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MULTIVARIATE PERIODIC WAVELETS

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Abstract. A general construction of a multiresolution analysis with a matrix dilation for periodic functions is described, together with a method of finding wavelet biorthogonal bases. The convergence of expansions with respect to these bases is studied.

§1. Introduction

Wavelet bases play an important role in solving numerous applied problems; also, they are an indispensable tool in approximation theory. In the late 1980s, a method of construction of wavelet bases for $L_2(\mathbb{R})$ was proposed in the papers [2] by Mallat and [3] by Meyer; this method is based on what is called *multiresolution analysis* (MRA in the sequel). The essence of this method is as follows. MRA is generated by a function (called a scaling function) with certain special properties. Starting with the scaling function, one creates another function (called a wavelet function) such that the shifts and dilations of it constitute a wavelet basis for $L_2(\mathbb{R})$ (see, e.g., [1, Chapter 5]).

Multivariate wavelet bases can be produced in a number of ways. First, it is possible to take the tensor product of several one-dimensional wavelet bases. This is simple, but the resulting multivariate system does not inherit all assets of the initial univariate bases. In particular, the localization property, which is of great value for applied problems, is not preserved. This can easily be demonstrated with the help of the Haar system. In the one-dimensional case, a basis function with a large index has a small support. But for the tensor product of two Haar systems (indexed doubly in a natural way), a basis function with an arbitrarily large absolute value of the index may have a support large in one direction.

Another approach to constructing a d-dimensional wavelet basis is to consider the tensor product of d one-dimensional MRAs. Such a structure is similar to that of a one-dimensional MRA and is generated by the tensor product of one-dimensional scaling functions. In this case, we have several wavelet functions whose shifts and dilations constitute a basis of $L_2(\mathbb{R}^d)$. A still more general definition of a multivariate multiresolution analysis was given by Meyer in [3]. By this definition, an MRA of $L_2(\mathbb{R}^d)$ is a collection of closed subspaces V_j, $j \in \mathbb{Z}$, of the space $L_2(\mathbb{R}^d)$ that satisfy the following conditions:

1) $V_j \subset V_{j+1}$ for any $j \in \mathbb{Z}$;
2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L_2(\mathbb{R}^d)$;
3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
4) $f(x_1,\dots,x_d) \in V_0 \iff f(2^j x_1,\dots,2^j x_d) \in V_j$;
5) there exists $\varphi \in V_0$ (the scaling function) such that the functions $\varphi(\cdot + k)$, $k \in \mathbb{Z}$, constitute an orthonormal basis of the space V_0.

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The problem of finding wavelet functions becomes more complex in the multidimensional case. Under various assumptions on the scaling function, this problem was considered by de Boor, DeVore, and Ron [4], Jia and Micchelli [5], [6], and Riemenschneider and Shen [7], [8]. In the most general case, a method for constructing wavelet functions was described explicitly by Jia and Shen in [11]. In Meyer's definition of an MRA, the scale factor is the diagonal matrix with 2's on the diagonal, i.e., dilations in all directions are the same. Other scale factors are also of interest for some applied problems. A more general approach to the multivariate MRA was given, e.g., in the book [18] by Wojtaszczyk. In that book, integral matrices satisfying some natural requirements played the role of the scale factor, and a general algorithm for construction of wavelet functions was presented. The problem reduces to finding a unitary matrix whose first row elements are given functions. Similarly, in order to find a biorthogonal pair of wavelet bases, we need to construct two matrices the first rows of which coincide with a given pair of orthogonal vector-valued functions. In [9], [10], Jia, Riemenschneider, and Shen presented a description of an algorithm for constructing orthogonal and biorthogonal compactly supported wavelet bases.

Usually, univariate periodic wavelets are defined by periodization of wavelets in $L_2(\mathbb{R})$. Such an approach to periodic objects is not quite natural; moreover, in the literature we find periodic wavelets (e.g., in the paper [12] by Chui and Mhaskar) that do not match this definition. Definitions of a multiresolution analysis for periodic functions (PMRA) has been proposed by many authors (Chui and Wang [13], Zheludev [14], Goh, Lee, Shen, and Tang [16], Petukhov [15], etc.). The most general definition of a PMRA in the spaces L_p, $1 < p < \infty$, and C was suggested by Skopina in [19], where also a description of the periodic wavelet bases was presented together with a method of constructing them. Moreover, in [19] conditions were found that ensure the convergence of a Fourier series with respect to a wavelet system. All descriptions were given in terms of Fourier coefficients. From the viewpoint of applications, this approach has an advantage as compared to the nonperiodic constructions, because the problems become discrete.

The present paper is devoted to description of multivariate PMRAs, to construction of orthogonal and biorthogonal wavelet bases, and to expansions with respect to such bases. As a scale factor, we take $(d \times d)$-matrices, where d is the dimension of the space. Let M be an integral matrix such that the absolute values of all of its eigenvalues exceed 1. We note that the operator with the matrix M, applied many times, provides dilation in all directions, because

$$\lim_{n\to+\infty} \|M^{-n}\| = 0. \tag{1}$$

This follows from the fact that the entire spectrum of M (in the finite-dimensional case the spectrum coincides with the set of eigenvalues) is located in the disk $|\lambda| \le r(M^{-1})$, where $r(M^{-1}) := \lim_{n\to\infty} \|M^{-n}\|^{1/n}$ is the spectral radius of M^{-1}, and there exists at least one point of the spectrum on the boundary of that disk (see, e.g., [17, Russian p. 267]). Since the absolute values of all eigenvalues of M^{-1} are strictly less than 1 and the set of eigenvalues is finite, we have $r(M^{-1}) < 1$. Therefore, the sequence $\|M^{-n}\|$ decays faster than a geometric progression.

§2. Notation and preliminary information

Throughout the paper, $\mathbb{N}$ is the set of positive integers, $\mathbb{R}^d$ is the d-dimensional Euclidean space, $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ are elements of $\mathbb{R}^d$ (vectors), $(x, y) = x_1y_1 + \cdots + x_dy_d$, $|x| = \sqrt{(x, x)}$, $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$, $\mathbb{Z}^d$ is the integral lattice in $\mathbb{R}^d$, $\mathbb{Z}_+ = \{x \in Z^1 : x \ge 0\}$, $\mathbb{T}^d = [0, 1)^d$ is the unit d-dimensional torus, and δ_{lk} is the Kronecker delta. By the space X we mean either $C(\mathbb{T}^d)$ or $L_p(\mathbb{T}^d)$, $1 \le p < \infty$;

$\widehat{f}(k) = \int_{\mathbb{T}^d} f(t)e^{-2\pi i(k,t)}\,dt$ is the kth Fourier coefficient of $f \in X$; $\langle f, g\rangle = \int_{\mathbb{T}^d} f\overline{g}$. We shall identify the functions defined on $\mathbb{T}^d = [0,1)^d$ with their periodic extensions to $\mathbb{R}^d$.

Let A be a nonsingular integral $(d\times d)$-matrix; we denote by $\|A\|$ the norm of A as an operator from $\mathbb{R}^d$ to $\mathbb{R}^d$, by A^* the matrix adjoint to A, and by $\det A$ the determinant of A. The unit $(d\times d)$-matrix is denoted by E_d. We shall say that two elements $k, n \in \mathbb{Z}^d$ are *congruent modulo* A (we write $k \equiv n \pmod A$) if $k - n = A\ell$, $\ell \in \mathbb{Z}^d$. The lattice $\mathbb{Z}^d$ is split into cosets with respect to this congruence. The number of cosets is equal to $|\det A|$ (see, e.g., [18, p. 107]). We fix an arbitrary representative in each coset, call these elements *digits*, and denote the set of digits by $D(A)$. For two sets K and L, we say that K is *congruent to* L *modulo* A (*modulo* $\mathbb{Z}^d$ if $A = E_d$) if K can be split into finitely many disjoint subsets, $K = \bigcup_{n=1}^N K_n$, in such a way that for some integral vectors $\ell_1, \dots, \ell_N$ we have $L = \bigcup_{n=1}^N (K_n + A\ell_n)$ and the sets $(K_n + A\ell_n)$ are mutually disjoint. Obviously, if K is congruent to L, then L is congruent to K. Congruence modulo $\mathbb{Z}^d$ means that we can "aggregate" L by shifting parts of K by integral vectors. Congruence modulo A means that we can "aggregate" L by shifting parts of K by vectors of the form $A\ell$ with integral ℓ. Obviously, if two measurable sets are congruent, then they are of equal measure.

Lemma 1. *If A is a nonsingular integral $(d\times d)$-matrix, then*

$$K := \bigcup_{r\in D(A)} (A^{-1}[0,1)^d + A^{-1}r)$$

is congruent to $[0,1)^d$ modulo $\mathbb{Z}^d$, and

$$\int_{\mathbb{T}^d} f = \sum_{r\in D(A)} \int_{A^{-1}[0,1)^d + A^{-1}r} f$$

for any $f \in L_1(\mathbb{T}^d)$.

Proof. Set $K_n := [n, n+1)^d \cap K$, $n \in \mathbb{Z}^d$. Clearly, $K_n \cap K_{n_1} = \emptyset$ for $n \neq n_1$, and $K = \bigcup_n K_n$. Since the set K is bounded, only finitely many of the sets K_n are nonempty. We show that $K_n \cap (K_{n_1} - \ell) = \emptyset$ for every $\ell \in \mathbb{Z}^d$. Let $u \in K_n \subset K$; this means that $u = A^{-1}v + A^{-1}r$ with $v \in [0,1)^d$ and $r \in D(A)$. Suppose there exists $u_1 \in K_{n_1}$, $n \neq n_1$, such that $u = u_1 - \ell$, $\ell \in \mathbb{Z}^d$, or (what is the same) the difference $u - u_1$ is integral. Let $u_1 = A^{-1}v_1 + A^{-1}r_1$, $v_1 \in [0,1)^d$, $r_1 \in D(A)$; then $A^{-1}(v - v_1) + A^{-1}(r - r_1) = \ell$, $\ell \in \mathbb{Z}^d$. Multiplying this relation by A from the right, we get $v - v_1 + r - r_1 = A\ell$, $\ell \in \mathbb{Z}^d$. Since the vectors r, r_1, and $A\ell$ are integral and the vectors v and v_1 are in $[0,1)^d$, this is possible only if $v = v_1$. However, since $r, r_1 \in D(A)$, these vectors cannot be congruent modulo A unless they coincide. Therefore, $r = r_1$, i.e., $u = u_1$, which contradicts the assumption $n \neq n_1$. Now we show that $[0,1)^d = \bigcup_n (K_n - n)$. Observe that $K_n - n \subset [0,1)^d$ for any n by the definition of K_n. Moreover, $(K_n - n) \cap (K_{n_1} - n_1) = \emptyset$. We check that for any $u \in [0,1)^d$ there exist $n \in \mathbb{Z}^d$ and $w \in K_n$ such that $u = w - n$. Multiplying the vector u by the matrix A from the left, we represent Au in the form $Au = p + v$, where $p \in \mathbb{Z}^d$, $v \in [0,1)^d$. The vector p is congruent to one of the digits, i.e., there are $r \in D(A)$ and $l \in \mathbb{Z}^d$ such that $p = r + Al$. Then $u = A^{-1}r + l + A^{-1}v$, whence $u - l \in K$. Therefore, $u - l \in K_n$ for some $n \in \mathbb{Z}^d$, and $n = -l$. Putting $w = u - l = u + n$, we get $w \in K_n$.

The second statement of the lemma readily follows from the 1-periodicity of the function f. □

Lemma 2 ([22]). *Let A be a nonsingular integral $(d\times d)$-matrix with $|\det A| > 1$. Then*

$$\sum_{s\in D(A^*)} e^{2\pi i(A^{-1}r,s)} = \begin{cases} |\det A| & \text{if } r \equiv \mathbf{0} \pmod A, \\ 0 & \text{if } r \not\equiv \mathbf{0} \pmod A. \end{cases} \tag{2}$$

For completeness, we include the proof of this lemma.

Proof. We set $m := |\det A| = |\det A^*|$. The cosets with respect to A^* form a group relative to addition; this group consists of m elements. If $r \equiv \mathbf{0} \pmod{A}$, i.e., $r = Al$ with $l \in \mathbb{Z}^d$, then (30) is obvious. If $r \not\equiv \mathbf{0} \pmod{A}$, we take any vector $a \in \mathbb{Z}^d$ such that $(A^{-1}r, a) \notin \mathbb{Z}$; then, in particular, $a \not\equiv \mathbf{0} \pmod{A^*}$. Consider the vectors a, $2a$, $3a$, $\dots$, and let m_1 be the minimum positive integer such that $m_1 a \equiv 0 \pmod{A^*}$. Then $1 < m_1 \le m$ (if none of the elements ka, $k = 1, \dots, m$, is congruent to zero, then they are mutually noncongruent, but the group in question has only m elements). Thus, the vectors a, $2a$, $\dots$, $m_1 a$ form a subgroup of m_1 elements. By the Lagrange theorem, m is divisible by m_1. Let $m = m_1 n$, and let $a_1 = 0, a_2, \dots, a_n$ be representatives of the elements of the corresponding quotient group. Then the elements $a_k + ja$, $k = 1, \dots, n$, $j = 1, \dots, m_1$, run over all of the group, and

$$\sum_{k=1}^{m} e^{2\pi i (A^{-1}r, s_k)} = \sum_{k=1}^{n} \sum_{j=1}^{m_1} e^{2\pi i (A^{-1}r, a_k + ja)} = \sum_{k=1}^{n} e^{2\pi i (A^{-1}r, a_k)} \sum_{j=1}^{m_1} e^{2\pi i (A^{-1}r, ja)}$$
$$= C \frac{1 - e^{2\pi i m_1 (A^{-1}r, a)}}{1 - e^{2\pi i (A^{-1}r, a)}}.$$

Since $m_1 a = A^* l$, $l \in \mathbb{Z}^d$, and $(A^{-1}r, a) \notin \mathbb{Z}$ by assumption, it follows that the ratio on the right is equal to zero. □

Corollary 3. *Under the assumptions of Lemma* 2, *the matrix* $\{e^{2\pi i (A^{-1}n, r)}\}_{\substack{n \in D(A), \\ r \in D(A^*)}}$ *is unitary up to the factor* $\sqrt{|\det A|}$.

Indeed, the inner product of the n_1th and n_2th columns is equal to

$$\sum_{r \in D(A^*)} e^{2\pi i (A^{-1}(n_1 - n_2), r)}.$$

By Lemma 2, this sum is equal to $|\det A|$ for $n_1 = n_2$, and otherwise it is equal to zero because $n_1 \not\equiv n_2 \pmod{A}$.

Lemma 4. *Let A be a nonsingular integral $(d \times d)$-matrix with $|\det A| > 1$. Then the set $\{r + A^j p\}$, $r \in D(A^j)$, $p \in D(A)$, is a set of digits for the matrix A^{j+1}.*

Proof. The number of all possible pairs (r, p) with $r \in D(A^j)$ and $p \in D(A)$ is equal to $|\det A|^{j+1}$. Thus, it suffices to prove that different pairs cannot give rise to vectors congruent modulo A^{j+1}. Let $r, r_1 \in D(A^j)$ and $p, p_1 \in D(A)$. Assume that $r + A^j p$ and $r_1 + A^j p_1$ are congruent modulo A^{j+1}; then $(r - r_1) + A^j (p - p_1) = A^{j+1} n$ for some $n \in \mathbb{Z}^d$. Multiplying both sides of this relation by A^{-j} form the left, we get $A^{-j}(r - r_1) = -(p - p_1) + An \in \mathbb{Z}^d$, i.e., $r \equiv r_1 \pmod{A^j}$. Since r and r_1 belong to the set $D(A^j)$, which contains only one representative of each coset, we have $r = r_1$. The relation $(p - p_1) = An$ implies that $p = p_1$. Thus, $r + A^j p$ and $r_1 + A^j p_1$ are congruent modulo A^{j+1} if and only if $r = r_1$ and $p = p_1$. □

Throughout the paper, M denotes a fixed integral $(d \times d)$-matrix such that the absolute values of all of its eigenvalues exceed 1; we put $m = |\det M|$. Obviously, for such a matrix we have $m \in \mathbb{Z}$, $m > 1$, and, as has already been noted,

$$\lim_{n \to \infty} |M^n x| = \infty \tag{3}$$

for all $x \in \mathbb{R}^d$, $x \neq \mathbf{0}$.

On the space X, we define a shift operator S_p^j, $p \in \mathbb{Z}^d$, $j \in \mathbb{Z}_+$, by the formula

$$S_p^j f(x) := f(x + M^{-j} p). \tag{4}$$

§3. PMRA and the scaling sequence

Definition 5. Let $V_j \subset X$, $j \in \mathbb{Z}_+$. The collection $\{V_j\}_{j=0}^{\infty}$ is called a PMRA of X if the following conditions (axioms) are fulfilled.

(MR1) $V_j \subset V_{j+1}$;
(MR2) $\overline{\bigcup_{j=0}^{\infty} V_j} = X$;
(MR3) $\dim V_j = m^j$;
(MR4) $\dim\{f \in V_j : S_n^j f = \lambda_n f \text{ for all } n \in \mathbb{Z}^d\} \le 1$ for any $\{\lambda_n\}_{n\in\mathbb{Z}^d}$,;
(MR5) $f \in V_j \iff S_n^j f \in V_j$ for all $n \in \mathbb{Z}^d$;
(MR6) a) $f \in V_j \implies f(M\cdot) \in V_{j+1}$,
b) $f \in V_{j+1} \implies \sum_{s\in D(M)} f(M^{-1}\cdot + M^{-1}s) \in V_j$.

Remark. Since f is periodic, in conditions (MR4) and (MR5) we can consider only the digits of the matrix M^j instead of all $n \in \mathbb{Z}^d$. So, these conditions can be replaced by

(MR4′) $\dim\{f \in V_j : S_r^j f = \lambda_r f \text{ for all } r \in D(M^j)\} \le 1$ for any $\{\lambda_r\}_{r\in D(M^j)}$;
(MR5′) $f \in V_j \iff S_r^j f \in V_j$ for all $n \in D(M^j)$.

Definition 6. Let V_j be a PMRA in X. A sequence $\{\varphi_j\}_{j=0}^{\infty}$ of functions $\varphi_j \in V_j$ is called a scaling sequence if $\{S_n^j \varphi_j\}_{n\in D(M^j)}$ is a basis for the space V_j.

Theorem 7. *Functions $\varphi_j \in X$, $j \in \mathbb{Z}_+$, constitute a scaling sequence for a* PMRA *of X if and only if*

(Φ1) $\widehat{\varphi}_0(k) = 0$ *for all* $k \ne 0$;

(Φ2) *for each $j \in \mathbb{Z}_+$ and each $n \in \mathbb{Z}^d$, there exists $k \equiv n \pmod{M^{*j}}$ such that $\widehat{\varphi}_j(k) \ne 0$;*

(Φ3) *for each $k \in \mathbb{Z}^d$, there exists $j \in \mathbb{Z}_+$ such that $\widehat{\varphi}_j(k) \ne 0$;*

(Φ4) *for each $j \in \mathbb{N}$ and every $n \in \mathbb{Z}^d$, there exists μ_n^j such that $\widehat{\varphi}_{j-1}(k) = \mu_n^j \widehat{\varphi}_j(k)$ for all $k \equiv n \pmod{M^{*j}}$;*

(Φ5) *for each $j \in \mathbb{Z}_+$ and every $n \in \mathbb{Z}^d$, there exists $\gamma_n^j \ne 0$ such that $\gamma_n^j \widehat{\varphi}_j(k) = \widehat{\varphi}_{j+1}(M^*k)$ for all $k \equiv n \pmod{M^{*j}}$.*

We preface the proof of the theorem with a series of auxiliary statements.

Lemma 8. *Suppose that $V_j \subset X$, $j \in \mathbb{Z}_+$, and that the axioms* (MR1), (MR2), (MR3), (MR5), *and* (MR6) *of Definition* 5 *are fulfilled; then $V_0 = \{\text{const}\}$.*

Proof. By (MR3), the space V_0 is one-dimensional. Let $f \in V_0$, and let $\|f\| \ne 0$. We show that $\widehat{f}(\mathbf{0}) \ne 0$. Suppose that $\widehat{f}(\mathbf{0}) = 0$. We introduce the following operator A: $Af = \sum_{s\in D(M)} f(M^{-1}\cdot + M^{-1}s)$, and set $g = Af$. By Lemma 1,

$$\begin{aligned}\widehat{g}(\mathbf{0}) &= \sum_{s\in D(M)} \int_{\mathbb{T}^d} f(M^{-1}x + M^{-1}s)\,dx \\ &= m \sum_{s\in D(M)} \int_{M^{-1}\mathbb{T}^d + M^{-1}s} f(z)\,dz = m\int_{\mathbb{T}^d} f(z)\,dz \\ &= m\widehat{f}(0).\end{aligned}$$

If $g_0 \in V_j$, then $g_1 := Ag_0 \in V_{j-1}$, $\dots$, $g_j := Ag_{j-1} \in V_0$. If $\widehat{g}_j(\mathbf{0}) \ne 0$, then this contradicts the fact that V_0 is one-dimensional, because $\widehat{f}(\mathbf{0}) = 0$. If $\widehat{g}_0(\mathbf{0}) = 0$, then the mean value of any function in V_j is zero, which contradicts (MR2).

Now we assume that $\widehat{f}(n) \ne 0$ for some $n \ne \mathbf{0}$. Set $f_1 := Af$. Since $f \in V_0$, we have $f \in V_1$ by (MR1). Consequently, $f_1 \in V_0$ by (MR6b). Since V_0 is one-dimensional,

$f_1 = \lambda f$. This implies that $\widehat{f_1}(n) = \lambda \widehat{f}(n)$. However, $\lambda = m$ because $\widehat{f_1}(0) = m\widehat{f}(0)$. On the other hand, a direct computation with the use of Lemma 1 yields

$$\begin{aligned}\widehat{f_1}(n) &= \sum_{s\in D(M)} \int_{\mathbb{T}^d} f(M^{-1}x + M^{-1}s) e^{2\pi i(x,n)}\, dx \\ &= m \sum_{s\in D(M)} \int_{M^{-1}\mathbb{T}^d + M^{-1}s} f(z) e^{2\pi i(Mz-s,n)}\, dz = m \int_{\mathbb{T}^d} f(z) e^{2\pi i(Mz,n)}\, dz \\ &= m\widehat{f}(M^*n).\end{aligned}$$

Therefore,

$$\widehat{f}(n) = \widehat{f}(M^*n) = \cdots = \widehat{f}(M^{l*}n) = \cdots .$$

Recalling that the Fourier coefficients of $f \in X$ tend to zero as the absolute value of their indices tends to infinity, and taking (3) into account, we arrive at a contradiction. □

On the space X, we define operators ω_n^j, $j \in \mathbb{Z}_+$, $n \in \mathbb{Z}^d$, by recursion:

$$\begin{aligned}\omega_n^0 f &:= f, \\ \omega_n^{j+1} f(x) &:= \frac{1}{m} \sum_{s\in D(M)} e^{-2\pi i(M^{-j-1}s,n)} \omega_n^j f(x + M^{-j-1}s).\end{aligned}$$

Lemma 9. *Suppose that $V_j \subset X$, $j \in \mathbb{Z}_+$, and that the axiom* (MR5) *of Definition* 5 *is fulfilled. If $f \in V_{j_0}$, then $\omega_n^j f \in V_{j_0}$ for all $j = 0, \dots, j_0$ and all $n \in \mathbb{Z}^d$, and*

$$\omega_n^j f \sim \sum_{m\in\mathbb{Z}^d} \widehat{f}(M^{*j}m + n) e^{2\pi i(M^{*j}m+n,\cdot)}, \tag{5}$$

i.e., $\widehat{\omega_n^j f}(k) = \widehat{f}(k)$ if $k \equiv n \pmod{M^{*j}}$, and $\widehat{\omega_n^j f}(k) = 0$ if $k \not\equiv n \pmod{M^{*j}}$.

Proof. We proceed by induction on j. The assertion for $j = 0$ is obvious. Suppose that $\omega_n^j f \in V_{j_0}$ for $0 \le j < j_0$, and that (5) is true for all $n \in \mathbb{Z}^d$. From (MR5) it follows that $\omega_n^j f(\cdot + M^{-j-1}s) = \omega_n^j f(\cdot + M^{-j_0}M^{j_0-j-1}s) \in V_{j_0}$. This implies the relation $\omega_n^{j+1} f \in V_{j_0}$. Next, we have

$$\begin{aligned}\widehat{\omega_n^{j+1} f}(k) &= \frac{1}{m} \int_{\mathbb{T}^d} \sum_{s\in D(M)} e^{-2\pi i(M^{-j-1}s,n)} \omega_n^j f(x + M^{-j-1}s) e^{-2\pi i(x,k)}\, dx \\ &= \frac{1}{m} \sum_{s\in D(M)} e^{-2\pi i(M^{-j-1}s,n)} \int_{\mathbb{T}^d} \omega_n^j f(t) e^{-2\pi i(t-M^{-j-1}s,k)}\, dt \\ &= \frac{1}{m} \sum_{s\in D(M)} e^{-2\pi i(M^{-j-1}s,n-k)} \widehat{\omega_n^j f}(k).\end{aligned}$$

If $k \equiv n \pmod{M^{*j+1}}$, then, obviously, the sum on the right is equal to m, whence $\widehat{\omega_n^{j+1} f}(k) = \widehat{\omega_n^j f}(k)$. If $k \not\equiv n \pmod{M^{*j+1}}$ and $n \equiv k \pmod{M^{*j}}$, i.e., $n - k = M^{*j}l$, where the vector l is not congruent to zero modulo M^*, then, by Lemma 2, the same sum is equal to zero. Finally, if $k \not\equiv n \pmod{M^{*j}}$, then, by the induction hypothesis, $\widehat{\omega_n^j f}(k) = 0$, whence $\widehat{\omega_n^{j+1} f}(k) = 0$. □

Lemma 10. *If V_j is a* PMRA *of X, then each space V_j possesses a basis $\{v_n^j\}_{n\in D(M^{*j})}$ with the following properties*:

(V1) $\widehat{v}_n^j(k) = 0$ *for all* $k \not\equiv n \pmod{M^{*j}}$;
(V2) *if* $\widehat{v}_n^j(k) \ne 0$, *then* $\widehat{v}_n^j(\ell) = \widehat{v}_n^{j+1}(\ell)$ *for all* $\ell \equiv k \pmod{M^{*j+1}}$;
(V3) $\widehat{v}_n^j(k) = \widehat{v}_{M^*n}^{j+1}(M^*k)$ *for all* $k \in \mathbb{Z}^d$.

For convenience, we set $\widehat{v}_l^j := \widehat{v}_n^j$ if $l \equiv n \pmod{M^j}$.

Proof. We use induction on j. The case of $j = 0$ is obvious because all integral vectors are congruent to each other if $j = 0$. Suppose that in the spaces V_j with $j = 0, \dots, j_0$ there exist bases satisfying (V1), (V2), and (V3). We introduce the spaces $V_j^{(n)} := \{f \in V_j : \widehat{f}(k) = 0 \text{ for all } k \not\equiv n \pmod{M^{*j}}\}$. If $F \in V_j$, then

$$F = \sum_{n \in D(M^{*j})} \omega_n^j F = \sum_{n \in D(M^{*j})} F_n, \quad F_n \in V_j^{(n)}.$$

This means that $V_j = \sum_{n \in D(M^{*j})} V_j^{(n)}$. Therefore,

$$m^j = \dim V_j \le \sum_{n \in D(M^{*j})} \dim V_j^{(n)}. \tag{6}$$

We find the dimension of $V_j^{(n)}$. If $f \in V_j^{(n)}$, then

$$f(x) \sim \sum_{m \in \mathbb{Z}^d} \widehat{f}(M^{*j} m + n) e^{2\pi i (M^{*j} m + n, x)}.$$

Applying the shift operator, we get

$$(S_p^j f)(x) \sim \sum_{m \in \mathbb{Z}^d} \widehat{f}(M^{*j} m + n) e^{2\pi i (M^{*j} m + n, x + M^{-j} p)} \sim e^{2\pi i (n, M^{-j} p)} f(x),$$

i.e., $S_p^j f(x) = e^{2\pi i (n, M^{-j} p)} f(x)$ for all $p \in \mathbb{Z}^d$. Using (MR4) and (6), we deduce that $\dim V_j^{(n)} = 1$.

We construct the basis $\{v_n^{j_0+1}\}$. If $\widehat{v}_k^{j_0}(k) \neq 0$, we set $v_k^{j_0+1} := \omega_k^{j_0+1} v_k^{j_0}$. Properties (V1) and (V2) are valid by Lemma 9. We check (V3). If $k \equiv \mathbf{0} \pmod{M^*}$ and $n \equiv k \pmod{M^{*j_0+1}}$, then

$$\widehat{v}_k^{j_0+1}(n) = \widehat{v}_k^{j_0}(n) = \widehat{v}_{M^{*-1}k}^{j_0-1}(M^{*-1} n) = \widehat{v}_{M^{*-1}k}^{j_0}(M^{*-1} n)$$

by the induction hypothesis. Thus, we have defined the basis functions with the indices k for which there exists $n \equiv k \pmod{M^{*j_0+1}}$ such that $\widehat{v}_k^{j_0}(n) \neq 0$. Now, suppose that $\widehat{v}_k^{j_0}(n) = 0$ for all $n \equiv k \pmod{M^{*j_0+1}}$ and $k \equiv \mathbf{0} \pmod{M^*}$. In this case we set $v_k^{j_0+1}(x) := v_{M^{*-1}k}^{j_0}(Mx)$. Using Lemma 1, we obtain

$$\begin{aligned}
\widehat{v}_k^{j_0+1}(n) &= \int_{\mathbb{T}^d} v_k^{j_0+1}(x) e^{-2\pi i (x,n)}\, dx = \int_{\mathbb{T}^d} v_{M^{*-1}k}^{j_0}(Mx) e^{-2\pi i (x,n)}\, dx \\
&= \sum_{s \in D(M)} \int_{M^{-1}\mathbb{T}^d + M^{-1}s} v_{M^{*-1}k}^{j_0}(Mx) e^{-2\pi i (x,n)}\, dx \\
&= \frac{1}{m} \sum_{s \in D(M)} \int_{\mathbb{T}^d + s} v_{M^{*-1}k}^{j_0}(t) e^{-2\pi i (t, M^{*-1} n)}\, dt \\
&= \widehat{v}_{M^{*-1}k}^{j_0}(M^{*-1} n).
\end{aligned}$$

Clearly, property (V3) is fulfilled, and property (V1) is valid by the induction hypothesis. Finally, let $\hat{v}_k^{j_0}(n) = 0$ for all $n \equiv k \pmod{M^{*j_0+1}}$ and $k \not\equiv \mathbf{0} \pmod{M^*}$. In this case, as $v_k^{j_0+1}$ we can use any nonzero element of the space $V_{j_0+1}^{(k)}$. Then property (V1) follows from the definition of $V_{j_0+1}^{(k)}$, and no verification is required for (V2) and (V3). $\square$

Remark. If only the axioms (MR1), (MR3), (MR4), and MR5 of Definition 5 are fulfilled for a sequence of subspaces $V_j \subset X$, $j \in \mathbb{Z}_+$, then, in each V_j, there exists a basis $\{v_n^j\}_{n\in D(M^{*j})}$ satisfying conditions (V1) and (V2). This can be proved by the same method, by using an arbitrary nonzero element of V_0 as v_0^0, and an arbitrary nonzero element of $V_{j_0+1}^k$ as $v_k^{j_0+1}$ in the case where $\widehat{v}_k^{j_0}(n) = 0$ for all $n \equiv k \pmod{M^{*j_0+1}}$ and $k \equiv \mathbf{0} \pmod{M^*}$.

Lemma 11. *If in each space $V_j \subset X$, $j \in \mathbb{Z}_+$, there exists a basis $\{v_n^j\}_{n\in D(M^{*j})}$ satisfying condition* (V1) *of Lemma* 10*, then the axiom* (MR4) *of Definition* 5 *is fulfilled.*

Proof. In view of the remark to Definition 5, it suffices to verify (MR4′). For $r \in D(M^j)$, let f be an eigenvector of the operator S_r^j, i.e., $S_r^j f = \lambda_r f$, and let $f = \sum_{n\in D(M^{*j})} \alpha_n^j v_n^j$. By condition (V1), the operator S_r^j acts on v_n^j as multiplication by $e^{2\pi i(M^{*-j}n,r)}$. Applying S_r^j to the function f, we get

$$S_r^j f(x) = \sum_{n\in D(M^{*j})} \alpha_n^j S_n^j v_n^j(x) = \sum_{n\in D(M^{*j})} \alpha_n^j e^{2\pi i(M^{*-j}n,r)} v_n^j(x).$$

Since f is an eigenvector of S_r^j, we have

$$0 = S_r^j f(x) - \lambda_r f(x) = \sum_{n\in D(M^{*j})} \alpha_n^j [e^{2\pi i(M^{*-j}n,r)} - \lambda_r] v_n^j(x).$$

It follows that $\alpha_n^j[e^{2\pi i(M^{*-j}n,r)} - \lambda_r] = 0$ for all $n \in D(M^{*j})$, because the v_n^j are linearly independent. Suppose that $\alpha_{n_0}^j \neq 0$ and $\alpha_{n_1}^j \neq 0$ for two different numbers $n_0 \neq n_1$. Subtracting, we get

$$(e^{2\pi i(M^{*-j}n_0,r)} - \lambda_r) - (e^{2\pi i(M^{*-j}n_1,r)} - \lambda_r) = 0,$$

or $e^{2\pi i(M^{*-j}(n_0-n_1),r)} = 1$. Now, let f be an eigenvector of all operators S_r^j, $r \in D(M^j)$. Summing the above identities over all r, we obtain

$$\sum_{r\in D(M^j)} e^{2\pi i(M^{*-j}(n_0-n_1),r)} = m^j. \tag{7}$$

On the other hand, since n_0 is not congruent to n_1 modulo M^{*j}, Lemma 2 yields

$$\sum_{r\in D(M^{*j})} e^{2\pi i(M^{*-j}(n_0-n_1),r)} = 0, \tag{8}$$

which contradicts (7). Thus, α_n^j may differ from zero only if $n = n_0$, so that f is proportional to $v_{n_0}^j$. Moreover, from the above arguments it follows that

$$\lambda_r = e^{2\pi i(M^{*-j}n_0,r)}, \quad r \in D(M^j). \tag{9}$$

Let $g = \sum_n \beta_n^j v_n^j$ be another eigenvector of all operators S_r^j, $r \in D(M^j)$. For this vector, again, only one of the β_n^j differs from zero. Suppose that $\beta_{n_1}^j \neq 0$, $n_1 \neq n_0$. As in (9), we have $\lambda_r = e^{2\pi i(M^{*-j}n_1,r)}$, $r \in D(M^j)$. Combining this with (9), we get $e^{2\pi i(M^{*-j}(n_0-n_1),r)} = 1$ for any $r \in D(M^j)$. Summing these relations over all r, we obtain (7), which contradicts (8). Therefore, g is also proportional to $v_{n_0}^j$, and the dimension of the subspace of all such functions does not exceed 1. □

Proposition 12. *Let V_j be a* PMRA *of X. A sequence $\{\varphi_j\}_{j=0}^\infty \subset X$ is a scaling sequence if and only if*

$$\varphi_j = \sum_{n\in D(M^{*j})} \alpha_n^j v_n^j, \quad \alpha_n^j \neq 0 \text{ for all } n \in D(M^{*j}), \tag{10}$$

where $\{v_n^j\}_{n\in D(M^{*j})}$ is the basis for V_j defined in Lemma 10.

Proof. The "only if" part. Let $\{\varphi_j\}_{j=0}^{\infty}$ be a scaling sequence, and let α_n^j, $n \in D(M^{*j})$, be the coefficients of the expansion of φ_j in the basis $\{v_n^j\}_{n\in D(M^{*j})}$. Applying the shift operator S_r^j, $r \in D(M^j)$, to φ_j, we obtain

$$S_r^j \varphi_j = \sum_{n\in D(M^{*j})} \alpha_n^j e^{2\pi i (M^{*-j}n, r)} v_n^j. \tag{11}$$

Suppose that $\alpha_{n_0}^j = 0$ for some n_0; then

$$V_j = \operatorname{span}\{S_r^j \varphi_j, r \in D(M^j)\} = \operatorname{span}\{v_n^j, n \in D(M^{*j}), n \neq n_0\},$$

which contradicts the minimality of the basis $\{v_n^j\}_{n\in D(M^{*j})}$.

The "if" part. Let φ_j be defined by formula (10). As above, we have

$$S_r^j \varphi_j = \sum_{n\in D(M^{*j})} \alpha_n^j e^{2\pi i (M^{*-j}n, r)} v_n^j, \quad r \in D(M^j).$$

We view this as a system of equations with the unknowns $\alpha_n^j v_n^j$. By Corollary 3, the matrix of this system is unitary (up to a factor). Since the functions $\alpha_n^j v_n^j$ constitute a basis, and any unitary transformation takes a basis to a basis, the functions $S_r^j \varphi_j$, $r \in D(M^j)$, constitute a basis for the space V_j. □

Corollary 13. *If $\{\varphi_j\}_{j=0}^{\infty}$ is a scaling sequence, then $\omega_n^j \varphi_j = \alpha_n^j v_n^j$, where $\alpha_n^j \neq 0$.*

This statement follows from (5) and the fact that $\dim V_j^{(n)} = 1$ (the $V_j^{(n)}$ are the spaces defined in Lemma 10).

Corollary 14. *In any* PMRA *there exists a scaling sequence.*

For the proof it suffices to set $\alpha_n^j = 1$ in (10).

Proof of Theorem 7. *The "only if" part.* Property (Φ1) follows from Lemma 8. To prove (Φ2) we use Corollary 13. The relation $\omega_n^j \varphi_j = \alpha_n^j v_n^j$ implies that $\widehat{\varphi}_j(k) = \widehat{\omega_k^j \varphi_j}(k) = \alpha_n^j \widehat{v}_n^j(k)$ for any $k \equiv n \pmod{M^{*j}}$. The existence of $k \equiv n \pmod{M^{*j}}$ such that $\widehat{v}_n^j(k) \neq 0$ follows from the relations $v_n^j \not\equiv 0$ (because v_n^j is a basis vector) and $\widehat{v}_n^j(\ell) = 0$ for all $\ell \not\equiv n \pmod{M^{*j}}$. Since $\alpha_n^j \neq 0$, we get $\widehat{\varphi}_j(k) \neq 0$. Property ($\Phi$3) will be proved by contradiction. Assume that $\widehat{\varphi}_j(k) = 0$ for all $j \in \mathbb{Z}_+$. This means that none of the functions φ_j involves the harmonic $e^{2\pi i(k,x)}$. In this case the same is true for the shifts $S_k^j \varphi_j$, i.e., the inner product $\langle f, e^{2\pi i(k,x)}\rangle$ is equal to zero for any $f \in \bigcup_{j=0}^{\infty} V_j$, which contradicts the completeness of the union of the spaces V_j (the axiom (MR4)). For the proof of (Φ4), we take an arbitrary $n \in \mathbb{Z}^d$. First, we analyze the case where there exists $k \equiv n \pmod{M^{*j}}$ such that $\widehat{\varphi}_{j-1}(k) \neq 0$. By Corollary 13, $\omega_n^j \varphi_j = \alpha_n^j v_n^j$ and $\omega_n^j \varphi_{j-1} = \omega_n^j \omega_n^{j-1} \varphi_{j-1} = \alpha_n^{j-1} \omega_n^j v_n^{j-1}$, $\alpha_n^j, \alpha_n^{j-1} \neq 0$. Therefore, by property (V2) in Lemma 10,

$$\widehat{\varphi}_{j-1}(\ell) = \frac{\alpha_n^{j-1}}{\alpha_n^j} \widehat{\varphi}_j(\ell)$$

for all $\ell \equiv n \pmod{M^{*j}}$. It remains to set $\mu_n^j = \alpha_n^{j-1}/\alpha_n^j$. If $\widehat{\varphi}_{j-1}(k) = 0$ for any $k \equiv n \pmod{M^{*j}}$, we set $\mu_n^j = 0$. For the proof of (Φ5), again we use Corollary 13. If $k \equiv n \pmod{M^{*j}}$, then

$$\widehat{\varphi}_{j+1}(M^*k) = \widehat{\omega_{M^*n}^{j+1} \varphi_{j+1}}(M^*k) = \alpha_{M^*n}^{j+1} \widehat{v}_{M^*n}^{j+1}(M^*k),$$

where $\alpha^{j+1}_{M^*n} \neq 0$. On the other hand, $\widehat{\varphi}_j(n) = \alpha^j_k \widehat{v}^j_n(k)$, $\alpha^j_n \neq 0$. Therefore, by property (V3) in Lemma 10, we get

$$\widehat{\varphi}_{j+1}(M^*k) = \frac{\alpha^{j+1}_{M^*n}}{\alpha^j_n}\widehat{\varphi}_j(k).$$

It remains to set $\gamma^j_n = \alpha^{j+1}_{M^*n}/\alpha^j_n$.

The "if" part. Suppose that the functions $\varphi_j \in X$ satisfy (Φ1)–(Φ5). Setting $V_j = \operatorname{span}\{S^j_n\varphi_j, n \in D(M^j)\}$, we show that $\{V_j\}_{j=0}^{\infty}$ is a PMRA of X and $\{\varphi_j\}_{j\in\mathbb{Z}_+}$ is a scaling sequence. First, we check (MR5). Let $f \in V_j$; then $f = \sum_{k\in D(M^j)} \alpha_k S^j_k \varphi_j$. Applying the shift operator S^j_p, $p \in D(M^j)$, we obtain

$$S^j_p f = \sum_{k\in D(M^j)} \alpha_k S^j_p S^j_k \varphi_j. \tag{12}$$

The periodicity of f implies that $S^j_p S^j_n f = S^j_r f$, where $r \in D(M^j)$, $r \equiv (p+n) \pmod{M^j}$. Substituting this in (12), we get $S^j_p f \in V_j$ by the definition of V_j. If $S^j_p f \in V_j$, then, as has already been proved, $f = S^j_{-p} S^j_p f \in V_j$. Now we prove (MR3). By analogy with (11), we use Lemma 9 to obtain

$$S^j_r \varphi_j = \sum_{n\in D(M^{*j})} e^{2\pi i(M^{*-j}n, r)} \omega^j_n \varphi_j. \tag{13}$$

It follows that $V_j = \operatorname{span}\{\omega^j_n\varphi_j, n \in D(M^{*j})\}$. We show that the functions $\omega^j_n\varphi_j$, $n \in D(M^{*j})$, constitute a basis for V_j. Suppose that these functions are linearly dependent. In this case there exist numbers α_k, $k \in D(M^{*j})$, $\alpha_{k_0} \neq 0$, such that $\sum_{k\in D(M^{*j})} \alpha_k \omega^j_k \varphi_j = 0$. By Lemma 9, this implies that $\widehat{\omega^j_n\varphi_j}(k) = \widehat{\varphi}_j(k) = 0$ for all $k \equiv k_0 \pmod{M^{*j}}$, which contradicts ($\Phi$2). For the proof of (MR3), it remains to note that the number of the functions $\omega^j_n\varphi_j$ is equal to m^j. Since the number of the functions $S^j_n\varphi_j, n \in D(M^j)$, is also equal to m^j, we have also proved that $\{S^j_n\varphi_j\}_{n\in D(M^j)}$ is a basis for the space V_j. For the proof of (MR1), we must check that $f \in V_j$ implies $f \in V_{j+1}$. It suffices to consider only the basis functions $\omega^j_n\varphi_j, n \in D(M^{*j})$. Lemmas 9 and 4 imply the relation

$$\omega^j_n\varphi_j = \sum_{p\in D(M^*)} \omega^{j+1}_{n+M^{*j}p}\omega^j_n\varphi_j. \tag{14}$$

Using (Φ4) and Lemma 9, and the fact that the sequence μ^{j+1}_n is M^{*j+1}-periodic with respect to the lower index, we obtain

$$\begin{aligned}
&\omega^{j+1}_{n+M^{*j}p}\omega^j_n\varphi_j \\
&\quad\sim \sum_{k\in\mathbb{Z}^d} \widehat{\omega^j_n\varphi_j}(M^{*j+1}k + n + M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)} \\
&\quad= \sum_{k\in\mathbb{Z}^d} \widehat{\varphi}_j(M^{*j+1}k + n + M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)} \\
&\quad= \sum_{k\in\mathbb{Z}^d} \mu^{j+1}_{M^{*j+1}k+n+M^{*j}p}\widehat{\varphi}_{j+1}(M^{*j+1}k + n + M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)} \\
&\quad= \mu^{j+1}_{n+M^{*j}p}\sum_{k\in\mathbb{Z}^d} \widehat{\varphi}_{j+1}(M^{*j+1}k + n + M^{*j}p)e^{2\pi i(M^{*j+1}k+n+M^{*j}p,x)} \\
&\quad\sim \mu^{j+1}_{n+M^{*j}p}\omega^{j+1}_{n+M^{*j}p}\varphi_{j+1}.
\end{aligned}$$

It follows that

$$\omega_n^j \varphi_j = \sum_{p \in D(M^*)} \mu_{n+M^{*j}p}^{j+1} \omega_{n+M^{*j}p}^{j+1} \varphi_{j+1}. \tag{15}$$

It remains to observe that, by Lemma 9, $\omega_{n+M^{*j}p}^{j+1} \varphi_{j+1} \in V_{j+1}$.

Before passing to the proof of the remaining properties, we show that V_j contains a basis satisfying the conditions listed in Lemma 10. By recursion on j, we introduce the following numbers α_n^j, $j \in \mathbb{Z}_+$, $n \in D(M^{*j})$: $\alpha_0^0 := 1$;

if $\mu_n^j \neq 0$, then $\alpha_n^j := \alpha_n^{j-1}/\mu_n^j$;

if $\mu_n^j = 0$, then $n \equiv \mathbf{0} \pmod{M^*}$, $\alpha_n^j := \alpha_{M^{*-1}n}^{j-1} \gamma_{M^{*-1}n}^{j-1}$;

if $\mu_n^j = 0$, $n \not\equiv \mathbf{0} \pmod{M^*}$, then $\alpha_n^j := 1$. Clearly, $\alpha_n^j \neq 0$ by construction. We put $v_n^j = \omega_n^j \varphi_j / \alpha_n^j$. As has already been shown, the $\omega_n^j \varphi_j$ constitute a basis for V_j. Therefore, $\{v_n^j\}_{n \in D(M^{*j})}$ is also a basis, and it is not difficult to check that properties (V1)–(V3) of Lemma 10 are fulfilled. Property (MR4) follows from Lemma 11. To prove (MR6), it suffices to show that this property is fulfilled for the functions $\{v_n^j\}_{n \in D(M^{*j})}$. Using Lemma 9 and (V3), we see that

$$\begin{aligned} v_n^j(Mx) &\sim \sum_{k \in \mathbb{Z}^d} \widehat{v}_n^j(k) e^{2\pi i (k, Mx)} = \sum_{k \in \mathbb{Z}^d} \widehat{v}_{M^*n}^{j+1}(M^*k) e^{2\pi i (M^*k, x)} \\ &= \sum_{l \in \mathbb{Z}^d, l \equiv \mathbf{0} \pmod{M^*}} \widehat{v}_{M^*n}^{j+1}(l) e^{2\pi i (l, x)} \\ &\sim v_{M^*n}^{j+1}(x). \end{aligned}$$

For the proof of (MR6a), it remains to note that $v_{M^*n}^{j+1} \in V_{j+1}$. We check (MR6b). First, we consider the case where $n \equiv \mathbf{0} \pmod{M^*}$. By (V3), we get

$$\sum_{k \in D(M)} v_n^{j+1}(M^{-1}x + M^{-1}k) = \sum_{k \in D(M)} v_{M^{*-1}n}^j (x + k) \in V_j.$$

Now, let $n \not\equiv \mathbf{0} \pmod{M^*}$. Then

$$\begin{aligned} &\sum_{k \in D(M)} v_n^{j+1}(M^{-1}x + M^{-1}k) \\ &\quad \sim \sum_{l \in \mathbb{Z}^d} \sum_{k \in D(M)} \widehat{v}_n^{j+1}(M^{*j+1}l + n) e^{2\pi i (M^{*j+1}l + n, M^{-1}x + M^{-1}k)} \\ &\quad = \sum_{l \in \mathbb{Z}^d} \widehat{v}_n^{j+1}(M^{*j+1}l + n) e^{2\pi i (M^{*j+1}l + n, M^{-1}x)} \sum_{k \in D(M)} e^{2\pi i (M^{*j}l + M^{*-1}n, k)}. \end{aligned}$$

But the last sum is equal to zero by Lemma 2, and so (MR6b) is fulfilled. It remains to prove (MR2). Since the trigonometric polynomials are dense in X, it suffices to check that any trigonometric polynomial can be approximated by functions in $\bigcup_{j=0}^{\infty} V_j$. It suffices to check this for an individual harmonic. Let $f_r(x) = e^{2\pi i (r,x)}$, $r \neq \mathbf{0}$. From (Φ3) it follows that there exists j_0 with $\widehat{\varphi}_{j_0}(r) \neq 0$. Since $\widehat{\varphi}_j(r) \neq 0$ by (Φ4), we have $\widehat{v}_r^j(r) \neq 0$ for all $j \geq j_0$. We define the functions h_j for $j \geq j_0$ by

$$h_j(x) := 1 - \frac{v_r^j(x)}{\widehat{v}_r^j(r)} e^{-2\pi i (r,x)}.$$

Then

$$\widehat{h}_j(n) = \delta_{n0} - \frac{\widehat{v}_r^j(r-n)}{\widehat{v}_r^j(r)}, \quad n \in \mathbb{Z}^d.$$

Thus, $\widehat{h}_j(n) \neq 0$ only if $n \equiv \mathbf{0} \pmod{M^{*j}}$, $n \neq \mathbf{0}$. Hence, $\widehat{h}_j(n) = 0$ for all $n \in M^{*j}(-1,1)^d$. We select a subsequence of embedded parallelepipeds $M^{*j_k}(-1,1)^d$. By (1), there exists n_0 such that $\|M^{*-n_0}\| \leq 1/2$. Setting $j_{k+1} = j_k + n_0$, we obtain $\|M^{*j_{k+1}}x\| \geq 2\|M^{*j_k}x\|$ for any $x \in \mathbb{R}^d$. For sufficiently large k, in each parallelepiped $M^{*j_k}(-1,1)^d$ we can inscribe a cube K_{j_k} with center at the origin, with edges parallel to the coordinate axes, and with integral side length a_k that monotonically increases to infinity as k increases. For convenience, in the sequel we shall write K_j for this subsequence. As has already been noted, $\widehat{h}_j(n) = 0$ for all integral n in K_j. Consequently, the partial sums over K_j of the Fourier series of h_j are equal to zero. Therefore, the corresponding Fejér means $\sigma_{K_j}(h_j)$ over these cubes are also equal to zero. We need the following identity for $j \geq j_0$:

$$h_j(x) = m^{j_0-j} \sum_{n \in D(M^{j-j_0})} h_{j_0}(x + M^{-j}n). \tag{16}$$

For the proof, we apply Lemma 9 to rearrange the right-hand side:

$$\begin{aligned}
&m^{j_0-j} \sum_{n \in D(M^{j-j_0})} h_{j_0}(x + M^{-j}n) \\
&\quad = m^{j_0-j} \sum_{n \in D(M^{j-j_0})} \Big[1 - \frac{v_r^{j_0}(x + M^{-j}n)}{\widehat{v}_r^{j_0}(r)} e^{-2\pi i(r, x+M^{-j}n)}\Big] \\
&\quad \sim -\frac{m^{j_0-j}}{\widehat{v}_r^{j_0}(r)} \sum_{n \in D(M^{j-j_0})} e^{-2\pi i(r, x+M^{-j}n)} \sum_{\substack{l \in \mathbb{Z}^d, \\ l \neq \mathbf{0}}} \widehat{v}_r^{j_0}(M^{*j_0}l + r) e^{2\pi i(M^{*j_0}l + r, x+M^{-j}n)} \\
&\quad = -\frac{m^{j_0-j}}{\widehat{v}_r^{j_0}(r)} \sum_{\substack{l \in \mathbb{Z}^d, \\ l \neq \mathbf{0}}} \widehat{v}_r^{j_0}(M^{*j_0}l + r) e^{2\pi i(M^{*j_0}l, x)} \sum_{n \in D(M^{j-j_0})} e^{-2\pi i(M^{*(j_0-j)}l, n)}.
\end{aligned}$$

By Lemma 2, the inner sum on the right is equal to m^{j-j_0} if $l \equiv \mathbf{0} \pmod{M^{j-j_0}}$, and is equal to zero otherwise. This means that in the outer sum only the terms with indices $l \equiv \mathbf{0} \pmod{M^{j-j_0}}$ are nonzero. This sum can be rewritten as

$$-\frac{m^{j_0-j}}{\widehat{v}_r^{j_0}(r)} \sum_{k \in \mathbb{Z}^d, k \neq \mathbf{0}} \widehat{v}_r^{j_0}(M^{*j}k + r) e^{2\pi i(M^{*j}k + r, x)} e^{-2\pi i(r,x)} \sim 1 - \frac{v_r^j(x)}{\widehat{v}_r^{j_0}(r)} e^{-2\pi i(r,x)}.$$

Since, by (V2), $\widehat{v}_r^{j_0}(r) = \widehat{v}_r^j(r)$ for $j \geq j_0$, and $\widehat{v}_r^{j_0}(r) \neq 0$, we arrive at (16). Now, (16) and the linearity of the Fejér means yield

$$\begin{aligned}
\|h_j\| &= \|h_j - \sigma_{K_j}(h_j)\| \\
&= \Big\| m^{j_0-j} \sum_{n \in D(M^{j-j_0})} [h_{j_0}(x + M^{-j}n) - \sigma_{K_j}(h_{j_0})(x + M^{-j}n)] \Big\| \\
&\leq m^{j_0-j} \sum_{n \in D(M^{j-j_0})} \|h_{j_0}(x + M^{-j}n) - \sigma_{K_j}(h_{j_0})(x + M^{-j}n)\| \\
&= m^{j_0-j} \sum_{n \in D(M^{j-j_0})} \|h_{j_0} - \sigma_{K_j}(h_{j_0})\|.
\end{aligned}$$

The last-written expression tends to zero as $j \to \infty$, because the Fejér means of a function in X converge to that function in norm (see, e.g., [23, Chapter 17, §1]). Thus, we have proved that

$$\lim_{j \to \infty} \Big\| f_r(x) - \frac{v_r^j(x)}{\widehat{v}_r^j(r)} \Big\| = \lim_{j \to \infty} \|h_j(x)\| = 0,$$

i.e., f_r is approximated by the functions $v_r^j(x)/\widehat{v}_r^j(r) \in V_j$. $\square$

From the proof of the theorem and the remark to Lemma 10, it is clear that if we exclude the axiom (MR6) from the definition of a PMRA, then a scaling sequence will be characterized by properties (Φ2)–(Φ4). This means that the following statement is true.

Theorem 15. *Let $\varphi_j \in X$, $j \in \mathbb{Z}_+$, and let $V_j = \operatorname{span}\{S_n^j \varphi_j, n \in D(M^{*j})\}$. The axioms* (MR1)–(MR5) *of Definition* 5 *are fulfilled for a collection of spaces $\{V_j\}_{j=0}^{\infty}$ if and only if the functions φ_j satisfy conditions* Φ2–Φ4 *of Theorem* 7.

A wide class of PMRAs of $L_2(\mathbb{T}^d)$ can be constructed by the following standard method. A scaling sequence is obtained by periodization of a function $\varphi \in L_2(\mathbb{R}^d)$ by the formulas

$$\varphi_j(x) = \sum_{k \in \mathbb{Z}^d} \varphi(M^j x + M^j k) \tag{17}$$

(we say that such a PMRA is generated by the function φ). Let $\varphi \in L_2(\mathbb{R}^d)$ be a scaling function of a nonperiodic MRA, i.e., the following conditions are fulfilled:

(i) there exist positive constants A, B such that

$$A \le \sum_{m \in \mathbb{Z}^d} |\widehat{\varphi}(\xi + m)|^2 < B \quad \text{for a.e. } \xi \in \mathbb{R}^d;$$

(ii) there exists a function $m_0 \in L_2(\mathbb{T}^d)$ such that

$$\widehat{\varphi}(M^* \xi) = m_0(\xi) \widehat{\varphi}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^d;$$

(iii) the function $\widehat{\varphi}$ is continuous at zero and $\widehat{\varphi}(\mathbf{0}) \neq 0$. If, moreover, we assume that φ decays sufficiently fast at infinity, for example,

$$\varphi(x) = O\left(\frac{1}{(1+|x|)^{d+\epsilon}}\right), \quad \epsilon > 0$$

(which is usually the case for the known MRA), then $\varphi_j \in L_2(\mathbb{T}^d)$, and, by the Poisson summation formula,

$$\varphi_j(x) = m^{-j} \sum_{m \in \mathbb{Z}} \widehat{\varphi}(M^{*-j} m) e^{2\pi i (m,x)}.$$

We check that the φ_j satisfy conditions (Φ1)–(Φ5) of Theorem 7. To prove (Φ1), we observe that, by (ii) and (iii), $m_0(n) = m_0(0) = 1$ for all $n \in \mathbb{Z}^d$. Therefore, if $\widehat{\varphi}_j(k) = m^{-j} \widehat{\varphi}(M^{*-j} k) \neq 0$ for some $k \in \mathbb{Z}^d$, $k \neq \mathbf{0}$, then $\widehat{\varphi}_j(M^{*\ell} k) = \widehat{\varphi}_j(k) \neq 0$ for all $\ell \in \mathbb{Z}_+$, which cannot be true for a function in $L_2(\mathbb{T}^d)$. Properties (Φ2) and (Φ3) follow (respectively) from (i) and (iii). It is not difficult to check Φ4) and (Φ5) if we set $\mu_k^j = m_0(M^{*-j-1} k)$, $\gamma_k^j = m$.

In [19], a condition was found under which a PMRA in $L_p(\mathbb{T}^1)$ or $C(\mathbb{T}^1)$ is generated by an integrable function. However, a PMRA can be generated by a nonintegrable function. For instance, the function

$$\varphi(x) = \frac{\sin \pi x}{\pi x} \tag{18}$$

is scaling; though $\varphi \notin L(\mathbb{R})$, periodization of φ is possible because the series (17) converges in the principal value sense. Petukhov [24] found a PMRA that is not generated by a function (in the above sense). In fact, the existence of a generating function have not yet been investigated adequately even in the one-dimensional case.

§4. Wavelet spaces

In this section, we follow a standard idea of constructing wavelet bases, in order to determine the wavelet spaces and the functions whose shifts constitute bases in these spaces. In the orthogonal case (which may occur only if $X = L_2(\mathbb{T}^d)$), the wavelet space is the orthogonal projection of the space V_{j+1} with kernel V_j. In the biorthogonal case we deal with two multiresolution analyses. The space V_{j+1} of one of them is projected orthogonally relative to the corresponding component $\widetilde{V}_j$ of the other.

We shall consider pairs of PMRA with the first component $\{X_j\}_{j=0}^{\infty}$ in $L_p(\mathbb{T}^d)$, $1 \le p \le \infty$ ($C(\mathbb{T}^d)$ for $p = \infty$) and the second component $\{\widetilde{V}_j\}_{j=0}^{\infty}$ in $L_q(\mathbb{T}^d)$, $1/p + 1/q = 1$ ($C(\mathbb{T}^d)$ for $p = 1$). Such a pair is called a (p, q)-pair.

Proposition 16. *Let $\{V_j\}_{j=0}^{\infty}$ and $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form a (p, q)-pair, and let $\varphi \in V_j$, $\widetilde{\varphi} \in \widetilde{V}_j$. The systems of functions $\{S_n^j\varphi\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}\}_{k\in D(M^j)}$ are biorthonormal if and only if*

$$\langle \omega_r^j\varphi, \omega_r^j\widetilde{\varphi}\rangle = m^{-j} \tag{19}$$

for all $r \in \mathbb{Z}^d$

Proof. By Lemma 9, we have

$$\begin{aligned}\langle S_n^j\varphi, S_k^j\widetilde{\varphi}\rangle &= \Big\langle \sum_{r\in D(M^{*j})} e^{2\pi i(M^{*-j}r,n)}\omega_r^j\varphi, \sum_{s\in D(M^{*j})} e^{2\pi i(M^{*-j}s,k)}\omega_s^j\widetilde{\varphi}\Big\rangle \\ &= \sum_{r\in D(M^{*j})} l \sum_{s\in D(M^{*j})} e^{2\pi i(M^{*-j}r,n)}e^{-2\pi i(M^{*-j}s,k)}\langle \omega_r^j\varphi, \omega_s^j\widetilde{\varphi}\rangle.\end{aligned}$$

Since the spectra of the functions $\omega_r^j\varphi$ and $\omega_s^j\widetilde{\varphi}$ are disjoint for $r \ne s$, the corresponding terms in the latter sum are equal to zero. Thus,

$$\langle S_n^j\varphi, S_k^j\widetilde{\varphi}\rangle = \sum_{r\in D(M^{*j})} e^{2\pi i(M^{*-j}r,n-k)}c_r, \tag{20}$$

where $c_r := \langle \omega_r^j\varphi, \omega_r^j\widetilde{\varphi}\rangle$. This and Lemma 2 ensure the "if" part. Now we assume that $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{k\in D(M^j)}$ are biorthonormal systems. We regard identities (20) with a fixed $n \in D(M^j)$ for all $k \in D(M^j)$ as a system of equations with the unknowns c_r. By Corollary 3, the solution $c_p = m^{-j}$, $p \in D(M^{*j})$, is unique. It remains to note that $c_p = c_{p+M^{*j}}$, i.e., $c_p = m^{-j}$ for all $p \in \mathbb{Z}^d$. □

Corollary 17. *Suppose that $\{V_j\}_{j=0}^{\infty}$ and $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form a (p, q)-pair with scaling sequences $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$, respectively, and let μ_n^j and $\widetilde{\mu}_k^j$ be the factors occurring in property (Φ4) in Theorem 7. If the systems $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthogonal, then*

$$\sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\overline{\widetilde{\mu}^j_{p+M^{*j-1}k}} = m \tag{21}$$

for all $p \in \mathbb{Z}^d$ and all $j \in \mathbb{Z}_+$.

Proof. As in (15), we have

$$\begin{aligned} m^{-j+1} &= \langle \omega_p^{j-1}\varphi_{j-1}, \omega_p^{j-1}\widetilde{\varphi}_{j-1}\rangle \\ &= \Big\langle \sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\omega^j_{p+M^{*j-1}k}\varphi_{j-1}, \sum_{l\in D(M^*)} \widetilde{\mu}^j_{p+M^{*j-1}l}\omega^j_{p+M^{*j-1}l}\widetilde{\varphi}_{j-1}\Big\rangle \\ &= \sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\overline{\widetilde{\mu}^j_{p+M^{*j-1}k}}\langle \omega^j_{p+M^{*j-1}k}\varphi_j, \omega^j_{p+M^{*j-1}k}\widetilde{\varphi}_j\rangle \\ &= m^{-j}\sum_{k\in D(M^*)} \mu^j_{p+M^{*j-1}k}\overline{\widetilde{\mu}^j_{p+M^{*j-1}k}}, \end{aligned}$$

and it remains to multiply both sides of this identity by m^j. □

Now we begin the construction of wavelet spaces and bases. First, we treat the orthogonal case. Let $\{V_j\}_{j=0}^\infty$ be a PMRA of $L_2(\mathbb{T}^d)$, with a scaling sequence $\{\varphi_j\}_{j=0}^\infty$ that generates the orthonormal shift bases $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ for the spaces V_j. We restrict ourselves to the case where all μ_k^j (the factors in property (Φ4) in Theorem 7) are real. Our goal is to find functions $\psi^{(\nu)}$, $\nu = 1,\dots,m-1$, in the space V_{j+1} such that the corresponding systems $\{S_n^j\psi_j^{(\nu)}\}_{n\in D(M^j)}$ are orthonormal and orthogonal to V_j, and complement $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ to a basis for the space V_{j+1}. For this we need to complete a matrix up to a unitary one starting with its first row. Suppose that real numbers $a_{00},\dots,a_{0m}$ (the first row of a future unitary matrix A) satisfy the condition

$$\sum_{r=1}^m a_{0r}^2 = 1. \tag{22}$$

If $a_{00} = 1$ and $a_{01},\dots,a_{0m} = 0$, we define A as the unit matrix. If $a_{00} \neq 1$, then the remaining elements of A are given by the Householder transformation:

$$a_{lk} = a_{kl} = -\frac{a_{0k}a_{0l}}{1-a_{00}},\ \ k\neq l, \quad a_{kk} = 1 - \frac{a_{0k}^2}{1-a_{00}}. \tag{23}$$

We fix $j\in\mathbb{Z}_+$ and $n\in D(M^{*j})$. Let s_k, $k=0,\dots,m-1$, be an enumeration of the set $D(M^*)$. We put $a_{0k} = \mu^{j+1}_{n+M^{*j}s_k}/\sqrt{m}$, $k=0,\dots,m-1$. By Corollary 17, these numbers satisfy (22); therefore, we can complete this row up to a unitary matrix A. Set $\alpha^{\nu,j}_{n+M^{*j}s_k} = \sqrt{m}a_{\nu k}$. If n runs over the entire set $D(M^{*j})$, then, by Lemma 4, the vectors $n+M^{*j}s_k$, $k=0,\dots,m-1$, run over the entire set $D(M^{*j+1})$, i.e., the numbers $\alpha_s^{\nu,j}$ are well defined for all $s\in D(M^{*j+1})$. We extend this sequence (with respect to the subindex) to $\mathbb{Z}^d$ by setting $\alpha_l^{\nu,j} = \alpha_s^{\nu,j}$ for all $l\equiv s \pmod{M^{*j+1}}$. For each $\nu = 1,\dots,m-1$, we introduce the wavelet functions $\psi^{(\nu)}$ via their Fourier coefficients $\widehat{\psi}_j^{(\nu)}(l) = \alpha_s^{\nu,j}\widehat{\varphi}_{j+1}(l)$, $l\in\mathbb{Z}^d$, and define the wavelet spaces to be

$$W_j^{(\nu)} := \operatorname{span}\{S_n^j\psi_j^{(\nu)}, n\in D(M^j)\}.$$

Theorem 18. *Let $\{V_j\}$ be a* PMRA *of $L^2(\mathbb{T}^d)$ with a scaling sequence $\{\varphi_j\}$, and let $S_n^j\varphi_j$, $n\in D(M^j)$, be an orthonormal system for any $j\in\mathbb{Z}_+$. Then*

$$V_{j+1} = V_j \oplus W_j^{(1)} \oplus \cdots \oplus W_j^{(m-1)}, \quad j\in\mathbb{Z}_+,$$

and $\{S_n^j\psi_j^{(\nu)}\}_{n\in D(M^j)}$ is an orthonormal basis for the space $W_j^{(\nu)}$, $\nu = 1,\dots,m-1$.

The proof of this theorem will be presented in a more general situation (Theorem 19).

Now we consider the biorthogonal case. Let a (p,q)-pair satisfy the assumptions of Corollary 17. To construct the wavelet functions, we need to supplement two suitable

rows up to two mutually inverse matrices. Let numbers $a_{00},\dots,a_{0m-1}$ and $\widetilde{a}_{00},\dots,\widetilde{a}_{0m-1}$ (the first rows of the future matrices A and $\widetilde{A}$, respectively) be such that

$$\sum_{r=0}^{m-1} a_{0r}\overline{\widetilde{a}_{0r}} = 1. \tag{24}$$

First we assume that $a_{00} = \overline{\widetilde{a}_{00}} \neq 1$. In this case, for $k,l = 1,\dots,m-1$, the remaining elements can be defined as follows:

$$a_{l0} = \overline{\widetilde{a}_{0l}}, \quad a_{lk} = \delta_{lk} - \frac{\overline{\widetilde{a}_{0l}}a_{0k}}{1-a_{00}}, \tag{25}$$

$$\widetilde{a}_{l0} = \overline{a_{0l}}, \quad \widetilde{a}_{lk} = \delta_{lk} - \frac{\overline{a_{0l}}\widetilde{a}_{0k}}{1-\overline{a_{00}}}. \tag{26}$$

It is easy to check that

$$A\widetilde{A}^* = E_m. \tag{27}$$

Now we assume that $a_{00}\widetilde{a}_{00} \neq 0$. Consider the numbers $a'_{0k} = Ca_{0k}$, $\widetilde{a}'_{0k} = \widetilde{a}_{0k}/\overline{C}$, $k = 0,\dots,m-1$, where C is chosen in such a way that $a'_{00} = \overline{\widetilde{a}'_{00}} \neq 1$. It suffices to set $C = \sqrt{\overline{\widetilde{a}_{00}}/a_{00}}$ and to choose a complex value of the square root for which $a'_{00} \neq 1$. Since the new rows satisfy all the requirements of the preceding case, we can complete these rows up to matrices A' and $\widetilde{A}'$ such that $A'\widetilde{A}'^* = E_m$. After replacing the first rows in these matrices by the initial ones, we obtain the required matrices A and $\widetilde{A}$. Finally, suppose that $a_{00}\widetilde{a}_{00} = 0$. From (24) it follows that there exists r_0 such that $a_{0r_0}\widetilde{a}_{0r_0} \neq 0$. Interchanging a_{0r_0} with a_{00} and $\widetilde{a}_{0r_0}$ with $\widetilde{a}_{00}$, we return to the preceding case. Supplementing the new rows up to mutually inverse matrices and interchanging the 0th and r_0th columns again, we obtain the required matrices A and $\widetilde{A}$.

Fix $j \in \mathbb{Z}_+$ and $n \in D(M^{*j})$. As above, let s_k, $k = 0,\dots,m-1$, be an enumeration of the set $D(M^*)$. We put $a_{0k} = \mu^{j+1}_{n+M^{*j}s_k}/\sqrt{m}$, $\widetilde{a}_{0k} = \widetilde{\mu}^{j+1}_{n+M^{*j}s_k}/\sqrt{m}$, $k = 0,\dots,m-1$. By Corollary 17, these numbers satisfy (24); therefore, we can complete these rows up to matrices A and $\widetilde{A}$ satisfying (27). Let $\alpha^{\nu,j}_{n+M^{*j}s_k} = \sqrt{m}a_{\nu k}$, $\widetilde{\alpha}^{\nu,j}_{n+M^{*j}s_k} = \sqrt{m}\widetilde{a}_{\nu k}$. From (27) it follows that

$$\sum_{k=0}^{m-1} \alpha^{\nu,j}_{n+M^{*j}s_k}\overline{\widetilde{\mu}^{j+1}_{n+M^{*j}s_k}} = 0, \quad \sum_{k=0}^{m-1} \widetilde{\alpha}^{\nu,j}_{n+M^{*j}s_k}\overline{\mu^{j+1}_{n+M^{*j}s_k}} = 0, \quad \nu = 1,\dots,m-1, \tag{28}$$

$$\sum_{k=0}^{m-1} \alpha^{l,j}_{n+M^{*j}s_k}\overline{\widetilde{\alpha}^{p,j}_{n+M^{*j}s_k}} = m\delta_{\nu l}, \quad l,\nu = 1,\dots,m-1. \tag{29}$$

If n runs over the entire set $D(M^{*j})$, then, by Lemma 4, the vectors $n + M^{*j}s_k$, $k = 0,\dots,m-1$, run over the entire set $D(M^{*j+1})$, i.e., the numbers $\alpha^{\nu,j}_s$, $\widetilde{\alpha}^{\nu,j}_s$ are well defined for all $s \in D(M^{*j+1})$. We extend these sequences (with respect to the subindex) to $\mathbb{Z}^d$ by setting $\alpha^{\nu,j}_l = \alpha^{\nu,j}_s$ and $\widetilde{\alpha}^{\nu,j}_l = \widetilde{\alpha}^{\nu,j}_s$ for all $l \equiv s \pmod{M^{*j+1}}$. For each $\nu = 1,\dots,m-1$, we introduce the wavelet functions $\psi^{(\nu)}$, $\widetilde{\psi}^{(\nu)}$ via their Fourier coefficients $\widehat{\psi}^{(\nu)}_j(l) = \alpha^{\nu,j}_s\widehat{\varphi}_{j+1}(l)$, $\widehat{\widetilde{\psi}}^{(\nu)}_j(l) = \widetilde{\alpha}^{\nu,j}_s\widehat{\widetilde{\varphi}}_{j+1}(l)$, $l \in \mathbb{Z}^d$, and define the wavelet spaces to be

$$W^{(\nu)}_j := \operatorname{span}\{S^j_n\psi^{(\nu)}_j, n \in D(M^j)\},$$
$$\widetilde{W}^{(\nu)}_j := \operatorname{span}\{S^j_n\widetilde{\psi}^{(\nu)}_j, n \in D(M^j)\}.$$

Observe that the orthogonal wavelets can be constructed by the general method described, but this is more complicated.

Theorem 19. *Suppose $\{V_j\}_{j=0}^{\infty}$ and $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form a (p,q)-pair with scaling sequences $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ (respectively) such that $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthonormal systems. Then:*

1) $W_j^{(\nu)} \subset V_{j+1}$, $\nu = 1,\dots,m-1$;
2) *for* $f \in V_{j+1}$ *we have* $f = f_0 + \sum_{\nu=1}^{m-1} f_\nu$, *where* $f_0 \in V_j$, $f_\nu \in W_j^{(\nu)}$;
3) $W_j^{(\nu)} \perp \widetilde{V}_j$, $\widetilde{W}_j^{(\nu)} \perp V_j$, $\nu = 1,\dots,m-1$;
4) $W_j^{(\nu)} \perp \widetilde{W}_j^{(\kappa)}$ *for* $\nu \neq \kappa$, $\nu,\kappa = 1,\dots,m-1$;
5) $(S_n^j\psi_j^{(\nu)}, S_k^j\widetilde{\psi}_j^{(\nu)}) = \delta_{nk}$, $\nu = 1,\dots,m-1$, $n,k \in D(M^j)$.

Proof. For fixed n and j, from (15) it follows that

$$\omega_n^j\varphi_j(x) = \sum_{l\in D(M^*)} \mu_{n+M^{*j}l}^{j+1}\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}(x). \tag{30}$$

Similarly,

$$\omega_n^j\psi_j^{(\nu)}(x) = \sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}(x), \quad \nu = 1,\dots,m-1. \tag{31}$$

We regard (30), (31) as a system of m equations with m unknowns $\{\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}\}$, $l \in D(M^*)$. By the construction of the numbers $\{a_{\nu k}\}$, the matrix of this system has an inverse. Consequently, its determinant is nonzero. Therefore, the unknowns $\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}$ can be expressed via $\omega_n^j\varphi_j \in V_j$ and $\omega_n^j\psi_j^{(\nu)} \in W_j^{(\nu)}$, which implies 1). On the other hand, as in (13), we have

$$S_r^j\psi_j^{(\nu)} = \sum_{n\in D(M^{*j})} e^{2\pi i(M^{*-j}n,r)}\omega_n^j\psi_j^{(\nu)}, \quad \nu = 1,\dots,m-1.$$

Using Corollary 3, we see that each $\omega_n^j\psi_j^{(\nu)}$ can be expressed via $S_r^j\psi_j^{(\nu)}$, $r \in D(M^j)$. For the proof of 2), it remains to note that $\{\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}\}$ is a basis of the space V_{j+1}. Moreover, since the functions $\omega_n^j\psi_j^{(\nu)}$, $n \in D(M^{*j})$, are linearly independent, the above arguments and property (MR3) in Definition 5 show that $\dim W_j^{(\nu)} = m^j$, i.e., both systems $\{\omega_n^j\psi_j^{(\nu)}\}_{n\in D(M^{*j})}$ and $\{S_r^j\psi_j^{(\nu)}\}_{r\in D(M^j)}$ are bases of $W_j^{(\nu)}$. For the proof of 3), it suffices to check that the basis functions of the space $\widetilde{V}_j$ are orthogonal to the basis functions of the space $W_j^{(\nu)}$, $\nu = 1,\dots,m-1$. Using (28) and Proposition 16, we obtain

$$\begin{aligned}
&\langle \omega_n^j\psi_j^{(\nu)}, \omega_k^j\widetilde{\varphi}_j\rangle \\
&\quad= \Big\langle \sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}, \sum_{k\in D(M^*)} \widetilde{\mu}_{n+M^{*j}k}^{j+1}\omega_{n+M^{*j}k}^{j+1}\widetilde{\varphi}_{j+1}\Big\rangle \\
&\quad= \sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\overline{\widetilde{\mu}_{n+M^{*j}l}^{j+1}}\langle \omega_{n+M^{*j}l}^{j+1}\varphi_{j+1}, \omega_{n+M^{*j}l}^{j+1}\widetilde{\varphi}_{j+1}\rangle \\
&\quad= m^{-j-1}\sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\overline{\widetilde{\mu}_{n+M^{*j}l}^{j+1}} \\
&\quad= 0.
\end{aligned}$$

In a similar way, invoking (29), we obtain

$$\langle \omega_n^j\psi_j^{(\nu)}, \omega_n^j\widetilde{\psi}_j^{(\kappa)}\rangle = m^{-j-1}\sum_{l\in D(M^*)} \alpha_{n+M^{*j}l}^{\nu,j}\overline{\widetilde{\alpha}_{n+M^{*j}l}^{\kappa,j}} = m^{-j}\delta_{\nu\kappa}.$$

This yields 4). Applying Proposition 16, we obtain 5). □

§5. Kotel′nikov–Shannon wavelets

We construct an example of a PMRA of $L^2(\mathbb{T}^2)$ with a scaling sequence formed by trigonometric polynomials with minimal possible symmetric spectra. A one-dimensional analog is the well-known Kotel′nikov–Shannon PMRA for which the sequence of Dirichlet kernels serves as a scaling sequence, and the generating function is given by (18). For the first time, expansions with respect to the one-dimensional Kotel′nikov–Shannon system were used for transmission of continuous information by communication channels.

We take the matrix $M = \left(\begin{smallmatrix} 2 & -1 \\ 2 & 1 \end{smallmatrix}\right)$ as the scale factor. Observing that $m = 4$, $M^* = \left(\begin{smallmatrix} 2 & 2 \\ -1 & 1 \end{smallmatrix}\right)$, and $M^{*-1} = \frac{1}{4}\left(\begin{smallmatrix} 1 & -2 \\ 1 & 2 \end{smallmatrix}\right)$, we fix the set $D(M^*)$ formed by the vectors $s_0 = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$, $s_1 = \left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$, $s_2 = \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$, and $s_3 = \left(\begin{smallmatrix} -1 \\ -1 \end{smallmatrix}\right)$. Let Ω_j denote the parallelogram $M^{*j}[-1,1]^2$ without vertices. We set $a_j := M^{*j}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $b_j := M^{*j}\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$. Let φ_j be the function with the Fourier coefficients defined as follows: $\widehat{\varphi}_0(k) := \delta_{k0}$ for all $k \in \mathbb{Z}^2$, and for $j \in \mathbb{N}$,

$$\widehat{\varphi}_j(k) := \begin{cases} 2^{-j} & \text{if } k \in \Omega_{j-1} \setminus \{a_{j-1}, b_{j-1}\}, \\ 2^{-j-1/2} & \text{if } k = a_{j-1} \text{ or } k = b_{j-1}, \\ 0 & \text{if } k \notin \Omega_{j-1}. \end{cases} \tag{32}$$

To show that this sequence is scaling, we need the following lemma.

Lemma 20. *The number of integral points in Ω_j is equal to $4^{j+1}+1$, all integral points in Ω_j except the point a_j are in different cosets of M^{*j+1}, and a_j is congruent to b_j modulo M^{*j+1}.*

Proof. First, we show that no integral point lies on the midlines of the parallelogram Ω_j except for the boundary points and zero. The midlines of Ω_j are the segments that join the point $M^{*j}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ with $M^{*j}\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$ and the point $M^{*j}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ with $M^{*j}\left(\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right)$. These segments pass through the origin and are symmetric with respect to the origin. Therefore, it suffices to prove the claim for the half-segments. We consider one of the half-segments (the argument is similar for the other one), i.e., we show, that no integral point lies on the interval $\left(\mathbf{0}, M^{*j}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$. Set $\left(\begin{smallmatrix} x_j \\ y_j \end{smallmatrix}\right) := M^{*j}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = M^*\left(\begin{smallmatrix} x_{j-1} \\ y_{j-1} \end{smallmatrix}\right)$. For all $j > 0$, the first coordinate of this vector is even, and the second is odd. Hence, a multiple of 2 cannot be a common divisor of x_j and y_j. From the formulas

$$x_{j-1} = \frac{x_j - 2y_j}{4}, \quad y_{j-1} = \frac{x_j + 2y_j}{4}$$

it is clear that if x_j and y_j have an odd common divisor, then x_{j-1} and y_{j-1} have the same common divisor. However, $x_0 = 1$ and $y_0 = 0$ are coprime. Consequently, by induction, x_j and y_j are coprime for any $j \in \mathbb{Z}_+$. For j fixed, we represent the segments $\left(\mathbf{0}, M^{*j}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$ in the parametric form $\left\{\begin{smallmatrix} x = tx_j \\ y = ty_j \end{smallmatrix}\right.$, where $t \in (0,1)$. Suppose that an integral point (x^0, y^0) belongs to this segment, i.e., there exists $t^0 \in (0,1)$ such that $x^0 = t^0 x_j$, $y^0 = t^0 y_j$, where $x^0, y^0, x_j, y_j \in \mathbb{Z}$. Then t^0 cannot be irrational. Nor can t^0 be rational, because if $t^0 = p/q$, then x_j and y_j are divisible by q and are not coprime, which is impossible.

We note that the set of integral points in $M^{*j}\mathbb{T}^2$ can be taken as a set of digits $D(M^{*j})$, because, for any two elements of $M^{*j}\mathbb{T}^2 \cap \mathbb{Z}^2$, their difference $M^{*j}r_1 - M^{*j}r_2$, where $r_1, r_2 \in \mathbb{T}^2$, can be congruent to zero modulo M^{*j} only if r_1 and r_2 coincide. However, the number of integral vectors in $M^{*j}\mathbb{T}^2$ is equal to m^j, i.e., coincides with the cardinality of $D(M^{*j})$. Since no integral point lies on the midlines of Ω_j except the boundary points and the point zero, the number of integral points in Ω_j is four times the number of integral inner points in $M^{*j}\mathbb{T}^2$ (there are $4^j - 1$ of them, because the digit corresponding to the zero coset is excluded from $M^{*j}[0,1)^2$), plus the number of

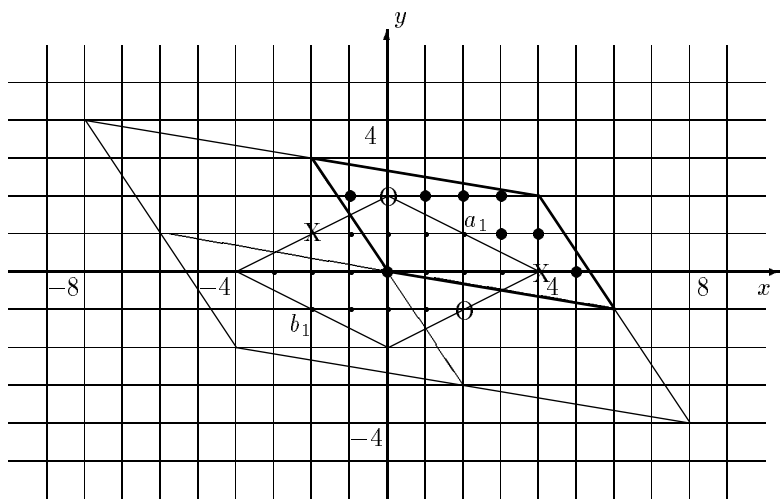

FIGURE 1. The inner and the outer parallelograms correspond to Ω_1 and Ω_2, respectively.

boundary points, plus 1 (the zero point). The number of boundary points is 4, because the endpoints of the midlines are integral, the vertices do not belong to Ω_j, and, since each edge differs from the corresponding midline by an integral shift, there are no other integral points on the boundary. Thus, the number of integral points in Ω_j is equal to $4(4^j-1)+1+4=4^{j+1}+1$.

Now we check that all points in Ω_j except a_j belong to different cosets of the matrix M^{*j+1}, and $a_j \equiv b_j \pmod{M^{*j+1}}$. By Lemma 1, the set $\bigcup_{k=0}^{3}\left(M^{*-1}[0,1)^2+M^{*-1}s_k\right)$ is congruent to $[0,1)^2$ modulo $\mathbb{Z}^2$. However, since

$$\bigcup_{k=0}^{3}(M^{*-1}[0,1)^2+M^{*-1}s_k)=\bigcup_{k=0}^{3}\left(M^{*-1}([0,1)^2+s_k)\right)=M^{*-1}[-1,1)^2,$$

we see that $M^{*-1}[-1,1)^2$ is congruent to $[0,1)^2$ modulo $\mathbb{Z}^2$. Fixing $j\in\mathbb{Z}_+$ and applying the operator M^{*j+1}, we conclude that the set $M^{*j}[-1,1)^2$ is congruent to the set $M^{*j+1}[0,1)^2$ modulo M^{*j+1}. Since all integral points in $M^{*j+1}[0,1)^2$ belong to different cosets of the matrix M^{*j+1}, all integral points in $M^{*j}[-1,1)^2$ also belong to different cosets of M^{*j+1}.

In Figure 1 the following regions are depicted: $M^*[-1,1]^2$ (the inner parallelogram), $M^{*2}[-1,1]^2$ (the outer parallelogram) and $M^{*2}[0,1]^2$ (the bold-face parallelogram). The integral points in the set Ω_1 are marked with small dots, and the points marked with heavy dots are those belonging to $M^{*2}[0,1]^2$ but not to Ω_1, i.e., the points congruent modulo M^{*2} to points in $M^{*1}[-1,1]^2$. Splitting the large parallelogram into four small ones (see Figure 1) and shifting them to the position of the right upper one, we see that the points that coincide after this translation are congruent to each other modulo M^{*4}. Also, we see that the points a_1 and b_1 are congruent modulo M^{*4}, and the points marked by X's (respectively, by O's) are also congruent. The set of integral points in $M^{*j}[-1,1)^2$ differs from the corresponding set Ω_j in the following way: Ω_j does not contain the points $M^{*j}\left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right)$, but contains the points $M^{*j}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $a_j=M^{*j}\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. It remains to observe that $M^{*j}\left(\begin{smallmatrix}-1\\-1\end{smallmatrix}\right)\equiv M^{*j}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \bmod M^{*j+1}$ and $a_j\equiv b_j \bmod M^{*j+1}$. $\square$

We prove that the sequence $\{\varphi_j\}_{j=0}^{\infty}$ defined by (32) satisfies the assumptions of Theorem 19. Property (Φ1) follows from the definition, and (Φ2) is implied by Lemma 20. Property (Φ3) follows from the fact that the absolute value of the eigenvalues of M is equal to 2, so the corresponding operator, applied many times, provides dilation in all directions (see the Introduction).

To check (Φ4), we find μ_k^j from the condition $\widehat{\varphi}_j(k) = \mu_k^{j+1}\widehat{\varphi}_{j+1}(k)$. Note that Ω_j is strictly contained in Ω_{j+1} for any $j \in \mathbb{Z}_+$, and all points of the boundary of Ω_j are inner points of Ω_{j+1}. For $j = 0$, the embedding in question is obvious. For greater j's, applying the operator M^{*j} to $\Omega_0 \subset \Omega_1$, we use the fact that, since the map M^{*j} is nonsingular, it preserves the required properties of the embedding. Taking all integral points in Ω_j except the point a_j for the role of the set $D(M^{*j+1})$, for $k \in D(M^{*j+1})$ we find the numbers μ_k^{j+1}: $\mu_k^{j+1} = 2$ if $k \in \Omega_{j-1}\backslash\{a_{j-1}, b_{j-1}\}$; $\mu_k^{j+1} = \sqrt{2}$ if $k = a_{j-1}$ or $k = b_{j-1}$, and $\mu_k^{j+1} = 0$ if $k \notin \Omega_{j-1}$. It is easily seen that the M^{*j+1}-periodic extension of μ_k^{j+1} with respect to the subindex is a sequence satisfying (Φ4). We can take $\gamma_k^j = 1/2$ for the role of the factors occurring in (Φ5). It is clear that $\gamma_k^j\widehat{\varphi}_j(n) = \widehat{\varphi}_{j+1}(M^*n)$ for all $j \in \mathbb{Z}_+$, $k \in \mathbb{Z}^2$, and $n \equiv k \pmod{M^{*j}}$.

Thus, the sequence of functions $\{\varphi_j\}_{j=0}^{\infty}$ satisfies all conditions required by Theorem 7. Hence, this sequence is scaling, and the functions φ_j are trigonometric polynomials with minimal possible symmetric spectra. It is easy to check that

$$\|\omega_r^j\varphi_j\|^2 = \sum_{l\in\mathbb{Z}^d} |\widehat{\varphi}_j(M^{*j}l + r)|^2 = 4^{-j}. \tag{33}$$

By Proposition 16, the system of functions $\{S_n^j\varphi\}_{n\in D(M^j)}$ is orthonormal.

Now we begin to construct the wavelet sequences. Fixing $j \in \mathbb{Z}_+$ we take the set $\Omega_{j-1}\backslash\{a_{j-1}\}$ as $D(M^{*j})$. As above, $D(M^*)$ consists of s_0, s_1, s_2, and s_3. Let $n \in D(M^{*j})$. If $n \neq b_{j-1}$, then $\mu_{n+M^{*j}s_0}^{j+1} = 2$, and $\mu_{n+M^{*j}s_k}^{j+1} = 0$ for $k = 1, 2, 3$, because the vectors $n + M^{*j}s_k$ do not belong to Ω_{j-1} and, moreover, are not congruent to any elements of Ω_{j-1} modulo M^{*j+1}. The corresponding unitary (4×4)-matrix is diagonal with 2's on the diagonal. If $n = b_{j-1}$, then $\mu_{n+M^{*j}s_k}^{j+1} = \sqrt{2}$ for $k = 0$, 3 (the vector $b_{j-1} + M^{*j}s_3$ is congruent to a_{j-1} modulo M^{*j+1}, which can readily be checked), and $\mu_{n+M^{*j}s_k}^{j+1} = 0$ for $k = 1$, 2, because the vectors $n + M^{*j}s_k$, $k = 1, 2$, do not belong to Ω_{j-1} and are not congruent to any elements of Ω_{j-1} modulo M^{*j+1}. The corresponding unitary matrix looks like this:

$$\begin{pmatrix} \sqrt{2}/2 & 0 & 0 & \sqrt{2}/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sqrt{2}/2 & 0 & 0 & -\sqrt{2}/2 \end{pmatrix}.$$

By Lemma 4, the vectors $n+M^{*j}s_k$ with $s_k \in D(M^*)$ and $n \in D(M^{*j})$ run over the set $D(M^{*j+1})$. Therefore, it suffices to find the Fourier coefficients of the wavelet functions for all integral vectors l that are congruent to the vectors $n + M^{*j}s_k$ modulo M^{*j+1}:

$$\widehat{\psi}_j^{(1)}(l) := \begin{cases} 2^{-j} & \text{if } l \equiv n + M^{*j}s_1 \pmod{M^{*j+1}}, \\ & \quad n \in D(M^{*j}),\ \ l \in \Omega_j\backslash\{a_j, b_j\}, \\ 2^{-j-1/2} & \text{if } l \equiv n + M^{*j}s_1 \pmod{M^{*j+1}}, \\ & \quad n \in D(M^{*j}),\ l = a_j,\ l = b_j, \\ 0 & \text{otherwise;} \end{cases} \tag{34}$$

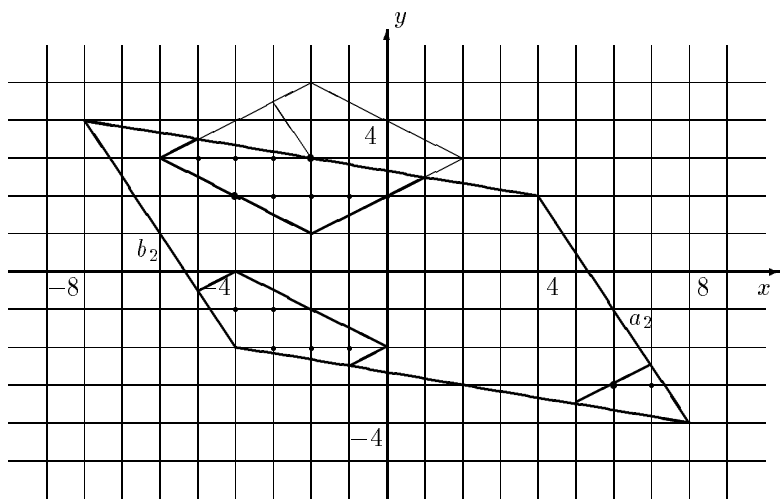

FIGURE 2. The spectrum of the function $\psi_2^{(1)}$ is marked by dots; at all points marked by heavy dots, the value of $\widehat{\psi}_2^{(1)}$ is equal to 2^{-2}.

$$\widehat{\psi}_j^{(2)}(l) := \begin{cases} 2^{-j} & \text{if } l \equiv n + M^{*j}s_2 \pmod{M^{*j+1}}, \\ & \quad n \in D(M^{*j}),\ \ l \in \Omega_j \backslash \{a_j, b_j\}, \\ 2^{-j-1/2} & \text{if } l \equiv n + M^{*j}s_2 \pmod{M^{*j+1}}, \\ & \quad n \in D(M^{*j}),\ l = a_j,\ l = b_j, \\ 0 & \text{otherwise;} \end{cases} \tag{35}$$

$$\widehat{\psi}_j^{(3)}(l) := \begin{cases} 2^{-j} & \text{if } l \equiv n + M^{*j}s_3 \pmod{M^{*j+1}}, \\ & \quad n \in D(M^{*j}),\ n \neq b_{j-1},\ \ l \in \Omega_j \backslash \{a_j, b_j\}, \\ 2^{-j-1/2} & \text{if } l \equiv n + M^{*j}s_3 \pmod{M^{*j+1}}, \\ & \quad n \in D(M^{*j}),\ n \neq b_{j-1},\ l = a_j,\ l = b_j, \\ 2^{-j-1/2} & \text{if } l \equiv b_{j-1} \pmod{M^{*j+1}}, \\ & \quad l \in \Omega_j \backslash \{a_j, b_j\}, \\ -2^{-j-1/2} & \text{if } l \equiv a_{j-1} \pmod{M^{*j+1}}, \\ & \quad l \in \Omega_j \backslash \{a_j, b_j\}, \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

The following regions are depicted in Figure 2: Ω_2 is the large parallelogram, and Ω_1 shifted by M^*s_1 is the small parallelogram. The entire spectrum of $\psi^{(1)}$ is located in the three separated regions inside the large parallelogram; these regions form a set congruent to the small parallelogram modulo M^{*2}. In a similar way, the spectra of $\psi^{(2)}$ and $\psi^{(3)}$ are depicted in Figures 3 and 4, respectively.

§6. Wavelet expansion of functions

We fix a (p, q)-pair satisfying the assumptions of Theorem 19. By that theorem, the following systems of wavelet functions are biorthonormal:

$$\{S_r^j \psi_j^{(\nu)}, j \in \mathbb{Z}_+, r \in D(M^j), \nu = 1, \dots, m-1\},$$
$$\{S_r^j \widetilde{\psi}_j^{(\nu)}, j \in \mathbb{Z}_+, r \in D(M^j), \nu = 1, \dots, m-1\}.$$

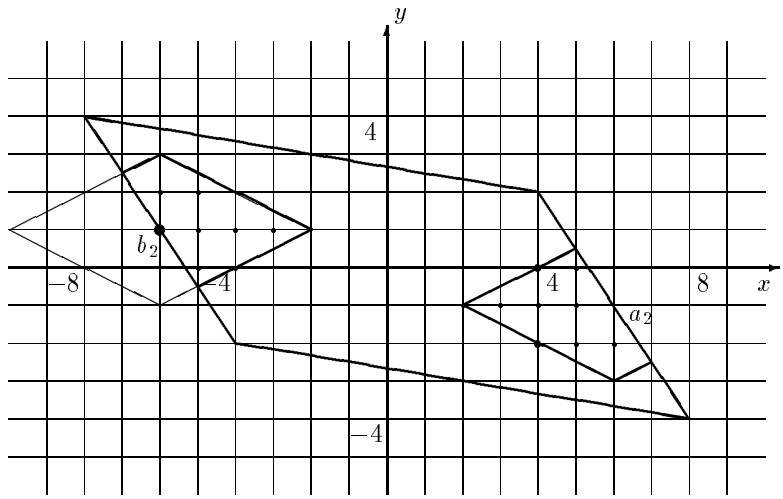

FIGURE 3. The spectrum of the function $\psi_2^{(2)}$ is marked by dots; at all points marked by heavy dots except for the point $b_2 = (-6, 1)$, the value of $\widehat{\psi}_2^{(2)}$ is equal to 2^{-2}, and $\widehat{\psi}_2^{(2)}(b_2) = 2^{-5/2}$.

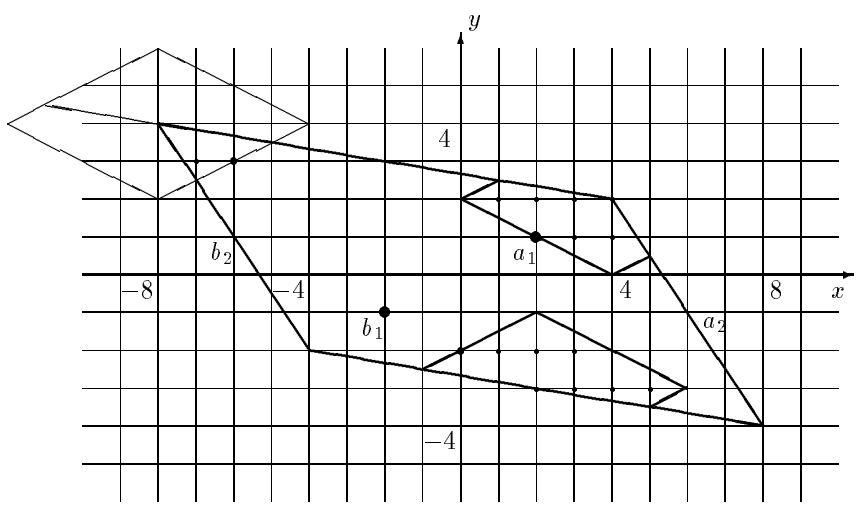

FIGURE 4. The spectrum of the function $\psi_2^{(3)}$ is marked by dots; at all points marked by heavy dots except for the points $a_1 = (2, 1)$ and $b_1 = (-2, -1)$, the value of $\widehat{\psi}_2^{(3)}$ is equal to 2^{-2}, and $\widehat{\psi}_2^{(3)}(b_1) = 2^{-5/2}$.

For $f \in L_p(\mathbb{T}^d)$, we can consider the corresponding Fourier expansions

$$\langle f, \widetilde{\varphi}_0 \rangle \varphi_0 + \sum_{j=0}^{\infty} \sum_{\nu=1}^{m-1} \sum_{r \in D(M^j)} \langle f, S_r^j \widetilde{\psi}_j^{(\nu)} \rangle S_r^j \psi_j^{(\nu)}. \tag{37}$$

Enumerating the sets of digits $D(M^j) = \{r_l\}_{l=0}^{m^j-1}$ in an arbitrary way, we denote the partial sums of the above series by $s_n(f)$; by the convergence of the series (3.7) we mean that of the sequence $s_n(f)$.

Theorem 21. *Suppose that $\{V_j\}_{j=0}^{\infty}$ and $\{\widetilde{V}_j\}_{j=0}^{\infty}$ form an $(\infty,1)$-pair with scaling sequences $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\widetilde{\varphi}_j\}_{j=0}^{\infty}$ such that the systems $\{S_n^j\varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j\widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthonormal. Let $\{\psi_j^{(\nu)}\}_{j=0}^{\infty}$ and $\{\widetilde{\psi}_j^{(\nu)}\}_{j=0}^{\infty}$, $\nu=1,\dots,m-1$, be the corresponding sequences of wavelet functions. If*

$$\sup_j \|\widetilde{\varphi}_j\|_1,\ \sup_{j,\nu}\|\widetilde{\psi}_j^{(\nu)}\|_1<\infty, \tag{38}$$

and there exists a monotone decreasing function K defined on $[0,\infty)$ and such that

$$\int_{\mathbb{R}^d} K(|x|)\,dx<\infty \tag{39}$$

and

$$|\varphi_j(x)|,\ |\psi_j^{(\nu)}(x)|\le K(|M^jx|) \tag{40}$$

for all $x\in\mathbb{T}^d$, then for every $f\in C(\mathbb{T}^d)$ the series (37) converges to f uniformly, and for every $g\in L(\mathbb{T}^d)$ the series

$$\langle f,\varphi_0\rangle\widetilde{\varphi}_0+\sum_{j=0}^{\infty}\sum_{\nu=1}^{m-1}\sum_{r\in D(M^j)}\langle f,S_r^j\psi_j^{(\nu)}\rangle S_r^j\widetilde{\psi}_j^{(\nu)} \tag{41}$$

converges to g in the norm of $L(\mathbb{T}^d)$.

Proof. Let $N=\kappa m^j+n$, $j\in\mathbb{Z}_+$, $\kappa=1,\dots,m-2$, $n=0,\dots,m^j-1$; then the partial sum $s_N(f,x)$ of the series (37) can be written as

$$\begin{aligned} s_N(f)&=\langle f,\widetilde{\varphi}_0\rangle\varphi_0+\sum_{i=0}^{j-1}\sum_{\nu=1}^{m-1}\sum_{r\in D(M^i)}\langle f,S_r^i\widetilde{\psi}_i^{(\nu)}\rangle S_r^i\psi_i^{(\nu)}\\ &\quad+\sum_{\nu=1}^{\kappa}\sum_{r\in D(M^j)}\langle f,S_r^j\widetilde{\psi}_j^{(\nu)}\rangle S_r^j\psi_j^{(\nu)}+\sum_{l=0}^{n}\langle f,S_{r_l}^j\widetilde{\psi}_j^{\kappa+1}\rangle S_{r_l}^j\psi_j^{\kappa+1}\\ &=s_N^{(0)}(f)+\sum_{\nu=1}^{\kappa}s_N^{(\nu)}(f)+s_N^{(\kappa+1)}(f). \end{aligned} \tag{42}$$

Since s_N^1 is a projection onto the space V_j, the sum $s_N^{(0)}(f)$ on the right can be reexpanded with respect to the shifts of the function φ_j:

$$s_N^{(0)}(f)=\sum_{r\in D(M^j)}\langle f,S_r^j\widetilde{\varphi}_j\rangle S_r^j\varphi_j. \tag{43}$$

Using (38), we obtain

$$\begin{aligned} |s_N^{(0)}(f,x)|&=\left|\int_{\mathbb{T}^d}f(t)\sum_{l=0}^{m^j-1}\widetilde{\varphi}_j(t+M^{-j}r_l)\varphi_j(x+M^{-j}r_l)\,dt\right|\\ &\le\|f\|_\infty\|\widetilde{\varphi}_j\|_1\sum_{l=0}^{m^j-1}|\varphi_j(x+M^{-j}r_l)|. \end{aligned}$$

We put

$$g_j(t)=\begin{cases}\varphi_j(M^{-j}t) & \text{if } t\in M^j\mathbb{T}^d,\\ 0 & \text{if } t\notin M^j\mathbb{T}^d.\end{cases}$$

Clearly,

$$|g_j(x)|\le K(|x|) \tag{44}$$

and

$$\varphi_j(x) = \sum_{k\in\mathbb{Z}^d} g_j(M^j x + M^j k) \tag{45}$$

for all $x \in \mathbb{R}^d$. Relations (44) and (45) yield

$$\sum_{l=0}^{m^j-1} |\varphi_j(x + M^{-j} r_l)| \le \sum_{l=0}^{m^j-1} \sum_{k\in\mathbb{Z}^d} K(|M^j x + M^j k + r_l|) = \sum_{k\in\mathbb{Z}^d} K(|M^j x + k|).$$

The monotonicity of K and relation (39) imply the uniform boundedness of the last sum. Thus, we have proved that, for $\nu = 0$, the operators $s_N^{(\nu)}$, which take $C(\mathbb{T}^d)$ to $C(\mathbb{T}^d)$, are uniformly bounded in norm. The uniform boundedness of the operators $s_N^{(\nu)}$ for $\nu = 1, \dots, m-1$ can be proved in a similar way. Therefore,

$$\|s_N(f)\| \le C, \tag{46}$$

where C is an absolute constant.

By property (MR2) in Definition 5, for any $\varepsilon > 0$ there exists $F \in V_{j_0}$ such that

$$\|f - F\|_\infty < \varepsilon.$$

By Theorem 19, we have $s_N(F) = F$ for $N \ge m^{j_0}$. Therefore, by (46),

$$|f - s_N(f)| = |f - F - s_N(f - F)| \le (C+1)\|f - F\| \le (C+1)\varepsilon.$$

This proves the first statement of the theorem. The second statement can be proved similarly; when estimating the sums $s_N^{(\nu)}(f)$, we must interchange the roles of φ_j, $\psi_j^{(\nu)}$, x and $\widetilde{\varphi}_j$, $\widetilde{\psi}_j$, t, respectively.

□

Theorem 22. *Suppose that $\{V_j\}_{j=0}^\infty$ and $\{\widetilde{V}_j\}_{j=0}^\infty$ form a (p,q)-pair with scaling sequences $\{\varphi_j\}_{j=0}^\infty$ and $\{\widetilde{\varphi}_j\}_{j=0}^\infty$ such that the systems $\{S_n^j \varphi_j\}_{n\in D(M^j)}$ and $\{S_k^j \widetilde{\varphi}_j\}_{n\in D(M^j)}$ are biorthonormal. Let $\{\psi_j^{(\nu)}\}_{j=0}^\infty$ and $\{\widetilde{\psi}_j^{(\nu)}\}_{j=0}^\infty$, $\nu = 1, \dots, m-1$, be the corresponding sequences of wavelet functions. If there exists a monotone decreasing function K defined on $[0,\infty)$, satisfying (39), and such that*

$$|\varphi_j(x)|,\ |\psi_j^{(\nu)}(x)|,\ |m^{-j}\widetilde{\varphi}_j(x)|,\ |m^{-j}\widetilde{\psi}_j^{(\nu)}(x)| \le K(|M^j x|) \tag{47}$$

for all $x \in \mathbb{R}^d$, then for every $f \in L_p(\mathbb{T}^d)$ the series (37) converges to f at each Lebesgue point for f.

Lemma 23 ([20, Lemma 2.7])**.** *Let K be a nonnegative monotone decreasing function defined on $[0,\infty)$ and satisfying (39). Then there exists a constant C depending only on the dimension d of the space and such that*

$$\sum_{k\in\mathbb{Z}^d} K(|x+k|)K(|y+k|) \le CK\left(\frac{|x-y|}{5}\right) \tag{48}$$

for all $x, y \in \mathbb{R}^d$.

Proof of Theorem 22. Let x be a Lebesgue point of f, and let $N = \kappa m^j + n$, $j \in \mathbb{Z}_+$, $\kappa = 1, \dots, m-2$, $n = 0, \dots, m^j - 1$. Since the space V_0 consists of constants only, we

have $s_N(h,x) = h$ if $h \equiv \text{const}$. Using (42) and (43), we obtain

$$
\begin{aligned}
f(x) - s_N(f,x) &\\
&= \int_{\mathbb{T}^d} (f(x) - f(x+t)) \sum_{r \in D(M^j)} S_r^j \widetilde{\varphi}_j(x+t) \rangle S_r^j \varphi_j(x)\, dt \\
&\quad + \sum_{\nu=1}^{\kappa} \int_{\mathbb{T}^d} (f(x) - f(x+t)) \sum_{r \in D(M^j)} S_r^j \widetilde{\psi}_j^{(\nu)}(x+t) S_r^j \psi_j^{(\nu)}(x)\, dt \\
&\quad + \int_{\mathbb{T}^d} (f(x) - f(x+t)) \sum_{l=0}^{n} S_{r_l}^j \widetilde{\psi}_j^{\kappa+1}(x+t) S_{r_l}^j \psi_j^{\kappa+1}(x)\, dt \\
&= I_0 + \sum_{\nu=1}^{\kappa} I_\nu + I_{\kappa+1}.
\end{aligned} \tag{49}
$$

Applying (45), (44) and the similar relations for $\widetilde{\varphi}_j$, we get

$$
\begin{aligned}
I_0 &\le m^j \int_{\mathbb{T}^d} |f(x) - f(x+t)| \\
&\qquad \times \sum_{r \in D(M^j)} \sum_{\ell \in \mathbb{Z}^d} K\left(M^j(x+t) + M^j \ell + r\right) \sum_{k \in \mathbb{Z}^d} K\left(M^j x + M^j k + r\right)\, dt \\
&= m^j \int_{\mathbb{R}^d} |f(x) - f(x+t)| \sum_{k \in \mathbb{Z}^d} K(M^j(x+t) + k) K\left(M^j x + k\right)\, dt.
\end{aligned}
$$

From Lemma 23 it follows that

$$
I_0 \le C m^j \int_{\mathbb{R}^d} |f(x) - f(x+t)| K\left(\frac{M^j t}{5}\right)\, dt.
$$

Combining this with a minor modification of Theorem 1.8 in [25], we see that

$$
I_0 \xrightarrow[j \to \infty]{} 0.
$$

For $\nu = 1, \ldots, m-1$, the relation

$$
I_2 \xrightarrow[j \to \infty]{} 0
$$

can be proved in a similar way. Recalling (49), we get

$$
\lim_{N \to \infty} s_N(f,x) = f(x).
$$

□

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