Алгебра и анализ
Том. 15 (2003), вып. 5

St. Petersburg Math. J.
Vol. 15 (2004), No. 5, Pages 715–732
S 1061-0022(04)00828-3
Article electronically published on August 2, 2004

ACTION OF HECKE OPERATORS ON THETA-FUNCTIONS WITH RATIONAL CHARACTERISTICS

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Abstract. The explicit formulas for the transformation of theta-functions of integral positive definite quadratic forms under the action of regular Hecke operators, obtained in the author's earlier paper (1996), are converted to transformation formulas for the theta-functions with rational characteristics (the theta-series) viewed as Siegel modular forms. As applications, sequences of invariant subspaces and eigenfunctions for all regular Hecke operators on spaces of theta-series are constructed.

§1. Introduction

The conjectural relationships between the zeta functions of algebraic varieties and the zeta functions of modular forms (which generalize the famous Shimura–Taniyama relations; see, e.g., [8] and [9]) suggest that it would be of interest to extend the rather scanty list of explicitly computed invariant subspaces and eigenfunctions for Hecke operators on spaces of Siegel modular forms. In the present paper, we consider, from this viewpoint, the spaces spanned by the theta-functions with rational characteristics.

Consider the *theta-function of genus g of a matrix Q with characteristic matrix V*, i.e., the function given by the series

$$\Theta(V,Z;Q)=\sum_{N\in\mathbb{Z}^r_g} e\bigl\{Z\cdot Q[N-V_2]+2\cdot {}^tV_1QN-{}^tV_1QV_2\bigr\}, \tag{1.1}$$

where $V=(V_1,V_2)\in(\mathbb{C}^r_g,\mathbb{C}^r_g)$, Z belongs to the *upper half-plane* $\mathbb{H}^g$ *of genus* $g\geq 1$,

$$\mathbb{H}^g=\left\{Z=X+iY\in\mathbb{C}^g_g\;\middle|\;{}^tZ=Z,\ Y>0\right\},$$

Q is the matrix of an integral positive definite quadratic form in $r\geq 1$ variables, i.e., an integral symmetric positive definite matrix of order r with even entries on the principal diagonal, and, for a complex square matrix M,

$$e\{M\}=\exp(\pi\sqrt{-1}\cdot\operatorname{trace}\{M\}). \tag{1.2}$$

The series (1.1) converges absolutely and uniformly on the compact subsets of the complex space $\mathbb{C}^r_{2g}\times\mathbb{H}^g$, thus determining a complex-analytic function on that space. The theta-functions have certain automorphic properties with respect to the modular transformations of the variables Z and V (see [5]). This makes it possible to define an action of the Hecke operators on theta-functions. In particular, in [6] explicit formulas were obtained that express the images of theta-functions under the regular Hecke operators as linear combinations (with constant coefficients) of similar theta-functions.

2000 *Mathematics Subject Classification.* Primary 11F27, 11F46, 11F60, 11F66.

Key words and phrases. Hecke operators, Siegel modular forms, theta functions of quadratic forms, zeta functions of modular forms.

Supported in part by RFBR (grant no. 02-01-00087) and by SFB 478 "Geometrische Structuren in der Mathematik", Westfälische Wilhelms-Universität Münster.

If all entries of the matrix V belong to the field $\mathbb{Q}$ of rational numbers, then the theta-function, which is called in this case the *theta-series of genus g of the quadratic form with the matrix Q* and is denoted by

$$\theta(Z, Q; V) = \Theta(V, Z, Q) \quad (V \in \mathbb{Q}^r_{2g}), \tag{1.3}$$

turns out to be a holomorphic (or Siegel) modular form of the variable $Z \in \mathbb{H}^g$ with respect to some congruence subgroup of the modular group $\Gamma^g = Sp_g(\mathbb{Z})$ of genus g.

Our first goal in the present paper is to convert the formulas obtained in [6] to explicit transformation formulas for the theta-series viewed as modular forms under the action of the regular Hecke operators for the corresponding congruence subgroup of Γ^g. In the special case where the matrix of characteristics has the form $V = (0, V_2)$ with a rational matrix V_2, this was also done in [6]. Our second purpose is to apply the resulting formulas in order to construct examples of eigenfunctions for all regular Hecke operators. In the case of $V = (0, V_2)$, series of examples of eigenfunctions were constructed in [7]. New examples of eigenfunctions of Hecke operators are of interest in view of possible relationships, generalizing the Shimura–Taniyama relations, between the zeta functions of modular forms and the zeta functions of algebraic varieties.

The present paper was started during my stay at Münster University (SFB 478 "Geometrische Strukturen in der Mathematik") and was continued at Heidelberg University (Heidelberg–Mannheim Schwergruppe "Arithmetik"). I am grateful to all my colleagues and especially Professor Jürgen Elstrodt and Professor Christopher Deninger of Münster University, Professor Eberhard Freitag and Professor Reiner Weissauer of Heidelberg University for kind invitations, fruitful discussions of applications of modular forms in Arithmetic, and excellent research conditions.

Notation. We reserve the letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ for the set of positive rational integers, the ring of rational integers, the field of rational numbers, and the field of complex numbers, respectively. $\mathbb{A}^m_n$ is the set of all $(m \times n)$-matrices with entries in a set $\mathbb{A}$.

If M is a matrix, ${}^tM$ always denotes the transpose of M; if the entries of M belong to $\mathbb{C}$, $\overline{M}$ is the matrix with complex conjugate entries. For a symmetric matrix Q we write $Q[M] = {}^tMQM$ if the product on the right is defined.

We denote by

$$\mathbb{E}^m = \left\{ Q = (q_{\alpha\beta}) \in \mathbb{Z}^m_m \;\middle|\; {}^tQ = Q,\ q_{11}, q_{22}, \dots, q_{mm} \in 2\mathbb{Z} \right\}$$

the set of all *even matrices* of order m, i.e., the set of matrices of integral quadratic forms q in m variables:

$$\mathbf{q}(X) = \frac{1}{2}\,{}^tXQX \quad ({}^tX = (x_1, \dots, x_m)).$$

We recall that the *level* q of an invertible matrix $Q \in \mathbb{E}^m$ (and of the corresponding form) is the smallest positive integer satisfying $qQ^{-1} \in \mathbb{E}^m$.

§2. Modular transformations of theta-functions and theta-series

Here we recall the basic transformation formulas for the theta-functions (1.1) of integral positive definite quadratic forms and then specialize these formulas to the case of rational characteristic.

First, we note that, since $U\mathbb{Z}_g^r = \mathbb{Z}_g^r$, for any integral unimodular matrix U of order r we have the identity

$$\begin{aligned}
(2.1)\quad &\Theta(V, Z; Q)\\
&= \sum_{N\in\mathbb{Z}_g^r} e\{Z\cdot Q[U][N - U^{-1}V_2] + 2\cdot {}^t(U^{-1}V_1)Q[U]N - {}^t(U^{-1}V_1)Q[U]U^{-1}V_2\}\\
&= \Theta(U^{-1}V, Z, Q[U]).
\end{aligned}$$

Next, the function $\Theta(V, Z; Q)$ regarded as a function of V is quasiperiodic with respect to the lattice $(Q^{-1}\mathbb{Z}_g^r, \mathbb{Z}_g^r) \subset \mathbb{C}_{2g}^r$. More precisely, for every $T_1 \in Q^{-1}\mathbb{Z}_g^r$ and every $T_2 \in \mathbb{Z}_g^r$ we have

$$\begin{aligned}
&\Theta((V_1 + T_1, V_2 + T_2), Z; Q)\\
(2.2)\qquad &= \sum_{N\in\mathbb{Z}_g^r} e\{Z\cdot Q[N - V_2 - T_2] + 2\cdot {}^t(V_1 + T_1)QN - {}^t(V_1 + T_1)Q(V_2 + T_2)\}\\
&= e\{{}^tV_1QT_2 - {}^tT_1QV_2 + {}^tT_1QT_2\}\Theta((V_1, V_2), Z; Q),
\end{aligned}$$

because, together with N, the matrix $N' = N - T_2$ ranges over the set $\mathbb{Z}_g^r$, and the matrices ${}^tT_1QN'$ are integral for all $N' \in \mathbb{Z}_g^r$.

Lemma 2.1. *Let Q be the matrix of a positive definite quadratic form in r variables, V a rational $r \times (2g)$-matrix, and d a common denominator of the entries of V, so that $V = d^{-1}L$ with $L \in \mathbb{Z}_{2g}^r$. Then the theta-series* (1.3) *satisfies the relation*

$$\theta(Z, Q; d^{-1}L + T) = \theta(Z, Q; d^{-1}L) \quad (L \in \mathbb{Z}_{2g}^r) \tag{2.3}$$

for every matrix

$$T = (T_1, T_2) \quad \textit{with } T_1 \in 2dQ^{-1}\mathbb{Z}_g^r \textit{ and } T_2 \in \mathbb{Z}_g^r \cap 2dQ^{-1}\mathbb{Z}_g^r.$$

Proof. If matrices V and T satisfy the assumptions of the lemma, then the matrices ${}^tV_1QT_2 = {}^t(dV_1)\cdot d^{-1}QT_2$, ${}^tT_1QV_2 = {}^t(d^{-1}QT_1)\cdot dV_2$, and ${}^tT_1QT_2 = {}^t(QT_1)T_2$ have integral even entries. Thus, relation (2.3) follows from (2.2). □

The following formulas for integral symplectic transformations of theta-functions with respect to the variable Z were proved in [5, Theorems 3.1 and 4.3].

Let Q be an even positive definite matrix of even order $r = 2k$, and let q be the level of Q. Then the theta-function $\Theta(V, Z; Q)$ of genus g of the matrix Q satisfies the functional equation

$$\begin{aligned}
\det(CZ + D)^{-k}\Theta\big(V\cdot {}^tM, (AZ + B)(CZ + D)^{-1}; Q\big)\\
= \boldsymbol{\chi}_Q(M)\Theta(V, Z; Q)
\end{aligned} \tag{2.4}$$

for every matrix $M = \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)$ in the group

$$\Gamma_0^g(q) = \left\{M = \begin{pmatrix} A & B\\ C & D\end{pmatrix} \in \mathbb{Z}_{2g}^{2g} \;\middle|\; {}^tMJ_gM = J_g,\ C \equiv 0 \pmod q\right\}. \tag{2.5}$$

Here

$$J_g = \begin{pmatrix} 0_g & 1_g\\ -1_g & 0_g\end{pmatrix}, \tag{2.6}$$

the character $\boldsymbol{\chi}_Q$ of the group (2.5) is determined by the conditions

$$\boldsymbol{\chi}_Q\left(\begin{pmatrix} A & B\\ C & D\end{pmatrix}\right) = \begin{cases} 1 & \text{if } q = 1,\\ \chi_Q(\det D) & \text{if } q > 1,\end{cases} \tag{2.7}$$

and χ_Q is the *character of the quadratic form with matrix* Q, i.e., the real Dirichlet character modulo q satisfying

$$\chi_Q(-1) = (-1)^k,$$
$$\chi_Q(p) = \left(\frac{(-1)^k \det Q}{p}\right) \quad \text{(the Legendre symbol)}$$

if p is an odd prime number not dividing q, and

$$\chi_Q(2) = \sum_{\mathbf{n}\in\mathbb{Z}^{2k}/2\mathbb{Z}^{2k}} \exp(\pi i\, {}^t\mathbf{n}Q\mathbf{n}/2)$$

if q is odd.

Theorem 2.2. *Let Q be an even positive definite matrix of even order $r = 2k$, and let q be the level of Q. Suppose L is an integral $(r \times (2g))$-matrix and $d \in \mathbb{N}$. Then, for every matrix $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)$ in the group $\Gamma_0^g(q) \cap \Gamma^g(h)$, where*

$$\Gamma^g(h) = \left\{ M \in \Gamma_0^g(1) \;\middle|\; M \equiv 1_{2g} \pmod{h} \right\}$$

is the principal congruence subgroup of the modular group $\Gamma^g = \Gamma_0^g(1)$ of arbitrary level h satisfying the conditions

$$h \equiv 0 \pmod{d}, \quad \frac{h}{2d^2}Q \in \mathbb{Z}_r^r, \tag{2.8}$$

the theta-series $\theta(Z, Q; d^{-1}L)$ of genus g of the matrix Q satisfies the functional equation

$$\begin{aligned} &\det(CZ + D)^{-k}\theta\big((AZ + B)(CZ + D)^{-1}, Q; d^{-1}L\big) \\ &\quad = \chi_Q(\det D)\theta(Z, Q; d^{-1}L) \end{aligned} \tag{2.9}$$

with the character χ_Q of the quadratic form with matrix Q.

Proof. If $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \Gamma_0^g(q)$, then, by (2.4) and (1.3) with $V = d^{-1}L$, we have

$$\det(CZ + D)^{-k}\theta\big((AZ + B)(CZ + D)^{-1}, Q; d^{-1}L\big) = \chi_Q(\det D)\theta\big(Z, Q; d^{-1}L \cdot {}^tM^{-1}\big).$$

If $M \in \Gamma^g(h)$, then ${}^tM^{-1} \in \Gamma^g(h)$, whence ${}^tM^{-1} = 1_g + hN$ with an integral matrix N. Thus, we can write $V \cdot {}^tM^{-1} = V + T$, where

$$T = (T_1, T_2) = hVN = \frac{h}{d}LN = \frac{h}{d}(L_1', L_2')$$

with integral $(r \times g)$-matrices L_1' and L_2'. By (2.8), we conclude that the matrices

$$(2d)^{-1}QT_1 = \frac{h}{2d^2}QL_1', \quad (2d)^{-1}QT_2 = \frac{h}{2d^2}QL_2', \quad \text{and} \quad T_2 = \frac{h}{d}L_2'$$

are integral. Then, from Lemma 2.1 it follows that

$$\theta\big(Z, Q; d^{-1}L \cdot {}^tM^{-1}\big) = \theta(Z, Q; d^{-1}L + T) = \theta(Z, Q; d^{-1}L). \qquad \square$$

Now we recall the definition of the Siegel modular forms of integral weights. The *general real positive symplectic group* $\mathbb{G}^g$ *of genus* g consists of all real symplectic matrices of order $2g$ with positive multipliers,

$$\mathbb{G}^g = GSp_g^+(\mathbb{R}) = \left\{ M \in \mathbb{R}_{2g}^{2g} \;\middle|\; {}^tMJ_gM = \mu(M)J_g,\ \mu(M) > 0 \right\}, \tag{2.10}$$

where J_g is the matrix (2.6). $\mathbb{G}^g$ is a real Lie group acting as a group of analytic automorphisms on the $(g(g + 1)/2)$-dimensional open complex variety $\mathbb{H}^g$ by the rule

$$\mathbb{G}^g \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto M\langle Z\rangle = (AZ + B)(CZ + D)^{-1} \quad (Z \in \mathbb{H}^g).$$

Acting on the upper half-plane $\mathbb{H}^g$, the general symplectic group operates also on the complex-valued functions F on $\mathbb{H}^g$ by *Petersson operators of integral weights* k,

$$\mathbb{G}^g \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : F \mapsto F|_k M = \det(CZ+D)^{-k} F(M\langle Z\rangle). \tag{2.11}$$

The Petersson operators obey the rules

$$F|_k MM' = (F|_k M)|_k M' \quad (M, M' \in \mathbb{G}^n). \tag{2.12}$$

Let S be a subgroup of $\mathbb{G}^g$ commensurable with the modular group Γ^g, let $\boldsymbol{\chi}$ be a *character* of S, i.e., a multiplicative homomorphism of S into the nonzero complex numbers with the kernel of finite index in S, and let k be an integer. A complex-valued function F on $\mathbb{H}^g$ is called a (*Siegel*) *modular form of weight* k *and character* $\boldsymbol{\chi}$ *for the group* S if the following conditions are fulfilled:

(1) F is a holomorphic function in $g(g+1)/2$ complex variables on $\mathbb{H}^g$;

(2) for every matrix $M \in S$, the function F satisfies the functional equation

$$F|_k M = \boldsymbol{\chi}(M)F, \tag{2.13}$$

where $|_k$ is the Petersson operator of weight k;

(3) if $n=1$, then every function $F|_k M$ with $M \in \Gamma^1$ is bounded on every subset of $\mathbb{H}^1$ of the form $\mathbb{H}^1_\varepsilon = \{x+iy \in \mathbb{H}^1 | y \geq \varepsilon\}$ with $\varepsilon > 0$.

Clearly, the set $\mathfrak{M}_k(S, \boldsymbol{\chi})$ of all modular forms of weight k and character $\boldsymbol{\chi}$ for the group S is a linear space over the field $\mathbb{C}$. Each space $\mathfrak{M}_k(S, \boldsymbol{\chi})$ has finite dimension over $\mathbb{C}$.

Corollary 2.3. *In the notation and under the assumptions of Theorem* 2.2, *the theta-series* $\theta(Z, Q; d^{-1}L)$ *of genus* g *of the matrix* Q *with characteristic* $V = d^{-1}L$ *is a Siegel modular form of weight* k *and character* $\boldsymbol{\chi}_Q$ *of the form* (2.7) *for the group* $\Gamma_0^g(q) \cap \Gamma^g(h)$:

$$\theta(Z, Q; d^{-1}L) \in \mathfrak{M}_k\big(\Gamma_0^g(q) \cap \Gamma^g(h), \boldsymbol{\chi}_Q\big).$$

In particular,

$$\theta(Z, Q; d^{-1}L) \in \mathfrak{M}_k\big(\Gamma^g([q,h]), \mathbf{1}\big) = \mathfrak{M}_k^g([q,h]), \tag{2.14}$$

where $[q,h]$ *is the least common multiple of* q *and* h, *and* $\mathbf{1}$ *denotes the unit character.*

§3. Action of the Hecke operators on theta-functions and theta-series

Here we recall the basic definitions concerning the Hecke operators and apply the general transformation formulas (obtained in [6]) for the action of regular Hecke operators on theta-functions to the case of theta-series viewed as Siegel modular forms.

Let Δ be a multiplicative semigroup, and let S be a subgroup of Δ such that every double coset SMS of Δ modulo S is a finite union of left cosets SM'. We consider the vector space over a field (say, the field $\mathbb{C}$ of complex numbers) that consists of all formal finite linear combinations with coefficients in $\mathbb{C}$ of symbols (SM) with $M \in \Delta$; the latter symbols are in one-to-one correspondence with the left cosets SM of the set Δ modulo S. The group S acts naturally on this space by right multiplication defined on the symbols (SM) by

$$(SM)\gamma = (SM\gamma) \quad (M \in \Delta, \gamma \in S).$$

We denote by

$$\mathcal{H}(S, \Delta) = HS_{\mathbb{C}}(S, \Delta)$$

the subspace of all S-invariant elements. The multiplication of elements of $\mathcal{H}(S,\Delta)$ given by the formula

$$\Big(\sum_\alpha a_\alpha(SM_\alpha)\Big)\Big(\sum_\beta b_\beta(SN_\beta)\Big)=\sum_{\alpha,\beta}a_\alpha b_\beta(SM_\alpha N_\beta)$$

does not depend on the choice of representatives $M_\alpha\in SM_\alpha$ and $N_\beta\in SN_\beta$, and turns $\mathcal{H}(S,\Delta)$ into an associative algebra over $\mathbb{C}$ with the unity element $(S1_S)$. This algebra is called *the Hecke–Shimura ring* or *HS-ring of Δ relative to S* (*over* $\mathbb{C}$). The elements

$$(M)=(M)_S=\sum_{M_i\in S\backslash SMS}(SM_i)\quad(M\in\Delta),\tag{3.1}$$

which are in one-to-one correspondence with the double cosets of Δ modulo S, belong to $\mathcal{H}(S,\Delta)$ and form a basis of the ring over $\mathbb{C}$. For brevity, the symbols (SM) and (M) will be referred to as *left* and *double classes* (*of Δ modulo S*), respectively.

We consider the semigroup

$$\Sigma^g=\mathbb{G}^g\cap\mathbb{Z}^{2g}_{2g}=\left\{M\in\mathbb{Z}^{2g}_{2g}\;\middle|\;{}^tMJ_gM=\mu(M)J_g,\ \mu(M)>0\right\}$$

and its subsemigroups

$$\begin{aligned}\Sigma^g_q&=\left\{M\in\Sigma^g\;\middle|\;\gcd(\mu(M),q)=1\right\},\\ \Sigma^g_0(q)&=\left\{M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Sigma^g_q\;\middle|\;C\equiv 0\pmod q\right\},\\ \Sigma^g(q)&=\left\{M\in\Sigma^g_0(q)\;\middle|\;M\equiv\begin{pmatrix}\mu(M)1_g&0\\0&1_g\end{pmatrix}\pmod q\right\}\end{aligned}\tag{3.2}$$

with $q\in\mathbb{N}$. It is easily seen that these semigroups satisfy the relation

$$\Sigma^g_0(q)=\Gamma^g_0(q)\Sigma^g(q)=\Sigma^g(q)\Gamma^g_0(q).\tag{3.3}$$

If S is a subgroup of finite index in the modular group Γ^g, then the pair S,Σ^g satisfies the conditions of the definition of HS-rings, and so does each pair S,Δ with every semigroup Δ satisfying $S\subset\Delta\subset\Sigma^g$. Therefore, we can define the corresponding Hecke–Shimura rings $\mathcal{H}(S,\Delta)$.

We shall say that a group S satisfying $\Gamma^g(q)\subset S\subset\Gamma^g$ is *q-symmetric* if $S\Sigma^g(q)=\Sigma^g(q)S$. For such S, the HS-ring

$$\mathcal{H}_{qr}(S)=\mathcal{H}(S,R(S))\quad\text{with}\quad R(S)=S\Sigma^g(q)=\Sigma^g(q)S$$

is called the *q-regular HS-ring of S*. In accordance with [2, Theorem 3.3.3], all q-regular HS-rings of a given genus n are isomorphic to each other. In particular, by (3.3), the group $\Gamma^g_0(q)$ is q-symmetric, and the q-regular HS-rings of the groups $\Gamma^g_0(q)$ and $\Gamma^g(q)$,

$$\begin{aligned}\mathcal{H}^g_0(q)&=\mathcal{H}_{qr}(\Gamma^g_0(q))=\mathcal{H}(\Gamma^g_0(q),\Sigma^g_0(q)),\\ \mathcal{H}^g(q)&=\mathcal{H}_{qr}(\Gamma^g(q))=\mathcal{H}(\Gamma^g(q),\Sigma^g(q))\end{aligned}\tag{3.4}$$

are isomorphic. This isomorphism can be defined as follows. Let

$$T'=\sum_\alpha a_\alpha(\Gamma^g_0(q)M_\alpha)\in\mathcal{H}^g_0(q),\tag{3.5}$$

where the left classes $(\Gamma^g_0(q)M_\alpha)$ with $a_\alpha\neq 0$ are pairwise distinct. By (3.3), without loss of generality we may assume that all the representatives M_α of the left cosets $\Gamma^g_0(q)M_\alpha$ belong to $\Sigma^g(q)$. Then, obviously, we have

$$T=\eta(T')=\sum_\alpha a_\alpha(\Gamma^g(q)M_\alpha)\in\mathcal{H}^g(q),\tag{3.6}$$

and the map η is a homomorphic embedding of the ring $\mathcal{H}_0^g(q)$ in $\mathcal{H}^g(q)$. In fact, η is a ring isomorphism, and the inverse isomorphism $\xi : \mathcal{H}^g(q) \mapsto \mathcal{H}_0^g(q)$ is determined by the condition

$$\xi : \mathcal{H}^g(q) \ni \sum_\beta b_\beta(\Gamma^g(q)N_\beta) \mapsto \sum_\beta b_\beta(\Gamma_0^g(q)N_\beta). \tag{3.7}$$

These isomorphisms of rings make it possible to transfer various constructions from one of the rings to another. For example, the *Zharkovskaya homomorphisms* $\Psi^{g,n} = \Psi^{g,n}_{k,\chi}$ of the ring $\mathcal{H}_0^g(q)$ to $\mathcal{H}_0^n(q)$, where $g > n \geq 1$, k is an integer, and χ a Dirichlet character modulo q satisfying $\chi(-1) = (-1)^k$, can be carried over to the corresponding HS-rings of principal congruence subgroups, so that we get homomorphisms

$$\Psi^{g,n} = \Psi^{g,n}_{k,\chi} : \mathcal{H}^g(q) \mapsto \mathcal{H}^n(q) \quad (g > n \geq 1) \tag{3.8}$$

satisfying the condition of commutativity of the diagram

$$\begin{array}{ccc} \mathcal{H}_0^g(q) & \xrightarrow{\eta} & \mathcal{H}^g(q) \\ {\scriptstyle \Psi^{g,r}}\big\downarrow & & {\scriptstyle \Psi^{g,r}}\big\downarrow \\ \mathcal{H}_0^n(q) & \xrightarrow{\eta} & \mathcal{H}^n(q) \end{array} \tag{3.9}$$

We recall that the Zharkovskaya map from genus g to genus n,

$$\Psi^{g,n} = \Psi^{g,n}_{k,\chi} : \mathcal{H}_0^g(q) \mapsto \mathcal{H}_0^n(q), \tag{3.10}$$

can be defined in the following way. Let $T' \in \mathcal{H}_0^g(q)$ be an element of the form (3.5). It can be assumed that each representative $M_\alpha \in \Gamma_0^g(q)\backslash\Sigma_0^g(q)$ is chosen in the form

$$M_\alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ 0 & D_\alpha \end{pmatrix} \quad \text{with} \quad D_\alpha = \begin{pmatrix} D'_\alpha & * \\ 0 & D''_\alpha \end{pmatrix} \text{ and } \; D'_\alpha \in \mathbb{Z}_n^n.$$

If $A_\alpha = \left(\begin{smallmatrix} A'_\alpha & * \\ * & * \end{smallmatrix}\right)$ and $B_\alpha = \left(\begin{smallmatrix} B'_\alpha & * \\ * & * \end{smallmatrix}\right)$ with $(r \times r)$-blocks A'_α and B'_α, then

$$M'_\alpha = \begin{pmatrix} A'_\alpha & B'_\alpha \\ 0 & D'_\alpha \end{pmatrix} \in \Sigma_0^r(q),$$

and we put

$$\Psi^{g,n}_{k,\chi}(T') = \sum_\alpha a_\alpha |\det D''_\alpha|^{-k} \chi^{-1}(|\det D''_\alpha|)(\Gamma_0^n(q)M'_\alpha).$$

Then, since the diagram (3.9) is commutative, we can define the Zharkovskaya map (3.8) by

$$\Psi^{g,n}(T) = (\eta\Psi'\xi)(T) \in \mathcal{H}^n(q) \quad (T \in \mathcal{H}^g(q))$$

with maps η and ξ given by (3.6) and (3.7), where Ψ' is the map (3.10).

The Hecke–Shimura rings act on modular forms and on theta-functions via the linear representations given by the *Hecke operators*. We shall need the Hecke operators of two related kinds: the Hecke operators for the groups $\Gamma_0^g(q)$ on theta functions (1.1) with arbitrary complex characteristics, and the Hecke operators for the groups $\Gamma^g(h)$ on theta functions with rational characteristics.

First, we consider the space $\mathcal{F} = \mathcal{F}(r, g)$ of all real-analytic functions

$$F = F(V, Z) : \mathbb{C}^r_{2g} \times \mathbb{H}^g \mapsto \mathbb{C}$$

with even $r = 2k$ and define the action of the semigroup $\Sigma_0^g(q)$ on $\mathcal{F}$ (where q is the level of an even positive definite matrix Q of order r) by the rule

$$\Sigma_0^g(q) \ni M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : F \mapsto F|M = j(M, Z)^{-1} F(V \cdot {}^tM, M\langle Z\rangle), \tag{3.11}$$

where

$$j(M,Z) = j_Q(M,Z) = \boldsymbol{\chi}_Q(M)\det(CZ+D)^k,$$

$\boldsymbol{\chi}_Q$ is the character (2.7), and $M\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$. It is easily seen that the factors $j(M,Z)$ satisfy

$$j(M, M'\langle Z\rangle)j(M',Z) = j(MM',Z)$$

for every $M, M' \in \Sigma_0^g(q)$ and $Z \in \mathbb{H}^g$. Consequently,

$$F|M|M' = F|MM' \quad (F \in \mathcal{F},\ M, M' \in \Sigma_0^g(q)). \tag{3.12}$$

This property of the operators $|M$ allows us to define the standard representation of the Hecke–Shimura ring $\mathcal{H}_0^g(q) = \mathcal{H}(\Gamma_0^g(q), \Sigma_0^g(q))$ on the subspace

$$\mathcal{F}(\Gamma_0^g(q)) = \left\{ F \in \mathcal{F} \;\middle|\; F|\gamma = F,\ \forall\gamma \in \Gamma_0^g(q) \right\}$$

of all $\Gamma_0^g(q)$-invariant functions in $\mathcal{F}$. Namely, if

$$T = \sum_\alpha a_\alpha(\Gamma_0^g(q)M_\alpha) \in \mathcal{H}_0^g(q) \tag{3.13}$$

is an element of the ring $\mathcal{H}_0^g(q)$ and $F \in \mathcal{F}(\Gamma_0^g(q))$, then the function

$$F|T = \sum_\alpha a_\alpha F|M_\alpha \tag{3.14}$$

does not depend on the choice of representatives $M_\alpha \in \Gamma_0^g(q)M_\alpha$ and still belongs to $\mathcal{F}(\Gamma_0^g(q))$. Clearly, the operators $|T$ are linear. The map $T \mapsto |T$ is linear. From (2.12) and the definition of multiplication in HS-rings it follows that this map satisfies $|T|T' = |TT'$. Thus, we get a linear representation of the ring $\mathcal{H}_0^g(q)$ on the space $\mathcal{F}(\Gamma_0^g(q))$. The operators $|T$ are called *Hecke operators*.

By (2.4), the theta-function $\Theta(V,Z;Q)$ with an even positive definite matrix Q of even order $r = 2k$ and level q, regarded as a function of V and Z, belongs to the space $\mathcal{F}(\Gamma_0^g(q))$. By the above, its image

$$\Theta(V,Z;Q)|T = \sum_\alpha a_\alpha j(M_\alpha, Z)^{-1}\Theta(V\cdot{}^tM_\alpha, M_\alpha\langle Z\rangle; Q) \tag{3.15}$$

under the Hecke operator corresponding to the element (3.14) does not depend on the choice of representatives $M_\alpha \in \Gamma_0^g(q)M_\alpha$ and still belongs to $\mathcal{F}(\Gamma_0^g(q))$. The formulas proved in [6, Theorem 4.1] express the images of a theta function under Hecke operators as linear combinations with constant coefficients of similar theta functions. In order to formulate that theorem, we must recall two related reductions.

By the definition of the semigroup $\Sigma_0^g(q)$, each matrix $M \in \Sigma_0^g(q)$ satisfies the relation ${}^tMJ_gM = \mu(M)J_g$, where $\mu(M)$ is a positive integer coprime with q called the *multiplier* of M. It satisfies

$$\mu(MM') = \mu(M)\mu(M') \quad (M, M' \in \Sigma_0^g(q)), \tag{3.16}$$

and

$$\mu(M) = 1 \Longleftrightarrow M \in \Gamma_0^g(q) \quad (M \in \Sigma_0^g(q)). \tag{3.17}$$

From (3.16) and (3.17) it follows that the function μ takes a fixed value on each left and double coset of $\Sigma_0^g(q)$ modulo $\Gamma_0^g(q)$, so that we can talk of *multipliers of cosets*. We say that a nonzero formal finite linear combination T of left or double cosets of $\Sigma_0^g(q)$ modulo $\Gamma_0^g(q)$ is *homogeneous of multiplier* $\mu(T) = \mu$ if all the cosets involved have the same multiplier μ. It is clear that every finite linear combination of cosets is a sum of *homogeneous components* having different multipliers, and these components are determined uniquely. In particular, this allows us to reduce the consideration of arbitrary Hecke operators $|T$ with $T \in \mathcal{H}_0^g(q)$ to the case where T is homogeneous.

Another reduction is related to a special choice of representatives in the left cosets $\Gamma_0^g(q)M$ contained in $\Sigma_0^g(q)$. By [2, Lemma 3.3.4], each of the left cosets contains a representative of the form

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \quad \text{with } A, B, D \in \mathbb{Z}_g^g,\ {}^tAD = \mu(M)1_g,\ {}^tBD = {}^tDB. \tag{3.18}$$

Such representatives are often convenient for computations with Hecke operators and will be referred to as *triangular representatives*.

The following particular case of Theorem 4.1 in [6] expresses the images of the theta-functions (1.1) under the action of regular Hecke operators in the form of linear combinations of similar theta-series with coefficients given explicitly in terms of certain trigonometric sums.

Theorem 3.1. *Let Q be an even positive definite matrix of even order $r = 2k$ and level q. Let*

$$T = \sum_\alpha a_\alpha(\Gamma_0^g(q)M_\alpha)$$

be a homogeneous element of the Hecke–Shimura ring $\mathcal{H}_0^g(q)$ with $\mu(T) = \mu$. We assume that if $g < r$, then T belongs to the image of the ring $\mathcal{H}_0^r(q)$ under the Zharkovskaya map (3.10) *from genus r to genus g with $k = r/2$ and with the character $\chi = \chi_Q$ defined in* §2,

$$T = \Psi^{r,g}_{r/2,\chi_Q}(\widetilde{T}) \quad \textit{with} \quad \widetilde{T} \in \mathcal{H}_0^r(q)\ (g < r).$$

Then the image of a theta-series of the form (1.5) *under the Hecke operator $|T$ can be written as*

$$\Theta(V, Z; Q)|T = \sum_{D \in \Delta(Q,\mu)/\Lambda} I(D, Q, \Psi^{g,r}(T))\Theta\big(\mu D^{-1}V, Z; \mu^{-1}Q[D]\big), \tag{3.19}$$

where

$$\Delta(Q,\mu) = \left\{ D \in \mathbb{Z}_r^r \,\middle|\, \det D = \pm\mu^{r/2}, \mu^{-1}Q[D] \in \mathbb{E}^r \right\}, \quad \Lambda = GL_r(\mathbb{Z}), \tag{3.20}$$

$$\Psi^{g,r}(T) = \Psi^{g,r}_{r/2,\chi_Q}(T) = \begin{cases} \Psi^{g,r}_{r/2,\chi_Q}(T) & \textit{if } g > r, \\ T & \textit{if } r = g, \\ \widetilde{T} \in (\Psi^{r,g}_{r/2,\chi_Q})^{-1}(T) & \textit{if } g < r. \end{cases}$$

Here, for the elements

$$T' = \xi(\widetilde{T}) = \sum_\beta b_\beta\left(\Gamma_0^r(q)\begin{pmatrix} A_\beta & B_\beta \\ 0 & D_\beta \end{pmatrix}\right) \in \mathcal{H}_0^r(q) \quad ({}^tA_\beta D_\beta = \mu 1_r),$$

written in the triangular form, the $I(D, Q, T')$ are trigonometric sums defined by the rule

$$I(D, Q, T') = \sum_{\beta; D\,{}^tD_\beta \equiv 0 \,(\mathrm{mod}\,\mu)} b_\beta |\det D_\beta|^{-r/2}\chi_Q^{-1}(|\det D_\beta|)e\{\mu^{-2}Q[D]\,{}^tD_\beta B_\beta\}, \tag{3.21}$$

where $e\{\cdots\}$ is the exponential (1.2).

Now, suppose that F belongs to the space $\mathfrak{M}_k(S) = \mathfrak{M}_k(S, \mathbf{1})$ of modular forms of integral weight k and of trivial character $\chi = \mathbf{1}$ for a subgroup S of finite index in the modular group Γ^g. Let

$$T = \sum_i a_\alpha(SM_\alpha) \in \mathcal{H}(S, \Sigma^g).$$

Then, from the definitions of modular forms and HS-rings and properties (2.12) and (2.13) of the Petersson operators (2.11), it follows easily that the function

$$F\|T = F\|_k T = \sum_\alpha a_\alpha F|_k M_\alpha$$

does not depend on the choice of representatives $M_\alpha \in SM_\alpha$ and still belongs to the space $\mathfrak{M}_k(S)$. These operators are also called the *Hecke operators* (*of weight k for the group S*). The Hecke operators corresponding to elements of regular HS-rings are said to be *regular*. The definition of multiplication in HS-rings and relations (2.12) imply that the map $T \mapsto \|T$ is a linear representation of the ring $\mathcal{H}(S)$ on the space $\mathfrak{M}_k(S)$.

The following theorem expresses the images of the theta-series (1.3) under the action of regular Hecke operators in the form of linear combinations of similar theta-series with coefficients given explicitly.

Theorem 3.2. *Let Q be an even positive definite matrix of even order $r = 2k$, let q be the level of Q, and let χ_Q be the corresponding Dirichlet character. Suppose L is an integral $(r \times 2g)$-matrix and d is a positive integer. Finally, let h be a positive integer satisfying* (2.8), *and $\hat{q} = [q, h]$ the least common multiple of q and h. Suppose that*

$$T = \sum_\alpha a_\alpha(\Gamma^g(\hat{q})M_\alpha) \in \mathcal{H}^g(\hat{q}) \tag{3.22}$$

is a homogeneous element with $\mu(T) = \mu$ and such that

$$T \in \Psi^{r,g}_{r/2,\chi_Q}(\mathcal{H}^r(\hat{q})) \quad \text{if} \;\; g < r. \tag{3.23}$$

Then the image of the theta-series $\theta(Z, Q; d^{-1}L)$ under the Hecke operator $\|T$ can be written in the form

$$\theta(Z, Q; d^{-1}L)\|T = \sum_{D \in \Delta(Q,\mu)/\Lambda} I(D, Q, \Psi^{g,r}(T))\Theta\left(Z, \mu^{-1}Q[D]; d^{-1}\mu D^{-1}L\langle\mu'\rangle\right), \tag{3.24}$$

where summation is as in (3.19), *$\Psi^{g,r}$ with $g \geq r$ is the Zharkovskaya operator* (3.8), *$\Psi^{g,r}T$ with $g < r$ denotes a homogeneous inverse image of T under the map $\Psi^{r,g}$: $\mathcal{H}^r(\hat{q}) \mapsto \mathcal{H}^g(\hat{q})$, and*

$$I(D, Q, T'') = I(D, Q, \xi(T'')) \quad (T'' \in \mathcal{H}^r(\hat{q})) \tag{3.25}$$

with the map $\xi : \mathcal{H}^r(\hat{q}) \mapsto \mathcal{H}^r_0(\hat{q})$ defined by (3.7) *and with the sums $I(D, Q, \xi(T''))$ defined by* (3.21). *Here, μ' stands for an integral inverse of μ modulo dh, and we set*

$$\langle a\rangle = \langle a\rangle_g = \begin{pmatrix} a1_g & 0_g \\ 0_g & 1_g \end{pmatrix}.$$

Proof. In order to apply Theorem 3.1, we need to compare the actions of the Hecke operators $|T$ and $\|T$ on $\Theta(V, Z, Q) = \theta(Z, Q; d^{-1}L)$. If $M \in \Sigma^g(\hat{q})$ (see (3.2)), then ${}^tM = \langle\mu\rangle + \hat{q}N$ with an integral matrix N. Thus, for $V = d^{-1}L = d^{-1}(L_1', L_2')$ with integral matrices L_1' and L_2', we have $V \cdot {}^tM = V\langle\mu\rangle + T$, where

$$T = (T_1, T_2) = \hat{q}VN = \frac{\hat{q}}{d}(L_1', L_2')N.$$

Since $\hat{q}$ is divisible by h, from (2.8) we see that the matrices

$$(2d)^{-1}QT_1 = \frac{h}{2d^2}QL_1'N, \quad (2d)^{-1}QT_2 = \frac{h}{2d^2}QL_2'N, \quad \text{and} \quad T_2 = \frac{h}{d}L_2'N$$

are integral. Now, from Lemma 2.1 it follows that

$$\Theta(V \cdot {}^tM, Z; Q) = \Theta\left(V\langle\mu\rangle + T, Z; Q\right) = \Theta\left(V\langle\mu\rangle, Z; Q\right).$$

Thus, by (3.15) with $V\langle\mu'\rangle$ in place of V, we have

$$\begin{aligned}\Theta(V\langle\mu'\rangle, Z; Q)|T &= \sum_{\alpha} a_\alpha j(M_\alpha, Z)^{-1}\Theta\left(V\langle\mu'\rangle\,{}^tM_\alpha, M_\alpha\langle Z\rangle; Q\right)\\ &= \sum_{\alpha} a_\alpha j(M_\alpha, Z)^{-1}\Theta\left(V\langle\mu'\rangle\langle\mu\rangle, M_\alpha\langle Z\rangle; Q\right)\\ &= \theta(Z, Q; V)\|T,\end{aligned}$$

and the theorem follows from Theorem 3.1. □

The next proposition answers, in particular, the following natural question: what homogeneous elements satisfy condition (3.23)?

Proposition 3.3. *Let $1 \le g < r$, and let T be a nonzero homogeneous element of the ring $\mathcal{H}^g = \mathcal{H}_0^g(q)$ (respectively, $\mathcal{H}^g(q)$) for a positive integer q with the multiplier $\mu(T) = \mu$. Then the relation $T \in \Psi^{r,g}(\mathcal{H}^r)$, where $\Psi^{r,g} = \Psi^{r,g}_{k,\chi} : \mathcal{H}^r \mapsto \mathcal{H}^g$ is the Zharkovskaya homomorphism with a positive integer k and a Dirichlet character χ modulo q, is fulfilled if and only if either $g \ge k$, or $g < k$ and $\chi(p) = 1$ for each prime number p occurring in the prime number factorization of μ in an odd degree.*

Proof. If μ is a power of a prime number, then for the rings $\mathcal{H} = \mathcal{H}_0(q)$ the claim follows from the definitions and [2, Proposition 2.4.19]. For composite μ, the claim follows in this case from [2, Theorem 3.3.12]. The case of $\mathcal{H} = \mathcal{H}(q)$ follows then from the definition of the Zharkovskaya map (see above) and [2, Theorem 3.3.3]. □

Sometimes, it is convenient to rewrite the above transformation formulas in a somewhat different form. For this, we need several preliminary remarks. We shall say that an even matrix Q' is *similar* to a nonsingular matrix $Q \in \mathbb{E}^m$, $Q' \sim Q$, if Q' can be written in the form $Q' = \mu^{-1}Q[D]$ with $D \in \Delta(Q, \mu)$, where μ is coprime with the level q of Q; if, moreover, μ is a perfect square, we say that Q' is *relative* to Q, $Q' \simeq Q$. It is easily seen that both relations are symmetric, reflexive, and transitive. Thus, the set of all nonsingular matrices of $\mathbb{E}^m$ is the disjoint union of the *similarity classes*

$$sm\{Q\} = \left\{Q' \in \mathbb{E}^m \;\middle|\; Q' \sim Q\right\} \quad (Q \in \mathbb{E}^m, \det Q \ne 0),$$

and each of the similarity classes is the disjoint union of the *relativity classes*, or *genera*,

$$gn\{Q\} = \left\{Q' \in \mathbb{E}^m \;\middle|\; Q' \simeq Q\right\} \quad (Q \in \mathbb{E}^m, \det Q \ne 0).$$

It is easily seen that all matrices in a similarity class of a matrix Q have the same signature, determinant, and level, coinciding with those of Q. Next, we recall that two matrices Q, Q' in $\mathbb{E}^m$ are said to be *equivalent*, $Q \cong Q'$, if $Q' = Q[U]$ with $U \in \Lambda = \Lambda^m = GL_m(\mathbb{Z})$. All matrices equivalent to Q form the *equivalence class of* Q,

$$eq\{Q\} = \left\{Q' = Q[U] \;\middle|\; U \in \Lambda\right\}.$$

In accordance with the reduction theory for integral quadratic forms, the similarity class and the genus of every nonsingular matrix $Q \in \mathbb{E}^m$ are finite unions of equivalence classes:

$$sm\{Q\} = \bigcup_{j=1}^{h(Q)} eq\{Q_j\}, \quad gn\{Q\} = \bigcup_{j=1}^{h^+(Q)} eq\{Q_j\}.$$

The numbers $h(Q) = h_s(Q)$ and $h^+(Q) = h_g(Q)$ will be called the *similarity class number* and the *genus class number* of Q, respectively.

Now we can reformulate Theorem 3.2.

Theorem 3.4. *In the notation and under the assumptions of Theorem* 3.2, *formula* (3.24) *can be rewritten as*

(3.26)
$$\theta(Z,Q;d^{-1}L)\|T=\sum_{j=1}^{h(Q)}\sum_{D\in R(Q,\mu Q_j)/E(Q_j)} I(D,Q,\Psi^{g,r}(T))\theta\left(Z,Q_j;d^{-1}\mu D^{-1}L\langle\mu'\rangle\right),$$

where $Q_1,\dots,Q_{h(Q)}$ *is a system of representatives of all different equivalence classes contained in the similarity class of* Q,

$$R(Q,Q')=\left\{M\in\mathbb{Z}_r^r\;\middle|\;Q[M]=Q'\right\}\quad(Q,Q'\in\mathbb{E}^r)$$

is the set of all integral representations of Q' *by* Q, *and*

$$E(Q')=R(Q',Q')\quad(Q'\in\mathbb{E}^r,\det Q'\neq 0)$$

denotes the group of all units of Q'. *In particular*, *if the similarity class number* $h(Q)$ *of* Q *is equal to* 1, *then formula* (3.24) *reshapes to*

$$\theta(Z,Q;d^{-1}L)\|T=\sum_{D\in R(Q,\mu Q)/E(Q)} I(D,Q,\Psi^{g,r}(T))\theta\left(Z,Q;d^{-1}\mu D^{-1}L\langle\mu'\rangle\right). \tag{3.27}$$

Proof. It is an easy consequence of the definitions that the sums (3.21) are independent of the choice of representatives in the decomposition of T' and satisfy

$$I(UDU',Q,T')=I(D,Q[U],T')\quad\text{for all}\ \ U,U'\in\Lambda=GL_r(\mathbb{Z}) \tag{3.28}$$

(see [6, Lemma 3.1]). From (1.6) and (3.21) we conclude that, in the sum on the right in (3.24), the term corresponding to a matrix $D\in\Delta(Q,\mu)$ depends only on the coset $D\Lambda$. On the other hand, each of the matrices $\mu^{-1}Q[D]$ with $D\in\Delta(Q,\mu)$ is similar to Q, whence it is equivalent to one of the matrices $Q_1,\dots,Q_{h(Q)}$, say, to $\mu^{-1}Q[D]\cong Q_j$. This means that $\mu^{-1}Q[D][U]=\mu^{-1}Q[DU]=Q_j$ with $U\in\Lambda$. Replacing D by DU, we may assume that $D\in R(Q,Q_j)$. If $D'=DU'$ is another such matrix, then, clearly, $Q_j[U']=Q_j$, whence $U'\in E(Q_j)$. This proves the required formulas. □

Similarly, from Theorem 3.2 and Proposition 3.3 we deduce the following formulas.

Theorem 3.5. *In the notation of Theorems* 3.2 *and* 3.4, *for each homogeneous element* $T\in\mathcal{H}^g(\hat q)$ *whose multiplier* $\mu(T)$ *is a perfect square*, *formula* (3.24) *can be rewritten as*

(3.29)
$$\theta(Z,Q;d^{-1}L)\|T=\sum_{j=1}^{h^+(Q)}\sum_{D\in R(Q,\mu Q_j)/E(Q_j)} I(D,Q,\Psi^{g,r}(T))\theta\left(Z,Q_j;d^{-1}\mu D^{-1}L\langle\mu'\rangle\right),$$

where $Q_1,\dots,Q_{h^+(Q)}$ *is a system of representatives of all different equivalence classes contained in the genus of* Q. *In particular*, *if the genus class number* $h^+(Q)$ *of* Q *is equal to* 1, *then formula* (3.24) *reshapes to*

$$\theta(Z,Q;d^{-1}L)\|T=\sum_{D\in R(Q,\mu Q)/E(Q)} I(D,Q,\Psi^{g,r}(T))\theta\left(Z,Q;d^{-1}\mu D^{-1}L\langle\mu'\rangle\right). \tag{3.30}$$

Note that homogeneous elements T whose multipliers are perfect squares are called *even*. Such elements generate a subring of the corresponding Hecke–Shimura ring, called the *even subring*.

We conclude this section with a summary of the known results concerning the computation of sums (3.21) and (3.25). We recall that the HS-ring $\mathcal{H}_0^r(q)$ is generated over

$\mathbb{C}$ by the following $r+1$ double classes of the form (3.1):

$$T^r(p) = \big(\mathrm{diag}(\underbrace{1,\dots,1}_{r},\underbrace{p,\dots,p}_{r})\big)_{\Gamma_0^r(q)}, \tag{3.31}$$

and

$$T_j^r(p^2) = \big(\mathrm{diag}(\underbrace{1,\dots,1}_{r-j},\underbrace{p,\dots,p}_{j},\underbrace{p^2,\dots,p^2}_{r-j},\underbrace{p,\dots,p}_{j})\big)_{\Gamma_0^r(q)} \quad (j=1,\dots,r), \tag{3.32}$$

where p runs over all prime numbers that do not divide q (see [2, Theorem 3.3.23]). It follows that the ring $\mathcal{H}^r(q)$ is generated over $\mathbb{C}$ by the classes

$$\widetilde{T}^r(p) = \eta(T^r(p)) \text{ and } \widetilde{T}_j^r(p^2) = \eta(T_j^r(p^2)) \quad (j=1,\dots,r), \tag{3.33}$$

where η is the map (3.6) and, again, p runs over all prime numbers that do not divide q. The even subring of $\mathcal{H}_0^r(q)$ is generated by the classes (3.32) and the class $(T^r(p))^2$ for all prime p that do not divide q, and the even subring of $\mathcal{H}^r(q)$ is generated by the η-images of these classes.

Proposition 3.6. *Let Q be an even positive definite matrix of even order r, let q be the level of Q, and let χ_Q be the Dirichlet character corresponding to Q. Then, for each prime number p that does not divide q, the sums* (3.21) *can be computed on some of the elements* (3.31)–(3.33) *by the following formulas*:

$$\begin{aligned} I(D,Q,\widetilde{T}^r(p)) &= I(D,Q,T^r(p)) \\ &= \begin{cases} p^{r/2}\prod_{j=1}^{r/2}(1+\chi_Q(p)p^{-j}) & \textit{if } D\in\Delta(Q,p)=\Lambda\begin{pmatrix}1_k & 0\\ 0 & p1_k\end{pmatrix}\Lambda,\\ 0 & \textit{otherwise}; \end{cases}\end{aligned} \tag{3.34}$$

$$\begin{aligned} I(D,Q,\widetilde{T}_{r-1}^r(p)) &= I(D,Q,T_{r-1}^r(p)) \\ &= \begin{cases} \chi_Q(p)p^{(2+r-r^2)/2} & \textit{if } D\in\Lambda D_{r-2,1}^r(p)\Lambda,\\ \alpha_r(p) & \textit{if } D\in\Lambda(p1_r),\\ 0 & \textit{otherwise}, \end{cases}\end{aligned} \tag{3.35}$$

where $\Lambda=\Lambda^r=GL_r(\mathbb{Z})$, $D_{r-2,1}^r(p)=\mathrm{diag}(1,\underbrace{p,\dots,p}_{r-2},p^2)$, *and*

$$\alpha_r(p)=\chi_Q(p)p^{(2+r-r^2)/2}\frac{(p^r-1)}{p-1}+p^{-r^2/2}(\chi_Q(p)p^{r/2}-1);$$

$$I(D,Q,\widetilde{T}_r^r(p^2)) = I(D,Q,T_r^r(p^2)) = \begin{cases} p^{-r^2/2} & \textit{if } D\in\Lambda(p1_r),\\ 0 & \textit{otherwise.}\end{cases} \tag{3.36}$$

Proof. In [3] (see formula (2.19) and Lemma 5.1 therein) the sums $\gamma(Q,D,T)$ similar to the sums (3.11) were defined and computed for $T=T^r(p)$. In [4, §2], the sums $\gamma(Q,D,T)$ were computed, in fact, for $T=T_r^r(p^2)=(p1_{2r})_{\Gamma_0^r(q)}$ and $T=T_{r-1}^r(p^2)$ (see also Lemma 3.3.32 in [2] for the presentation of $T_{r-1}^r(p^2)$ used in [4]). The definitions of the sums imply directly that $I(D,Q,T)=\mu^{r/2}\gamma(Q,\mu D^{-1},T)$. The rest is clear. □

Observe that the formulas of Proposition 3.4 determine the sums $I(D,Q,T)$ for all generators of the rings $\mathcal{H}_0^2(q)$ and $\mathcal{H}^2(q)$. We do not know any other results on computation of the sums (3.21) or (3.25).

§4. Action of Hecke operators on average theta-series

Here we shall show that the transformation formulas of the preceding section simplify if we replace the theta-series $\theta(Z,Q;V)$ by their linear combinations of the form

$$\theta(Z,Q;\sigma,\tau,d^{-1}L)=\sum_{\substack{S\in\mathbb{Z}_r^r/dh\mathbb{Z}_r^r\\ t\in\mathbb{Z}/dh\mathbb{Z}}}\sigma(\det S)\tau(t)\theta(Z,Q;d^{-1}SL\langle t\rangle), \tag{4.1}$$

where σ and τ are homomorphisms of the multiplicative semigroup $\mathbb{Z}/dh\mathbb{Z}$ into the complex numbers; the series (4.1) will be called *average theta-series*. This will allow us to show that some of the average theta-series are common eigenfunctions for all regular Hecke operators.

Theorem 4.1. *In the notation and under the assumptions of Theorem* 3.2, *the image of the average theta-series* (4.1) *under the Hecke operator* $\|T$ *corresponding to an arbitrary homogeneous element* $T\in\mathcal{H}^g(\hat{q})$ *with* $\mu(T)=\mu$ *satisfying* (3.23) *can be written in the form*

$$\begin{aligned}\theta(Z,Q;\sigma,\tau,d^{-1}L)\|T\\ =\overline{\sigma}(\mu)^k\tau(\mu)\sum_{D\in\Delta(Q,\mu)/\Lambda}I(D,Q,\Psi^{g,r}(T))\theta\left(Z,\mu^{-1}Q[D];\sigma,\tau,d^{-1}L\right),\end{aligned} \tag{4.2}$$

where the bar means complex conjugation and the remaining notation is the same as in (3.24).

Proof. Multiplying both sides of (3.24) with $SL\langle t\rangle$ in place of L by $\sigma(\det S)\tau(t)$ and summing over $S\in\mathbb{Z}_r^r/dh\mathbb{Z}_r^r$ and $t\in\mathbb{Z}/dh\mathbb{Z}$, we get the relation

$$\begin{aligned}&\theta(Z,Q;\sigma,\tau,d^{-1}L)\|T\\ &\quad=\sum_{\substack{S\in\mathbb{Z}_r^r/dh\mathbb{Z}_r^r\\ t\in\mathbb{Z}/dh\mathbb{Z}}}\sigma(\det S)\tau(t)\sum_{D\in\Delta(Q,\mu)/\Lambda}I(D,Q,\Psi^{g,r}(T))\\ &\qquad\qquad\times\theta(Z,\mu^{-1}Q[D];d^{-1}\mu D^{-1}SL\langle t\rangle\langle\mu'\rangle)\\ &\quad=\sum_{D\in\Delta(Q,\mu)/\Lambda}\sum_{\substack{S\in\mathbb{Z}_r^r/dh\mathbb{Z}_r^r\\ t\in\mathbb{Z}/dh\mathbb{Z}}}\sigma(\det(\mu D^{-1})'\det(\mu D^{-1}S))\tau(t\mu'\mu)\\ &\qquad\qquad\times\theta(Z,\mu^{-1}Q[D];d^{-1}\mu D^{-1}SL\langle t\mu'\rangle),\end{aligned}$$

where $\det(\mu D^{-1})'\det(\mu D^{-1})\equiv 1 \pmod{dh}$, whence we see that

$$\sigma(\det(\mu D^{-1}))=\sigma^{-1}(\mu^r\mu^{-r/2})=\overline{\sigma}(\mu)^k.$$

In the last-written sum, we replace summations over S and t by summations over $\mu D^{-1}S$ and $t\mu'$, respectively. This allows us to rewrite the corresponding double sum in the form

$$\begin{aligned}&=\overline{\sigma}(\mu)^k\tau(\mu)\sum_{D\in\Delta(Q,\mu)/\Lambda}I(D,Q,\Psi^{g,r}(T))\\ &\qquad\times\sum_{\substack{S\in\mathbb{Z}_r^r/dh\mathbb{Z}_r^r\\ t\in\mathbb{Z}/dh\mathbb{Z}}}\sigma(\det S)\tau(t)\theta(Z,\mu^{-1}Q[D];d^{-1}SL\langle t\rangle),\end{aligned}$$

which proves formula (4.2). □

Similarly, Theorem 3.4 and Theorem 3.5 imply the following transformation formulas for average theta-series.

Theorem 4.2. *In the notation and under the assumptions of Theorems* 3.2 *and* 3.4, *formulas* (4.2) *can be rewritten as*

$$\theta(Z,Q;\sigma,\tau,d^{-1}L)\|T = \overline{\sigma}(\mu)^k\tau(\mu)\sum_{j=1}^{h(Q)}\sum_{D\in R(Q,\mu Q_j)/E(Q_j)} I(D,Q,\Psi^{g,r}(T))\theta\left(Z,Q_j;\sigma,\tau,d^{-1}L\right). \tag{4.3}$$

In particular, if the similarity class number $h(Q)$ *of* Q *is equal to* 1, *then formula* (4.2) *reshapes to*

$$\theta(Z,Q;\sigma,\tau,d^{-1}L)\|T = \overline{\sigma}(\mu)^k\tau(\mu)\Bigg(\sum_{D\in R(Q,\mu Q)/E(Q)} I(D,Q,\Psi^{g,r}(T))\Bigg)\theta\left(Z,Q;\sigma,\tau,d^{-1}L\right), \tag{4.4}$$

so that the averaged theta-series (4.1) *is an eigenfunction for the Hecke operator* $\|T$ *with the eigenvalue*

$$\overline{\sigma}(\mu)^k\tau(\mu)\Bigg(\sum_{D\in R(Q,\mu Q)/E(Q)} I(D,Q,\Psi^{g,r}(T))\Bigg). \tag{4.5}$$

Theorem 4.3. *In the notation of Theorems* 3.2 *and* 3.5, *for each homogeneous element* $T\in\mathcal{H}^g(\hat{q})$ *whose multiplier* $\mu(T)$ *is a perfect square, formula* (4.2) *can be rewritten as*

$$\theta(Z,Q;\sigma,\tau,d^{-1}L)\|T = \overline{\sigma}(\mu)^k\tau(\mu)\sum_{j=1}^{h^+(Q)}\sum_{D\in R(Q,\mu Q_j)/E(Q_j)} I(D,Q,\Psi^{g,r}(T))\theta\left(Z,Q_j;\sigma,\tau,d^{-1}L\right), \tag{4.6}$$

where $Q_1,\dots,Q_{h^+(Q)}$ *is a system of representatives of all different equivalence classes contained in the genus of* Q. *In particular, if the genus class number* $h^+(Q)$ *of* Q *is equal to* 1, *then formula* (3.24) *takes the form* (4.4), *so that the average theta-series* (4.1) *is an eigenfunction for the Hecke operator* $\|T$ *with the eigenvalue* (4.5).

§5. Eigenfunctions and eigenvalues

The theorems of §4 show that, under certain conditions on an element $T\in\mathcal{H}^g(\hat{q})$, the corresponding Hecke operator $\|T$ maps the linear space spanned by the average theta-series $\theta(Z,Q_i;\sigma,\tau,d^{-1}L)$ for $i=1,\dots,h(Q)$ or $i=1,\dots,h^+(Q)$ (with fixed σ, τ, and $d^{-1}L$) into itself. More precisely, under the same assumptions, formula (4.3) can be rewritten in the form

$$\mathbf{\Theta}\|T=\overline{\sigma}(\mu)^k\tau(\mu)C\left(\Psi^{g,r}(T)\right)\mathbf{\Theta}, \tag{5.1}$$

where $\mathbf{\Theta}=\mathbf{\Theta}(Z,Q;\sigma,\tau,d^{-1}L)$ is the column with the entries $\theta(Z,Q_i;\sigma,\tau,d^{-1}L)$ for $i=1,\dots,h(Q)$, and where, for a homogeneous element $T'\in\mathcal{H}^r(\hat{q})$ with $\mu(T')=\mu$, $C(T')$ is the complex square matrix of order $h(Q)$ given by

$$C(T')=(c_{i,j}(T'))=\Bigg(\sum_{D\in R(Q_i,\mu Q_j)/E(Q_j)} I(D,Q_i,T')\Bigg). \tag{5.2}$$

Similarly, formula (4.6) can be rewritten as

$$\mathbf{\Theta}^+\|T=\overline{\sigma}(\mu)^k\tau(\mu)C^+\left(\Psi^{g,r}(T)\right)\mathbf{\Theta}^+, \tag{5.3}$$

where

$$\mathbf{\Theta}^+=\mathbf{\Theta}^+(Z,Q;\sigma,\tau,d^{-1}L)$$

is the column with entries $\theta(Z, Q_i; \sigma, \tau, d^{-1}L)$ for $i = 1, \dots, h^+(Q)$, and where, for a homogeneous element T' in the even subring of the ring $\mathcal{H}^r(\hat{q})$ with $\mu(T') = \mu$, $C^+(T')$ is the complex square matrix of order $h^+(Q)$ with the entries

$$c_{i,j}(T') = \sum_{D \in R(Q_i, \mu Q_j)/E(Q_j)} I(D, Q_i, T').$$

The following lemma reduces the treatment of eigenfunctions and eigenvalues of Hecke operators on the spaces of average theta-series to the particular case of spaces spanned by usual theta-series

$$\theta^r(Z, Q_i) = \sum_{N \in \mathbb{Z}_r^r} e\{ZQ_i[N]\} = \theta(Z, Q_i; \mathbf{0}) \quad (i = 1, \dots, h(Q)), \tag{5.4}$$

of genus r of the matrices Q_i, where $\mathbf{0}$ stands for the zero $(r \times 2r)$-matrix.

Lemma 5.1. *Let Q be an even positive definite matrix of even order $r = 2k$, let q be the level of Q, and let $Q_1, \dots, Q_{h(Q)}$ (respectively, $Q_1, \dots, Q_{h^+(Q)}$) be a system of representatives of all different equivalence classes contained in the similarity class of Q (respectively, in the genus of Q). Let T' be a nonzero homogeneous element of the ring $\mathcal{H}_0^r(q)$ (respectively, of the even subring of $\mathcal{H}_0^r(q)$) with the multiplier $\mu(T) = \mu$. Suppose that a linear combination*

$$F_0 = \sum_{i=1}^{h(Q)} a_i \theta^r(Z, Q_i) \quad \Big(\text{respectively, } F_0^+ = \sum_{i=1}^{h^+(Q)} a_i \theta^r(Z, Q_i)\Big)$$

of theta-series (5.4) *is an eigenfunction for the Hecke operator corresponding to an element T' with eigenvalue $\lambda(T')$ (respectively, $\lambda^+(T')$):*

$$F_0 \| T' = \lambda(T') F_0 \quad (\text{respectively, } F_0^+ \| T' = \lambda^+(T') F_0^+). \tag{5.5}$$

Then the corresponding linear combination

$$F = \sum_{i=1}^{h(Q)} a_i \theta(Z, Q_i; \sigma, \tau, d^{-1}L) \quad \Big(\text{respectively, } F^+ = \sum_{i=1}^{h^+(Q)} a_i \theta(Z, Q_i; \sigma, \tau, d^{-1}L)\Big)$$

of the average theta-series (4.1) *is an eigenfunction of the Hecke operator $\|T$ for each element $T \in \mathcal{H}^g(\hat{q})$ satisfying*

$$\Psi^{g,r}(T) = \eta(T'),$$

where $\hat{q}$ is defined as in Theorem 3.2, *and η is the isomorphism* (3.6), *with the eigenvalue $\overline{\sigma}(\mu)^k \tau(\mu) \lambda(T')$ (respectively, $\overline{\sigma}(\mu)^k \tau(\mu) \lambda^+(T')$) :*

$$F \| T = \overline{\sigma}(\mu)^k \tau(\mu) \lambda(T') F \quad (\text{respectively, } F^+ \| T = \overline{\sigma}(\mu)^k \tau(\mu) \lambda^+(T') F^+). \tag{5.6}$$

Proof. Consider, e.g., the first case. Let $\mathbf{\Theta}_0$ be the column with the entries $\theta^r(Z, Q_i)$ for $i = 1, \dots, h(Q)$. Then Theorem 4.2 in the form (5.1) for the series (5.4) and $T = \eta(T')$ shows that $\mathbf{\Theta}_0 \| T' = \mathbf{\Theta}_0 \| \eta(T') = C(\eta(T')) \mathbf{\Theta}_0$. On the other hand, relation (5.5) means that $(\mathbf{a}\mathbf{\Theta}_0) \| T' = \lambda(T') \mathbf{a}\mathbf{\Theta}_0$, where $\mathbf{a}$ is the row $(a_1, \dots, a_{h(Q)})$. These relations imply the identity

$$\mathbf{a} C(\eta(T')) \mathbf{\Theta}_0 = (\mathbf{a}\mathbf{\Theta}_0) \| T' = \lambda(T') \mathbf{a}\mathbf{\Theta}_0. \tag{5.7}$$

It is easily seen that the entries of the column $\mathbf{\Theta}_0$ are linearly independent over the field $\mathbb{C}$. Indeed, clearly, the coefficients $r(Q_i, A) = \#(R(Q_i, A))$ with $A \in \mathbb{E}^r,\ A \geq 0$ of the Fourier expansions

$$\theta^r(Z, Q_i) = \sum_{A \in \mathbb{E}^r, A \geq 0} r(Q_i, A) e\{ZA\}$$

satisfy $r(Q_i, Q_j) = 0$ if $i \neq j$ and $r(Q_i, Q_i) > 0$, whence $\det(r(Q_i, Q_j)) \neq 0$. Therefore, identity (5.7) implies the matrix relation

$$\mathbf{a}C(\eta(T')) = \lambda(T')\mathbf{a}. \tag{5.8}$$

Multiplying both sides of (5.1) from the right by the row $\mathbf{a}$ and using (5.8), we get

$$\mathbf{a}\mathbf{\Theta}\|T = \overline{\sigma}(\mu)^k\tau(\mu)\mathbf{a}C(\eta(T'))\mathbf{\Theta} = \overline{\sigma}(\mu)^k\tau(\mu)\lambda(T')\mathbf{a}\mathbf{\Theta},$$

which is relation (5.6). □

Now, consider the linear spaces

$$\mathfrak{T}^g(Q) = \mathfrak{T}^g(Q, \sigma, \tau, d^{-1}L) = \left\{ \sum_{i=1}^{h(Q)} c_i\theta(Z, Q_i; \sigma, \tau, d^{-1}L) \;\middle|\; c_i \in \mathbb{C} \right\} \tag{5.9}$$

and

$$\mathfrak{T}^g_+(Q) = \mathfrak{T}^g_+(Q, \sigma, \tau, d^{-1}L) = \left\{ \sum_{i=1}^{h^+(Q)} c_i\theta(Z, Q_i; \sigma, \tau, d^{-1}L) \;\middle|\; c_i \in \mathbb{C} \right\} \tag{5.10}$$

spanned over $\mathbb{C}$ by the average theta-series of genus g with fixed σ, τ, and $d^{-1}L$ and corresponding to representatives of different equivalence classes contained in the similarity class and the genus of Q, respectively. By Theorems 3.2 and 4.2, the first of these spaces is mapped into itself by every Hecke operator $\|T$ corresponding to the elements of $\mathcal{H}^g(\hat{q})$ that satisfy (3.23). By Theorem 4.3, the second space is invariant under all Hecke operators corresponding to the elements of the even subring of $\mathcal{H}^g(\hat{q})$ generated by all homogeneous elements with quadratic multipliers. The following theorem gives a positive answer to the natural question about the possibility of simultaneous diagonalization of these operators.

Theorem 5.2. *For each positive definite matrix Q of even order r, and for fixed $g \geq 1$, σ, τ, and $d^{-1}L \in \mathbb{Q}^r_{2g}$, there exists a basis of the space $\mathfrak{T}^g(Q)$ (respectively, $\mathfrak{T}^{+g}(Q)$) that consists of common eigenfunctions for all Hecke operators corresponding to the elements of $\mathcal{H}^g(\hat{q})$ that satisfy condition (3.23) (respectively, the elements of the even subring of $\mathcal{H}^g(\hat{q})$).*

Proof. Consider, e.g., the case of spaces (5.9). The consideration of spaces (5.10) is based on the same arguments. By [2, Theorem 5.2.14], the space of all linear combinations with complex coefficients of the theta-series $\theta^r(Z, Q_1), \dots, \theta^r(Z, Q_{h(Q)})$ is invariant under all Hecke operators corresponding to the elements of $\mathcal{H}^r_0(q)$, and this space has a basis of common eigenfunctions for all such operators. Then, by Lemma 5.1, the same is true for the space $\mathfrak{T}^g(Q)$ and for the Hecke operators corresponding to the elements of $\mathcal{H}^g(\hat{q})$ that satisfy (3.23). □

As to the explicit construction of the eigenfunctions, in the case where the order r of Q is 2, all eigenfunctions and the corresponding eigenvalues can easily be found with the help of Lemma 5.1 and [7, Theorem 12]. In the general case, for the moment we can prove only the following.

Theorem 5.3. *For each positive definite matrix Q of even order r, and for fixed σ, τ, and $d^{-1}L \in \mathbb{Q}^r_{2g}$, the genus theta-series*

$$\sum_{i=1}^{h^+(Q)} e(Q_i)^{-1}\theta(Z, Q_i; \sigma, \tau, d^{-1}L), \tag{5.11}$$

where $Q_1, \dots, Q_{h^+(Q)}$ are representatives of different equivalence classes contained in the genus of Q, and $e(Q_i) = \#E(Q_i)$ is the number of units of Q_i, is an eigenfunction for all the Hecke operators corresponding to the elements of the even subring of the ring $\mathcal{H}^g(\hat{q})$.

Proof. This follows from Lemma 5.1 and [1, Theorem 4.5.1]. □

We note that Lemma 5.1 and [1, Theorem 4.5.1] allow us also to determine the Satake p-parameters of the even subring of the ring $\mathcal{H}^g(\hat{q})$ that correspond to the eigenfunction (5.11) for all prime p not dividing $\hat{q}$, and consequently to compute explicitly the regular part of the standard zeta function of this eigenfunction in terms of restricted Dirichlet L-series.

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Received 23/APR/2003

Translated by THE AUTHOR