t^4, t^5, t^6, t^7)$, then $\mathcal{C}$ is not symplectically simple.*

Proof. It is clear that, perturbing $\mathcal{C}$, we can ensure that the restriction of ω to the tangent space W of the 5-jet of $\mathcal{C}$ be nondegenerate and the restriction of ω to the tangent space V of $\mathcal{C}$ (V is 4-dimensional) be also nondegenerate, $W \subset V$. Thus, we bring the 5-jet of $\mathcal{C}$ to the form

$$(q_1 = t^4 + o(t^4), p_1 = t^5; q_{>1} = p_{>1} = 0).$$

The restriction of ω to the skew-orthogonal complement of W in V is nondegenerate, which means that we can bring the 7-jet of $\mathcal{C}$ to the form

$$(q_1 = t^4 + o(t^4), q_2 = t^6 + o(t^6); p_1 = t^5 + o(t^5), p_2 = \alpha t^7, q_{>2} = p_{>2} = 0).$$

We show that the parameter α is a module, i.e., for distinct α's these curves are symplectically nonequivalent. Indeed, if such an equivalence existed, then so would its linear part. Clearly, $P_2 = p_2/b$. If $Q_2 = o(t^5)$, then necessarily Q_2 is a linear combination of q_2 and p_2. However, the term p_2 is immaterial for us, and if we delete it, then the change will remain symplectic. Therefore, we may assume that $Q_2 = bq_2$. Now we observe that the subspace with the coordinates (q_2, p_2) is invariant with respect to our transformation. Consequently, the skew-orthogonal complement of the subspace (q_2, p_2) is also invariant. Arguing in a similar way, we see that changes of the following form can be used $(a, b, c \neq 0)$:

$$\begin{aligned} Q_1 &= aq_1, & P_1 &= \frac{p_1}{a}, \\ Q_2 &= bq_2, & P_2 &= \frac{p_2}{b}, \\ t &= cT. \end{aligned}$$

Performing these changes, we easily check that the required result cannot be achieved. □

Now we easily see that all the remaining curves adjoin one of the two families presented above.

§5. Symplectic invariants of singularities

We show that all curves occurring in Lemma 6 are symplectically distinct.

Theorem 3. *For each curve $\mathcal{C}$ as in Lemma 6, the order of tangency of $\mathcal{C}$ with any Lagrangian surface $\mathcal{L}$ does not exceed the order of tangency of $\mathcal{C}$ with the* (*Lagrangian*) *surface $\mathcal{L}_0 := \{p = 0\}$.*

Proof. We argue by contradiction. Suppose that for some curve $\mathcal{C}$ there exists a Lagrangian surface $\mathcal{L}$ such that the order of tangency of $\mathcal{C}$ with $\mathcal{L}$ is higher than the order of tangency of $\mathcal{C}$ with $\mathcal{L}_0$. Then the tangent space of $\mathcal{L}$ at the origin is $\{p = 0\}$.

Indeed, suppose $\mathcal{L}$ is given by equations of the form

$$Aq_1 + Bq_2 + Cq_3 + Dp_1 + Ep_2 + Fp_3 + R(p, q) = 0,$$

where $R(p, q)$ denotes terms of higher order. Substituting $\mathcal{C}$, we obtain $At^3 + Bt^4 + Ct^5 + R_2t^6 + o(t^6)$. By assumption, this is at least $o(t^7)$. Therefore, $A = B = C = 0$, which implies our assertion.

In view of this assertion, $\mathcal{L}$ is given (locally) by the equation $p = \partial_q S$. For short, we let $q_1 = x$, $q_2 = y$, and $q_3 = z$. Then

$$S(x, y, z) = A(x) + B(x)y + C(x)y^2 + D(x)y^3 + E(x)yz + F(x)z^2 + G(x)y^2z + R(x)z,$$

because the higher terms y^4, y^3z, yz^2, and z^3 contribute $O(t^{12})$, $O(t^{12})$, $O(t^9)$, and $O(t^{10})$, respectively.

On $\mathcal{C}$ we have $y^3 = x^4$, $yz = x^3$, $z^2 = x^2y$, and $y^2 = xz$; therefore,

$$\begin{aligned}
\frac{\partial S}{\partial x} &= A' + B'y + C'y^2 + D'y^3 + E'yz + F'z^2 + G'y^2z + R'z \\
&= (A' + x^3E' + x^4D') + y(B' + x^2F' + x^3G') + z(R' + xC'), \\
\frac{\partial S}{\partial y} &= B + 2Cy + 3Dy^2 + Ez + 2Gyz \\
&= (B + 2x^3G) + y(2C) + z(E + 3xD), \\
\frac{\partial S}{\partial z} &= Ey + 2Fz + Gy^2 + R \\
&= R + y(E) + z(2F + xG).
\end{aligned}$$

Both curves (1) and (2) in Lemma 6 can be written as $(t^3, t^4, t^5; t^a, 0, 0)$, where $a = 7$ or $a = 8$. Then, by assumption, for one of these a we have

$$\frac{\partial S}{\partial x} - t^a = o(t^a), \quad \frac{\partial S}{\partial y} = o(t^a), \quad \text{and} \quad \frac{\partial S}{\partial z} = o(t^a).$$

First, we consider the curve (1) ($a = 7$). The above formulas show that t^7 in $\partial_x S$ can only come from x in $B'(x)$, which means that $B(x)$ contains x^2 with a nonzero coefficient. But this means that $\partial_y S$ contains t^6 with a nonzero coefficient, because t^6 cannot be "killed" with the help of the expressions in the other parentheses. In this case, $\partial_y S \neq o(t^7)$.

Now we consider the curve (2) ($a = 8$). The above formulas show that t^8 in $\partial_x S$ can only come from x in $R'(x)$ or from the constant term in $C'(x)$, which means that $R(x)$ contains x^2 with a nonzero coefficient, or $C(x)$ contains x with a nonzero coefficient. But this implies that either $\partial_z S$ contains t^6 with a nonzero coefficient, or $\partial_y S$ contains t^7 with a nonzero coefficient, because these terms cannot be "killed" with the help of the expressions in the other parentheses. In this case, $\partial_z S \neq o(t^8)$ or $\partial_y S \neq o(t^8)$. A contradiction. □

Remark. The idea of the construction of such an invariant is borrowed from [2].

Part II. Simple singularities of multigerms

§6. A pair of intersecting lines

Suppose that

$$\mathcal{F} = (\mathcal{C}_1, \mathcal{C}_2) \underset{\mathrm{RL}}{\sim} ((t, 0), (0, t)). \tag{6.1}$$

Lemma 10. *If the restriction of ω to the tangent space of the multigerm* (6.1) *is nonzero, then*

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((q_1 = t, q_{>1} = p = 0), (p_1 = t, q = p_{>1} = 0)).$$

Proof. The argument is similar to that used in the proof of Lemma 1. Suppose that $\mathcal{F}$ lies on a 2-dimensional submanifold $\mathcal{M}^2 \subset \mathbb{C}^{2n}$ and has the form $((t, 0), (0, t))$ in some coordinates (x, y). Then $\mathcal{F}$ is given by the equation $xy = 0$, and we can set $\mathbf{v} := x\,\partial_x + y\,\partial_y$. □

Lemma 11. *If the restriction of ω to the tangent space of the multigerm* (6.1) *is zero, then*

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((q_1 = t, q_{>1} = p = 0), (q_2 = t, q_{\neq 2} = p = 0)).$$

Proof. It is obvious that a symplectic transformation brings the 1-jet of $\mathcal{F}$ to the required form. Now we project $\mathcal{F}$ to the q-subspace along the p-subspace. A diffeomorphism Φ of the q-coordinates brings the projection to the form $((q_1 = t, q_{>1} = 0), (q_2 = t, q_{\neq 2} = 0))$. Φ induces a symplectomorphism of $\mathbb{C}^n$. For $\mathcal{C}_1$ we have $p_{>2} = f(q_1)$. Correspondingly, for $\mathcal{C}_2$, we have $p_{>2} = g(q_2)$ and $f, g = o(q)$. Then, obviously, $\mathcal{F}$ lies on the 4-dimensional submanifold

$$\mathcal{M} := \{q_{>2} = 0, p_{>2} = f(q_1) + g(q_2)\}.$$

The restriction of ω to $\mathcal{M}$ is of the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. By the Darboux–Givental theorem, we obtain

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((t, 0; f(t), g(t)), (0, t; u(t), v(t))),$$

where $f, g, u, v = O(t^2)$.

Applying the change (2.4) with

$$S(q_1, q_2) = A(q_1) + B(q_2) + q_1 C(q_2) + q_2 D(q_1),$$

where A, B, C, and D will be chosen later in such a way that all of them be $O(q^2)$, we obtain

$$\begin{cases} \frac{\partial S}{\partial q_1} = A'(q_1) + C(q_2) + q_2 D'(q_1), \\ \frac{\partial S}{\partial q_2} = B'(q_2) + D(q_1) + q_1 C'(q_2). \end{cases}$$

Along $\mathcal{C}_1$, we have

$$\frac{\partial S}{\partial q_1} = A'(q_1) \quad \text{and} \quad \frac{\partial S}{\partial q_2} = D(q_1).$$

Along $\mathcal{C}_2$, we have

$$\frac{\partial S}{\partial q_1} = C(q_2) \quad \text{and} \quad \frac{\partial S}{\partial q_2} = B'(q_2).$$

Taking

$$A'(q_1) = f(q_1), \quad B'(q_2) = v(q_2), \quad C(q_2) = u(q_2), \quad \text{and} \quad D(q_1) = g(q_1),$$

we bring $\mathcal{F}$ to the required form. □

§7. A pair of nonsingular curves $\mathcal{C}_1$, $\mathcal{C}_1$ touching each other

As was shown in [8], in this case

$$\mathcal{F} \underset{\mathrm{RL}}{\sim} ((t, 0), (t, t^a)). \tag{7.1}$$

Lemma 12. *Suppose the restriction of ω to the tangent space of the multigerm* (7.1) *is nonzero. Then*

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((q_1 = t, q_{>2} = p = 0), (q_1 = t, p_1 = t^a, q_{>1} = p_{>1} = 0)).$$

Proof. The argument repeats the proof of Lemma 10 with the only difference that (in some coordinates on the corresponding 2-manifold) $\mathcal{F}$ is given by the equation $(x^a - y)y = 0$, and in order to bring ω to the standard form we use the vector field $\mathbf{v} = x\,\partial_x + ay\,\partial_y$. □

Lemma 13. *Suppose the restriction of ω to the tangent space of the multigerm* (7.1) *is zero. Then $\mathcal{F}$ is symplectically equivalent to one of the following multigerms*:

(1) $((q_1 = t, q_{>1} = p = 0), (q_1 = t, q_2 = t^a, q_{>2} = p = 0))$,
(2) $((q_1 = t, q_{>1} = p = 0), (q_1 = t, q_2 = t^a, p_1 = t^b, q_{>2} = p_{>1} = 0))$, *where* $a < b < 3a - 1$.

Remark. Putting $b = 3a - 1$ in the normal form 2, we obtain a multigerm symplectically equivalent to the normal form 1.

Proof. Suppose that $\mathcal{C}_1$ already has the required form. First, we reduce the problem to the 4-dimensional case. We observe that a symplectic transformation brings the a-jet to the required form. We project $\mathcal{F}$ to the q-subspace along the p-subspace. By the theorem on the RL-classification, a diffeomorphism Φ of the q-subspace brings the projection to the form

$$((q_1 = t, q_{>1} = 0), (q_1 = t, q_2 = t^a, q_{>2} = 0)).$$

Φ induces a symplectomorphism of $\mathbb{C}^n$. Along the second component, we have $p_i = q_2 f_i(q_1)$, $i = 1, 2$. Thus, $\mathcal{F}$ lies on the 4-dimensional submanifold

$$\mathcal{M} := \{q_{>2} = 0, p_3 = q_2 f_3(q_1), \dots, p_n = q_2 f_n(q_1)\}.$$

The restriction of ω to $\mathcal{M}$ is of the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ (because $dq_{>2} = 0$). The Darboux–Givental theorem yields

$$\mathcal{F} = ((t, 0; 0, 0), (t, t^a; f(t), g(t))), \quad q_{>2} = p_{>2} = 0.$$

For simplicity, we let $q_1 = x$ and $q_2 = y$. Clearly, along $\mathcal{C}_2$ we have $p_1 = y \cdot z(x)$ and $p_2 = y \cdot w(x)$, where $z, w = O(x)$.

The change (2.4) with $S(x, y) = y^2 B(x) + y^3 C(x)$ yields

$$\begin{cases} \frac{\partial S}{\partial x} = y^2 B'(x) + y^3 C'(x), \\ \frac{\partial S}{\partial y} = 2yB(x) + 3y^2 C(x). \end{cases}$$

Along $\mathcal{F}$ we have $x^a y = y^2$, whence

$$\begin{cases} \frac{\partial S}{\partial x} = y(x^a B' + x^{2a} C'), \\ \frac{\partial S}{\partial y} = y(2B + 3x^a C). \end{cases}$$

Observe that both expressions above vanish along $\mathcal{C}_1$. Now we find $B(x)$ from the condition $2B(x) + 3x^a C(x) = w(x)$, with $C(x)$ to be chosen later. Differentiating this identity, we obtain

$$2(B'(x) + x^a C'(x)) = w'(x) - 3ax^{a-1} C(x) - x^a C'(x).$$

After this change, $p_2 = 0$ along $\mathcal{C}_2$. For p_1, we have

$$\begin{aligned} p_1 = f(t) - \frac{\partial S}{\partial x} &= y(z(x) - (x^a B'(x) + x^{2a} C'(x))) \\ &= y\Big(z - \frac{1}{2} x^a w'(x) + \frac{1}{2} x^{2a-1}(3aC + xC')\Big). \end{aligned}$$

For every function $h(x)$, the equation $3aC(x) + xC'(x) = h(x)$ is solvable (locally). Thus, along $\mathcal{C}_2$, we obtain

$$p_1 = y \sum_{r=1}^{2a-2} c_r x^r = \sum_{r=a+1}^{3a-2} d_r t^r.$$

Let b be the minimum degree of a nonzero monomial in p_1 of $\mathcal{C}_2$. We bring the $(3a-2)$-jet of $\mathcal{F}$ to the required form. Consider the following submanifold $\mathfrak{M} \subset J^{3a-2}$ ($c_1, \dots, c_{2a-2}$ are coordinates on $\mathfrak{M}$):

$$\mathfrak{M} := \Big\{(t, 0; 0, 0), \Big(t, t^a; t^b + \sum_{r=1}^{3a-b-2} c_r t^{b+r}, 0\Big)\Big\}.$$

Suppose that we have deleted the monomials of degree less than $b + r$ (except t^b). We show how we can delete t^{b+r}. In the tangent space at the identity of the group of $(3a-2)$-jets of the coordinate changes of the form considered, we construct a smooth family of vectors the images of which under the action on $\mathfrak{M}$ have the form $(0, 0; t^{b+r} + o(t^{b+r}), 0)$. The function $H = p_1 q_1^{r+1}$ generates the vector field $(-q_1^{r+1}, 0; (r+1)p_1 q_1^r, 0)$. The action on the second component of $\mathfrak{M}$ yields the vector $(-t^{r+1}, 0; (r+1)t^{b+r} + o(t^{b+r}), 0)$. It is

also clear that the vector $(t^{r+1}, at^{a+r}; bt^{b+r} + o(t^{b+r}), 0)$ is the image of some vector in the tangent space of R. Finally, the image of the vector field with the Hamiltonian $H = -q_1^r q_2 p_2$ is $(0, t^{a+r}; 0, 0)$. We easily obtain the required vector as a linear combination (with constant coefficients!) of the above vectors.

Thus, we have brought the $(3a-2)$-jet of $\mathcal{F}$ to the required form. Changing the q-coordinates, we bring the projection of $\mathcal{C}_2$ to the q-subspace to the form (t, t^a). This change preserves the projection of $\mathcal{C}_1$, and also induces a symplectomorphism preserving the $(3a-2)$-jet of $\mathcal{F}$. However, the above-said implies that a change of the form (2.4) brings $\mathcal{F}$ to the required form. □

Lemma 14. *The normal forms indicated in Lemma* 13 *are distinct.*

Proof. The normal forms in question are discriminated by the maximum possible order of tangency of $\mathcal{C}_2$ with the germs of 2-dimensional Lagrangian surfaces $\mathcal{L}$ at the origin containing $\mathcal{C}_1$. We show that this order of tangency is equal to b. Suppose that there is a surface $\mathcal{L}$ having a higher order of tangency with $\mathcal{C}_2$.

First we show that the tangent space of $\mathcal{L}$ at the origin is given by the condition $\{p = 0\}$. Suppose that $\mathcal{L}$ is given by equations of the form

$$Aq_1 + Bq_2 + Cp_1 + Dp_2 + R(p, q) = 0,$$

where $R(p, q)$ is the remainder term of the first order Taylor expansion. Since $\mathcal{C}_1 \subset \mathcal{L}$, all monomials of the form q_1^k occur in the Taylor expansion with zero coefficient. In particular, this means that $A = 0$. For $\mathcal{C}_2$, we have $Bt^a + Ct^b + o(t^a) = o(t^b)$, whence $B = 0$. Thus, $A = B = 0$, and the tangent space has the equation $\{p = 0\}$. This implies that $\mathcal{L}$ is given (locally) by the equation $\{p = \partial_q S\}$.

For convenience, we set $q_1 = x$ and $q_2 = y$. Since $3a > b$, it suffices to set

$$S(x, y) = A(x) + yB(x) + y^2C(x) + y^3D(x).$$

(Along $\mathcal{C}_2$, y^4 gives at least $t^{3a} = o(t^b)$.) We have

$$\begin{cases} \frac{\partial S}{\partial x} = A' + yB' + y^2C' + y^3D', \\ \frac{\partial S}{\partial y} = B + 2yC + 3y^2D. \end{cases}$$

Since $\mathcal{C}_1 \subset \mathcal{L}$, we have $A'(x) = B(x) = 0$. Thus, along $\mathcal{C}_2$, where $x^a = y$, we have

$$\begin{cases} \frac{\partial S}{\partial x} = y(x^aC' + x^{2a}D'), \\ \frac{\partial S}{\partial y} = y(2C + 3x^aD). \end{cases}$$

By assumption, $\partial_y S = o(t^b)$ and $\partial_x S - t^b = o(t^b)$. Since $3a > b$, $C'(x)$ contains x^{b-2a} with a nonzero coefficient. Consequently, $C(x)$ contains x^{b-2a+1}. Then, along $\mathcal{C}_2$, $\partial_y S$ necessarily contains t^{b-a+1}, which we cannot "kill" by the second term because $2a > b - a + 1$ (or, what is the same, $b < 3a - 1$). This is a contradiction, since $t^{b-a+1} \neq o(t^b)$. □

§8. The multigerms $\mathcal{F}$ consisting of a nonsingular component $\mathcal{C}_{\text{nons}}$ and a component $\mathcal{C}_{\text{sing}}$ of multiplicity 2

We denote the tangent vectors of $\mathcal{C}_{\text{nons}}$ and $\mathcal{C}_{\text{sing}}$ by $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.

8.1. Case where $\omega(\mathbf{v}_1, \mathbf{v}_2) \neq 0$. From [8] it follows that up to RL-equivalence there exist two series satisfying this condition:

$$((t, 0, 0), (0, t^2, t^{2m+1})), \tag{8.1}$$

$$((t, 0), (t^{2m+1}, t^2)). \tag{8.2}$$

8.1.1. First, suppose that $\mathcal{F} \underset{\mathrm{RL}}{\sim} (8.1)$.

Lemma 15. *Suppose $\omega(\mathbf{v}_1, \mathbf{v}_2) \neq 0$, and, moreover, the restriction of ω to the tangent space of $\mathcal{C}_{\mathrm{sing}}$ is nonzero. Then*

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((0, t; t, 0), (t^2, 0; t^{2m+1}, 0)) \quad (p_{>2} = q_{>2} = 0).$$

Proof. We may assume that

$$\mathcal{C}_{\mathrm{sing}} = (q_1 = t^2; p_1 = t^{2m+1}),$$

and the 1-jet of $\mathcal{C}_{\mathrm{nons}}$ is of the form $(at, t; t, 0)$. We project $\mathcal{F}$ to the (q_1, p_1)-plane. From [8] it follows that in some coordinates (x, y) the projection is given by the equation $x(x^{2m+1} - y^2) = 0$. As before, the diffeomorphism of (x, y) generated by the vector field $\mathbf{v} = 2x\,\partial_x + (2m+1)y\,\partial_y$ brings the restriction of ω to the form $dx \wedge dy$. The induced symplectomorphism of $\mathbb{C}^{2n}$ yields

$$\mathcal{F} = ((q_1 = t^2, p_1 = t^{2m+1}, p_{>1} = q_{>1} = 0), (q_1 = 0, q_2 = t + f(t); p_1 = t, p_2 = g(t))).$$

We project $\mathcal{F}$ to the subspace $(q_2, \dots, q_n; p_2, \dots, p_n)$. By the Darboux–Givental theorem, a symplectic transformation (and, possibly, some right change) brings the projection to the form $(t, 0, \dots, 0)$. This transformation yields

$$\mathcal{F} = ((t^2, 0; t^{2m+1}, 0), (0, t; t + f(t), 0)),$$

where $f(t) = o(t)$. It is easy to check that the change (2.4) with $S(q_1, q_2) = q_1 f(q_2)$ brings $\mathcal{F}$ to the required form. □

Remark. We easily see that the above normal form is symplectically equivalent to

$$((t^2, t^{2m+1}; t^{2m+1}, 0), (0, 0; t, 0)).$$

Lemma 16. *Suppose that $\omega(\mathbf{v}_1, \mathbf{v}_2) \neq 0$, but the restriction of ω to the tangent space of $\mathcal{C}_{\mathrm{sing}}$ is zero. Then $\mathcal{F}$ is symplectically equivalent to one of the following forms*:

(1) $((t^2, t^{2m+1}; t^{2k+1}, 0), (0, 0; t, 0))$, $m < k < 3m$;
(2) $((t^2, t^{2m+1}; 0, 0), (0, 0; t, 0))$.

Remark. The list of singular components coincides with the list obtained by Arnol′d in [2].

Proof. We may assume that $\mathcal{C}_{\mathrm{nons}}$ has the standard form. Clearly, we can bring the $(2m+1)$-jet of $\mathcal{C}_{\mathrm{sing}}$ to the form

$$(q_1 = t^2, q_2 = t^{2m+1}, q_{>2} = p = 0),$$

and $\mathcal{C}_{\mathrm{nons}} = (p_1 = t, q = p_{>1} = 0)$. We project $\mathcal{F}$ to the q-subspace. A diffeomorphism Φ of the q-subspace brings the projection of $\mathcal{C}_{\mathrm{sing}}$ to the form $(t^2, t^{2m+1}, 0, \dots, 0)$, and the differential of Φ at the origin is the identity. Then Φ induces a symplectomorphism of $\mathbb{C}^n$ linear in p, and the differential at the origin is also the identity. This implies that $\mathcal{C}_{\mathrm{nons}}$ survives this transformation. Also, observe that along $\mathcal{C}_{\mathrm{sing}}$ we have $p_{i>2} = f_i(q_1, q_2)$ and $f_i(q_1, q_2) = o(1)$. Consequently, $\mathcal{F}$ lies on the 4-dimensional submanifold

$$\mathcal{M} := \{q_{>2} = 0, p_3 = f_3(q_1, q_2), \dots, p_n = f_n(q_1, q_2)\}.$$

Since the restriction of ω to $\mathcal{M}$ is of the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, the Darboux–Givental theorem implies that

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((t^2, t^{2m+1}; f(t), g(t)), (0, 0; t, 0)), \quad f, g = o(t^{2m+1}).$$

As was shown in [2], a change of the form (2.4) yields

$$\mathcal{C}_{\text{sing}} = \Big(t^2, t^{2m+1}; \sum_{r=1}^{2m} c_r t^{2(m+r)+1}, 0\Big).$$

It is obvious that $\mathcal{C}_{\text{nons}}$ survives this change. Let $2k+1$ be the minimum degree of a monomial occurring in the expansion of p_1 with a nonzero coefficient. We show how the terms of higher order can be "deleted". Suppose that we have already deleted all terms of degree less than $2(k+r)+1$. The $(2(k+r)+1)$-jet of $\mathcal{C}_{\text{sing}}$ has the form

$$(t^2, t^{2m+1}; t^{2k+1} + \alpha t^{2(k+r)+1}, 0).$$

We must construct a family of Hamiltonian vector fields (and also a family of vector fields on $(\mathbb{C}, 0)$) smoothly depending on α and such that these fields give the vector $(0, 0; t^{2(k+r)+1}, 0)$ under the action on the space of jets. At the points of the family considered, the image under the action on the space of jets of the vector field with the Hamiltonian $H = p_1 q_1^{r+1}$ has the form $(t^{2(r+1)}, 0; -(r+1)t^{2(k+r)+1}, 0)$. The vector

$$(2t^{2+2r}, (2m+1)t^{2m+2r+1}; (2k+1)t^{2(k+r)+1}, 0)$$

is the image of some vector in the tangent space at the identity of R. Finally, the Hamiltonian $H = q_1^r q_2 p_2$ yields $(0, t^{2m+2r+1}; 0, 0)$. We easily obtain the required vector field as a linear combination of the vector fields indicated above.

We also observe the following. First, $\mathcal{C}_{\text{nons}}$ survives the actions of the symplectomorphisms constructed. Second, if we consider jets of an arbitrarily high order, then we see that the corresponding transformations do not change q_1, q_2, and p_2.

After we have deleted the monomials up to the degree $6m+1$, we can delete the remaining "tail" by using the form (2.4). □

8.1.2. Finally, suppose that $\mathcal{F} \underset{\text{RL}}{\sim}$ (8.2).

Lemma 17. *Suppose that $\omega(\mathbf{v}_1, \mathbf{v}_2) \neq 0$. Then $\mathcal{F} \underset{\text{Sp}}{\sim} ((t^2; t^{2m+1}), (0; t))$.*

Proof. The proof is similar to that of Lemma 10. □

8.2. Case where $\omega(\mathbf{v}_1, \mathbf{v}_2) = 0$.

Lemma 18. *Suppose that*

$$\mathcal{F} \underset{\text{RL}}{\sim} ((t^2, t^3), (0, t))$$

and the restriction of ω to the tangent space of $\mathcal{F}$ at the origin is zero. Then $\mathcal{F}$ is not symplectically simple.

Proof. We assume that $\mathcal{C}_{\text{nons}}$ has the standard form. Then we bring the 3-jet of $\mathcal{F}$ to the form $((t^2, t^3; 0, 0), (0, t; 0, 0))$. We project $\mathcal{F}$ to the q-subspace. By [8], a diffeomorphism Φ of the q-subspace brings the projection to the form

$$((q_1 = t^2, q_2 = t^3, q_{>2} = 0), (q_2 = t, q_{\neq 2} = 0)).$$

Φ induces a symplectomorphism of $\mathbb{C}^n$ linear in p. Consequently, $p = 0$ along $\mathcal{C}_{\text{nons}}$, and $p = o(t^3)$ along $\mathcal{C}_{\text{sing}}$. Then we have $p_{>2} = f(q_1, q_2)$ along $\mathcal{C}_{\text{sing}}$, and $f(q_1, q_2) = 0$ along $\mathcal{C}_{\text{nons}}$. Thus, $\mathcal{F}$ lies on the 4-dimensional submanifold

$$\mathcal{M} := \{q_{>2} = 0, p_{>2} = f(q_1, q_2)\}.$$

The restriction of ω to $\mathcal{M}$ is of the form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Applying the Darboux–Givental theorem, we reduce the problem to the 4-dimensional case. Thus,

$$\mathcal{F} = ((t^2, t^3; f(t), g(t)), (0, t; 0, 0)) \quad f, g = o(t^3). \tag{8.3}$$

Applying a change of the form (2.4), we ensure that $f, g = o(t^4)$. (In order to delete the monomial in p_2 proportional to t^4, we take $S(q_1, q_2)$ proportional to $q_1^2 q_2$.) The 5-jet has the form

$$((t^2, t^3; at^5, bt^5), (0, t; 0, 0)).$$

There is no loss of generality in assuming that $a \neq 0$. After a right change of the form $t = kT$ and a linear symplectic change, we may assume that $a = 1$. Now we consider a one-parameter family of 5-jets of the form (8.3) with parameter b. In much the same way as in the proof of Lemma 8, we show that at none of the points in this family the tangent space of the orbit contains the tangent vector of the family, i.e., the vector $((0, 0; 0, t^5), (0, 0; 0, 0))$. This implies that the parameter b is a module. □

Lemma 18 implies that if $\omega(\mathbf{v}_1, \mathbf{v}_2) = 0$, then $\mathcal{C}_{\text{sing}} \underset{\text{RL}}{\sim} (t^2, t^3)$. Suppose that

$$\mathcal{F} \underset{\text{RL}}{\sim} ((t^2, t^3, 0), (0, 0, t)). \tag{8.4}$$

Lemma 19. *Suppose that $\omega(\mathbf{v}_1, \mathbf{v}_2) = 0$, while the restriction of ω to the tangent space of $\mathcal{C}_{\text{sing}}$ is nonzero. Then $\mathcal{F}$ is symplectically equivalent to one of the following two normal forms*:

$$1.\ ((t^2, 0; t^3, 0), (t, t; 0, 0)), \quad 2.\ ((t^2, 0; t^3, 0), (0, t; 0, 0)).$$

Remark. It is easily seen that the normal form 1 is symplectically equivalent to

$$((t^2, 0; t^3, t^3), (0, t; 0, 0)).$$

Proof. Suppose that $\mathcal{C}_{\text{sing}}$ has the standard form, i.e., $\mathcal{C}_{\text{sing}} = (q_1 = t^2; p_1 = t^3)$. There are two cases: the tangent space of $\mathcal{C}_{\text{sing}}$ either contains or does not contain a vector such that the value of ω at this vector and at the tangent vector of $\mathcal{C}_{\text{nons}}$ is nonzero. Depending on which of these cases occurs, we obtain

$$\mathcal{C}_{\text{nons}} = (\varepsilon t + f(t), t; g(t), 0) \quad f, g = o(t), \ \ \varepsilon \in \{0, 1\}.$$

The change (2.4) with $S(q_1, q_2) = g(q_2) q_1$ preserves $\mathcal{C}_{\text{sing}}$, and after this change we have $p_1 = 0$ along $\mathcal{C}_{\text{nons}}$. Possibly, something is added to the coordinate p_2 of $\mathcal{C}_{\text{nons}}$, but we easily compensate this by using the change (2.4) with $S = S(q_2)$. In order to bring q_1 to the required form, we proceed as follows. We transpose the coordinates q_1 and p_1 and change the sign of one of them. Obviously, this transformation is symplectic. After that, we use the same arguments as when bringing p_1 to the required form, then transpose p_1 and q_1 once again, and change the sign of the same coordinate as before. The change constructed preserves $\mathcal{C}_{\text{sing}}$ and brings $\mathcal{F}$ to the required form. □

Lemma 20. *Suppose that $\omega(\mathbf{v}_1, \mathbf{v}_2) = 0$ and that the restriction of ω to the tangent space of $\mathcal{C}_{\text{sing}}$ is zero. Then $\mathcal{F}$ is symplectically equivalent to one of the following normal forms*:

$$1.\ ((t^2, t^3; t^5, 0), (0, 0; 0, t)), \qquad 2.\ ((t^2, t^3; 0, 0), (0, 0; 0, t)),$$
$$3.\ ((t^2, t^3, 0; t^5, 0, 0), (0, 0, t; 0, 0, 0)), \qquad 4.\ ((t^2, t^3, 0; 0, 0, 0), (0, 0, t; 0, 0, 0)).$$

Proof. We assume that $\mathcal{C}_{\text{nons}}$ has the standard form. There are two cases: the tangent space of $\mathcal{C}_{\text{sing}}$ either contains or does not contain a vector such that the value of ω at this vector and the tangent vector of $\mathcal{C}_{\text{nons}}$ is nonzero. In these cases, we bring the 3-jet to one of the following two forms:

$$\begin{cases} ((t^2, t^3, 0, \dots, 0; 0, \dots, 0), (0, \dots, 0; 0, t, 0, \dots, 0)), \\ ((t^2, t^3, 0, \dots, 0; 0, \dots, 0), (0, 0, t, 0, \dots, 0; 0, \dots, 0)). \end{cases}$$

A diffeomorphism Φ of the q-subspace with identity 1-jet at the origin brings the first component to the form $(t^2, t^3, 0, \dots, 0)$ and does not affect the second component. Φ induces a symplectomorphism of $\mathbb{C}^n$ with identical 1-jet; consequently, Φ preserves the second component. Since any function $o(t^3)$ has the form $f(q_1, q_2)$, we can apply the Darboux–Givental theorem to reduce the problem to the 4- and the 6-dimensional case, respectively.

In the first case, we have

$$\mathcal{F} = ((t^2, t^3; f(t), g(t)), (0, 0; 0, t)), \quad f, g = o(t^3).$$

The further argument repeats the proof of Lemma 16 (it is easily seen that the vectors indicated there give a zero vector under the action on the nonsingular component), and we obtain the normal forms 1 and 2.

In the second case, we have

$$\mathcal{F} = ((t^2, t^3, 0; f(t), g(t), h(t)), (0, 0, t; 0, 0, 0)), \quad f, g, h = o(t^3).$$

There is a function $\phi = \phi(q_1, q_2)$ such that along $\mathcal{C}_{\text{sing}}$ we have $h(t) = \phi(q_1, q_2)$. The change (2.4) with $S = q_3\phi(q_1, q_2)$ does not affect $\mathcal{C}_{\text{nons}}$, and after this change we obtain $p_3 = 0$ along $\mathcal{C}_{\text{sing}}$. The projection of $\mathcal{F}$ to the subspace $(q_1, q_2; p_1, p_2)$ has the form $(t^2, t^3; o(t^3), o(t^3))$. To this projection we apply the argument of Arnol′d [2]; this leads to the normal forms 3 and 4. (We could also apply the argument used in the proof of Lemma 16.) □

Now it remains to consider the case where the tangent vectors of the components of $\mathcal{F}$ are collinear. From [8] it follows that up to RL-equivalence there are two multigerms:

$$((t^2, t^3, t^4), (t, 0, 0)), \tag{8.5}$$

$$((t^2, t^3), (t, 0)). \tag{8.6}$$

Certainly, the restriction of ω to the tangent space of $\mathcal{C}_{\text{sing}}$ is nonzero by Lemma 18.

Lemma 21. *Suppose that the restriction of ω to the tangent spaces of the singular components of the multigerms* (8.5) *and* (8.6) *is nonzero. Then these multigerms are symplectically equivalent, respectively, to*

$$1.\ ((t^2, t^4; t^3, 0), (t, 0; 0, 0)); \quad 2.\ ((t^2; t^3), (t; 0)).$$

Proof. First, we consider (8.6). As we have already done several times, we consider a 2-dimensional submanifold $\mathcal{M}$ containing $\mathcal{F}$. In some coordinates (x, y) on $\mathcal{M}$, $\mathcal{F}$ is given by the equation $(x^3 - y^2)y = 0$. Further, we apply the homotopy method and the Darboux–Givental theorem, using the fact that the vector field $\mathbf{v} = 2x\,\partial_x + 3y\,\partial_y$ is tangent to $\mathcal{F}$.

Now we turn to (8.5). Assume that $\mathcal{C}_{\text{nons}}$ has the standard form. Clearly, we can bring the 4-jet to the form

$$((t^2, 0; t^3, t^4), (t, 0; 0, 0)), \quad q_{>2} = p_{>2} = 0.$$

We project $\mathcal{F}$ to the subspace $(q_1, q_3, \dots, q_n; p_1, p_3, \dots, p_n)$. The projection is RL-equivalent to (8.6), so that we can bring it to the normal form. Thus, we obtain

$$\mathcal{F} = ((t^2, f(t); t^3, t^4 + g(t)), (t, 0; 0, 0)), \quad f, g = o(t^4).$$

Let $f(t) = at^5 + o(t^5)$. We get rid of at^5. For this, in the space of 5-jets we consider the corresponding 1-dimensional submanifold with coordinate a on it. In the tangent space at the identity of the group of 5-jets of symplectomorphisms, we consider the vector with the Hamiltonian $H = q_1p_1p_2$. The corresponding vector has the form $(q_1p_2, q_1p_1; -p_1p_2, 0)$. Obviously, the image of this vector under the action on our submanifold has the form

$(0, t^5; 0, 0)$. Integrating the corresponding right-invariant vector field, we obtain the required change, and $\mathcal{C}_{\text{nons}}$ survives this change. Now we can find a function $\phi = \phi(p_1, p_2)$ such that $q_2 = \phi(p_1, p_2)$ along $\mathcal{C}_{\text{sing}}$. We define $S(p_1, p_2) = \int_0^{p_2} \phi(p_1, x)dx$. Then the change $Q = q - \partial_p S$ yields

$$\mathcal{C}_{\text{sing}} = (t^2 + o(t^2), 0; t^3 + o(t^3), t^4 + o(t^4)).$$

Applying the arguments used when analyzing (8.6), we obtain

$$\mathcal{F} = ((t^2, 0; t^3, t^4 + h(t)), (t, 0; 0, 0)), \quad h(t) = o(t^4).$$

Clearly, $h(t) = \psi(q_1, p_1)$, and $\psi(q_1, 0) = 0$. Integrating the vector field with the Hamiltonian $H = -\psi(q_1, p_1)q_2$, we "delete the tail" in the coordinate p_2 of $\mathcal{C}_{\text{sing}}$. Transposing q_2 and p_2 and changing the sign of the latter coordinate, we obtain the required normal form 1. □

§9. About other two-component multigerms

9.1. Here, we show that there are no other simple multigerms with two components.

Lemma 22. *A multigerm $\mathcal{F}$ consisting of a nonsingular component $\mathcal{C}_{\text{nons}}$ and a component $\mathcal{C}_{\text{sing}}$ of multiplicity 3 is not simple.*

Proof. We assume that $\mathcal{C}_{\text{nons}}$ has the standard form. The most general situation is as follows: $\omega(\mathbf{v}_1, \mathbf{v}_2) \neq 0$. Consequently, we may assume that the 3-jet of $\mathcal{F}$ has the form

$$((t, 0, \dots, 0; 0, \dots, 0), (0, \dots, 0; t^3, 0, \dots, 0)).$$

We consider the 4-jet of $\mathcal{F}$. In the general case, the restriction of ω to the tangent space of this 4-jet is nonzero, and at the same time, from the viewpoint of RL-equivalence the situation $((t, 0, 0, 0), (0, t^3, t^4, t^5))$ is the most general. Thus, we may assume that the 4-jet has the form

$$((t, 0, \dots, 0), (t^4, t^4, 0, \dots, 0; t^3, 0, \dots, 0)).$$

In the general case, the 5-jet of $\mathcal{F}$ is equivalent to

$$((t, 0; 0, 0), (t^4 + at^5, t^4; t^3, bt^5)), \quad q_{>2} = p_{>2} = 0.$$

We fix an arbitrary $a_0 \neq 0$ and show that a_0 has a neighborhood U such that the 5-jet at a_0 is not equivalent to the 5-jet in the same family at a if $a \in U \setminus \{a_0\}$. This will imply that a is a module.

Assume the contrary, i.e., there is a number a arbitrarily close to a_0 and such that the corresponding 5-jets are symplectically equivalent. We consider a symplectomorphism and right changes realizing this equivalence. The linear part A of this symplectomorphism at the origin is also a symplectomorphism and also brings the 5-jet to the required form. It follows that A takes the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ to vectors proportional to them. Consequently, the subspace spanned by the vectors $(1, 0; 0, 0)$ and $(0, 0; 1, 0)$ is invariant with respect to A. Therefore, its skew-orthogonal complement is also invariant with respect to A. This implies that, in the coordinates, A has the form $Q_1 = \alpha q_1$, $P_1 = p_1/\alpha$. (We are not interested in the form of the other coordinates.) We still have a right transformation of $\mathcal{C}_{\text{sing}}$. Its 2-jet is of the form $t = kT$, because otherwise there arises a monomial in p_1 proportional to t^4. Then we have conditions on α and k:

$$\alpha k^4 = \frac{k^3}{\alpha} = 1, \quad a\alpha k^5 = a_0.$$

The first two relations imply that $\alpha = k^3$ and $k^7{=}1$, whence $a_0/a = k^8 = k$. Consequently, if a is sufficiently close to a_0, then $k = 1$ and $a = a_0$. A contradiction. □

9.2. Now suppose that our multigerm $\mathcal{F}$ consists of two singular components.

Lemma 23. *A multigerm $\mathcal{F}$ consisting of two components of multiplicity 2 is not simple.*

Proof. Without loss of generality, we may assume that $\omega(\mathbf{v}_1, \mathbf{v}_2) \neq 0$. Then we bring the 2-jet of $\mathcal{F}$ to the form

$$((t_1^2, 0, \dots, 0; 0, \dots, 0), (0, \dots, 0; t_2^2, 0, \dots, 0)).$$

In the general case, the restrictions of ω to the tangent spaces of the components of $\mathcal{F}$ are nonzero, which means that in the general case the projection of the 3-jet to the (q_1, p_1)-plane has the form $((t_1^2; t_1^3), (at_2^3, t_2^2))$. After that, we argue as in the proof of Lemma 22. □

§10. Multigerms consisting of three nonsingular components

10.1. Suppose that

$$\mathcal{F} = (\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) \underset{\mathrm{RL}}{\sim} ((t, 0, 0), (0, t, 0), (0, 0, t)). \tag{10.1}$$

Lemma 24. *Suppose that the value of ω at the tangent vectors of some two components is nonzero. Then $\mathcal{F}$ is symplectically equivalent to a multigerm on the following list*:

1. $((t, 0; 0, 0), (0, 0; t, 0), (t, t; t, 0))$,
2. $((t, 0; 0, 0), (0, 0; t, 0), (t, t; 0, 0))$,
3. $((t, 0; 0, 0), (0, 0; t, 0), (0, t; 0, 0))$.

Proof. There is no loss of generality in assuming that the components mentioned in the statement are $\mathcal{C}_1$ and $\mathcal{C}_2$. Then a symplectic transformation brings them to the form

$$\mathcal{C}_1 = (q_1 = t, q_{>1} = p = 0), \quad \mathcal{C}_2 = (p_1 = t, q = p_{>1} = 0).$$

We project $\mathcal{F}$ to the subspace $(q_2, \dots, q_n; p_2, \dots, p_n)$. The components $\mathcal{C}_1$ and $\mathcal{C}_2$ are projected to zero, and the projection of $\mathcal{C}_3$ is RL-equivalent to $(t, 0, \dots, 0)$. The Darboux–Givental theorem yields $\mathcal{C}_3 = (q_2 = t, q_{>2} = p_{>1} = 0)$.

Let $\mathbf{v}_i$ denote the tangent vector of $\mathcal{C}_i$ at the origin. Three cases are possible:

1. $\omega(\mathbf{v}_1, \mathbf{v}_3) \neq 0$ and $\omega(\mathbf{v}_2, \mathbf{v}_3) \neq 0$;
2. $\omega(\mathbf{v}_1, \mathbf{v}_3) = 0$ and $\omega(\mathbf{v}_2, \mathbf{v}_3) \neq 0$;
3. $\omega(\mathbf{v}_1, \mathbf{v}_3) = \omega(\mathbf{v}_2, \mathbf{v}_3) = 0$.

We do not consider case 1 because after renumbering the components it reduces to case 2. The above implies that we can obtain

$$\mathcal{F} = ((t, 0; 0, 0), (0, 0; t, 0), (\varepsilon_1 t + f(t), t; \varepsilon_2 t + g(t), 0)), \quad f, g = o(t), \ \ \varepsilon_{1,2} \in \{0, 1\}.$$

Here, case 1 corresponds to $\varepsilon_1 = \varepsilon_2 = 1$, case 2 corresponds to $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$, and case 3 corresponds to $\varepsilon_1 = \varepsilon_2 = 0$. After the change (2.4) with $S(q_1, q_2) = q_1 g(q_2)$, the coordinate p_1 of $\mathcal{C}_3$ will be equal to $\varepsilon_2 t$, and the coordinate p_2 will change somehow, but we can compensate this by the change (2.4) with $S = S(q_2)$. In order to bring q_1 to the required form, we transpose the coordinates q_1 and p_1 and change the sign of one of them. After that, we apply the arguments that were used for bringing p_1 to the required form, and then transpose the same coordinates once again and change the sign of one of them. □

Lemma 25. *Suppose that the value of ω at any pair of the tangent vectors of the components of the multigerm* (10.1) *is zero. Then*

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((t, 0, 0; 0, 0, 0), (0, t, 0; 0, 0, 0), (0, 0, t; 0, 0, 0)).$$

Proof. Lemma 11 allows us to assume that $\mathcal{C}_1 = (t, 0; 0, 0)$ and $\mathcal{C}_2 = (0, t; 0, 0)$. Then we bring the 1-jet of $\mathcal{C}_3$ to the form

$$(q_3 = t, q_{\neq 3} = p = 0).$$

We project $\mathcal{F}$ to the subspace $(q_3, \dots, q_n; p_3, \dots, p_n)$. The components $\mathcal{C}_1$ and $\mathcal{C}_2$ are projected to zero, and $\mathcal{C}_3$ is projected to a regular curve. By the Darboux–Givental theorem, a symplectic transformation brings this projection to the form $(q_3 = t, q_{>3} = p_{>2} = 0)$. Thus, the problem is reduced to the 6-dimensional case. Now we project $\mathcal{F}$ to the 3-dimensional subspace (q_1, q_2, q_3). A diffeomorphism Φ of this subspace brings the projection to the form (10.1). Φ induces a symplectomorphism of $\mathbb{C}^n$ linear in p. Thus, we have $\mathcal{C}_3 = (0, 0, t; f(t), g(t), h(t))$, where $f, g, h = o(t)$. We apply the change (2.4) with

$$S(q_1, q_2, q_3) = q_1 f(q_3) + q_2 g(q_3) + \int_0^{q_3} h(x)dx.$$

It is easily seen that this change preserves $\mathcal{C}_1$ and $\mathcal{C}_2$ and brings $\mathcal{C}_3$ to the required form. □

10.2. Now suppose that

$$\mathcal{F} \underset{\mathrm{RL}}{\sim} ((t, 0), (0, t), (t, t)). \tag{10.2}$$

Lemma 26. *Suppose that the restriction of ω to the tangent space of the multigerm* (10.2) *is nonzero. Then*

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((q_1 = t, q_{>1} = p = 0), (p_1 = t, q = p_{>1} = 0), (q_1 = p_1 = t, q_{>2} = p_{>2} = 0)).$$

Proof. The proof is similar to that of Lemma 10. □

Lemma 27. *If the restriction of ω to the tangent space of the multigerm* (10.2) *is zero, then $\mathcal{F}$ is not symplectically simple.*

Proof. In the general case, we easily bring the 2-jet of $\mathcal{F}$ to the form

$$((t, 0; 0, 0), (0, t; 0, 0), (t, t; at^2, bt^2)).$$

We assume that $a \neq 0$. Then, performing simple linear transformations, we can put $a = 1$. Consider a 1-parameter family of 2-jets of multigerms (b is a parameter in this family). We easily see that if $b \neq 0$, then the tangent space of the orbit of $\mathcal{F}$ does not contain the tangent vector of the family, i.e., the vector $((0, 0; 0, 0), (0, 0; 0, 0), (0, 0; 0, t^2))$. This implies that $\mathcal{F}$ is not simple. □

10.3. Now suppose that $\mathcal{F}$ is RL-equivalent to one of the following two multigerms:

$$((t, 0, 0), (0, t, 0), (t, 0, t^a)), \tag{10.3}$$

$$((t, 0), (0, t), (t, t^a)). \tag{10.4}$$

Lemma 27 implies that the restriction of ω to the tangent space of the first two components is nonzero. (Otherwise, $\mathcal{F}$ adjoins the multigerm of Lemma 27 and, consequently, is not simple.)

Lemma 28. *Suppose that the restriction of ω to the tangent space of the multigerm* (10.4) *is nonzero. Then* $\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((t; 0), (0; t), (t; t^a))$.

Proof. The proof is similar to that of Lemma 10. □

Lemma 29. *Suppose that the restriction of ω to the tangent space of the first two components of the multigerm* (10.3) *is nonzero, and, furthermore, that the restriction of ω to the tangent space of the first and third components of $\mathcal{F}$ is nonzero. Then*

$$\mathcal{F} \underset{\mathrm{Sp}}{\sim} ((t,0;0,0),(0,t;t,0),(t,0;t^a,0)).$$

Remark. It is easy to check that the given normal form is equivalent to

$$((t,0;0,0),(0,0;t,0),(t,t^a;t^a,0)).$$

Proof. By Lemma 12, we can simultaneously bring the first and the third component to the form

$$\mathcal{C}_1 = (q_1 = t, q_{>1} = p = 0), \quad \mathcal{C}_3 = (q_1 = t, p_1 = t^a, q_{>1} = p_{>1} = 0).$$

We project our multigerm (10.3) to the (q_1, p_1)-plane. The projection is RL-equivalent to (10.4), because $\omega(\mathbf{v}_1, \mathbf{v}_2) \neq 0$ by assumption. Therefore, by Lemma 28, a symplectic transformation brings this projection to the form $((t;0),(0;t),(t;t^a))$. Now we project $\mathcal{F}$ to the subspace $(q_2,\dots,q_n;p_2,\dots,p_n)$. The projections of $\mathcal{C}_1$ and $\mathcal{C}_3$ are zero, while the projection of $\mathcal{C}_2$ is a regular curve. (Otherwise, as it follows from [8], we would have $\mathcal{F} \underset{\mathrm{RL}}{\sim}$ (10.4).) Consequently, we can use the Darboux–Givental theorem to bring the projection of $\mathcal{C}_3$ to the form $(q_2 = t, q_{>2} = p = 0)$. All together, this means that a symplectic transformation and a right change yield

$$\mathcal{F} = ((t,0;0,0),(0,t;t+f(t),0),(t,0;t^a,0)), \quad f(t) = o(t).$$

Finally, the change (2.4) with $S(q_1,q_2) = f(q_2)q_1$ obviously does not affect $\mathcal{C}_1$ and $\mathcal{C}_3$ and brings $\mathcal{C}_2$ to the required form. □

Now suppose that the restriction of ω to the tangent space of the first and third components of the multigerm (10.3) is zero.

Lemma 30. *Suppose that the restriction of ω to the tangent space of the first two components of the multigerm* (10.3) *is nonzero, and, furthermore, that the restriction of ω to the tangent space of $\mathcal{C}_1$ and $\mathcal{C}_3$ is zero. Then $\mathcal{F}$ is symplectically equivalent to a multigerm on the following list*:

(1) $((q_1 = t, q_{>1} = p = 0), (p_1 = t, p_{>1} = q = 0), (q_1 = t, q_2 = t^a, q_{>2} = p = 0))$;
(2) $((q_1 = t, q_{>1} = p = 0), (p_1 = t, p_{>1} = q = 0), (q_1 = t, q_2 = t^a, p_1 = t^b, q_{>2} = p_{>1} = 0))$, *and* $a < b < 3a - 1$.

Proof. By Lemma 10, we may assume that

$$\mathcal{C}_1 = (q_1 = t, q_{>2} = p = 0), \quad \mathcal{C}_2 = (p_1 = t, p_{>2} = q = 0).$$

Then, obviously, we can bring the a-jet of $\mathcal{C}_3$ to the form $(q_1 = t, q_2 = t^a, q_{>2} = p = 0)$. We project $\mathcal{F}$ to the q-subspace along the p-subspace. A diffeomorphism Φ of the q-subspace brings the projections of $\mathcal{C}_1$ and $\mathcal{C}_3$ to the form $((q_1 = t, q_{>1} = 0),\ (q_1 = t, q_2 = t^a, q_{>2} = 0))$, and the 1-jet of Φ at the origin is the identity transformation. Then Φ induces a symplectomorphism of $\mathbb{C}^n$, also with the identity 1-jet at the origin. Consequently, the symplectomorphism constructed preserves $\mathcal{C}_2$. The further argument repeats the proof of Lemma 13. (The changes used there are easily seen to preserve $\mathcal{C}_2$.) □

§11. About other multigerms with at least three components

Lemma 31. *If a multigerm $\mathcal{F}$ consists of two nonsingular components and one singular component, then $\mathcal{F}$ is not symplectically simple.*

Proof. From the viewpoint of RL-equivalence, in this situation the multigerm

$$((t,0,0,0),(0,t,0,0),(0,0,t^2,t^3))$$

is generic. In the generic case, the tangent vectors of the nonsingular components at the origin are not skew-orthogonal, and the tangent vector of the singular component also is not skew-orthogonal to the tangent vectors of the nonsingular components. This means that we can bring the nonsingular components to the form $((t,0;0,0),(0,0;t,0))$, and we can bring the 2-jet of the singular component to the form $(t^2,t^2;t^2,0)$. In the generic case, the 3-jet of the singular component is equivalent to $(q_1=t^2, p_1=t^2+at^3)$. (We are not interested in the form of the other coordinates.) As in the proof of Lemma 23, we show that the parameter a is a module. □

Lemma 32. *If a multigerm $\mathcal{F}$ contains at least four nonsingular components, then $\mathcal{F}$ is not symplectically simple.*

Proof. In the generic case, we may assume that the first two components have the form

$$((q_1=t, q_{>1}=p=0),(p_1=t, p_{>1}=q=0)).$$

Again in the generic case, the tangent vectors of the other components are not skew-orthogonal to the tangent vectors of the first two components. This implies that we can bring the 1-jet of $\mathcal{F}$ to the following form (we indicate only q_1 and p_1 because we are not interested in the form of the other coordinates):

$$((q_1=t;p_1=0),(q_1=0;p_1=t),(q_1=t_1;p_1=t_1),(q_1=t_2;p_1=at_2)).$$

Now it remains to observe that, in the plane, four lines through the origin have a continuous invariant: the cross-ratio. □

If $\mathcal{F} \underset{\mathrm{RL}}{\sim} ((t,0,0),(t,t^2,0),(t,0,t^2))$, then $\mathcal{F}$ is not symplectically simple because $\mathcal{F}$ adjoins the multigerm from Lemma 27. This (and also [8]) implies that there exist no other symplectically simple multigerms.

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