Алгебра и анализ
Том. 15 (2003), вып. 3

St. Petersburg Math. J.
Vol. 15 (2004), No. 3, Pages 393–418
S 1061-0022(04)00814-3
Article electronically published on March 30, 2004

$\mathrm{SL}_2(\mathbb{R})$, EXPONENTIAL HERGLOTZ REPRESENTATIONS, AND SPECTRAL AVERAGING

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Abstract. We revisit the concept of spectral averaging and point out its relationship with one-parameter subgroups of $\mathrm{SL}_2(\mathbb{R})$ and the corresponding Möbius transformations. In particular, we identify exponential Herglotz representations as the basic ingredient for the absolute continuity of averaged spectral measures with respect to Lebesgue measure; the associated spectral shift function turns out to be the corresponding density for the averaged measure. As a by-product of our investigations we unify the treatment of rank-one perturbations of selfadjoint operators and that of selfadjoint extensions of symmetric operators with deficiency indices $(1,1)$. Moreover, we derive separate averaging results for absolutely continuous, singular continuous, and pure point measures and conclude with an averaging result for the κ-continuous part (with respect to the κ-dimensional Hausdorff measure) of singular continuous measures.

1. Introduction

Usually, spectral averaging is associated with an integration of the spectral measure of a one-parameter family of selfadjoint operators with respect to a parameter, typically a coupling constant or a boundary condition parameter (cf., e.g., (1.13), (1.18), and (3.41)). This is done with the expectation in mind to prove the absolute continuity of the integrated (averaged) spectral measure with respect to Lebesgue measure. Moreover, one is interested in establishing the universality of spectral averaging, provided that the averaging process is carried out over the entire parameter space.

In this paper we revisit this range of ideas. In particular, the following items are dealt with in depth:

- The intimate relationship between spectral averaging, $\mathrm{SL}_2(\mathbb{R})$, and Möbius transformations is clarified in detail.
- The exponential Herglotz representation theorem is shown to be the underlying reason for the absolute continuity of averaged spectral measures with respect to Lebesgue measure. In particular, this identifies the spectral shift function as the density of the absolutely continuous averaged spectral measure.
- Various existing results on the universality of spectral averaging are extended. In particular, we do not assume the existence of a spectral gap (or the boundedness from below) of the associated selfadjoint operators.
- Conditions for the (non)universality of spectral averaging are identified.
- A unified treatment of selfadjoint rank-one perturbations of a selfadjoint operator and selfadjoint extensions of a densely defined closed symmetric operator with deficiency indices $(1,1)$ is presented.

2000 *Mathematics Subject Classification.* Primary 34B20, 47A11; Secondary 34L05, 47A10.
Key words and phrases. Spectral averaging, $\mathrm{SL}_2(\mathbb{R})$, Möbius transformations, Herglotz representations.

- Separate averaging for the point spectrum, the absolutely continuous, and the singular continuous spectrum is discussed.
- A partial result for averaging the κ-continuous part (with respect to the κ-dimensional Hausdorff measure) of singular continuous measures is derived.

Next, we illustrate these ideas in two canonical cases: the rank-one perturbation theory and the theory of selfadjoint extensions of symmetric operators with deficiency indices $(1,1)$.

Let A be a selfadjoint operator in a separable complex Hilbert space $\mathcal{H}$, and P an orthogonal rank-one projection in $\mathcal{H}$. We introduce two Herglotz functions, M and N, associated with the pair (A,P):

$$M(z) = \operatorname{tr}(P(A-z)^{-1}P), \quad z \in \mathbb{C}_+, \tag{1.1}$$

and

$$N(z) = \operatorname{tr}(P(zA+I)(A-z)^{-1}P), \quad z \in \mathbb{C}_+, \tag{1.2}$$

with $\mathbb{C}_+$ the open upper complex half-plane. One then has the Herglotz representations

$$M(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}, \quad z \in \mathbb{C}_+, \tag{1.3}$$

with μ a probability measure on $\mathbb{R}$, $\mu(\mathbb{R}) = 1$, and

$$N(z) = B + \int_{\mathbb{R}} d\nu(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \quad z \in \mathbb{C}_+, \tag{1.4}$$

with $B \in \mathbb{R}$ and ν a Borel measure satisfying

$$\int_{\mathbb{R}} \frac{d\nu(\lambda)}{1+\lambda^2} < \infty. \tag{1.5}$$

Actually, a short computation reveals that

$$N(z) = z + (1+z^2)M(z) \tag{1.6}$$

and

$$d\nu(\lambda) = (1+\lambda^2)\, d\mu(\lambda), \quad B = \operatorname{Re}(N(i)) = 0. \tag{1.7}$$

Thus, (1.4) simplifies to

$$N(z) = \int_{\mathbb{R}} (1+\lambda^2) d\mu(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \quad z \in \mathbb{C}_+. \tag{1.8}$$

If A is unbounded and $\operatorname{ran}(P) \cap \operatorname{dom}(A) = \{0\}$, then the measure ν is infinite.

Lemma 1.1. *Consider the one-parameter family of selfadjoint operators*

$$A_t = A + tP, \quad t \in \mathbb{R}, \tag{1.9}$$

with resolvents

$$(A_t - z)^{-1} = (A-z)^{-1} - \frac{1}{M(z) + (1/t)} (A-z)^{-1} P (A-z)^{-1}, \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+, \tag{1.10}$$

and M given by (1.1). *Moreover, we introduce*

$$M_t(z) = \frac{M(z)}{tM(z)+1}, \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+. \tag{1.11}$$

Then M_t is the corresponding M-function associated with the pair (A_t, P) (cf. (1.1)*). Next, we denote by μ_t the measure in* (1.3) *associated with M_t, and by Δ a bounded*

Borel set on $\mathbb{R}$. *Then averaging* μ_t *yields a measure absolutely continuous with respect to Lebesgue measure*:

$$\int_{t_1}^{t_2} dt\, \mu_t(\Delta) = \int_\Delta d\lambda\, [\xi(\lambda; A_{t_2}, A) - \xi(\lambda; A_{t_1}, A)], \tag{1.12}$$

where $\xi(\cdot, B, A)$ *is the spectral shift function associated with the pair* (B, A) *of selfadjoint operators. Moreover, spectral averaging is universal in the sense that*

$$\int_{-\infty}^{\infty} dt\, \mu_t(\Delta) = |\Delta| \tag{1.13}$$

with $|\cdot|$ *denoting Lebesgue measure on* $\mathbb{R}$.

Remark 1.2. The proof of the lemma is well known and can be found in [26] and [44]. In fact, (1.12) is a particular case of the Birman–Solomyak spectral averaging formula [6] proven in the mid-seventies.

Lemma 1.3. *Assume that* A *is an unbounded selfadjoint operator and that*

$$\operatorname{ran}(P) \cap \operatorname{dom}(A) = 0. \tag{1.14}$$

Then the operator-valued functions

$$R_t(z) = (A - z)^{-1} - \frac{1}{N(z) + (1/t)}(A - i)(A - z)^{-1}P(A + i)(A - z)^{-1}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+, \tag{1.15}$$

with N *given by* (1.2) *represent the resolvents of selfadjoint operators* A_t, $t \in \mathbb{R}$. *Moreover, the function*

$$N_t(z) = \frac{N(z) - t}{tN(z) + 1}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+, \tag{1.16}$$

is the N-*function* (*in the sense of* (1.15)) *of the pair* (A_t, P) (*cf.* (1.8)). *The family* $\{A_t\}_{t\in\mathbb{R}}$ *is a one-parameter family of selfadjoint extensions of a closed symmetric densely defined operator* $\dot{A}$ *with deficiency indices* $(1, 1)$,

$$\dot{A} = A\big|_{\operatorname{dom}(\dot{A})}, \quad \operatorname{dom}(\dot{A}) = \bigcap_{t\in\mathbb{R}} \operatorname{dom}(A_t). \tag{1.17}$$

In particular, $\lim_{t\to 0} A_t = A$ *in the strong resolvent sense. Next, we denote by* ν_t *the measure in* (1.8) *associated with* N_t *and by* Δ *a bounded Borel set on* $\mathbb{R}$. *Then averaging* ν_t *yields a measure absolutely continuous with respect to Lebesgue measure*:

$$\frac{1}{\pi}\int_{t_1}^{t_2} \frac{dt}{1 + t^2}\, \nu_t(\Delta) = \int_\Delta d\lambda\, [\xi(\lambda; A_{t_2}, A) - \xi(\lambda; A_{t_1}, A)]. \tag{1.18}$$

Moreover, spectral averaging is universal in the sense that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{1 + t^2}\, \nu_t(\Delta) = |\Delta|. \tag{1.19}$$

Remark 1.4. The resolvent formula (1.15) is due to Krein [35] and Naimark [39]. The proof of the transformation law (1.16) can be found, for instance, in [21]. In the case of boundary condition dependence, (1.18) was proved by Javrjan [30] for a semibounded Schrödinger operator. Javrjan's method can easily be adapted to the case of arbitrary selfadjoint operators A having a spectral gap. The treatment of the general case of A with $\operatorname{spec}(A) = \mathbb{R}$ needs some additional information on the spectral shift function theory in the case of trace class resolvent differences (see also [37]). In this case the spectral shift function should be viewed as a path-dependent homotopy invariant characteristic of the perturbation (see, [49, Chapter 8, Section 8]), and the proof of (1.18) requires minor additional effort.

In the case of perturbation theory the transformation (1.11) can be represented in the form

$$M_t(z) = g_t(M(z)), \tag{1.20}$$

where $\{g_t\}_{t\in\mathbb{R}}$ is a one-parameter group of automorphisms of the open upper half-plane $\mathbb{C}_+$:

$$g_t \circ g_s = g_{t+s}, \quad s, t \in \mathbb{R}, \tag{1.21}$$

with

$$g_t(z) = \frac{z}{tz+1}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+. \tag{1.22}$$

In the case of selfadjoint extension theory the transformation (1.16) can be written as

$$N_t(z) = f_t(N(z)), \tag{1.23}$$

where $\{f_t\}_{t\in\mathbb{R}}$ is a one-parameter family of automorphisms of $\mathbb{C}_+$,

$$f_t(z) = \frac{z-t}{tz+1}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+. \tag{1.24}$$

The family $\{f_t\}_{t\in\mathbb{R}}$ of transformations is not a one-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$. However, by the change of parametrization $t \mapsto \tan(t)$, the group law (1.21) can be restored with

$$g_t(z) = f_{\arctan(t)}(z), \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+. \tag{1.25}$$

In either case, the one-parameter subgroup $\{g_t\}_{t\in\mathbb{R}}$ of automorphisms of $\mathbb{C}_+$ gives rise to a dynamical system on a certain "phase space" of measures as discussed in Section 3.

We continue with an intuitive explanation of how exponential Herglotz representations, and hence spectral shift functions, naturally enter the averaging process (1.12), (1.18). Both M_t in Lemma 1.1 and N_t in Lemma 1.3 (the latter after reparametrizing $t \mapsto \tan(t)$) are of the type

$$M_t(z) = \frac{a_t M_0(z) + b_t}{c_t M_0(z) + d_t} = \frac{d}{dt} \operatorname{Ln}(c_t M_0(z) + d_t), \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+. \tag{1.26}$$

Here M_0 represents M and N in Lemmas 1.1 and 1.3, respectively, and $\operatorname{Ln}(\cdot)$ denotes the logarithm on the standard infinitely sheeted Riemann surface branched at zero and infinity. Moreover, the coefficients a_t, b_t, c_t, d_t are all real-valued satisfying

$$\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}\bigg|_{t=0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } a_t d_t - b_t c_t = 1, \quad t \in \mathbb{R}. \tag{1.27}$$

Since M_0 is a Herglotz function, so is M_t for each $t \in \mathbb{R}$. Similarly, $c_t M_0 + d_t$ is a Herglotz or anti-Herglotz function and thus M_t and $c_t M_0 + d_t$ admit Herglotz and exponential Herglotz representations of the type

$$M_t(z) = B_t + \int_{\mathbb{R}} d\omega_t(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \tag{1.28}$$

$$\operatorname{Ln}(c_t M_0(z) + d_t)) = C_t + \int_{\mathbb{R}} d\lambda\, \xi_t(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \tag{1.29}$$

where $B_t, C_t \in \mathbb{R}$,

$$\omega_t((\lambda_1,\lambda_2]) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda_1+\delta}^{\lambda_2+\delta} d\lambda\ \mathrm{Im}(M_t(\lambda+i\varepsilon)), \tag{1.30}$$

$$\xi_t(\lambda) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\left(\mathrm{Ln}(c_t M_0(\lambda+i\varepsilon)+d_t)\right) \quad \text{for a.e. } \lambda\in\mathbb{R}, \tag{1.31}$$

and

$$\int_{\mathbb{R}}\frac{d\omega_t(\lambda)}{1+\lambda^2} < \infty, \quad \xi_t(\cdot)\in L^\infty(\mathbb{R}),\ \ t\in\mathbb{R}. \tag{1.32}$$

Thus, for any bounded Borel set $\Delta\subset\mathbb{R}$ we formally obtain

$$\begin{aligned}\int_{t_1}^{t_2} dt\,\omega_t(\Delta) &= \frac{1}{\pi}\int_\Delta d\lambda\int_{t_1}^{t_2} dt\,\lim_{\varepsilon\downarrow 0}\frac{d}{dt}\mathrm{Im}\left(\mathrm{Ln}(c_t M_0(\lambda+i\varepsilon)+d_t)\right)\\ &= \int_\Delta d\lambda\int_{t_1}^{t_2} dt\,\frac{d}{dt}\xi_t(\lambda)\\ &= \int_\Delta d\lambda\,[\xi_{t_2}(\lambda)-\xi_{t_1}(\lambda)],\end{aligned} \tag{1.33}$$

freely interchanging integrals, limits, and differentiation. Once rigorously established, (1.33) proves that averaging ω_t over the interval $[t_1,t_2]$ yields a measure absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$, and the resulting density is related to the spectral shift function $\xi = \xi_{t_2}-\xi_{t_1}$. Moreover, in the case of perturbations discussed in Lemma 1.1, it can be shown that

$$\xi_{t_2}(\lambda)-\xi_{t_1}(\lambda)\to 1 \quad \text{as } t_1\downarrow -\infty \text{ and } t_2\uparrow\infty \tag{1.34}$$

and hence the universal behavior (1.13),

$$\int_{-\infty}^{\infty} dt\,\omega_t(\Delta) = |\Delta|, \tag{1.35}$$

emerges. The case of selfadjoint extensions discussed in Lemma 1.3 requires some additional periodicity considerations with respect to t but in the end also yields the universality (1.19) of spectral averaging. However, yet another case of one-parameter subgroups of SL$_2$(ℝ) considered in the following sections shows that universality cannot be taken for granted and may in fact fail. The material in Sections 2 and 3 will justify the formal procedures in (1.33).

Before describing the content of each section we briefly review the historical development of this subject, which appears to be less well known. To the best of our knowledge, the credit for the first paper on spectral averaging belongs to Javrjan [29] (see also the subsequent paper [30]), who considered half-line Schrödinger operators on $(0,\infty)$ and averaged over the boundary condition parameter at $x=0$ as early as in 1966. The next step was due to Birman and Solomyak [6] in 1975. They considered trace class perturbations of selfadjoint operators and averaged over the coupling constant parameter (by using the differentiation formula for operator-valued functions due to Daleckiĭ and S. Kreĭn [14]). In 1987, Aleksandrov [1] appears to have been the first to consider spectral averaging of a measure and separately averaging of its singular part in connection with the boundary behavior of inner functions in the unit disk. More recent treatments of spectral averaging can be found in Birman and Pushnitski [5], Gesztesy and Makarov [23], [25], Gesztesy, Makarov, and Naboko [22] (the latter references contain a discussion of an operator-valued version of the Birman–Solomyak averaging formula), Gesztesy, Makarov, and Motovilov [24], and Simon [44], [45].

Starting with the early eighties, the concept of spectral averaging developed into an important tool in investigations of disordered systems, in particular, in connection with random Schrödinger and Jacobi operators. In 1983, Carmona [8] (see also [9]), apparently unaware of previous results by Javrjan and Birman and Solomyak, used spectral averaging over boundary condition parameters to prove the existence of an absolutely continuous (a.c.) component in random and deterministic Schrödinger operators. For some random systems he also proved that the remainder of the spectrum consists of eigenvalues dense in certain intervals with exponentially localized eigenfunctions. In 1984, Kotani also used this approach to link the existence of pure point spectrum and exponentially decaying eigenfunctions with the positivity of the Lyapunov exponent [33] (his paper was only published in 1986). Kotani's work inspired new proofs of exponential localization by Delyon, Lévy, and Souillard [18], [19], Simon and Wolff [46], Simon [43], Delyon, Simon, and Souillard [20], and Kotani and Simon [34] for one- and quasi-one-dimensional as well as multidimensional Anderson models (the latter for large disorder or sufficiently high energy) and one-dimensional random Schrödinger operators. In all these references spectral averaging over coupling constants played a crucial role. This is especially transparent in the paper by Simon and Wolff [46], who used results by Aronszajn [2] and Donoghue [21] as their point of departure to study the variation of singular spectra under rank-one perturbations of selfadjoint operators. This was also discussed in Simon's survey [44]. (For textbook presentations of spectral averaging in this context we refer to [10, Section VIII.2], [40, Section 13].) Subsequently, Gordon [27], [28] used spectral averaging in his studies of eigenvalues embedded in the essential spectrum. Spectral averaging was also used to prove exponential localization for the one-dimensional Poisson model by Stolz; see [48]. A more general approach, involving two-parameter spectral averaging, has recently been employed to prove exponential localization in the Poisson and random displacement models in dimension one by Buschmann and Stolz [7]. The latter approach was again used by Sims and Stolz [47] in their discussion of exponential localization for the one-dimensional random displacement model and in a one-dimensional model of wave propagation in a random medium. Combes and Hislop [11] used averaging of spectral families to prove a Wegner-type estimate for a family of Anderson and Poisson-like multidimensional random Hamiltonians. Moreover, spectral averaging in the spirit of Birman and Solomyak was treated by Combes, Hislop, and Mourre [12] in their discussion of perturbations of singular spectra and exponential localization for certain multidimensional random Schrödinger operators, and by Combes, Hislop, Klopp, and Nakamura [13] in their study of the Wegner estimate and the integrated density of states.

In Section 2 we collect basic facts on $\mathrm{SL}_2(\mathbb{R})$ and Möbius transformations, as needed in the subsequent sections. Section 3, the principal section of this paper, then develops spectral averaging for spectral measures as well as for the associated absolutely continuous, singular continuous, and pure point parts (with respect to Lebesgue measure). Finally, in Section 4 we obtain a partial result concerning spectral averaging of the κ-continuous part (with respect to the κ-dimensional Hausdorff measure) of the singular continuous part of measures.

2. Preliminaries on $\mathrm{SL}_2(\mathbb{R})$ and on Möbius transformations

$\mathrm{SL}_2(\mathbb{R})$ denotes the group of 2×2 real matrices with determinant equal to 1. By definition, its Lie algebra, $\mathrm{sl}_2(\mathbb{R})$, consists of the matrices X such that $e^{tX} \in \mathrm{SL}_2(\mathbb{R})$ for all $t \in \mathbb{R}$ (cf., e.g., [36, Chapter VI]). Therefore, $\mathrm{sl}_2(\mathbb{R})$ consists of all 2×2 real matrices

X with zero trace, $\mathrm{tr}(X) = 0$. The following three matrices then form a basis for $\mathrm{sl}_2(\mathbb{R})$:

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

and one verifies the commutation relations

$$[X_2, X_1] = 2X_1, \quad [X_1, X_3] = X_2, \quad [X_3, X_2] = 2X_3. \tag{2.2}$$

If $X \in \mathrm{sl}_2(\mathbb{R})$, then the map $t \mapsto e^{tX}$, $t \in \mathbb{R}$, is a one-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$ and all one-parameter continuous subgroups can be obtained in that way.

For future reference we denote the automorphisms of the open complex upper half-plane $\mathbb{C}_+$ by $\mathrm{Aut}(\mathbb{C}_+)$:

$$\mathrm{Aut}(\mathbb{C}_+) = \{g \colon \mathbb{C}_+ \to \mathbb{C}_+ \mid g \text{ is biholomorphic (i.e., a conformal self-map of } \mathbb{C}_+)\}. \tag{2.3}$$

$\mathrm{Aut}(\mathbb{C}_+)$ becomes a group with respect to compositions of maps. For simplicity, this group is denoted by the same symbol.

To fix the notational setup, we now introduce the following convention.

Hypothesis 2.1. Given $\alpha, \beta, \gamma \in \mathbb{R}$, we represent an element $X = X(\alpha, \beta, \gamma) \in \mathrm{sl}_2(\mathbb{R})$ by

$$X = \alpha X_1 + \beta X_2 + \gamma X_3 = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix} \tag{2.4}$$

and denote by

$$\begin{aligned} g_t(z) &= \frac{a_t z + b_t}{c_t z + d_t}, \quad (t, z) \in \mathbb{R} \times \mathbb{C}_+, \\ g_0(z) &= z, \quad z \in \mathbb{C}_+, \end{aligned} \tag{2.5}$$

the corresponding one-parameter group of automorphisms of the open upper half-plane $\mathbb{C}_+$ such that

$$\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = e^{tX} \in \mathrm{SL}_2(\mathbb{R}), \quad t \in \mathbb{R}. \tag{2.6}$$

We briefly recall a few facts in connection with Möbius (i.e., linear fractional) transformations (2.5). Let M be a Möbius transformation of the type

$$M(z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C} \cup \{\infty\}, \ a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0. \tag{2.7}$$

Then,

(i) M maps $\mathbb{R} \cup \{\infty\}$ onto itself if and only if M admits a representation of the form (2.7), where $a, b, c, d \in \mathbb{R}$ and $|ad - bc| = 1$.

(ii) M maps $\mathbb{C}_+$ onto itself if and only if M admits a representation of the form (2.7), where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

We also recall that $\mathrm{Aut}(\mathbb{C}_+)$ is isomorphic to $\mathrm{SL}_2(\mathbb{R})/\{I_2, -I_2\}$ (here I_2 denotes the identity matrix in $\mathbb{R}^2$).

Assuming $\det(M) = ad - bc = 1$ in (2.7), one can use $\mathrm{tr}(M) = (a + d)$ to classify M as

elliptic if $(a + d) \in \mathbb{R}$ and $|a + d| < 2$;

parabolic if $(a + d) = \pm 2$;

hyperbolic if $(a + d) \in \mathbb{R}$ and $|a + d| > 2$;

loxodromic if $(a + d) \in \mathbb{C} \backslash \mathbb{R}$.

On the other hand, assuming $\left(\begin{smallmatrix} a_t & b_t \\ c_t & d_t \end{smallmatrix}\right) = e^{tX}$, $t \in \mathbb{R}$, with $\operatorname{tr}(X) = 0$ and $X = \left(\begin{smallmatrix} \beta & \alpha \\ \gamma & -\beta \end{smallmatrix}\right)$, one can use $\det(X) = -\alpha\gamma - \beta^2$ to classify the one-parameter subgroups of Möbius transformations in (2.5) and distinguish three cases:

$$\begin{aligned} &\textit{Case I} \ : \det(X) > 0 \text{ (cyclic subgroup)};\\ &\textit{Case II} \ : \det(X) = 0;\\ &\textit{Case III} \ : \det(X) < 0 \text{ (hyperbolic subgroup)}. \end{aligned}$$

Lemma 2.2. *We assume Hypothesis 2.1 and let* $(t, z) \in \mathbb{R} \times \mathbb{C}_+$.

(i) *If* $\det(X) > 0$, *then*

$$g_t(z) = \frac{\big(\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t)\big)z + \frac{\alpha}{\omega}\sin(\omega t)}{\big(\frac{\gamma}{\omega}\sin(\omega t)\big)z + \cos(\omega t) - \frac{\beta}{\omega}\sin(\omega t)}, \tag{2.8}$$

where $\omega = \sqrt{\det(X)} > 0$.

(ii) *If* $\det(X) = 0$, *then*

$$g_t(z) = \frac{\big(1 + \beta t\big)z + \alpha t}{\gamma t z + (1 - \beta t)}. \tag{2.9}$$

(iii) *If* $\det(X) < 0$, *then*

$$g_t(z) = \frac{\big(\cosh(\omega t) + \frac{\beta}{\omega}\sinh(\omega t)\big)z + \frac{\alpha}{\omega}\sinh(\omega t)}{\big(\frac{\gamma}{\omega}\sinh(\omega t)\big)z + \cosh(\omega t) - \frac{\beta}{\omega}\sinh(\omega t)}, \tag{2.10}$$

where $\omega = \sqrt{|\det(X)|} > 0$.

Proof. Since $\operatorname{tr}(X) = 0$, every entry a_t, b_t, c_t, and d_t of the matrix e^{tX} in (2.6) is a solution of the initial value problem

$$\ddot{y} + \det(X) y = 0, \tag{2.11}$$

$$y(0) = 1,\ \dot{y}(0) = \beta \quad \text{for } y(t) = a_t, \tag{2.12}$$

$$y(0) = 0,\ \dot{y}(0) = \alpha \quad \text{for } y(t) = b_t, \tag{2.13}$$

$$y(0) = 0,\ \dot{y}(0) = \gamma \quad \text{for } y(t) = c_t, \tag{2.14}$$

$$y(0) = 1,\ \dot{y}(0) = -\beta \quad \text{for } y(t) = d_t, \tag{2.15}$$

where the dot $\cdot$ denotes d/dt. Solving the initial value problems (2.11), (2.12)–(2.15) proves (2.8)–(2.10). □

Remark 2.3. If $\gamma = 0$ in (2.4), the subgroup g_t is a group of linear transformations of $\mathbb{C}_+$. If $\gamma \in \mathbb{R}\backslash\{0\}$, the subgroup g_t corresponds to the case of linear fractional transformations of $\mathbb{C}_+$. If $\gamma \in \mathbb{R}\backslash\{0\}$, the automorphism $g_t(z)$ is a linear function in z if and only if $t \in (\pi/\omega)\mathbb{Z}$ in case I and $t = 0$ in cases II and III, respectively. In other words,

$$\gamma \in \mathbb{R}\backslash\{0\} \text{ if and only if } c_t \neq 0 \text{ for } \begin{cases} t \in \mathbb{R}\backslash\{(\pi/\omega)\mathbb{Z}\} & \text{in case I,} \\ t \in \mathbb{R}\backslash\{0\} & \text{in cases II, III.} \end{cases} \tag{2.16}$$

Moreover, suppose that $t \in \mathbb{R}\backslash\{(\pi/\omega)\mathbb{Z}\}$, that is, g_t is not the identity transformation ($g_t(z) \neq z$). Then case I consists of elliptic Möbius transformations. Case II always corresponds to parabolic Möbius transformations, and as long as $t \neq 0$, case III corresponds to hyperbolic Möbius transformations.

Remark 2.4. The selfadjoint rank-one perturbations tP of the selfadjoint operator A discussed in Lemma 1.1 correspond to the case where $\det(X) = 0$ with $\alpha = \beta = 0$, $\gamma = 1$ as one readily verifies upon comparison with (1.11). Similarly, the selfadjoint extensions of a closed symmetric densely defined operator $\dot{A}$ with deficiency indices $(1,1)$ discussed in Lemma 1.3 correspond to the case where $\det(X) = 1$, $\omega = 1$ with $\alpha = -1$, $\beta = 0$, $\gamma = 1$ as can easily be seen upon comparison with (1.16) and the change of parametrization $t \mapsto \tan(t)$ in (1.25).

Remark 2.5. The geometry of the trajectories $\bigcup_{t\in\mathbb{R}}\{g_t(z)\}$, $z \in \mathbb{C}_+$, of the one-parameter groups of automorphisms (2.8)–(2.10) can be understood in terms of the trajectories $\bigcup_{t\in\mathbb{R}}\{F_t(z)\}$, $z \in \mathbb{C}_+$, of the map F_t given by

$$F_t(z) = \frac{(1+\beta t)z + \alpha t}{\gamma t z + (1-\beta t)}, \quad (\alpha,\beta,\gamma) \in \mathbb{R}^3. \tag{2.17}$$

In fact, we have the following representations ($X = X(\alpha,\beta,\gamma)$; cf. (2.4)):

$$g_t(z) = \begin{cases} F_{\tan(\sqrt{\det(X)}t)/\sqrt{\det(X)}}(z) & \text{if } \det(X) = -\alpha\gamma - \beta^2 > 0, \\ F_t(z) & \text{if } \det(X) = -\alpha\gamma - \beta^2 = 0, \\ F_{\tanh(\sqrt{|\det(X)|}t)/\sqrt{|\det(X)|}}(z) & \text{if } \det(X) = -\alpha\gamma - \beta^2 < 0. \end{cases} \tag{2.18}$$

Therefore, the trajectories of the groups (2.8)–(2.10) can be described by

$$\bigcup_{t\in\mathbb{R}}\{g_t(z)\} = \bigcup_{t\in\mathbb{R}}\{F_t(z)\} \quad \text{in cases I and II} \tag{2.19}$$

and by

$$\bigcup_{t\in\mathbb{R}}\{g_t(z)\} = \bigcup_{|t|<|\det(X)|^{-1/2}}\{F_t(z)\} \subsetneqq \bigcup_{t\in\mathbb{R}}\{F_t(z)\} \quad \text{in case III.} \tag{2.20}$$

We observe that F_t is a one-parameter group of transformations of $\mathbb{C}_+$ with respect to t, that is, $F_{t+s} = F_t \circ F_s$ for all $s,t \in \mathbb{R}$, if and only if $\alpha\gamma + \beta^2 = 0$.

Next, we denote by $\log(\cdot)$ the branch of the logarithm on the cut plane $\Pi = \mathbb{C}\backslash[0,\infty)$ by assuming

$$0 < \arg(\log(z)) < 2\pi \quad \text{for } z \in \Pi. \tag{2.21}$$

In addition, we extend $\log(\cdot)$ to the upper rim $\partial_+\Pi$ of Π by

$$\lim_{\varepsilon\downarrow 0}\log(x+i\varepsilon) \in \mathbb{R}, \quad x > 0, \tag{2.22}$$

and hence obtain

$$\mathrm{Im}(\log(x)) = \pi, \quad x < 0. \tag{2.23}$$

Analytic continuation of the branch $\log(\cdot)$ defined above then leads to the infinitely sheeted Riemann surface $\mathcal{R}$ of the logarithm with branch points of infinite order at zero and infinity. We denote the resulting analytic function on $\mathcal{R}$ by $\mathrm{Ln}(\cdot)$. For future reference we also introduce the nth sheet $\mathcal{S}_n$ of $\mathcal{R}$. We use the convention $\mathcal{S}_0 = \Pi \cup \partial_+\Pi$. Then $\mathrm{Ln}\colon v \mapsto w = \mathrm{Ln}(v)$ maps the interior $\mathrm{int}(\mathcal{S}_n)$ of each sheet $\mathcal{S}_n$ biholomorphically onto the strip $2\pi n < \mathrm{Im}(z) < 2\pi(n+1)$ and

$$v \in \mathcal{S}_n \text{ if and only if } 2\pi n \le \arg(w) < 2\pi(n+1), \quad n \in \mathbb{Z}. \tag{2.24}$$

Assuming Hypothesis 2.1 with $\gamma \neq 0$, we shall subsequently denote by

$$\begin{aligned} &\overleftrightarrow{c_t z + d_t} \text{ the lift of the trajectory } t \mapsto c_t z + d_t \text{ to } \mathcal{R} \\ &\text{with } \overleftrightarrow{c_0 z + d_0} = d_0 = 1 \in \partial\mathcal{S}_0,\ z \in \mathbb{C}_+. \end{aligned} \tag{2.25}$$

Lemma 2.6. *We assume Hypothesis 2.1 with* $\gamma \in \mathbb{R}\backslash\{0\}$ (*cf.* (2.16)) *and let* $-\infty < t_1 < t_2 < \infty$, $z \in \mathbb{C}_+$. *Moreover, we recall the convention* (2.25). *Then,*

$$\int_{t_1}^{t_2} dt \ \mathrm{Im}(g_t(z)) = \frac{1}{\gamma} \mathrm{Im}\left(\mathrm{Ln}(\overleftrightarrow{c_{t_2} z + d_{t_2}}) - \mathrm{Ln}(\overleftrightarrow{c_{t_1} z + d_{t_1}})\right). \tag{2.26}$$

Proof. Since the entries of the matrix (2.6) solve the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix}\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}, \quad t \in \mathbb{R}, \tag{2.27}$$

we obtain

$$\dot{c}_t = \gamma a_t - \beta c_t, \quad \dot{d}_t = \gamma b_t - \beta d_t \tag{2.28}$$

and

$$a_t = \frac{\dot{c}_t + \beta c_t}{\gamma} \quad \text{and} \quad b_t = \frac{\dot{d}_t + \beta d_t}{\gamma}. \tag{2.29}$$

Thus,

$$a_t z + b_t = \frac{\beta}{\gamma}(c_t z + d_t) + \frac{1}{\gamma}(\dot{c}_t z + \dot{d}_t), \tag{2.30}$$

and hence

$$g_t(z) = \frac{a_t z + b_t}{c_t z + d_t} = \frac{\beta}{\gamma} + \frac{1}{\gamma}\frac{\dot{c}_t z + \dot{d}_t}{c_t z + d_t} = \frac{\beta}{\gamma} + \frac{1}{\gamma}\frac{d}{dt}\mathrm{Ln}(\overleftrightarrow{c_t z + d_t}). \tag{2.31}$$

Integrating (2.31) from t_1 to t_2 and taking imaginary parts of the resulting expression proves (2.26). □

3. Dynamical systems on a space of measures

As shown below, each one-parameter subgroup $\{e^{tX}\}_{t\in\mathbb{R}}$ of $\mathrm{SL}_2(\mathbb{R})$, or, what is the same, each one-parameter group $\{g_t\}_{t\in\mathbb{R}}$ of automorphisms of the open upper half-plane $\mathbb{C}_+$ generates a dynamical system $\{g_t^*\}_{t\in\mathbb{R}}$ on the (phase) space $\mathcal{M} = [0,\infty) \times \mathbb{R} \times \Omega$. Here Ω denotes the space of Borel measures μ on $\mathbb{R}$ with the property

$$\int_{\mathbb{R}} \frac{d\mu(\lambda)}{1+\lambda^2} < \infty. \tag{3.1}$$

Let $\{g_t\}_{t\in\mathbb{R}}$ be a one-parameter subgroup of $\mathrm{Aut}(\mathbb{C}_+)$, the group of automorphisms of $\mathbb{C}_+$:

$$g_t(z) = \frac{a_t z + b_t}{c_t z + d_t}, \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+. \tag{3.2}$$

Given a point $(A_0, B_0, \mu_0) \in \mathcal{M}$, we introduce the Herglotz function

$$M_0(z) = A_0 z + B_0 + \int_{\mathbb{R}} d\mu_0(\lambda)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \quad z \in \mathbb{C}_+, \tag{3.3}$$

where

$$\mu_0((\lambda_1,\lambda_2)) + \frac{1}{2}\mu_0(\{\lambda_1\}) + \frac{1}{2}\mu_0(\{\lambda_2\}) = \frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\lambda_1}^{\lambda_2} d\lambda \ \mathrm{Im}(M_0(\lambda + i\varepsilon)). \tag{3.4}$$

Since $g_t \in \mathrm{Aut}(\mathbb{C}_+)$ for each $t \in \mathbb{R}$, the one-parameter family of the functions

$$M_t(z) = g_t(M_0(z)), \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+, \tag{3.5}$$

is a one-parameter family of Herglotz functions. Therefore, M_t admits the representation

$$M_t(z) = A_t z + B_t + \int_{\mathbb{R}} d\mu_t(\lambda)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \quad (t,z) \in \mathbb{R} \times \mathbb{C}_+, \tag{3.6}$$

for a unique triple $(A_t, B_t, \mu_t) \in \mathcal{M}$. We introduce the map

$$g_t^* : \mathcal{M} \to \mathcal{M}, \ \ (A_0, B_0, \mu_0) \mapsto (A_t, B_t, \mu_t), \quad t \in \mathbb{R}. \tag{3.7}$$

Then

$$g_{t+s}^* = g_t^* \circ g_s^*, \quad s, t \in \mathbb{R}. \tag{3.8}$$

Thus, $\{g_t^*\}_{t\in\mathbb{R}}$ determines a dynamical system on $\mathcal{M}$ as claimed.

We note that

$$M_t(i) = A_t i + B_t + i \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} \tag{3.9}$$

and thus,

$$A_t = \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} - \operatorname{Im}(M_t(i)), \ \ B_t = \operatorname{Re}(M_t(i)), \quad t \in \mathbb{R}. \tag{3.10}$$

Moreover, if $c_t \neq 0$ in (3.2), then $A_t = 0$, and hence

$$\int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} = \operatorname{Im}(M_t(i)) \quad \text{if } c_t \neq 0. \tag{3.11}$$

For the remainder of this section it is convenient to introduce the following assumptions.

Hypothesis 3.1. Assume Hypothesis 2.1 and suppose that

$$\gamma \in \mathbb{R}\backslash\{0\}, \text{ or equivalently, } c_t \neq 0 \text{ for } \begin{cases} t \in \mathbb{R}\backslash\{(\pi/\omega)\mathbb{Z}\} & \text{in case I,} \\ t \in \mathbb{R}\backslash\{0\} & \text{in cases II, III.} \end{cases} \tag{3.12}$$

The following statement is a variant of the exponential Herglotz representation theorem due to Aronszajn–Donoghue [3] (see also [4]).

Lemma 3.2. *We assume Hypothesis 3.1 and let* $(z,t) \in \mathbb{C}_+ \times \mathbb{R}$. *Moreover, we recall the convention* (2.25). *Given a Herglotz function* M_0 *with* $\operatorname{Im}(M_0(i)) \neq 0$, *we introduce the function*

$$N_t(z) = \operatorname{Ln}(\overleftrightarrow{c_t M_0(z) + d_t}). \tag{3.13}$$

Then $N_t(\cdot)$ *is analytic on* $\mathbb{C}_+$ *and it can be represented in the form*

$$N_t(z) = \operatorname{Re}(N_t(i)) + \int_{\mathbb{R}} d\lambda\, \xi_t(\lambda) \bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \bigg), \tag{3.14}$$

where

$$\xi_t(\lambda) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(\operatorname{Ln}(\overleftrightarrow{c_t M_0(\lambda + i\varepsilon) + d_t}) \big) \quad \textit{for a.e. } \lambda \in \mathbb{R}. \tag{3.15}$$

Proof. Since $M_0(i) \neq 0$, the expression $c_t M_0(z) + d_t$ never vanishes, and hence the lift $\overleftrightarrow{c_t M_0(z) + d_t}$ is well defined as a point on $\mathcal{R}$. To set the stage, we assume that $\overleftrightarrow{c_t M_0(z) + d_t}$ is a point on the nth sheet $\mathcal{S}_n$ of $\mathcal{R}$ for some (and hence for all) $z \in \mathbb{C}_+$,

$$2\pi n \leq \arg(\overleftrightarrow{c_t M_0(z) + d_t}) < 2\pi(n+1), \quad n \in \mathbb{Z}. \tag{3.16}$$

Then, by the definition of $\operatorname{Ln}(\cdot)$ on $\mathcal{R}$ we obtain

$$N_t(z) = \log(c_t M_0(z) + d_t) + 2\pi i n, \tag{3.17}$$

where $\log(\cdot)$ denotes the branch (2.21), (2.22) on $\mathcal{S}_0 = \Pi \cup \partial_+ \Pi$.

Given $t \in \mathbb{R}$, there are three possible outcomes for N_t depending on whether $c_t > 0$, $c_t < 0$, or $c_t = 0$. If $c_t > 0$, the function $c_t M_0(z) + d_t$ is a Herglotz function and thus,

$$N_t(z) = \operatorname{Re}(N_t(i)) + \int_{\mathbb{R}} d\lambda\, \eta_t(\lambda) \bigg(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \bigg) + 2\pi i n, \tag{3.18}$$

where

$$\eta_t(\lambda) = \frac{1}{\pi} \operatorname{Im}\big(\log(c_t m_0(\lambda) + d_t)\big) \text{ for a.e. } \lambda \in \mathbb{R} \tag{3.19}$$

and

$$m_0(\lambda) = \lim_{\varepsilon \downarrow 0} M_0(\lambda + i\varepsilon) \text{ for a.e. } \lambda \in \mathbb{R}. \tag{3.20}$$

Since

$$\frac{1}{\pi} \int_{\mathbb{R}} d\lambda \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) = i, \quad z \in \mathbb{C}_+, \tag{3.21}$$

one can rewrite (3.18) in the form (3.14) with

$$\begin{aligned} \xi_t(\lambda) &= \eta_t(\lambda) + 2n \\ &= \frac{1}{\pi} \operatorname{Im}\big(\log(c_t m_0(\lambda) + d_t) + 2\pi n i\big) \\ &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(\operatorname{Ln}(\overleftrightarrow{c_t M_0(\lambda + i\varepsilon) + d_t})\big). \end{aligned} \tag{3.22}$$

This proves (3.14) and (3.15) in the case where $c_t > 0$.
If $c_t < 0$, we obtain

$$\begin{aligned} N_t(z) &= \log(c_t M_0(z) + d_t) + 2\pi i n \\ &= \log(|c_t| M_0(z) - d_t) + 2\pi i n + \pi i. \end{aligned} \tag{3.23}$$

Using the Herglotz representation theorem for $|c_t| M_0(z) - d_t$, we arrive at

$$N_t(z) = \operatorname{Re}(N_t(i)) + \int_{\mathbb{R}} d\lambda\, \eta_t(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) + 2\pi i n + \pi i, \tag{3.24}$$

where

$$\eta_t(\lambda) = \frac{1}{\pi} \operatorname{Im}(\log(|c_t| m_0(\lambda) - d_t)) \quad \text{for a.e. } \lambda \in \mathbb{R}. \tag{3.25}$$

Thus, one again arrives at (3.20). Hence (3.14) applies with

$$\begin{aligned} \xi_t(\lambda) &= \eta_t(\lambda) + 2n - 1 \\ &= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \operatorname{Im}\big(\log(c_t M_0(\lambda + i\varepsilon) + d_t) + 2\pi n i\big) \\ &= \frac{1}{\pi} \operatorname{Im}\big(\operatorname{Ln}(\overleftrightarrow{c_t m_0(\lambda) + d_t})\big). \end{aligned} \tag{3.26}$$

Finally, if $c_t = 0$, then $N_t(z)$ is a constant with respect to z. Thus,

$$\begin{aligned} N_t(z) &= \log(d_t) + 2\pi i(n-1) \\ &= \log|d_t| + i(2\pi(n-1) + \arg(d_t)), \quad \operatorname{Im}(z) > 0, \end{aligned} \tag{3.27}$$

which proves (3.14) with

$$\xi_t(\lambda) = 2(n-1) + \pi^{-1} \arg(d_t) \in \mathbb{Z}, \tag{3.28}$$

a λ-independent integer constant. □

Now we can prove the absolute continuity of the measure associated with the Herglotz representation of the integrated (averaged) Herglotz function

$$M_{t_1,t_2}(z) = \int_{t_1}^{t_2} dt\, M_t(z), \quad \operatorname{Im}(z) > 0, \ \ t_1, t_2 \in \mathbb{R}, \ \ t_1 < t_2. \tag{3.29}$$

Theorem 3.3. *We assume Hypothesis 3.1 and let $z \in \mathbb{C}_+$, $t_j \in \mathbb{R}$, $j = 1, 2$, $t_1 < t_2$. Moreover, we recall the convention* (2.25)*. Then the integrated Herglotz function*

$$M_{t_1,t_2}(z) = \int_{t_1}^{t_2} dt\, M_t(z) \tag{3.30}$$

admits the Herglotz representation

$$M_{t_1,t_2}(z) = B_{t_1,t_2} + \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right). \tag{3.31}$$

Here $B_{t_1,t_2} \in \mathbb{R}$ and the measure μ_{t_1,t_2} is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ with Radon–Nikodym derivative (*density*) *a bounded function $\frac{d\mu_{t_1,t_2}}{d\lambda} = \xi_{t_1,t_2} \in L^\infty(\mathbb{R})$. In fact,*

$$\xi_{t_1,t_2}(\lambda) = \frac{1}{\gamma}\big(\xi_{t_2}(\lambda) - \xi_{t_1}(\lambda)\big), \tag{3.32}$$

where

$$\xi_t(\lambda) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \operatorname{Im}\big(\operatorname{Ln}(\overleftrightarrow{c_t M_0(\lambda + i\varepsilon) + d_t})\big), \quad t \in \mathbb{R}. \tag{3.33}$$

Proof. By Lemma 2.6, $\operatorname{Im}(M_{t_1,t_2})$ admits the representation

$$\begin{aligned} \operatorname{Im}(M_{t_1,t_2}(z)) &= \int_{t_1}^{t_2} dt\, \operatorname{Im}(g_t(M_t(z))) \\ &= \frac{1}{\gamma} \operatorname{Im}\big(\operatorname{Ln}(\overleftrightarrow{c_{t_2} M_0(z) + d_{t_2}}) - \operatorname{Ln}(\overleftrightarrow{c_{t_1} M_0(z) + d_{t_1}})\big). \end{aligned} \tag{3.34}$$

Hence, $\operatorname{Im}(M_{t_1,t_2})$ is uniformly bounded on $\mathbb{C}_+$, which proves that M_{t_1,t_2} has no linear term in its Herglotz representation. Moreover, by Fatou's theorem, the boundedness of $\operatorname{Im}(M_{t_1,t_2})$ on $\mathbb{C}_+$ ensures the absolute continuity of the measure μ_{t_1,t_2} in (3.31) with respect to Lebesgue measure on $\mathbb{R}$. Hence, (3.32) is a consequence of (3.34). ☐

Corollary 3.4. *In addition to the hypotheses of Theorem 3.3, we assume that $\gamma > 0$ and $t_1 < 0 < t_2$. If $\det(X) > 0$ we assume in addition that*

$$-\frac{\pi}{2\sqrt{\det(X)}} < t_1 < 0 < t_2 < \frac{\pi}{2\sqrt{\det(X)}}. \tag{3.35}$$

Then the density (3.32) *has the form*

$$\xi_{t_1,t_2}(\lambda) = \frac{1}{\gamma} + \frac{1}{\gamma\pi} \operatorname{Im}\left(\log\left(\frac{\Theta(t_2) m_0(\lambda) + 1}{-\Theta(t_1) m_0(\lambda) - 1} \right)\right) \quad \textit{for a.e. } \lambda \in \mathbb{R}, \tag{3.36}$$

where

$$m_0(\lambda) = \lim_{\varepsilon \downarrow 0} (\gamma M_0(\lambda + i\varepsilon) - \beta) \quad \textit{for a.e. } \lambda \in \mathbb{R} \tag{3.37}$$

and

$$\Theta(t) = \lim_{s \to \sqrt{\det(X)}} \frac{\tan(st)}{s} = \begin{cases} \frac{\tan(\sqrt{\det(X)}t)}{\sqrt{\det(X)}} & \textit{if } \det(X) > 0, \\ t & \textit{if } \det(X) = 0, \\ \frac{\tanh(\sqrt{|\det(X)|}t)}{\sqrt{|\det(X)|}} & \textit{if } \det(X) < 0, \end{cases} \quad t \in \mathbb{R}. \tag{3.38}$$

Remark 3.5. Define

$$T_1(X) = \begin{cases} -\frac{\pi}{2\sqrt{\det(X)}}, & \det(X) > 0, \\ -\infty, & \det(X) \le 0, \end{cases} \qquad T_2(X) = \begin{cases} \frac{\pi}{2\sqrt{\det(X)}}, & \det(X) > 0, \\ \infty, & \det(X) \le 0. \end{cases} \tag{3.39}$$

Then the density (3.32) has the form

$$\xi_{T_1(X),T_2(X)}(\lambda) = \frac{1}{\gamma} + \begin{cases} 0 & \text{if } \det(X) \geq 0, \\ \frac{1}{\gamma\pi}\operatorname{Im}\left(\log\left(\frac{\pi m_0(\lambda)+2\sqrt{|\det(X)|}}{\pi m_0(\lambda)-2\sqrt{|\det(X)|}}\right)\right) & \text{if } \det(X) < 0. \end{cases} \tag{3.40}$$

Next, we discuss the following technical result.

Lemma 3.6. *We assume Hypothesis 3.1 and let* $t_j \in \mathbb{R} \cup \{-\infty, \infty\}$, $j = 1, 2$, $t_1 < t_2$. *Moreover, we denote by* μ_{t_1,t_2} *the Borel measure in the Herglotz representation* (3.31) *of the integrated Herglotz function* (3.30). *Then for any bounded Borel set* $\Delta \subset \mathbb{R}$ *the function* $t \mapsto \mu_t(\Delta)$ *is measurable and one has*

$$\int_{t_1}^{t_2} dt\, \mu_t(\Delta) = \mu_{t_1,t_2}(\Delta). \tag{3.41}$$

Proof. The proof is based on the following representation:

$$\int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda)\, f(\lambda) = \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\, f(\lambda), \tag{3.42}$$

which is valid for a wide function class of f to be specified below. We split the proof into four steps. First, we establish (3.42) for functions f of the type

$$f(\lambda) = \phi_\varepsilon(\lambda - \delta), \quad \varepsilon > 0, \ \delta \in \mathbb{R}, \tag{3.43}$$

where $\phi_\varepsilon(\lambda)$ is an approximate identity:

$$\phi_\varepsilon(\lambda) = \varepsilon^{-1}\phi(\varepsilon^{-1}\lambda) \quad \text{with } \phi(\lambda) = \frac{1}{\pi}\frac{1}{1+\lambda^2}, \ \lambda \in \mathbb{R}. \tag{3.44}$$

Second, we prove (3.42) for functions f that can be represented as a convolution of ϕ_ε with a C_0^∞-function k:

$$f(\lambda) = (\phi_\varepsilon * k)(\lambda), \quad k \in C_0^\infty(\mathbb{R}), \ \varepsilon > 0. \tag{3.45}$$

Third, we prove (3.42) for $f \in C_0^\infty(\mathbb{R})$. Finally, to prove assertion (3.41), we establish (3.42) for characteristic functions of finite intervals.

Step I. Let $z = \delta + i\varepsilon \in \mathbb{C}_+$. By the representation (3.6),

$$\operatorname{Im}\big(g_t(M_0(\delta + i\varepsilon))\big) = A_t\varepsilon + \varepsilon\int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{(\lambda-\delta)^2+\varepsilon^2}. \tag{3.46}$$

Since $\gamma \neq 0$, we have $A_t = 0$ for almost all $t \in \mathbb{R}$ by (2.16) and hence

$$\int_{t_1}^{t_2} dt\, \operatorname{Im}\big(g_t(M_0(\delta + i\varepsilon))\big) = \varepsilon\int_{t_1}^{t_2} dt \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{(\lambda-\delta)^2+\varepsilon^2}. \tag{3.47}$$

On the other hand, we infer

$$\int_{t_1}^{t_2} dt\, \operatorname{Im}\big(g_t(M_0(\delta + i\varepsilon))\big) = \varepsilon\int_{\mathbb{R}} \frac{d\mu_{t_1,t_2}(\lambda)}{(\lambda-\delta)^2+\varepsilon^2} \tag{3.48}$$

by applying Theorem 3.3. Comparing (3.47) and (3.48) proves (3.42) for functions of the type (3.43), (3.44).

Step II. Let $k \in C_0^\infty(\mathbb{R})$ and let $\operatorname{supp}(k) \subset [\delta_1, \delta_2]$ for some $-\infty < \delta_1 < \delta_2 < \infty$.

We start with two observations. Given $\varepsilon > 0$, the function $k_\varepsilon(\lambda, \delta) = \phi_\varepsilon(\lambda - \delta)k(\delta)$, $(\lambda, \delta) \in \mathbb{R} \times [\delta_1, \delta_2]$, is summable with respect to the product measure $d\mu_t \times d\delta$, $t \in [t_1, t_2]$, as well as with respect to the product measure $d\mu \times d\delta$:

$$k_\varepsilon \in L^1(\mathbb{R} \times [\delta_1, \delta_2]; d\mu_t \times d\delta), \quad t \in [t_1, t_2] \tag{3.49}$$

and

$$k_\varepsilon \in L^1(\mathbb{R}\times[\delta_1,\delta_2]; d\mu\times d\delta). \tag{3.50}$$

Moreover, we claim that the function F_ε (cf. (3.45)),

$$F_\varepsilon(t,\delta) = \int_{\mathbb{R}} d\mu_t(\lambda)\,\phi_\varepsilon(\lambda-\delta)f(\delta), \quad t\in[t_1,t_2], \tag{3.51}$$

is summable on $[t_1,t_2]\times[\delta_1,\delta_2]$:

$$F_\varepsilon \in L^1([t_1,t_2]\times[\delta_1,\delta_2]; dt\times d\delta). \tag{3.52}$$

To prove (3.52), we note that, by the representation (3.6), formula (3.46) is valid again. Thus, for any $\varepsilon>0$, the function h given by

$$h(t,\delta) = A_t\varepsilon + \varepsilon\int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{(\lambda-\delta)^2+\varepsilon^2} \tag{3.53}$$

is continuous on $[t_1,t_2]\times[\delta_1,\delta_2]$. Hence $h(t,\delta)f(\delta)$ is also continuous on $[t_1,t_2]\times[\delta_1,\delta_2]$ and thus bounded. Since $A_t=0$ a.e., $F_\varepsilon(t,\delta)$ is measurable and essentially bounded on $[t_1,t_2]\times[\delta_1,\delta_2]$. This proves (3.52).

Finally, the validity of (3.42) for the function class (3.45) follows from the subsequent chain of equalities:

$$\begin{aligned}
&\int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda)\,(\phi_\varepsilon * k)(\lambda)\\
&= \int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda) \int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{since } \operatorname{supp}(k)\subset[\delta_1,\delta_2])\\
&= \int_{t_1}^{t_2} dt \int_{\delta_1}^{\delta_2} d\delta \int_{\mathbb{R}} d\mu_t(\lambda)\,\phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by (3.49) and Fubini's theorem})\\
&= \int_{\delta_1}^{\delta_2} d\delta \int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda)\,\phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by (3.52) and Fubini's theorem})\\
&= \int_{\delta_1}^{\delta_2} d\delta \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\,\phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by step I})\\
&= \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda) \int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)k(\delta) \quad (\text{by (3.50) and Fubini's theorem})\\
&= \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\,(\phi_\varepsilon * k)(\lambda) \quad (\text{since } \operatorname{supp}(k)\subset[\delta_1,\delta_2]).
\end{aligned} \tag{3.54}$$

Step III. Let $f\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}(f)\subset[\delta_1,\delta_2]$. One then infers

$$\lim_{\varepsilon\downarrow 0}(\phi_\varepsilon * f)(\lambda) = \lim_{\varepsilon\downarrow 0}\int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)f(\delta) = f(\lambda) \tag{3.55}$$

uniformly with respect to λ as long as λ varies in a compact set $\Lambda\subset\mathbb{R}$. With $\Lambda = [\delta_1-1,\delta_2+1]$ we obtain the estimate

$$\left|\int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)f(\delta)\right| \le \max_{\delta\in\operatorname{supp}(f)}|f(\delta)|(\delta_2-\delta_1)\frac{1}{\pi}\frac{\varepsilon}{\operatorname{dist}^2(\lambda,[\delta_1,\delta_2])}, \quad \lambda\in\mathbb{R}\backslash\Lambda. \tag{3.56}$$

Thus, there exists a constant $C=C(\delta_1,\delta_2)$ such that

$$|(\phi_\varepsilon * f)(\lambda)| = \left|\int_{\delta_1}^{\delta_2} d\delta\,\phi_\varepsilon(\lambda-\delta)f(\delta)\right| \le C\frac{\varepsilon}{1+\lambda^2}, \quad \lambda\in\mathbb{R}\backslash\Lambda. \tag{3.57}$$

Since

$$\sup_{t\in[t_1,t_2]} \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} < \infty, \tag{3.58}$$

the uniform convergence (3.55) combined with the estimate (3.57) and the result of Step II proves (3.42) for $f \in C_0^\infty(\mathbb{R})$.

Step IV. Let Δ be a finite interval and $f_1(\lambda) \geq f_2(\lambda) \geq \cdots$ a monotone sequence of nonnegative functions, $f_n \in C_0^\infty(\mathbb{R})$, $n \in \mathbb{N}$, converging pointwise to the characteristic function of the interval Δ as n approaches infinity:

$$\lim_{n\to\infty} f_n(\lambda) = \chi_\Delta(\lambda), \quad \lambda \in \mathbb{R}. \tag{3.59}$$

By the dominated convergence theorem we then obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\, f_n(\lambda) = \int_{\mathbb{R}} d\mu_{t_1,t_2}(\lambda)\, \chi_\Delta(\lambda) = \mu_{t_1,t_2}(\Delta) \tag{3.60}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) = \int_{\mathbb{R}} d\mu_t(\lambda)\, \chi_\Delta(\lambda) = \mu_t(\Delta), \quad t \in [t_1,t_2]. \tag{3.61}$$

Since

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) \leq \int_{\mathbb{R}} d\mu_t(\lambda)\, f_1(\lambda) \\ &\leq \max_{s\in\operatorname{supp}(f_1)} \big((1+s^2) f_1(s)\big) \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} \\ &\leq \max_{s\in\operatorname{supp}(f_1)} \big((1+s^2) f_1(s)\big) \sup_{t\in\mathbb{R}} \bigg(\int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1+\lambda^2} \bigg), \end{aligned} \tag{3.62}$$

we obtain

$$\lim_{n\to\infty} \int_{t_1}^{t_2} dt \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) = \int_{t_1}^{t_2} dt \lim_{n\to\infty} \int_{\mathbb{R}} d\mu_t(\lambda)\, f_n(\lambda) = \int_{t_1}^{t_2} dt\, \mu_t(\Delta), \tag{3.63}$$

where we have used the dominated convergence theorem once again. By Step III and by (3.60) this proves (3.42) for $f(\lambda) = \chi_\Delta(\lambda)$.

The extension from the case of bounded intervals Δ to the case of bounded Borel sets Δ is now straightforward and the proof is complete. □

Given a general Herglotz function M of the type

$$M(z) = Az + B + \int_{\mathbb{R}} d\mu(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right), \quad A \geq 0,\ B \in \mathbb{R},\ z \in \mathbb{C}_+, \tag{3.64}$$

we introduce the following subsets of $\mathbb{R}$:

$$\mathcal{A}(M) = \big\{\lambda \in \mathbb{R} \mid \lim_{\varepsilon\downarrow 0} M(\lambda + i\varepsilon) \in \mathbb{C}_+\big\}, \tag{3.65}$$

$$\begin{aligned} \mathcal{P}(M) = &\big\{\lambda \in \mathbb{R} \mid \lim_{\varepsilon\downarrow 0} \varepsilon \operatorname{Im}(M(\lambda + i\varepsilon)) \in (0,\infty)\big\} \\ &\cup \big\{\lambda \in \mathbb{R} \mid \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \operatorname{Im}(M(\lambda + i\varepsilon)) \in (0,\infty)\big\}, \end{aligned} \tag{3.66}$$

$$\mathcal{S}(M) = \mathbb{R}\backslash\{\mathcal{A}(M) \cup \mathcal{P}(M)\}. \tag{3.67}$$

By the results of Aronszajn [2], Donoghue [21], and Simon and Wolff [46], the subsets (3.65)–(3.67) are invariant with respect to the entire family $\{g \circ M\}_{g\in\operatorname{Aut}(\mathbb{C}_+)}$ of Herglotz functions:

$$\mathcal{A}(M) = \mathcal{A}(g\circ M), \quad \mathcal{P}(M) = \mathcal{P}(g\circ M), \quad \mathcal{S}(M) = \mathcal{S}(g\circ M), \quad g \in \operatorname{Aut}(\mathbb{C}_+). \tag{3.68}$$

Strictly speaking, these results were obtained for Herglotz functions that are the Stieltjes transforms of finite Borel measures. For the sake of completeness we prove this invariance in the case of general Herglotz functions. The invariance of the set $\mathcal{A}(M)$ is obvious from (2.8)–(2.10). The invariance of the set $\mathcal{S}(M)$ is then a consequence of the invariance of $\mathcal{P}(M)$. In order to prove the invariance of the set $\mathcal{P}(M)$ we need some additional considerations.

We start with recalling the following well-known result.

Lemma 3.7 (see, e.g., [2], [3], [46])**.** *Let M be a Herglotz function with representation* (3.64)*. Then, for any $\lambda_0 \in \mathbb{R}$ we have*

$$\lim_{\varepsilon\downarrow 0}(-i\varepsilon)M(\lambda_0+i\varepsilon)=\mu(\{\lambda_0\}). \tag{3.69}$$

In particular,

$$\lim_{\varepsilon\downarrow 0}\varepsilon\,\mathrm{Im}(M(\lambda_0+i\varepsilon))=\mu(\{\lambda_0\}) \tag{3.70}$$

and

$$\lim_{\varepsilon\downarrow 0}\varepsilon\,\mathrm{Re}(M(\lambda_0+i\varepsilon))=0. \tag{3.71}$$

Definition 3.8. A Herglotz function M of the type (3.64) is said to have a normal derivative at a point $\lambda \in \mathbb{R}$ if the following two limits exist (finitely):

(i) $M(\lambda)=\lim_{\varepsilon\downarrow 0}M(\lambda+i\varepsilon)\in\mathbb{C}$.

(ii) $M'(\lambda)=\lim_{\varepsilon\downarrow 0}(M(\lambda+i\varepsilon)-M(\lambda))/(i\varepsilon)\in\mathbb{C}$.

Lemma 3.9. *Let M be a Herglotz function with representation* (3.64)*. Assume*, *in addition*, *that*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\,\mathrm{Im}(M(\lambda_0+i\varepsilon))\in(0,\infty) \tag{3.72}$$

for some point $\lambda_0 \in \mathbb{R}$. Then M has a real normal boundary value at λ_0 and a strictly positive normal derivative at λ_0:

$$M(\lambda_0)=\lim_{\varepsilon\downarrow 0}M(\lambda_0+i\varepsilon)\in\mathbb{R}, \tag{3.73}$$

$$M'(\lambda_0)=\lim_{\varepsilon\downarrow 0}\frac{M(\lambda_0+i\varepsilon)-M(\lambda_0)}{i\varepsilon}\in(0,\infty). \tag{3.74}$$

Proof. Let $\mathcal{I}$ be a finite open interval containing λ_0. We decompose M as $M=M_1+M_2$, where

$$M_1(z)=Az+B+\int_{\mathbb{R}\backslash\mathcal{I}}d\mu(\lambda)\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right)-\int_{\mathcal{I}}d\mu(\lambda)\,\frac{\lambda}{1+\lambda^2}, \tag{3.75}$$

$$M_2(z)=\int_{\mathcal{I}}\frac{d\mu(\lambda)}{\lambda-z}. \tag{3.76}$$

Clearly,

$$M_1(\lambda_0)=\lim_{\varepsilon\downarrow 0}M_1(\lambda_0+i\varepsilon)\in\mathbb{R} \tag{3.77}$$

and

$$M_1'(\lambda_0)=\begin{cases}\lim_{\varepsilon\downarrow 0}\frac{M_1(\lambda_0+i\varepsilon)-M_1(\lambda_0)}{i\varepsilon}>0 & \text{if } A\neq 0 \text{ or } \mu(\mathbb{R}\backslash\mathcal{I})\neq 0,\\ 0 & \text{if } A=0 \text{ and } \mu(\mathbb{R}\backslash\mathcal{I})=0.\end{cases} \tag{3.78}$$

Hypothesis (3.72) and formula (3.76) imply

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\,\mathrm{Im}(M_1(\lambda_0+i\varepsilon))=0 \tag{3.79}$$

and hence

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\operatorname{Im}(M(\lambda_0+i\varepsilon)) &= \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\operatorname{Im}(M_2(\lambda_0+i\varepsilon)) = \lim_{\varepsilon\downarrow 0}\int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0)^2+\varepsilon^2}\\ &= \int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0)^2}\in[0,\infty),\end{aligned} \tag{3.80}$$

where we have used the monotone convergence theorem at the last step. Since $\mathcal{I}$ is a finite interval and $\int_{\mathcal{I}} d\mu(\lambda)\,(\lambda-\lambda_0)^{-2}<\infty$ by (3.72) and (3.80), an application of the dominated convergence theorem then yields

$$\lim_{\varepsilon\downarrow 0}\operatorname{Re}(M_2(\lambda_0+i\varepsilon)) = \lim_{\varepsilon\downarrow 0}\int_{\mathcal{I}} d\mu(\lambda)\,\frac{(\lambda-\lambda_0)}{(\lambda-\lambda_0)^2+\varepsilon^2} = \int_{\mathcal{I}}\frac{d\mu(\lambda)}{\lambda-\lambda_0}\in\mathbb{R}. \tag{3.81}$$

Thus,

$$M_2(\lambda_0)=\lim_{\varepsilon\downarrow 0}M_2(\lambda_0+i\varepsilon)\in\mathbb{R}, \tag{3.82}$$

and combining (3.77) and (3.82) then proves (3.73). Applying the dominated convergence theorem once again yields

$$\begin{aligned}M_2'(\lambda_0) &= \lim_{\varepsilon\downarrow 0}\frac{M_2(\lambda_0+i\varepsilon)-M_2(\lambda_0)}{i\varepsilon} = \lim_{\varepsilon\downarrow 0}\int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0-i\varepsilon)(\lambda-\lambda_0)}\\ &= \begin{cases}\int_{\mathcal{I}}\frac{d\mu(\lambda)}{(\lambda-\lambda_0)^2}>0 & \text{if } \mu(\mathcal{I})\neq 0,\\ 0 & \text{if } \mu(\mathcal{I})=0.\end{cases}\end{aligned} \tag{3.83}$$

Since by the hypothesis (3.72), either $A>0$ or $\mu(\mathbb{R})\neq 0$ in the Herglotz representation (3.64) of M, combination of (3.78) and (3.83) proves (3.74). ☐

Lemma 3.10. *Let M be a Herglotz function of the type* (3.64)*. Then*

$$\mathcal{P}(M)=\mathcal{P}(g\circ M),\quad g\in\operatorname{Aut}(\mathbb{C}_+). \tag{3.84}$$

Proof. It suffices to prove the inclusion

$$\mathcal{P}(M)\subset\mathcal{P}(g\circ M),\quad g\in\operatorname{Aut}(\mathbb{C}_+). \tag{3.85}$$

Moreover, any automorphism $g\in\operatorname{Aut}(\mathbb{C}_+)$ admits the representation

$$g=g_1\circ f\circ g_2, \tag{3.86}$$

where $g_j\in\operatorname{Aut}(\mathbb{C}_+)$, $j=1,2$, are linear transformations of $\mathbb{C}_+$ and

$$f(z)=-\frac{1}{z},\quad z\in\mathbb{C}_+. \tag{3.87}$$

Since, obviously, $\mathcal{P}(M)$ is invariant under linear transformations of $\mathbb{C}_+$, it suffices to establish the inclusion

$$\mathcal{P}(M)\subset\mathcal{P}(f\circ M). \tag{3.88}$$

Let $\lambda\in\mathcal{P}(M)$. By the definition of $\mathcal{P}(M)$ either

$$\lim_{\varepsilon\downarrow 0}\varepsilon\operatorname{Im}(M(\lambda+i\varepsilon))\in(0,\infty) \tag{3.89}$$

or

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\operatorname{Im}(M(\lambda+i\varepsilon))\in(0,\infty). \tag{3.90}$$

In the case of (3.89) we obtain

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\operatorname{Im}(f\circ M(\lambda+i\varepsilon)) = \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\frac{\operatorname{Im}(M(\lambda+i\varepsilon))}{|M(\lambda+i\varepsilon)|^2}\in(0,\infty) \tag{3.91}$$

by using (3.69)–(3.71). Therefore, $\lambda \in \mathcal{P}(f \circ M)$. Next, we consider the case (3.90). By Lemma 3.9, $M(\lambda) = \lim_{\varepsilon\downarrow 0} M(\lambda + i\varepsilon) \in \mathbb{R}$ and $M(z)$ has a positive normal derivative at the point λ. If $M(\lambda) \neq 0$, then

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \operatorname{Im}((f\circ M)(\lambda+i\varepsilon)) = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} \operatorname{Im}\left(\frac{1}{M(\lambda)} - \frac{1}{M(\lambda+i\varepsilon)}\right) = \frac{M'(\lambda)}{(M(\lambda))^2} > 0. \tag{3.92}$$

If $M(\lambda) = 0$, then

$$\lim_{\varepsilon\downarrow 0} \varepsilon \operatorname{Im}((f\circ M)(\lambda+i\varepsilon)) = \lim_{\varepsilon\downarrow 0} \varepsilon \operatorname{Im}\left(-\frac{1}{M(\lambda+i\varepsilon)}\right) = \frac{1}{M'(\lambda)} > 0. \tag{3.93}$$

Thus, (3.90) implies $\lambda \in \mathcal{P}(f \circ M)$. Therefore, in both cases (3.89) and (3.90), we have $\lambda \in \mathcal{P}(f \circ M)$, which proves (3.88) and hence (3.84). □

The following result provides a spectral characterization of the invariant sets $\mathcal{A}(M)$, $\mathcal{S}(M)$, and $\mathcal{P}(M)$ (see [46] for a strategy of the proof). We recall that a measure μ on $\mathbb{R}$ is supported by the set $\mathcal{T} \subseteq \mathbb{R}$ if $\mu(\mathbb{R}\backslash\mathcal{T}) = 0$.

Lemma 3.11. *Suppose that M is a Herglotz function of the type* (3.64), *and that $g \in \operatorname{Aut}(\mathbb{C}_+)$. Moreover, assume that μ_g is the measure in the Herglotz representation of $g \circ M$ and that*

$$\mu_g = \mu_g^{\rm ac} + \mu_g^{\rm sc} + \mu_g^{\rm pp}, \quad g \in \operatorname{Aut}(\mathbb{C}_+), \tag{3.94}$$

is the Lebesgue decomposition of μ_g into its absolutely continuous, singular continuous, and pure point parts. Then $\mu_g^{\rm ac}$, $\mu_g^{\rm sc}$, and $\mu_g^{\rm pp}$ are supported by the sets $\mathcal{A}(M)$, $\mathcal{S}(M)$, and $\mathcal{P}(M)$. Moreover, for any point $\lambda \in \mathcal{P}(M)$ there exists an automorphism $g \in \operatorname{Aut}(\mathbb{C}_+)$ such that

$$\mu_g^{\rm pp}(\{\lambda\}) > 0. \tag{3.95}$$

Remark 3.12. Originally, the set $\mathcal{P}(M)$ was introduced in the context of rank-one perturbations in [46] by

$$\mathcal{P}(M) = \big\{\lambda \in \mathbb{R} \mid \mu(\{\lambda\}) > 0\big\} \cup \left\{\lambda \in \mathbb{R} \,\Big|\, \int_{\mathbb{R}} \frac{d\mu(s)}{(s-\lambda)^2} < \infty\right\}. \tag{3.96}$$

Naively one might think that the set $\mathbb{R}\backslash\mathcal{A}(M)$ coincides (modulo Lebesgue null sets) with the complement of the support of the absolutely continuous component $\mu^{\rm ac}$ of the measure μ associated with the Herglotz function M. Thus, one might erroneously conclude that

$$|\operatorname{supp}(\mu^{\rm ac}) \cap (\mathbb{R}\backslash\mathcal{A}(M))| = 0. \tag{3.97}$$

The following counterexample illustrates the situation.

Example 3.13. Let $K \subset [0,1]$ be a closed nowhere dense set of positive Lebesgue measure and let

$$M(z) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{\lambda - z}, \quad d\mu(\lambda) = \chi_{[0,1]\backslash K}(\lambda)\, d\lambda, \tag{3.98}$$

where χ_Λ denotes the characteristic function of a set $\Lambda \subseteq \mathbb{R}$. Then

$$\operatorname{supp}(\mu) = \operatorname{supp}(\mu^{\rm ac}) = [0,1] \tag{3.99}$$

but

$$|\operatorname{supp}(\mu^{\rm ac}) \cap (\mathbb{R}\backslash\mathcal{A}(M))| = |K| > 0. \tag{3.100}$$

Thus, (3.97) is not valid in general.

Combining the results of Corollary 3.4, Remark 3.5, and Lemma 3.6, we can formulate the following spectral averaging theorems.

Theorem 3.14. *Assume Hypothesis 3.1 and let* $t_j \in \mathbb{R} \cup \{-\infty, \infty\}$, $j = 1, 2$, $t_1 < t_2$. *Moreover, suppose that* M_0 *is a Herglotz function of the type* (3.64) *and that* M_t, $t \in \mathbb{R}$, *is the one-parameter family of Herglotz functions given by* (3.5) *and* (3.6). *In addition, we denote by* μ_t^{ac}, μ_t^{sc}, *and* μ_t^{pp} *the absolutely continuous, singular continuous, and pure point parts in the Lebesgue decomposition of* μ_t *in* (3.6):

$$\mu_t = \mu_t^{\mathrm{ac}} + \mu_t^{\mathrm{sc}} + \mu_t^{\mathrm{pp}}, \quad t \in \mathbb{R}. \tag{3.101}$$

Then the averaged measures

$$\int_{t_1}^{t_2} dt\, d\mu_t, \quad \int_{t_1}^{t_2} dt\, d\mu_t^{\mathrm{ac}}, \quad \int_{t_1}^{t_2} dt\, d\mu_t^{\mathrm{sc}}, \quad \int_{t_1}^{t_2} dt\, d\mu_t^{\mathrm{pp}} \tag{3.102}$$

are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$. *More precisely, given a bounded Borel set* $\Delta \subset \mathbb{R}$, *the functions* $t \mapsto \mu_t(\Delta)$, $t \mapsto \mu_t^{\mathrm{ac}}(\Delta)$, $t \mapsto \mu_t^{\mathrm{sc}}(\Delta)$, *and* $t \mapsto \mu_t^{\mathrm{pp}}(\Delta)$ *are measurable and*

$$\int_{t_1}^{t_2} dt \left\{ \begin{array}{c} \mu_t(\Delta) \\ \mu_t^{\mathrm{ac}}(\Delta) \\ \mu_t^{\mathrm{sc}}(\Delta) \\ \mu_t^{\mathrm{pp}}(\Delta) \end{array} \right\} = \left\{ \begin{array}{l} \mu_{t_1,t_2}(\Delta), \\ \mu_{t_1,t_2}(\Delta \cap \mathcal{A}), \\ \mu_{t_1,t_2}(\Delta \cap \mathcal{S}), \\ \mu_{t_1,t_2}(\Delta \cap \mathcal{P}), \end{array} \right\} \tag{3.103}$$

where μ_{t_1,t_2} *is the absolutely continuous measure in the Herglotz representation* (3.31) *of the integrated Herglotz function* (3.30) *in Theorem 3.3 and* $\mathcal{A}(M_0)$, $\mathcal{P}(M_0)$, *and* $\mathcal{S}(M_0)$ *are the invariant sets* (3.65)–(3.67) *associated with the Herglotz function* M_0. *In particular,*

$$|\{t \in \mathbb{R} \mid \mu_t^{\mathrm{sc}}(\mathbb{R}) \neq 0\}| = 0 \quad \text{if } |\mathcal{S}(M_0)| = 0, \tag{3.104}$$

$$|\{t \in \mathbb{R} \mid \mu_t^{\mathrm{pp}}(\mathbb{R}) \neq 0\}| = 0 \quad \text{if } |\mathcal{P}(M_0)| = 0. \tag{3.105}$$

Proof. Equation (3.41) implies the result (3.103) since

$$\begin{aligned} \mu_t(\Delta \cap \mathcal{A}(M_0)) &= \mu_t^{\mathrm{ac}}(\Delta \cap \mathcal{A}(M_0)), \\ \mu_t(\Delta \cap \mathcal{S}(M_0)) &= \mu_t^{\mathrm{sc}}(\Delta \cap \mathcal{S}(M_0)), \\ \mu_t(\Delta \cap \mathcal{P}(M_0)) &= \mu_t^{\mathrm{pp}}(\Delta \cap \mathcal{P}(M_0)) \end{aligned} \tag{3.106}$$

and $\mathcal{A}(M_0)$, $\mathcal{S}(M_0)$, and $\mathcal{P}(M_0)$ are Borel sets (cf. [17]). □

Remark 3.15. The "life-time" $|\{t \in \mathbb{R} \mid \mu_t^{\mathrm{sing}}(\mathbb{R} \backslash \mathcal{A}(M_0)) \neq 0\}|$ is never zero whenever $|\mathcal{S}(M_0) \cup \mathcal{P}(M_0)| \neq 0$. Here

$$\mu_t^{\mathrm{sing}} = \mu_t^{\mathrm{sc}} + \mu_t^{\mathrm{pp}}. \tag{3.107}$$

As concrete examples show (cf. [15]), it may be finite or infinite depending upon the choice of the Herglotz function M_0.

Remark 3.16. Example 3.13 shows that the sets $\mathrm{supp}(\mu_t^{\mathrm{ac}})$ and $\mathbb{R} \backslash \mathcal{A}$ may have nontrivial intersection of positive Lebesgue measure and that

$$|\{t \in \mathbb{R} \mid \mu_t^{\mathrm{sing}}(\mathrm{supp}(\mu_t^{\mathrm{ac}})) \neq 0\}| \neq 0 \tag{3.108}$$

in general.

As a consequence of the previous theorem we get the following global result.

Theorem 3.17. *Under the hypotheses of Theorem 3.14, suppose that $\mathcal{A}(M_0)$, $\mathcal{S}(M_0)$, and $\mathcal{P}(M_0)$ are the invariant sets associated with the Herglotz function M_0. Then for any bounded Borel set $\Delta \subset \mathbb{R}$ the following results hold*:

(i) *If* $\det(X) > 0$, *then*

$$|\gamma| \int_{-\pi/\left(2\sqrt{\det(X)}\right)}^{\pi/\left(2\sqrt{\det(X)}\right)} dt \begin{Bmatrix} \mu_t(\Delta) \\ \mu_t^{\mathrm{ac}}(\Delta) \\ \mu_t^{\mathrm{sc}}(\Delta) \\ \mu_t^{\mathrm{pp}}(\Delta) \end{Bmatrix} = \begin{Bmatrix} |\Delta|, \\ |\Delta \cap \mathcal{A}(M_0)|, \\ |\Delta \cap \mathcal{S}(M_0)|, \\ |\Delta \cap \mathcal{P}(M_0)|. \end{Bmatrix} \tag{3.109}$$

(ii) *If* $\det(X) = 0$, *then*

$$|\gamma| \int_{-\infty}^{\infty} dt \begin{Bmatrix} \mu_t(\Delta) \\ \mu_t^{\mathrm{ac}}(\Delta) \\ \mu_t^{\mathrm{sc}}(\Delta) \\ \mu_t^{\mathrm{pp}}(\Delta) \end{Bmatrix} = \begin{Bmatrix} |\Delta|, \\ |\Delta \cap \mathcal{A}(M_0)|, \\ |\Delta \cap \mathcal{S}(M_0)|, \\ |\Delta \cap \mathcal{P}(M_0)|. \end{Bmatrix} \tag{3.110}$$

(iii) *If* $\det(X) < 0$, *then*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t(\Delta) = \int_{\Delta} d\lambda\, \xi(\lambda), \tag{3.111}$$

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{\mathrm{ac}}(\Delta) = \int_{\Delta \cap \mathcal{A}} d\lambda\, \xi(\lambda) \tag{3.112}$$

with

$$\xi(\lambda) = 1 + \frac{1}{\pi} \operatorname{Im}\left(\log\left(\frac{\pi m_0(\lambda) + 2\sqrt{|\det(X)|}}{\pi m_0(\lambda) - 2\sqrt{|\det(X)|}} \right) \right) \quad \textit{for a.e. } \lambda \in \mathbb{R} \tag{3.113}$$

and m_0 given by (3.37). *Moreover,*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{\mathrm{sc}}(\Delta) = |\Delta \cap \mathcal{S}(M_0) \cap \mathcal{R}(M_0)| \tag{3.114}$$

and

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{\mathrm{pp}}(\Delta) = |\Delta \cap \mathcal{P}(M_0) \cap \mathcal{R}(M_0)|, \tag{3.115}$$

where

$$\mathcal{R}(M_0) = \left\{ \lambda \in \mathbb{R} \mid \lim_{\varepsilon \downarrow 0} M_0(\lambda + i\varepsilon) \in \mathbb{R} \textit{ and } |m_0(\lambda)| < (2/\pi)\sqrt{|\det(X)|} \right\}. \tag{3.116}$$

Remark 3.18. Theorem 3.17 shows the universality of the averaging over the corresponding period in the case of cyclic groups g_t (associated with the one-parameter subgroups e^{tX} of SL$_2(\mathbb{R})$), where $\det(X) > 0$. It also yields the universality of the averaging over the entire parameter space in the limiting cases corresponding to $\det(X) = 0$. However, the theorem also shows that averaging in the case of hyperbolic one-parameter subgroups g_t with $\det(X) < 0$ depends on the initial Herglotz function M_0.

Remark 3.19. An analogous result concerning the decomposition of Lebesgue measure $|\cdot|$ restricted to $\mathcal{A}$ and $|\cdot|$ restricted to $\mathbb{R}\backslash\mathcal{A}$ into integrals of the measures μ_t^{ac} and $\mu_t^{\mathrm{sing}} = \mu_t^{\mathrm{sc}} + \mu_t^{\mathrm{pp}}$ on the unit circle first appeared in [1]. In the case of selfadjoint rank-one perturbations of selfadjoint operators (which is a special case of (3.110) as observed in Remark 2.4), (3.110) appeared in [17].

4. Spectral averaging and Hausdorff measures

Lebesgue's decomposition (3.101) of measures is a particular case of a more general result in the theory of decomposing measures with respect to Hausdorff measures. This result states that for each $\kappa \in [0,1]$, a Borel measure μ can be decomposed uniquely as

$$\mu = \mu^{\kappa\text{-c}} + \mu^{\kappa\text{-s}}, \tag{4.1}$$

where $\mu^{\kappa\text{-c}}$ is κ-continuous with respect to the κ-dimensional Hausdorff measure h^κ (i.e., $\mu^{\kappa\text{-c}}$ gives zero weight to sets with zero κ-dimensional Hausdorff measure h^κ) and $\mu^{\kappa\text{-s}}$ is κ-singular with respect to the κ-dimensional Hausdorff measure h^κ (i.e., $\mu^{\kappa\text{-s}}$ is supported on a set of zero κ-dimensional Hausdorff measure h^κ). For more details on the decomposition (4.1), we refer the reader to [38], [41], and [42].

We recall that the κ-dimensional Hausdorff (outer) measure h^κ, $\kappa \in [0,1]$, of a set $S \subset \mathbb{R}$ is defined as

$$h^\kappa(S) = \lim_{\delta\downarrow 0} \inf_{\delta\text{-covers}} \sum_{n\in\mathbb{N}} |I_n(\delta)|^\kappa, \tag{4.2}$$

where the infimum is taken over the countable collections $\{I_n(\delta)\}_{n\in\mathbb{N}}$ of intervals, the δ-covers, such that

$$S \subset \bigcup_{n\in\mathbb{N}} I_n(\delta) \quad \text{and} \quad |I_n(\delta)| < \delta \text{ for all } n \in \mathbb{N}. \tag{4.3}$$

We also recall that the Hausdorff dimension of a set S is defined by

$$\dim_H(S) = \inf\{\kappa \in [0,1] \mid h^\kappa(S) = 0\}. \tag{4.4}$$

Our goal in this section is to obtain partial results concerning spectral averaging of the κ-continuous part $\mu_t^{\text{sc},\kappa\text{-c}}$ with respect to h^κ, $\kappa \in (0,1)$, of the singular continuous part μ_t^{sc} (3.101) (with respect to Lebesgue measure) of the measure μ_t associated with the family of Herglotz functions $M_t = g_t(M_0)$:

$$\mu_t^{\text{sc}} = \mu_t^{\text{sc},\kappa\text{-c}} + \mu_t^{\text{sc},\kappa\text{-s}}, \quad t \in \mathbb{R}. \tag{4.5}$$

Here g_t is a one-parameter group of automorphisms of $\operatorname{Aut}(\mathbb{C}_+)$.

We introduce the following hypothesis.

Hypothesis 4.1. Suppose M_0 is a Herglotz function of the type (3.64), $\kappa \in (0,1)$,

$$\mathcal{S}_\kappa(M_0) = \big\{\lambda \in \mathbb{R} \mid \liminf_{\varepsilon\downarrow 0} \varepsilon^{\kappa-1} \operatorname{Im}(M_0(\lambda + i\varepsilon)) \in (0,\infty)\big\}, \tag{4.6}$$

and assume that the set $\mathcal{A}_\kappa(M_0)$ defined by

$$\mathcal{A}_\kappa(M_0) = \bigcup_{\kappa'\in[\kappa,1)} \mathcal{S}_{\kappa'}(M_0) \tag{4.7}$$

is a Borel set of positive Lebesgue measure.

We note that, by Hypothesis 4.1,

$$\mathcal{A}_\kappa(M_0) \subseteq \mathcal{S}(M_0), \tag{4.8}$$

where $\mathcal{S}(M_0)$ is the invariant set (3.67) associated with the Herglotz function M_0.

Lemma 4.2. *We assume Hypothesis 4.1 and the hypotheses of Theorem 3.14. Moreover, we suppose that*

$$\mu_t^{\text{sc}} = \mu_t^{\text{sc},\kappa\text{-}c} + \mu_t^{\text{sc},\kappa\text{-}s}, \quad t \in \mathbb{R}, \tag{4.9}$$

is the decomposition of the measure μ_t^{sc} (3.101) *such that* $\mu^{\text{sc},\kappa\text{-}c}$ *is* κ*-continuous and* $\mu^{\text{sc},\kappa\text{-}s}$ *is* κ*-singular* (*with respect to the* κ*-dimensional Hausdorff measure* h^κ). *Then, for* *any bounded Borel set* $\Delta \subset \mathbb{R}$ *and* $0 \neq |t| < \pi/\big(2\sqrt{\det(X)}\big)$ *in case* I, *and* $0 \neq t \in \mathbb{R}$ *in cases* II *and* III, *we have*

$$\mu_t^{\rm sc}(\Delta \cap \mathcal{A}_\kappa(M_0)) = \mu_t^{{\rm sc},\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)). \tag{4.10}$$

Proof. We note that

$$0 < \liminf_{\varepsilon\downarrow 0} \varepsilon^{\kappa-1}\,\mathrm{Im}(M_0(\lambda+i\varepsilon)) \text{ (possibly equal to } +\infty) \text{ for } \lambda \in \mathcal{A}_\kappa(M_0). \tag{4.11}$$

Using the estimate

$$\mathrm{Im}(M_t(z)) = \frac{\mathrm{Im}(M_0(z))}{|c_t M_0(z) + d_t|^2} \leq \frac{1}{c_t^2\,\mathrm{Im}(M_0(z))}, \quad z \in \mathbb{C}_+ \tag{4.12}$$

(we recall that $c_t \neq 0$ and $d_t \in \mathbb{R}$ by hypothesis) we infer

$$0 \leq \limsup_{\varepsilon\downarrow 0} \varepsilon^{1-\kappa}\,\mathrm{Im}(M_t(\lambda+i\varepsilon)) < \infty, \quad \lambda \in \mathcal{A}_\kappa(M_0). \tag{4.13}$$

It is known (cf. [16, Lemma 3.2]) that (4.13) implies

$$0 \leq \limsup_{\delta\downarrow 0} \frac{\mu_t(\lambda-\delta,\lambda+\delta)}{\delta^\kappa} < \infty, \quad \lambda \in \mathcal{A}_\kappa(M_0). \tag{4.14}$$

Hence, by a result of Rogers and Taylor [41], [42] (see also [16, Theorem 2.1]), the measure $\mu_t \restriction \mathcal{A}_\kappa(M_0)$ is κ-continuous, which proves (4.10) since

$$\mu_t \restriction \mathcal{A}_\kappa(M_0) = \mu_t^{\rm sc} \restriction \mathcal{A}_\kappa(M_0) \tag{4.15}$$

by (4.8). □

Remark 4.3. In general, we can neither state that $\mathcal{A}_\kappa(M_0)$ is a Borel set (cf. Hypothesis 4.1), nor that

$$\mu_t^{\rm sc}(\Delta) = \mu_t^{{\rm sc},\kappa\text{-c}}(\Delta \cap \mathcal{A}_\kappa(M_0)), \quad t \neq 0. \tag{4.16}$$

It was pointed out to us by Barry Simon that a different but not unrelated discussion of singular continuous measures for continuous and discrete half-line Schrödinger operators, based on the asymptotic behavior of solutions, was recently provided in [32] (following a previous result in [31]).

As a consequence we get the following result.

Corollary 4.4. *We assume the hypotheses of Lemma 4.2. Then for any bounded Borel set* $\Delta \subset \mathbb{R}$ *the following results hold*:

(i) *If* $\det(X) > 0$, *then*

$$|\gamma| \int_{-\pi/\left(2\sqrt{\det(X)}\right)}^{\pi/\left(2\sqrt{\det(X)}\right)} dt\, \mu_t^{{\rm sc},\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)) = |\Delta \cap \mathcal{A}_\kappa(M_0)|. \tag{4.17}$$

(ii) *If* $\det(X) = 0$, *then*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{{\rm sc},\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)) = |\Delta \cap \mathcal{A}_\kappa(M_0)|. \tag{4.18}$$

(iii) *If* $\det(X) < 0$, *then*

$$|\gamma| \int_{-\infty}^{\infty} dt\, \mu_t^{{\rm sc},\kappa\text{-}c}(\Delta \cap \mathcal{A}_\kappa(M_0)) = |\Delta \cap \mathcal{A}_\kappa(M_0) \cap \mathcal{R}(M_0)|, \tag{4.19}$$

where

$$\begin{aligned} \mathcal{R}(M_0) = \big\{\lambda \in \mathbb{R} \,\big|\, &\lim_{\varepsilon\downarrow 0} M_0(\lambda+i\varepsilon) \in \mathbb{R} \\ &\textit{and } \big|\lim_{\varepsilon\downarrow 0}(\gamma M_0(\lambda+i\varepsilon) - \beta)\big| < (2/\pi)\sqrt{|\det(X)|}\big\}. \end{aligned} \tag{4.20}$$

Even though Corollary 4.4 appears to be a new result, it cannot be regarded as a complete analog of (3.103), since first of all we have no results for the singular part $\mu_t^{\mathrm{sc},\kappa-\mathrm{s}}$, and second, we were not able to remove the set $\mathcal{A}_\kappa(M_0)$ on the left-hand sides of (4.17)–(4.19). We hope our present attempt will encourage future work in this direction.

Acknowledgments. We are indebted to Vadim Kostrykin, Yuri Latushkin, Alexander Motovilov, and Barry Simon for stimulating discussions and hints to pertinent literature. Moreover, we are grateful to the referee for a careful reading of the manuscript.

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