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BACKWARD UNIQUENESS FOR THE HEAT OPERATOR IN A HALF-SPACE

L. ESCAURIAZA, G. SEREGIN, AND V. ŠVERÁK

Abstract. A backward uniqueness result is proved for the heat operator with variable lower order terms in a half-space. The main point of the result is that the boundary conditions are not controlled by the assumptions.

§1. Introduction

In this paper, which can be thought of as a continuation of [3] and [4], we deal with the following backward uniqueness problem for the heat operator. Let $\mathbb{R}^n_+ = \{x = (x_i) \in \mathbb{R}^n : x_n > 0\}$, and let $Q_+ = \mathbb{R}^n_+ \times]0,1[$. We consider a vector-valued function $u : Q_+ \to \mathbb{R}^n$, assuming that it is "sufficiently regular" and satisfies the conditions

$$|\partial_t u + \Delta u| \le c_1(|\nabla u| + |u|) \quad \text{in } Q_+ \tag{1.1}$$

for some $c_1 > 0$, and

$$u(\cdot, 0) = 0 \quad \text{in } \mathbb{R}^n_+. \tag{1.2}$$

Do (1.1) and (1.2) imply $u \equiv 0$ in Q_+? We prove that the answer is in the positive if we impose natural restrictions on the growth of u at infinity. For instance, we may assume that

$$|u(x,t)| \le e^{M|x|^2} \tag{1.3}$$

for all $(x,t) \in Q_+$ and for some $M > 0$. The natural regularity assumptions under which (1.1)–(1.3) can be considered are, e.g., as follows:

$$\begin{aligned} &u \text{ and the distributional derivatives } \partial_t u, \nabla^2 u \text{ are square} \\ &\text{integrable over the bounded subdomains of } Q_+. \end{aligned} \tag{1.4}$$

We can formulate our main result.

Theorem 1.1. *In the notation introduced above, assume that u satisfies conditions* (1.1)–(1.4). *Then $u \equiv 0$ in Q_+.*

This extends the main result of [3] and [4], where an analog of Theorem 1.1 was proved for $Q_+ = (\mathbb{R}^n \setminus B(R)) \times]0,T[$. Here, as usual, $B(R)$ denotes the n-dimensional ball of radius R with center at the origin.

Such results are of interest in control theory; see, e.g., [8]. Also, as was explained in [10], results of this type are helpful in regularity theory for the Navier–Stokes equations. By combining the methods developed in [10] and the boundary regularity results proved in [9], it can be shown that the so-called Leray–Hopf solutions to the Navier–Stokes equations in a regular domain $\Omega \in \mathbb{R}^3$ that satisfy the homogeneous Dirichlet boundary

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condition and belong to the space $L_\infty(0,1;L_3(\Omega;\mathbb{R}^3))$, are regular up to the boundary of Ω. A detailed proof of this result will be presented elsewhere.

As in [3] and [4], the proof of Theorem 1.1 is based on two Carleman-type inequalities. The first is essentially the same as the one used in [3] and [4] (see also [1], [5], and [11]); it is of the form

$$\begin{aligned}\int_{\mathbb{R}^n\times]0,2[} & h^{-2a}(t)e^{-\frac{|x|^2}{4t}}(a\frac{|w|^2}{t}+|\nabla w|^2)\,dxdt \\ & \le c_2\int_{\mathbb{R}^n\times]0,2[} h^{-2a}(t)e^{-\frac{|x|^2}{4t}}|\partial_t w+\Delta w|^2\,dxdt.\end{aligned} \tag{1.5}$$

Here $h(t)=te^{\frac{1-t}{3}}$, and c_2 is a positive universal constant. Inequality (1.5) is fulfilled for any $w\in C_0^\infty(\mathbb{R}^n\times]0,2[)$ and any positive number a.

The second inequality is anisotropic in a sense:

$$\begin{aligned}\int_{(\mathbb{R}^n_+ + e_n)\times]0,1[} & t^2e^{2\phi(x,t)}\Big(a\frac{|w|^2}{t^2}+\frac{|\nabla w|^2}{t}\Big)\,dxdt \\ & \le c_3\int_{(\mathbb{R}^n_+ + e_n)\times]0,1[} t^2e^{2\phi(x,t)}|\partial_t w+\Delta w|^2\,dxdt.\end{aligned} \tag{1.6}$$

Here $\phi=\phi^{(1)}+\phi^{(2)}$, $\phi^{(1)}(x,t)=-\frac{|x'|^2}{8t}$, $\phi^{(2)}(x,t)=a(1-t)\frac{x_n^{2\alpha}}{t^\alpha}$, $\alpha\in]1/2,1[$ is fixed, $x'=(x_1,x_2,\dots,x_{n-1})$ (so that $x=(x',x_n)$), $c_3(\alpha)>0$, and $e_n=(0,0,\dots,0,1)$. Inequality (1.6) is fulfilled for any $w\in C_0^\infty((\mathbb{R}^n_+ + e_n)\times]0,1[)$ and any $a>a_0(\alpha)$.

Our concluding remark is that Theorem 1.1 is true for functions $u:Q_+\to\mathbb{R}^m$ with $1\le m<+\infty$. This is an easy exercise, which is left to the reader.

Our paper is organized as follows. In §2 we prove Theorem 1.1, taking Carleman's inequalities for granted. The proofs of the latter are given in §§3 and 4, respectively.

§2. Proof of Theorem 1.1

In what follows, we always assume that the function u is extended by zero to the negative values of t.

We start with several lemmas. The first of them plays the crucial role in our approach. It enables us to apply the powerful technique of Carleman's inequalities.

Lemma 2.1. *Suppose that conditions* (1.1), (1.2), *and* (1.4) *are fulfilled. There exists an absolute positive constant $A_0<1/32$ with the following properties. If*

$$|u(x,t)|\le e^{A|x|^2} \tag{2.1}$$

for all $(x,t)\in Q_+$ and some $A\in[0,A_0]$, then there are constants $\beta(A)>0$, $\gamma(c_1)\in]0,1/12[$, and $c_4(c_1,A)>0$ such that

$$|u(x,t)|\le c_4e^{4A|x'|^2}e^{-\beta\frac{x_n^2}{t}} \tag{2.2}$$

for all $(x,t)\in(\mathbb{R}^n_+ + 2e_n)\times]0,\gamma[$.

Proof. Referring to the regularity theory for solutions of parabolic equations (see [7]), we may assume that

$$|u(x,t)|+|\nabla u(x,t)|\le c_5e^{2A|x|^2} \tag{2.3}$$

for all $(x,t)\in(\mathbb{R}^n_+ + e_n)\times]0,1/2[$.

Fixing $x_n>2$ and $t\in]0,\gamma[$, we define a new function v by the usual parabolic scaling

$$v(y,s)=u(x+\lambda y,\lambda^2 s-t/2).$$

The function v is well defined on the set $Q_\rho = B(\rho)\times]0,2[$, where $\rho = (x_n - 1)/\lambda$ and $\lambda = \sqrt{3t} \in]0, 1/2[$. Then relations (1.1), (1.2), and (2.3) take the following form:

$$|\partial_s v + \Delta v| \le c_1\lambda(|\nabla v| + |v|) \quad \text{a.e. in } Q_\rho; \tag{2.4}$$

$$|v(y,s)| + |\nabla v(y,s)| \le c_5 e^{4A|x|^2} e^{4A\lambda^2|y|^2} \tag{2.5}$$

for $(y,s) \in Q_\rho$;

$$v(y,s) = 0 \tag{2.6}$$

for $y \in B(\rho)$ and $s \in]0, 1/6]$.

In order to apply inequality (1.5), we choose two smooth cut-off functions such that

$$\phi_\rho(y) = \begin{cases} 0 & \text{if } |y| > \rho - 1/2, \\ 1 & \text{if } |y| < \rho - 1, \end{cases}$$

and

$$\phi_t(s) = \begin{cases} 0 & \text{if } 7/4 < s < 2, \\ 1 & \text{if } 0 < s < 3/2. \end{cases}$$

All functions take values in $[0,1]$. Moreover, the function ϕ_ρ satisfies $|\nabla^k \phi_\rho| < C_k$, $k = 1, 2$. We put $\eta(y,s) = \phi_\rho(y)\phi_t(s)$ and $w = \eta v$. From (2.4) it follows that

$$|\partial_s w + \Delta w| \le c_1\lambda(|\nabla w| + |w|) + \chi c_6(|\nabla v| + |v|). \tag{2.7}$$

Here c_6 is a positive constant depending on c_1 and C_k only, $\chi(y,s) = 1$ if $(y,s) \in \omega = \{\rho - 1 < |y| < \rho\}\times]0,2[\cup\{|y| \le \rho - 1\}\times]3/2, 2[$, and $\chi(y,s) = 0$ if $(y,s) \notin \omega$. Obviously, the function w has a compact support in $\mathbb{R}^n\times]0,2[$, and we may use inequality (1.5). As a result, we have

$$I \equiv \int_{Q_\rho} h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|w|^2 + |\nabla w|^2)\, dyds \le c_2 10(c_1^2\lambda^2 I + c_6^2 I_1), \tag{2.8}$$

where

$$I_1 = \int_{Q_\rho} \chi(y,s) h^{-2a}(s) e^{-\frac{|y|^2}{4s}} (|v|^2 + |\nabla v|^2)\, dyds.$$

Choosing $\gamma = \gamma(c_1)$ sufficiently small, we may assume that $c_2 10 c_1^2\lambda^2 \le 1/2$, and then (2.8) implies

$$I \le c_7(c_1) I_1. \tag{2.9}$$

On the other hand, if $A < 1/32$, then

$$8A\lambda^2 - \frac{1}{4s} < -\frac{1}{8s} \tag{2.10}$$

for $s \in]0, 2]$. By (2.5) and (2.10), we have

$$\begin{aligned} I_1 &\le c_5^2 e^{8A|x|^2} \int_0^2 \int_{B(\rho)} \chi(y,s) h^{-2a}(s) e^{-\frac{|y|^2}{8s}}\, dyds \\ &\le c_8 e^{8A|x|^2} \Big[h^{-2a}(3/2) + \int_0^2 h^{-2a}(s) e^{-\frac{(\rho-1)^2}{8s}}\, ds \Big]. \end{aligned} \tag{2.11}$$

Now, taking (2.11) into account, we deduce the estimate

$$\begin{aligned} D &\equiv \int_{B(1)}\int_{\frac12}^1 |w|^2\,dyds = \int_{B(1)}\int_{\frac12}^1 |v|^2\,dyds \\ &\le c_9\int_{Q_\rho} h^{-2a}(s)e^{-\frac{|y|^2}{4s}}(|w|^2+|\nabla w|^2)\,dyds \\ &\le c_{10}(c_1)e^{8A|x|^2}\Big[h^{-2a}(3/2)+\int_0^2 h^{-2a}(s)e^{-\frac{\rho^2}{32s}}\,ds\Big] \\ &= c_{10}e^{8A|x|^2-2\beta\rho^2}\Big[h^{-2a}(3/2)e^{2\beta\rho^2}+\int_0^2 h^{-2a}(s)e^{2\beta\rho^2-\frac{\rho^2}{32s}}\,ds\Big]. \end{aligned}$$

We can take $\beta = 8A < 1/256$ and then choose

$$a = \beta\rho^2/\ln h(3/2).$$

Since $\rho \ge x_n$, this leads to the estimate

$$D \le c_{10}e^{8A|x'|^2}e^{-\beta\rho^2}\Big[1+\int_0^2 g(s)\,ds\Big],$$

where $g(s) = h^{-2a}(s)e^{-\frac{\rho^2}{64s}}$. It is easy to check that $g'(s) \ge 0$ for $s \in]0,2[$ if $\beta < \frac{1}{96}\ln h(3/2)$. Thus, we obtain

$$D \le 2c_{10}e^{8A|x'|^2}e^{-\beta\rho^2} \le 2c_{10}e^{8A|x'|^2}e^{-\frac{\beta x_n^2}{12t}}. \tag{2.12}$$

On the other hand, regularity theory yields

$$|v(0,1/2)|^2 = |u(x,t)|^2 \le c_{10}'D. \tag{2.13}$$

Combining (2.12) and (2.13), we complete the proof of Lemma 2.1. □

The next lemma will be deduced from a consequence of Lemma 2.1 and the second Carleman inequality (see (1.6)).

Lemma 2.2. *Suppose that u satisfies conditions* (1.1), (1.2), (1.4), *and* (2.1). *There exists a number $\gamma_1(c_1,c_3) \in]0,\gamma/2]$ such that*

$$u(x,t) = 0 \tag{2.14}$$

for all $x \in \mathbb{R}^n_+$ and all $t \in]0,\gamma_1[$.

Proof. As usual, Lemma 2.1 and regularity theory allow us to assume that

$$|u(x,t)| + |\nabla u(x,t)| \le c_{11}(c_1,A)e^{8A|x'|^2}e^{-\beta\frac{x_n^2}{2t}} \tag{2.15}$$

for all $x \in \mathbb{R}^n_+ + 3e_n$ and all $t \in]0,\gamma/2]$.

Using scaling, we define $v(y,s) = u(\lambda y, \lambda^2 s - \gamma_1)$ for $(y,s) \in Q_+$; here $\lambda = \sqrt{2\gamma_1}$. This function satisfies the following conditions:

$$|\partial_s v + \Delta v| \le c_1\lambda(|\nabla v| + |v|) \quad \text{a.e. in } Q_+; \tag{2.16}$$

$$v(y,s) = 0 \tag{2.17}$$

for all $y \in \mathbb{R}^n_+$ and all $s \in]0,1/2[$;

$$|\nabla v(y,s)| + |v(y,s)| \le c_{11}e^{8A\lambda^2|y'|^2}e^{-\frac{\beta\lambda^2 y_n^2}{2(\lambda^2 s-\gamma_1)}} \le c_{11}e^{8A\lambda^2|y'|^2}e^{-\beta\frac{y_n^2}{2s}} \tag{2.18}$$

for all $1/2 < s < 1$ and all $y \in \mathbb{R}^n_+ + \frac{3}{\lambda}e_n$. Since $A < 1/32$ and $\lambda \le \sqrt{\gamma} \le 1/\sqrt{12}$, inequality (2.18) can be reduced to the form

$$|\nabla v(y,s)| + |v(y,s)| \le c_{11}e^{\frac{|y'|^2}{48}}e^{-\beta\frac{y_n^2}{2s}} \tag{2.19}$$

with the same y and s as in (2.18).

We fix two smooth cut-off functions such that

$$\psi_1(y_n) = \begin{cases} 0 & \text{if } y_n < \frac{3}{\lambda} + 1, \\ 1 & \text{if } y_n > \frac{3}{\lambda} + 2, \end{cases}$$

and

$$\psi_2(r) = \begin{cases} 0 & \text{if } r > -1/2, \\ 1 & \text{if } r < -3/4. \end{cases}$$

Invoking the functions $\phi^{(1)}$ and $\phi^{(2)}$) (see (1.6)), we set

$$\phi_B(y_n, s) = \frac{1}{a}\phi^{(2)}(y_n, s) - B = (1-s)\frac{y_n^{2\alpha}}{s^\alpha} - B,$$

where $\alpha \in]1/2, 1[$ is fixed, $B = \frac{2}{a}\phi^{(2)}(\frac{3}{\lambda} + 2, 1/2)$, and

$$\eta(y_n, s) = \psi_1(y_n)\psi_2(\phi_B(y_n, s)/B), \quad w(y, s) = \eta(y_n, s)v(y, s).$$

Although the function w is not compactly supported in Q_+^1, Lemma 2.1 and the special structure of the weight in (1.6) allow us to claim the validity of (1.6) for w. As a result, we have

$$\int_{Q_+^1} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} (|w|^2 + |\nabla w|^2)\, dyds$$
$$\leq c_3 \int_{Q_+^1} s^2 e^{2\phi^{(1)}} e^{2a\phi_B} |\partial_s w + \Delta w|^2\, dyds.$$

Arguing as in the proof of Lemma 2.1, we can choose $\gamma_1(c_1, c_3)$ so small that

$$I \equiv \int_{Q_+^1} s^2 e^{2a\phi_B} (|w|^2 + |\nabla w|^2) e^{-\frac{|y'|^2}{4s}}\, dyds$$
$$\leq c_{12}(c_1, c_3) \int_{(\mathbb{R}_+^n + (\frac{3}{\lambda}+1)e_n)\times]1/2,1[} \chi(y_n, s)(sy_n)^2 e^{2a\phi_B} (|v|^2 + |\nabla v|^2) e^{-\frac{|y'|^2}{4s}}\, dyds,$$

where $\chi(y_n, s) = 1$ if $(y_n, s) \in \omega$, $\chi(y_n, s) = 0$ if $(y_n, s) \notin \omega$, and

$$\omega \equiv \{(y_n, s) : y_n > 1, 1/2 < s < 1, \phi_B(y_n, s) < -D/2\},$$

where $D = -2\phi_B(3/\lambda + 3/2, 1/2) > 0$. Now, we estimate the right-hand side of the above inequality with the help of (2.19). We find

$$I \leq c_{13} e^{-Da} \int_{\frac{3}{\lambda}+1}^{+\infty} \int_{1/2}^{1} (y_n s)^2 e^{-\beta \frac{y_n^2}{s}}\, dy_n ds \int_{\mathbb{R}^{n-1}} e^{(\frac{1}{24} - \frac{1}{4s})|y'|^2}\, dy'.$$

Passing to the limit as $a \to +\infty$, we see that $v(y, s) = 0$ if $1/2 \leq s < 1$ and that $\phi_B(y_n, s) > 0$. Using unique continuation through spatial boundaries (see [2]), we show that $v(y, s) = 0$ if $\in \mathbb{R}_+^n$ and $0 < s < 1$. Lemma 2.2 is proved. □

Now, Theorem 1.1 can be deduced from Lemmas 2.1 and 2.2 with the help of more or less standard arguments. We include them merely for completeness.

Lemma 2.3. *Suppose that u satisfies all conditions of Lemma 2.2. Then $u \equiv 0$ in Q_+.*

Proof. By Lemma 2.2, $u(x, t) = 0$ for all $x \in \mathbb{R}_+^n$ and all $t \in]0, \gamma_1[$. Using scaling, we introduce $u^{(1)}(y, s) = u(\sqrt{1-\gamma_1}y, (1-\gamma_1)s + \gamma_1)$. It is easy to check that the function $u^{(1)}$ is well defined in Q_+ and satisfies all conditions of Lemma 2.2 with the same constants c_1 and A. Therefore, $u^{(1)}(y, s) = 0$ for $y_n > 0$ and $0 < s < \gamma_1$. This means that $u(x, t) = 0$ for all $x_n > 0$ and all $0 < t < \gamma_2 = \gamma_1 + (1-\gamma_1)\gamma_1$. Then, we introduce the function

$$u^{(2)}(y, s) = u(\sqrt{1-\gamma_2}y, (1-\gamma_2)s + \gamma_2), \quad (y, s) \in Q_+,$$

and apply Lemma 2.2. After k steps, we see that $u(x,t)=0$ for all $x_n>0$ and all $0<t<\gamma_{k+1}$, where $\gamma_{k+1}=\gamma_k+(1-\gamma_k)\gamma_1\to 1$. Lemma 2.3 is proved. □

Proof of Theorem 1.1. Assume that $A_0<M$. Then $\lambda^2\equiv A_0/2M<1/2$. Introducing the function $v(y,s)=u(\lambda y,\lambda^2 s)$, $(y,s)\in Q_+$, we see that v satisfies all conditions of Lemma 2.3 with the constants c_1 and $A=\frac{1}{2}A_0$. Therefore, $u(x,t)=0$ for $x_n>0$ and all $0<t<\frac{A_0}{2M}$. Now we can repeat the arguments of Lemma 2.3, replacing γ_1 with $\frac{A_0}{2M}$ and A with M, and finishing the proof of the theorem. □

§3. Proof of the first Carleman inequality

In our proof we employ the standard techniques used in the L_2-theory of Carleman inequalities (see, e.g., [6] and [11]).

Let u be an arbitrary function of class $C_0^\infty(\mathbb{R}^N\times]0,2[)$. We set $\phi(x,t)=-\frac{|x|^2}{8t}-(a+1)\ln h(t)$ and $v=e^\phi u$. Then

$$Lv\equiv e^\phi(\partial_t u+\Delta u)=\partial_t v-v\Delta\phi-2\nabla v\cdot\nabla\phi+\Delta v+(|\nabla\phi|^2-\partial_t\phi)v.$$

The main trick in the approach mentioned above is the decomposition of the operator tL into symmetric and skew-symmetric parts. We put

$$tL=S+A, \tag{3.1}$$

where

$$Sv\equiv t(\Delta v+(|\nabla\phi|^2-\partial_t\phi)v)-\frac{1}{2}v \tag{3.2}$$

and

$$Av\equiv\frac{1}{2}(\partial_t(tv)+t\partial_t v)-t(v\Delta\phi+2\nabla v\cdot\nabla\phi). \tag{3.3}$$

Obviously,

$$\begin{aligned}\int t^2e^{2\phi}|\partial_t u+\Delta u|^2\,dxdt&=\int t^2|Lv|^2\,dxdt\\&=\int|Sv|^2\,dxdt+\int|Av|^2\,dxdt+\int[S,A]v\cdot v\,dxdt,\end{aligned} \tag{3.4}$$

where $[S,A]=SA-AS$ is the commutator of S and A. Simple calculations show that

$$\begin{aligned}I&\equiv\int[S,A]v\cdot v\,dxdt\\&=4\int t^2\big[\phi_{,ij}v_{,i}\cdot v_{,j}+\phi_{,ij}\phi_{,i}\phi_{,j}|v|^2\big]\,dxdt\\&\quad+\int t^2|v|^2(\partial_t^2\phi-2\partial_t|\nabla\phi|^2-\Delta^2\phi)\,dxdt\\&\quad+\int t|\nabla v|^2\,dxdt-\int t|v|^2(|\nabla\phi|^2-\partial_t\phi)\,dxdt.\end{aligned} \tag{3.5}$$

Here and in what follows, we adopt the convention that summation is done over the repeated Latin indices running from 1 to n. The partial derivatives in spatial variables are denoted by the comma in lower indices, i.e., $v_{,i}=\frac{\partial v}{\partial x_i}$, $\nabla v=(v_{i,j})$, etc. With the function ϕ as above, we have

$$I=(a+1)\int t^2\Big[-\Big(\frac{h'(t)}{h(t)}\Big)'-\frac{h'(t))}{th(t)}\Big]|v|^2\,dxdt=\frac{a+1}{3}\int t|v|^2\,dxdt. \tag{3.6}$$

Using the simple identity

$$|\nabla v|^2 = \frac{1}{2}(\partial_t + \Delta)|v|^2 - v \cdot (\partial_t v + \Delta v), \tag{3.7}$$

we find

$$\int t^2|\nabla v|^2\,dxdt = -\int t|v|^2\,dxdt - \int t^2 v \cdot Lv\,dxdt + \int t^2|v|^2(|\nabla\phi|^2 - \partial_t\phi)\,dxdt. \tag{3.8}$$

In our case,

$$|\nabla\phi|^2 - \partial_t\phi = -|\nabla\phi|^2 + (a+1)\frac{h'(t)}{h(t)}.$$

The latter relation (together with (3.6)) implies the estimate

$$\int t^2(|\nabla v|^2 + |v|^2|\nabla\phi|^2)\,dxdt \le 3I - \int t^2 v \cdot Lv\,dxdt \le b_1 \int t^2|Lv|^2\,dxdt, \tag{3.9}$$

where b_1 is a positive universal constant. Since

$$e^{\phi}|\nabla u| \le |\nabla v| + |v||\nabla\phi|, \tag{3.10}$$

from (3.4)–(3.10) it follows that

$$\begin{aligned}\int h^{-2a}(t)(th^{-1}(t))^2\Big((a+1)\frac{|u|^2}{t} + |\nabla u|^2\Big)e^{-\frac{|x|^2}{4t}}\,dxdt \\ \le b_2 \int h^{-2a}(t)(th^{-1}(t))^2|\partial_t u + \Delta u|^2 e^{-\frac{|x|^2}{4t}}\,dxdt,\end{aligned}$$

where b_2 is a positive universal constant. Inequality (1.5) is proved.

§4. Proof of the second Carleman inequality

Let $u \in C_0^\infty(Q_+^1)$, and let

$$\phi = \phi^{(1)} + \phi^{(2)}, \quad \phi^{(1)}(x,t) = -\frac{|x'|^2}{8t}, \quad \phi^{(2)}(x,t) = a(1-t)\frac{x_n^{2\alpha}}{t^\alpha}, \tag{4.1}$$

where $Q_+^1 = (\mathbb{R}_+^n + e_n)\times]0,1[$, $x = (x', x_n)$, $\alpha \in]1/2, 1]$ is fixed, and a is a positive parameter.

We are going to use formulas (3.1)–(3.5) for the new functions u, v, and ϕ. Now, all integrals in those formulas are taken over Q_+^1.

First, we observe that

$$\begin{gathered}\nabla\phi = \nabla\phi^{(1)} + \nabla\phi^{(2)}, \\ \nabla\phi^{(1)}(x,t) = -\frac{x'}{4t}, \quad \nabla\phi^{(2)}(x,t) = 2\alpha a\frac{1-t}{t^\alpha}x_n^{2\alpha-1}e_n.\end{gathered} \tag{4.2}$$

Therefore,

$$\nabla\phi^{(1)} \cdot \nabla\phi^{(2)} = 0, \quad |\nabla\phi|^2 = |\nabla\phi^{(1)}|^2 + |\nabla\phi^{(2)}|^2. \tag{4.3}$$

Moreover,

$$\begin{aligned}&\nabla^2\phi = \nabla^2\phi^{(1)} + \nabla^2\phi^{(2)}, \\ &\phi^{(1)}_{,ij} = \begin{cases} -\frac{\delta_{ij}}{4t}, & 1 \le i,j \le n-1, \\ 0, & i = n \text{ or } j = n, \end{cases} \\ &\phi^{(2)}_{,ij} = \begin{cases} 0, & i \ne n \text{ or } j \ne n, \\ 2\alpha(2\alpha-1)a\frac{1-t}{t^\alpha}x_n^{2\alpha-2}, & i = n \text{ and } j = n. \end{cases}\end{aligned} \tag{4.4}$$

In particular, (4.4) implies

$$\phi_{,ij}\phi_{,i}\phi_{,j} = -\frac{1}{4t}|\nabla\phi^{(1)}|^2 + 2\alpha(2\alpha-1)a\frac{1-t}{t^\alpha}x_n^{2\alpha-2}|\nabla\phi^{(2)}|^2 \geq -\frac{1}{4t}\frac{|x'|^2}{16t^3}. \tag{4.5}$$

Using (4.3)–(4.5), we write the integral I in (3.5) as follows:

$$I = I_1 + I_2 + \int t|\nabla v|^2\,dxdt, \tag{4.6}$$

where

$$\begin{aligned} I_s = 4\int t^2\big[\phi^{(s)}_{,ij}v_{,i}\cdot v_{,j} + \phi^{(s)}_{,ij}\phi^{(s)}_{,i}\phi^{(s)}_{,j}|v|^2\big]\,dxdt \\ + \int t^2|v|^2\Big(\partial_t^2\phi^{(s)} - 2\partial_t|\nabla\phi^{(s)}|^2 - \Delta^2\phi^{(s)} - \frac{1}{t}|\nabla\phi^{(s)}|^2 + \frac{1}{t}\partial_t\phi^{(s)}\Big)\,dxdt, \\ s = 1, 2. \end{aligned}$$

Direct calculations yield

$$I_1 = -\int t(|\nabla v|^2 - |v_{,n}|^2)\,dxdt,$$

whence

$$I = \int t|v_{,n}|^2\,dxdt + I_2. \tag{4.7}$$

Now, our aim is to estimate I_2 from below. Since $\alpha \in]1/2, 1[$, we can suppress the first integral in the expression for I_2. As a result, we have

$$I_2 \geq \int t^2|v|^2(A_1 + A_2 + A_3)\,dxdt, \tag{4.8}$$

where

$$\begin{aligned} A_1 &= -\partial_t|\nabla\phi^{(2)}|^2, \\ A_2 &= A_1 - \Delta^2\phi^{(2)} - \frac{1}{t}|\nabla\phi^{(2)}|^2, \\ A_3 &= \partial_t^2\phi^{(2)} + \frac{1}{t}\partial_t\phi^{(2)}. \end{aligned}$$

For A_2, we find

$$A_2 \geq \frac{1-t}{t^\alpha}x_n^{2\alpha-4}a(2\alpha-1)\Big[\frac{4\alpha^2 a x_n^{2\alpha+2}}{t^{\alpha+1}} - 2\alpha(2\alpha-2)(2\alpha-3)\Big].$$

Since $x_n \geq 1$ and $0 < t < 1$, we see that $A_2 > 0$ for all $a \geq 2$. Hence, (4.7) and (4.8) imply the inequality

$$I \geq \int t^2|v|^2(A_1 + A_3)\,dxdt. \tag{4.9}$$

It is not difficult to check that

$$A_3 \geq a(2\alpha-1)\frac{x_n^{2\alpha}}{t^{\alpha+2}}. \tag{4.10}$$

On the other hand,

$$-\partial_t|\nabla\phi^{(2)}|^2 - \frac{1}{t}|\nabla\phi^{(2)}|^2 \geq (2\alpha-1)\frac{1-t}{t^{2\alpha+1}}4\alpha^2a^2x_n^{2(2\alpha-1)} \geq 0$$

so that

$$A_1 \geq \frac{1}{t}|\nabla\phi^{(2)}|^2. \tag{4.11}$$

Combining (4.9)–(4.11), from (3.4) we deduce the estimate

$$\begin{aligned}\int t^2|Lv|^2\,dxdt &\geq I\\ &\geq a(2\alpha-1)\int \frac{x_n^{2\alpha}}{t^\alpha}|v|^2\,dxdt+\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt\\ &\geq a(2\alpha-1)\int |v|^2\,dxdt+\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt.\end{aligned}\tag{4.12}$$

Recalling (3.7), we can find the following analog of (3.8):

$$\int t|\nabla v|^2\,dxdt=-\frac{1}{2}\int |v|^2\,dxdt-\int tv\cdot Lv\,dxdt+\int t|v|^2(|\nabla\phi|^2-\partial_t\phi)\,dxdt.\tag{4.13}$$

By the special structure of ϕ, we have

$$\begin{aligned}|\nabla\phi|^2-\partial_t\phi &= |\nabla\phi^{(1)}|^2-\partial_t\phi^{(1)}+|\nabla\phi^{(2)}|^2-\partial_t\phi^{(2)}\\ &= -|\nabla\phi^{(1)}|^2+|\nabla\phi^{(2)}|^2-\partial_t\phi^{(2)},\end{aligned}$$

which allows us to reduce (4.13) to the form

$$\begin{aligned}&\int \big(t|\nabla v|^2+t|v|^2(|\nabla\phi^{(1)}|^2+|\nabla\phi^{(2)}|^2)\big)\,dxdt\\ &\quad=\int t\big(|\nabla v|^2+|v|^2|\nabla\phi|^2\big)\,dxdt\\ &\quad=-\frac{1}{2}\int |v|^2\,dxdt-\int tv\cdot Lv\,dxdt\\ &\qquad+2\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt-\int t|v|^2\partial_t\phi^{(2)}\,dxdt.\end{aligned}\tag{4.14}$$

But

$$-t\partial_t\phi^{(2)}\leq a\frac{x_n^{2\alpha}}{t^\alpha}$$

and, by (3.10) and (4.14),

$$\frac{1}{2}\int te^{2\phi}|\nabla u|^2\leq-\int v\cdot(tLv)\,dxdt+2\int t|v|^2|\nabla\phi^{(2)}|^2\,dxdt+a\int\frac{x_n^{2\alpha}}{t^\alpha}|v|^2\,dxdt.\tag{4.15}$$

The Cauchy–Schwarz inequality, (4.12), and (4.15) imply the required inequality (1.6).

References

[1] L. Escauriaza, *Carleman inequalities and the heat operator*, Duke Math. J. **104** (2000), no. 1, 113–127. MR **2001m:**35135

[2] L. Escauriaza and F. J. Fernández, *Unique continuation for parabolic operators* (to appear).

[3] L. Escauriaza, G. Seregin, and V. Šverák, *On backward uniqueness for parabolic equations*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **288** (2002), 100–103. MR **2003i:**35225

[4] ———, *On backward uniqueness for parabolic equations*, Arch. Rational Mech. Anal. (to appear).

[5] L. Escauriaza and L. Vega, *Carleman inequalities and the heat operator*. II, Indiana Univ. Math. J. **50** (2001), 1149–1169. MR **2003b:**35088

[6] L. Hörmander, *Linear partial differential operators*, Grundlehren Math. Wiss., vol. 116, Springer-Verlag, Berlin, etc., 1963. MR **28:**4221

[7] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural′tseva, *Linear and quasilinear equations of parabolic type*, "Nauka", Moscow, 1967; English transl., Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968. MR **39:**3159a

[8] S. Micu and E. Zuazua, *On the lack of null-controllability of the heat equation on the half-space*, Port. Math. (N.S.) **58** (2001), no. 1, 1–24. MR **2002a:**93011

[9] G. Seregin, *Local regularity of suitable weak solutions to the Navier–Stokes equations near the boundary*, J. Math. Fluid Mech. **4** (2002), no. 1, 1–29. MR **2003a:**35152

[10] G. Seregin and V. Šverák, *The Navier–Stokes equations and backward uniqueness*, Nonlinear Problems of Mathematical Physics and Related Topics. Vol. 2 (in Honor of Prof. O. A. Ladyzhenskaya), Kluwer Acad./Plenum Publ., 2002, pp. 359–370.

[11] D. Tataru, *Carleman estimates, unique continuation, and applications*, Notes downloadable from http://math.berkeley.edu/tataru/ucp.html.

Dipartimento di Matemáticas, UPV/EHU, Bilbao, Spain
E-mail address: mtpeszul@lq.ehu.es

St. Petersburg Branch, Steklov Mathematical Institute, Russian Academy of Sciences, Fontanka 27, St. Petersburg 191011, Russia
E-mail address: seregin@pdmi.ras.ru

School of Mathematics, University of Minnesota, Minneapolis, MN
E-mail address: sverak@math.umn.edu

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