Алгебра и анализ
Том. 15 (2003), вып. 5

St. Petersburg Math. J.
Vol. 15 (2004), No. 5, Pages 733–752
S 1061-0022(04)00829-5

WEIGHTED BERNSTEIN-TYPE INEQUALITIES, AND EMBEDDING THEOREMS FOR THE MODEL SUBSPACES

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Abstract. Weighted estimates are obtained for the derivatives in the model (shift-coinvariant) subspaces K^p_Θ, generated by meromorphic inner functions Θ of the Hardy class $H^p(\mathbb{C}^+)$. It is shown that the differentiation operator acts from K^p_Θ to a space $L^p(w)$, where the weight w depends on the function $|\Theta'|$, the rate of growth of the argument of Θ along the real line.

As an application of the weighted Bernstein-type inequalities, new Carleson-type theorems on embeddings of the subspaces K^p_Θ in $L^p(\mu)$ are proved. Also, results on the compactness of such embeddings are obtained, and properties of measures μ for which the norms $\|\cdot\|_{L^p(\mu)}$ and $\|\cdot\|_p$ are equivalent on a given model subspace K^p_Θ, are established.

Introduction

Let Θ be an inner function in the upper half-plane $\mathbb{C}^+$, that is, a bounded function analytic in $\mathbb{C}^+$ and such that $\lim_{y\to 0+}|\Theta(x+iy)| = 1$ for almost all $x \in \mathbb{R}$ with respect to Lebesgue measure. With each inner function Θ, we associate the model subspace $K^p_\Theta = H^p \cap \Theta\overline{H^p}$ of the Hardy class H^p in the upper half-plane. It is well known that for $p \in (1,\infty)$ the subspaces K^p_Θ are the only subspaces of H^p that are coinvariant with respect to the semigroup of shifts $(U_t)_{t\ge 0}$, where $(U_t f)(x) = e^{itx}f(x)$. The shift-coinvariant subspaces are of great importance in function theory and in operator theory (see [1, 2]); in particular, they play a key role in the construction of the Nagy–Foiaş functional model for contractions in a Hilbert space.

If $\Theta = B$ is a Blaschke product with simple zeros $\{z_n\}$, then for $1 < p \le \infty$ the model subspace K^p_B coincides with the closed linear span in $L^p(\mathbb{R})$ of the proper rational fractions with poles in the set $\{\overline{z}_n\}$ in the lower half-plane. The Paley–Wiener space PW^p_a, which consists of the entire functions of exponential type at most a the restrictions of which to the real line $\mathbb{R}$ are in $L^p(\mathbb{R})$, is also closely related to the shift-coinvariant subspaces. Namely, if $\Theta(z) = \exp(iaz)$, $a > 0$, then $K^p_\Theta = PW^p_a \cap H^p$.

Consider the differentiation operator $D: f \mapsto f'$, $f \in K^p_\Theta$. It is well known that every $f \in PW^p_a$, $1 \le p \le \infty$, satisfies the Bernstein inequality $\|f'\|_p \le a\|f\|_p$ (see [3]). Dyakonov [4, 5] obtained a description of the inner functions Θ such that the operator D is bounded as an operator from K^p_Θ to $L^p(\mathbb{R})$, that is, a Bernstein-type inequality is valid for the nontangential boundary values:

$$\|f'\|_p \le C(p,\Theta)\|f\|_p, \quad f \in K^p_\Theta.$$

2000 *Mathematics Subject Classification.* Primary 47B32, 30B50.

Key words and phrases. Hardy class, inner function, shift-coinvariant subspace, Bernstein-type inequality.

The work was partially supported by RFBR grant no. 03-01-00377 and by the grant for Leading Scientific Schools no. NSH-2266.2003.1.

The operator $D : K^p_\Theta \to L^p(\mathbb{R})$, $1 < p \le \infty$, is bounded if and only if $\Theta' \in L^\infty(\mathbb{R})$ in the sense of nontangential boundary values, or, equivalently, $\Theta' \in H^\infty(\mathbb{C}^+)$. Moreover, there is a constant $C = C(p)$ such that

$$\|f'\|_p \le C\|\Theta'\|_\infty \|f\|_p, \quad f \in K^p_\Theta. \tag{1}$$

Also, in [5] the problem of compactness of the differentiation operator was studied; the operator $D : K^p_\Theta \to L^p(\mathbb{R})$, $1 < p < \infty$, is compact if and only if $\Theta' \in C_0(\mathbb{R})$ (by $C_0(\mathbb{R})$ we denote the class of functions that are continuous on $\mathbb{R}$ and vanish at infinity).

The condition $\Theta' \in L^\infty$ implies that, up to multiplication by a unimodular constant, the inner function Θ is of the form $\Theta(z) = \exp(iaz)B(z)$, where $a \ge 0$, B is a Blaschke product with zeros $z_n = x_n + iy_n$ such that $\lim_{n\to\infty} |z_n| = \infty$, and $\inf_n y_n > 0$. In this case Θ is meromorphic in the entire complex plane $\mathbb{C}$; thus, there is a well-defined branch of the argument of the function Θ on $\mathbb{R}$, that is, a smooth and monotone increasing function φ such that $\Theta(t) = \exp(2i\varphi(t))$. We also have $|\Theta'(t)| = 2\varphi'(t)$, $t \in \mathbb{R}$, and

$$\varphi'(t) = \frac{a}{2} + \sum_n \frac{y_n}{|t - z_n|^2}.$$

In what follows we assume that Θ is a meromorphic function of the form described above, and we fix the notation a, $z_n = x_n + iy_n$, and φ.

For $p = \infty$, an essentially stronger weighted inequality was obtained by Levin [6, 29]. Namely,

$$\|f'/\Theta'\|_\infty \le \|f\|_\infty, \quad f \in K^\infty_\Theta, \tag{2}$$

for any meromorphic inner function Θ. Also, in [6] more general results were proved concerning estimates of the derivatives at a point on the real axis where the function Θ (not necessarily meromorphic) has a finite angular derivative. Inequality (2) was independently rediscovered by Borwein and Erdelyi [7, 8] and also by Li, Mohapatra, and Rodriguez [9] in the case where Θ is a finite Blaschke product. It should be mentioned that some other weighted estimates of the derivatives in model subspaces were obtained in [10, 11].

In the present paper we prove certain analogs of the weighted Bernstein-type inequality (2) for L^p-norms with $p < \infty$. We are concerned with inequalities of the form

$$\|f'w\|_p \le C(p, \Theta, w)\|f\|_p, \quad f \in K^p_\Theta, \tag{3}$$

where the weight w is correlated to the function Θ (for example, $w = |\Theta'|^{-\alpha}$). Weighted inequalities of the form (3) may be true even in the case where $\Theta' \notin L^\infty(\mathbb{R})$, and the Bernstein inequality for the L^p-norms without weight is not fulfilled.

At the same time, the following theorem shows that even in the case of inner functions with bounded derivatives, weighted estimates can be obtained to generalize Dyakonov's inequality (1) and refine it quantitatively.

Theorem 1. *Let $1 \le p < \infty$, and let $\Theta' \in L^\infty$. There exists a positive constant $C = C(p)$ such that*

$$\int_{\mathbb{R}} \frac{|f'(t)|^p}{|\Theta'(t)|^{p-\frac{1}{2}-\sigma}}\, dt \le C \frac{\|\Theta'\|_\infty^{\frac{1}{2}+\sigma}}{\sigma^p} \|f\|_p^p, \quad f \in K^p_\Theta, \tag{4}$$

for each $\sigma \in (0, 1/2]$.

Remarks. 1. Inequality (4) is stronger than inequality (1) for the L^p-norms without weight, because it may happen that $\inf_{\mathbb{R}} |\Theta'| = 0$ or $\Theta' \in C_0(\mathbb{R})$. We note that estimate (4) is of interest even for the spaces K^p_Θ generated by finite Blaschke products, that is, for the spaces of rational fractions with fixed poles. By applying Theorem 1, it is easy to prove the sufficiency of the condition $\Theta' \in C_0(\mathbb{R})$ in the compactness theorem for the differentiation operator.

2. A particular case of Theorem 1 ($p = 2$, $\sigma = 1/2$) was proved in the paper [12] by the author.

3. The exponent $p - \frac{1}{2}$ is sharp in the following sense: for any $p \in (1, \infty)$ there exists an inner function Θ such that the left-hand side in (4) cannot be estimated from above by $\|f\|_p^p$ if $\sigma \le 0$. For example, if we take Θ to be a finite Blaschke product and $f(z) = (z - \overline{z}_1)^{-1}$, then the integral on the left in (4) diverges for $\sigma = 0$. Indeed, $|f'(t)| \asymp t^{-2} \asymp |\Theta'(t)|$, $t \to \infty$ (we write $g \asymp h$ if $C_1 g \le h \le C_2 g$ for some positive constants C_1 and C_2).

However, for some special classes of inner functions the exponent $p - \frac{1}{2}$ may be improved. We say that an inner function Θ satisfies the *connected level set condition* if the level set $\{z \in \mathbb{C}^+ : |\Theta(z)| < \varepsilon\}$ is connected for some $\varepsilon \in (0,1)$ (such inner functions are sometimes referred to as one-component inner functions). Applying the arguments similar to the proof of Theorem 1, and A. B. Aleksandrov's characterization of the functions satisfying the connected level set condition (see [13]), it is not difficult to prove the following estimate.

Theorem 2. *Let B be an infinite Blaschke product satisfying the connected level set condition, let $B' \in L^\infty$, and let $1 \le p < \infty$. Then for any $\sigma \in (0, 1]$ we have*

$$\int_{\mathbb{R}} \frac{|f'(t)|^p}{|B'(t)|^{p-\sigma}}\, dt \le C(B,p)\frac{\|B'\|_\infty^\sigma}{\sigma^p}\|f\|_p^p, \quad f \in K_B^p.$$

Recently, the author showed that if a function Θ satisfies the connected level set condition and the point at infinity belongs to the boundary spectrum of Θ, then a stronger inequality similar to (2) is true, namely, $\|f'/\Theta'\|_p \le C(p,\Theta)\|f\|_p$, $f \in K_\Theta^p$, $p \in (1, \infty)$. Moreover, we do not need the assumption $\Theta' \in L^\infty$ here. The proof of this result will appear elsewhere.

Now, suppose that the inner function Θ satisfies the condition $\inf_n y_n > 0$, which is weaker than the condition $\Theta' \in L^\infty$. In this case we also have a weighted Bernstein-type inequality.

Theorem 3. *Suppose $\inf_n y_n = M > 0$, $1 < p < \infty$, $\sigma \in (0, 1/2]$. Put*

$$w(t) = [|\Theta'(t)|^p + |\Theta'(t)|^{p-\frac{1}{2}-\sigma}]^{-1}.$$

Then there is a constant $C = C(p)$ such that

$$\int_{\mathbb{R}} |f'(t)|^p w(t)\, dt \le C(\sigma^{-p} M^{-\frac{1}{2}-\sigma} + 1)\|f\|_p^p, \quad f \in K_\Theta^p.$$

Generally speaking, the summand $|\Theta'|^p$ in the definition of w^{-1} cannot be replaced by $|\Theta'|^\alpha$ with $\alpha < p$, which means that the exponent p is sharp. The assumptions of Theorem 3 cannot be relaxed: in the sequel we give an example of a meromorphic Blaschke product with zeros approaching the real axis and such that the inequality $\|f' w^{1/p}\|_p \le C\|f\|_p$, $f \in K_\Theta^p$, fails for any weight w of the form $|\Theta'|^{-\alpha}$ or $[|\Theta'|^\alpha + |\Theta'|^\beta]^{-1}$, $\alpha, \beta > 0$.

Weighted Bernstein-type inequalities turn out to be an efficient tool for obtaining new Carleson-type embedding theorems for model subspaces, that is, for a description of the measures μ such that $K_\Theta^p \subset L^p(\mu)$, $1 \le p < \infty$. Let $\mathcal{C}_p(\Theta)$ denote the class of all measures μ such that we have the embedding $K_\Theta^p \subset L^p(\mu)$, or, equivalently, $\|f\|_{L^p(\mu)} \le C\|f\|_p$, $f \in K_\Theta^p$. The problem of description of the class $\mathcal{C}_p(\Theta)$ for a given inner function Θ was posed by Cohn in [14]. At present, this problem is solved only for some fairly special classes of inner functions, in particular, for the functions satisfying the connected level set condition. In the paper [15] by Volberg and Treil, a condition was found that ensures the embedding in question in the general case: we have the embedding $K^p_\Theta \subset L^p(\mu)$ if there exist $\varepsilon \in (0,1)$ and $C > 0$ such that $\mu(S(x,h)) \le Ch$ for all squares $S(x,h) = [x, x+h] \times [0,h]$ satisfying

$$S(x,h) \cap \{z \in \mathbb{C}^+ : |\Theta(z)| < \varepsilon\} \ne \varnothing.$$

In other words, the measure μ satisfies the Carleson condition $\mu(S(x,h)) \le Ch$ for the squares $S(x,h)$ of a certain special form (we denote the class of such measures by $\mathcal{C}(\Theta)$). However, in [16] it was shown that this condition is necessary only for the inner functions satisfying the connected level set condition. Embedding theorems for the shift-coinvariant subspaces were considered also in [16]–[18]. Compactness of such embeddings was studied in [10, 19]. The study of the measures for which the corresponding L^p-norm is equivalent to the natural norm on a given model subspace is also of great importance; see [10, 19, 20] (in §6 some known facts about equivalent norms will be discussed).

The following embedding theorem for measures with support on the real axis follows immediately from Theorem 1. We assume (as in the Volberg–Treil theorem) that the measure μ satisfies the Carleson condition on the intervals of a special form, with length controlled by the growth of the argument of the function Θ. Here, in contrast to the Volberg–Treil theorem, it suffices to check the Carleson condition only for a countable set of intervals. We note also that the conditions on the measure depend essentially on the exponent p. Below the symbol $|I|$ denotes the length of the interval I.

Theorem 4. *Let $\Theta' \in L^\infty$, let μ be a measure on $\mathbb{R}$, and let $1 < p < \infty$, $1/p + 1/q = 1$, $\delta > 0$. Assume that $\mathbb{R} = \bigcup_{k\in\mathbb{Z}} I_k$, where the intervals I_k satisfy*

$$\sup_k |I_k| \biggl(\int_{I_k} |\Theta'(t)|^{\frac{q+1}{2}-\delta}\, dt \biggr)^{p/q} < \infty. \tag{5}$$

1. *If $\mu(I_k) = O(|I_k|)$ (i.e., $\mu(I_k) \le C|I_k|$), then $\mu \in \mathcal{C}_p(\Theta)$.*
2. *If $\mu(I_k) = o(|I_k|)$ (i.e., $\lim_{|k|\to\infty} \mu(I_k)/|I_k| = 0$), then the operator that embeds K^p_Θ in $L^p(\mu)$ is compact.*

We construct examples to show that for certain classes of inner functions Theorem 4 describes a class of measures that is larger than $\mathcal{C}(\Theta)|_{\mathbb{R}}$ (the set of measures in $\mathcal{C}(\Theta)$ with support on the line $\mathbb{R}$). Thus, Theorem 4 provides nontrivial examples of embeddings that cannot be obtained from the Volberg–Treil theorem.

A similar statement is true for the meromorphic inner functions with zeros separated away from the real axis.

Theorem 4′. *Let Θ be a meromorphic inner function such that $\inf_n y_n > 0$, and let μ be a Borel measure on the line $\mathbb{R}$. Assume that $\mathbb{R} = \bigcup_k I_k$, where the intervals I_k satisfy*

$$\sup_k |I_k| \biggl(\int_{I_k} \bigl(|\Theta'(t)|^{\frac{q+1}{2}-\delta} + |\Theta'(t)|^q \bigr)\, dt \biggr)^{p/q} < \infty \tag{6}$$

for some $\delta > 0$. Then $K^p_\Theta \subset L^p(\mu)$ if $\mu(I_k) = O(|I_k|)$, and the embedding operator is compact if $\mu(I_k) = o(|I_k|)$.

If, moreover, the supremum in (5) (respectively, in (6)) is sufficiently small (smaller than some constant $\varepsilon(\Theta, p, \delta) > 0$) and $\mu(I_k) \asymp |I_k|$, then the norms $\|\cdot\|_{L^p(\mu)}$ and $\|\cdot\|_p$ are equivalent on K^p_Θ.

Another application of the weighted Bernstein-type inequalities is related to perturbations of orthogonal bases of reproducing kernels. Let $|\alpha| = 1$, and let $t_k \in \mathbb{R}$ be such that $\{t_k\} = \{t : \Theta(t) = \alpha\}$. We denote by $k(\cdot, t_k)$ the reproducing kernel of the space

K^2_Θ corresponding to the point t_k. It is well known (see de Branges [21] and Clark [22]) that for all but at most one α the system $\{k(\cdot, t_k)\}$ forms an orthogonal basis in K^2_Θ, and

$$\|f\|_2^2 = 2\pi \sum_k \frac{|f(t_k)|^2}{|\Theta'(t_k)|}, \quad f \in K^2_\Theta.$$

In other words, the embedding $K^2_\Theta \subset L^2(\mu)$, where $\mu = 2\pi \sum_k \delta_{t_k}/|\Theta'(t_k)|$, is isometric (here, δ_x is the Dirac measure at the point x).

Let $s_k \in \mathbb{R}$; by $\langle s_k, t_k\rangle$ we denote the interval with endpoints s_k and t_k. The following theorem gives conditions (in terms of the closeness of s_k to t_k) ensuring that the measure $\nu = \sum_k \delta_{s_k}/|\Theta'(s_k)|$ belongs to the class $\mathcal{C}_2(\Theta)$ or determines an equivalent norm on the space K^2_Θ.

Theorem 5. *Suppose* $\Theta' \in L^\infty$, $\delta < 1/2$, *and*

$$d(\{s_k\}) = \sup_k \int_{\langle s_k, t_k\rangle} |\Theta'(t)|^\delta \, dt < \infty.$$

Then $K^2_\Theta \subset L^2(\nu)$. *If, moreover,* $d(\{s_k\}) < r$, *where* $r = r(\Theta, \delta)$ *is a positive constant, then the norms* $\|\cdot\|_{L^2(\nu)}$ *and* $\|\cdot\|_2$ *are equivalent on* K^2_Θ.

Remark. Similar perturbation results for bases of reproducing kernels were obtained by Cohn in [10] in the case where the connected level set condition is fulfilled (and the "smallness" of the perturbation is expressed in similar terms). Let Θ be a meromorphic inner function satisfying the connected level set condition. Then there is $\varepsilon = \varepsilon(\Theta)$ such that for any sequence $s_k \in \mathbb{R}$ satisfying the inequality $\int_{\langle s_k, t_k\rangle} |\Theta'(t)|\, dt < \varepsilon$ the measure ν determines an equivalent norm on the space K^2_Θ. Thus, for the functions with a connected level set we can take $\delta = 1$ in Theorem 5. It should be mentioned that the proof of Cohn's theorem also involves certain Bernstein-type inequalities.

We show that, generally speaking, for inner functions with bounded derivatives the constant 1/2 in Theorem 5 cannot be replaced with a larger one.

The paper is divided into 7 sections. In §1 we discuss a general method for obtaining weighted Bernstein-type inequalities; this method reduces the problem to the study of certain singular integral operators (in particular, the Calderón commutator). A series of preliminary pointwise and integral estimates for the reproducing kernels of model subspaces and for their derivatives is presented in §2. On the basis of these estimates, in §3 we prove Theorem 1; also, here we give a new proof of the theorem on compactness of the differentiation operator. Other weighted Bernstein-type inequalities (Theorems 2 and 3) are proved in §4 and §5, respectively. As an application of the weighted Bernstein-type inequalities, in §6 we obtain new versions of embedding theorems (Theorems 4 and 4′) and also present examples that show the sharpness of the sufficient conditions obtained in a certain class of model subspaces. Small perturbations of bases of reproducing kernels are discussed in §7 (see Theorem 5).

The author is deeply grateful to V. P. Havin for the formulation of the problem and permanent interest in this work, and also to K. M. Dyakonov and V. I. Vasyunin for valuable discussions and remarks.

§1. Generalized Bernstein-type inequality

In what follows, an exceptionally important role is played by the reproducing kernels of model subspaces. The reproducing kernel of the space K^2_Θ corresponding to the point $w \in \mathbb{C}^+$ is of the form

$$k(z, w) = \frac{i}{2\pi} \cdot \frac{1 - \overline{\Theta(w)}\Theta(z)}{z - \overline{w}},$$

and for any $f \in K^2_\Theta$ we have

$$f(w) = \int_{\mathbb{R}} f(t)\overline{k(t,w)}\,dt, \quad w \in \mathbb{C}^+. \tag{7}$$

Note that $k(\cdot, w) \in H^p$, $1 < p \le \infty$, and identity (7) is also true for any function $f \in K^p_\Theta$ with $1 \le p < \infty$. If the function Θ is analytic in a neighborhood of a point $x \in \mathbb{R}$, then $k(\cdot, x) \in H^p$, $1 < p \le \infty$, and (7) remains true for $w = x$. (Ahern and Clark showed in [23] that for $w = x$ formula (7) is true for any $f \in K^2_\Theta$ whenever Θ has a finite angular derivative at the point x.) We shall need the following formulas for the L^2-norms of the reproducing kernels:

$$\|k(\cdot, w)\|_2^2 = \frac{1 - |\Theta(w)|^2}{4\pi \Im w}, \quad w \in \mathbb{C}^+,$$

and $\|k(\cdot, x)\|_2^2 = |\Theta'(x)|/2\pi$, $x \in \mathbb{R}$ (a series of estimates for the L^p-norms of reproducing kernels can be found in [16, 17]).

If Θ is a meromorphic inner function, then any element of the model subspace K^p_Θ is also meromorphic in the entire complex plane $\mathbb{C}$, and its poles are in the set $\{\overline{z}_n\}$. In this case, the value of a function $f \in K^p_\Theta$, $1 \le p < \infty$, at a point $x \in \mathbb{R}$ may be recovered by formula (7), namely,

$$f(x) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{1 - \overline{\Theta(t)}\Theta(x)}{t - x} f(t)\,dt. \tag{8}$$

Proposition 1. *Suppose* $1 \le p < \infty$, $0 < \delta \le \frac{1}{2p}$, *and* $\varepsilon > 0$. *Let* $w \ge 0$ *be a function satisfying the following conditions*:

1) $\sup_{x\in\mathbb{R}}[w(x)\sup_{|t-x|<\varepsilon}|k'_x(t,x)|^p] = A_1 < \infty$;

2) $\sup_{x\in\mathbb{R}} w(x)(\varphi'(x))^p = A_2 < \infty$;

3) *if* $p > 1$, *then*

$$\sup_{x\in\mathbb{R}} w(x)\biggl(\int_{|t-x|>\varepsilon} \frac{|\Theta(t) - \Theta(x)|^q}{|t-x|^{q+1-\delta q}}\,dt\biggr)^{p/q} = A_3 < \infty,$$

where $1/p + 1/q = 1$. *For* $p = 1$ *we replace* 3) *by*

3′)

$$\sup_{x\in\mathbb{R}}\biggl[w(x)\sup_{|t-x|>\varepsilon}\biggl|\frac{\Theta(t) - \Theta(x)}{t - x}\biggr|^{1-\delta}\biggr] = A_3 < \infty.$$

Then there is a constant $C = C(p)$ *such that*

$$\int_{\mathbb{R}} |f'(x)|^p w(x)\,dx \le C(\varepsilon^p A_1 + A_2 + \varepsilon^{-\delta p}(\delta p)^{-1}A_3)\|f\|_p^p, \quad f \in K^p_\Theta.$$

Proof. The proof of this proposition is a modification of the arguments used in [5]. Differentiating the representation (8), we obtain

$$f'(x) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(t)\overline{\Theta(t)}\Phi(t,x)\,dt, \quad x \in \mathbb{R}, \tag{9}$$

where

$$\Phi(t,x) = 2\pi i\Theta(t)(\overline{k(t,x)})'_x = \frac{\Theta(t) - \Theta(x) - \Theta'(x)(t-x)}{(t-x)^2}, \quad t \in \mathbb{R}.$$

Since Θ admits analytic extension across the real axis $\mathbb{R}$, the function $\Phi(\cdot, x)$ is continuous on $\mathbb{R}$ (we assume that $\Phi(x,x) = \Theta''(x)/2$).

We denote $f\overline{\Theta}$ by g and fix $\varepsilon > 0$. We split the integral (9) into three parts:

$$2\pi i f'(x) = I_1 g(x) + \Theta'(x) I_2 g(x) + I_3 g(x), \tag{10}$$

where

$$I_1 g(x) = \int_{|t-x|<\varepsilon} g(t)\Phi(t,x)\,dt, \quad I_2 g(x) = \int_{|t-x|\ge\varepsilon} \frac{g(t)}{x-t}\,dt,$$

and

$$I_3 g(x) = \int_{|t-x|\ge\varepsilon} g(t)\frac{\Theta(t)-\Theta(x)}{(t-x)^2}\,dt.$$

We show that the integral operators I_j, $j = 1,2,3$, are bounded operators from $L^p(\mathbb{R})$ ($H^p(\mathbb{R})$) to $L^p(w(t)dt)$. It is easily seen that, by condition 1),

$$\int_{\mathbb{R}} |I_1 g(x)|^p w(x)\,dx \le (2\varepsilon)^p A_1 \|g\|_p^p.$$

Next,

$$|I_2 g(x)| \le \sup_{\delta>0} \left| \int_{|t-x|\ge\delta} \frac{g(t)}{x-t} \right| = \pi \mathcal{H}_M g(x).$$

It is well known that the maximal Hilbert transformation $\mathcal{H}_M$ is bounded as an operator from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$, $1 < p < \infty$, and also from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$ (see [1, 24]). It should be noted that $\mathcal{H}_M$ is not bounded on $L^1(\mathbb{R})$. But in our case $g = \overline{\Theta} f \in \overline{H^1}$ because $f \in K_\Theta^1$. Thus,

$$\int_{\mathbb{R}} |\Theta'(x) I_2 g(x)|^p w(x)\,dx \le (2\pi)^p A_2 K_p^p \|g\|_p^p,$$

where K_p is the norm of the maximal Hilbert transformation on H^p, $1 \le p < \infty$.

Now we estimate $I_3 g$. By the Hölder inequality,

$$|I_3 g(x)|^p w(x) \le A_3 \int_{|t-x|\ge\varepsilon} \frac{|g(t)|^p}{|t-x|^{1+\delta p}}\,dt.$$

Consequently,

$$\int_{\mathbb{R}} |I_3 g(x)|^p w(x) dx \le \frac{2\varepsilon^{-\delta p}}{\delta p} A_3 \|g\|_p^p. \qquad \square$$

Remark. The integral representation (9) allows us to reduce the study of weighted estimates for the differentiation operator in K_Θ^p to that of certain singular integral operators. Namely,

$$2\pi i f'(x) = -\Theta'(x)(\mathcal{H}g)(x) + C_\Theta^1 g(x),$$

where $g = \overline{\Theta} f$, $(\mathcal{H}g)(x) = \mathrm{P.V.}\int \frac{g(t)}{t-x} dt$ is the Hilbert transform of the function g (up to the constant factor $1/\pi$), whereas C_Θ^1 is what is called a *first order Calderón commutator*,

$$C_\Theta^1 f(x) = \mathrm{P.V.} \int \frac{\Theta(t)-\Theta(x)}{(t-x)^2} f(t) dt.$$

It is well known that the commutator C_Θ^1 is bounded in $L^p(\mathbb{R})$, $1 < p < \infty$, if $\Theta' \in L^\infty(\mathbb{R})$ (see [25, 26]). In this case, boundedness follows from the general theory of the Calderón–Zygmund operators. We recall that a singular integral operator with kernel $w(t,x)$ is called a Calderón–Zygmund operator if $|w(t,x)| \le C|t-x|^{-1}$,

$$|w(t,x) - w(t',x)| \le C|t-x|^{-1-\delta}|t-t'|^\delta,$$

and

$$|w(t,x) - w(t,x')| \le C|t-x|^{-1-\delta}|x-x'|^\delta$$

for some $\delta > 0$ whenever $2|t-t'| \le |t-x|$ and $2|x-x'| \le |t-x|$.

One of the proofs of the Dyakonov theorem (see [4, 5]) is based on the boundedness of the commutator C_Θ^1. We are interested, however, in the weighted estimates of the form $\|f'/|\Theta'|^\alpha\|_p \le C(p,\Theta,\alpha)\|f\|_p$. In this case we cannot apply the general theory of the

Calderón–Zygmund operators, because the operator with the kernel $|\Theta'(x)|^{-\alpha}\frac{\Theta(t)-\Theta(x)}{(t-x)^2}$ may be bounded on L^p, and simultaneously may fail to be Calderón–Zygmund.

§2. Preliminary estimates of reproducing kernels

We shall need some technical propositions related to estimates of reproducing kernels and their derivatives. In the following lemmas we assume that Θ is a meromorphic inner function with zeros $z_n = x_n + iy_n$, and that $M = \inf_n y_n > 0$. Recall that $|\Theta'(t)| = 2\varphi'(t)$. In this section, the symbols C, C_1, C_2 denote various absolute (independent of Θ) positive constants.

Lemma 1. *If* $x, t \in \mathbb{R}$, *and* $|t-x| \le M$ *or* $|t-x| \le \sqrt{\frac{M}{4\varphi'(x)}}$, *then*

$$\frac{\varphi'(x)}{4} \le \varphi'(t) \le 4\varphi'(x).$$

Proof. Observe that $y_n \ge M$ and $\varphi'(t) \ge M|t-z_n|^{-2}$ for any n. Consequently, the conditions of the lemma imply that

$$|t-x| \le \max(y_n, |t-z_n|/2),$$

whence $1/2 \le |t-z_n|^{-1}|x-z_n| \le 2$, and therefore, $1/4 \le \varphi'(t)/\varphi'(x) \le 4$. □

Lemma 2. *We have*

$$\sup_{t\in\mathbb{R}} |k(t,x)| \le C_1[\varphi'(x) + (\varphi'(x))^{1/2}M^{-1/2}], \tag{11}$$

and if $\varphi' \in L^\infty$, *then*

$$\sup_{t\in\mathbb{R}} |k(t,x)| \le C_2(\varphi'(x))^{1/2}\|\varphi'\|_\infty^{1/2}. \tag{12}$$

Proof. Estimate (11) follows immediately from Lemma 1, and (12) is a consequence of formula (8) and the Cauchy inequality. □

Lemma 3. $\sup_{|t-x|\le M} |k'_x(t,x)| \le C\varphi'(x)(\varphi'(x) + M^{-1})$, $x \in \mathbb{R}$.

Proof. Note that $|k'_x(t,x)| = |\Phi(t,x)| \le \max|\Theta''(s)|/2$, $s \in \langle t, x\rangle$. It is easily seen that $|\Theta''(s)| \le 4(\varphi'(s))^2 + 2|\varphi''(s)|$, and

$$|\varphi''(s)| = \left|\sum_n \frac{2y_n(s-x_n)}{|s-z_n|^4}\right| \le \sum_n \frac{1}{|s-z_n|^2} \le \frac{\varphi'(s)}{M}.$$

Thus, $|k'_x(t,x)| \le C(\varphi'(s))^2 + \varphi'(s)/M)$, where $s \in \langle t, x\rangle$. It remains to apply Lemma 1. □

The next two lemmas will be used for estimating the third summand in (10).

Lemma 4. *Let* $\varepsilon, \delta, y > 0$, *and let* $\psi(t) = \arctan(t/y)$. *Then*

$$\int_{|t-x|\ge\varepsilon} \frac{|\sin(\psi(t)-\psi(x))|}{|t-x|^{2+\delta}}\,dt \le C\frac{\psi'(x)}{\delta}\left(\frac{1}{\varepsilon^\delta} + \frac{1}{y^\delta}\right), \quad x \in \mathbb{R}.$$

Proof. Without loss of generality we assume that $x \ge 0$. Repeatedly, we shall make use of the fact that $\psi'(x) = y/(x^2+y^2) \asymp 1/y$ if $x \le Cy$, and $\psi'(x) \asymp y/x^2$ if $x \ge Cy$. We denote by Δ the interval $|x-t| < \varepsilon$ and split the integral to be estimated into three parts I_1, I_2, and I_3 that are the integrals over the sets $(-\infty, 0] \setminus \Delta$, $[0, x/2] \setminus \Delta$, and $[x/2, \infty) \setminus \Delta$, respectively. Changing the variable in the first integral, we get

$$I_1 = \int_{\max(0,\varepsilon-x)}^{\infty} \frac{|\sin(\arctan(t/y) + \arctan(x/y))|}{(t+x)^{2+\delta}}\,dt.$$

If $x \le 2y$, then

$$I_1 \le \int_{\max(0,\varepsilon-x)}^{\infty} \frac{t+x}{y(t+x)^{2+\delta}}\,dt \le \frac{1}{y\delta\varepsilon^\delta} \le C_1\psi'(x)$$

(we have used the inequalities $|\arctan u| \le |u|$ and $|\sin u| \le |u|$, $u \in \mathbb{R}$). If $x > 2y$, then

$$\int_0^{2y} \frac{|\sin(\psi(t)+\psi(x))|}{|t+x|^{2+\delta}}\,dt \le \frac{2y}{x^{2+\delta}} \le C_2\frac{\psi'(x)}{y^\delta}.$$

To estimate the integral over the set $[2y,\infty)$, we observe that $\arctan u = \pi/2 - 1/u + O(1/u^2)$, $u \ge 2$, whence $|\sin(\psi(t)+\psi(x))| \le C_3(y/x + y/t)$, $x, t \in [2y,\infty)$. Consequently,

$$\int_{2y}^{\infty} \frac{|\sin(\psi(t)+\psi(x))|}{|t+x|^{2+\delta}}\,dt \le C_3\int_{2y}^{\infty}\frac{y}{tx(t+x)^{1+\delta}}\,dt \le C_4\frac{\psi'(x)}{\delta y^\delta}.$$

Next,

$$I_3 = \int_{[x/2,\infty)\setminus\Delta} \frac{|\sin(\psi(t)-\psi(x))|}{|t-x|^{2+\delta}}\,dt \le \psi'(x/2)\int_{[x/2,\infty)\setminus\Delta}\frac{dt}{|t-x|^{1+\delta}} \le C_5\frac{\psi'(x)}{\delta\varepsilon^\delta}.$$

Note that the set $[0,x/2]\setminus\Delta$ is empty if $x < \varepsilon$. In this case $I_2 = 0$. Now let $x \ge \varepsilon$; then, using the identity $\arctan u - \arctan v = \arctan\frac{u-v}{1+uv}$, $u, v > 0$, we obtain

$$I_2 \le \int_0^{x/2} \frac{y\,dt}{(tx+y^2)|x-t|^{1+\delta}} \le C_6\frac{y}{x^{2+\delta}}\int_0^{x/2}\frac{dt}{t+y^2/x} = C_6\frac{y}{x^{2+\delta}}\log\left(1+\frac{x^2}{2y^2}\right).$$

If $x \le 2y$, then $I_2 \le C_7x^{-\delta}/y \le C_8\varepsilon^{-\delta}\psi'(x)$. Finally, if $x > 2y$, then

$$I_2 \le C_9\frac{y}{x^2y^\delta}\left(\frac{y}{x}\right)^\delta \log\frac{x}{y} \le C_{10}\frac{\psi'(x)}{\delta y^\delta}$$

(the second inequality follows from the trivial estimate $u^{-\delta}\log u \le (e\delta)^{-1}$, $u \ge 1$). ☐

Lemma 5. *Let $\varepsilon, \delta > 0$. Then*

$$\int_{|t-x|\ge\varepsilon} \frac{|\Theta(t)-\Theta(x)|}{|t-x|^{2+\delta}}\,dt \le C\frac{\varphi'(x)}{\delta}\left(\frac{1}{\varepsilon^\delta}+\frac{1}{M^\delta}\right), \quad x \in \mathbb{R}.$$

Proof. Since

$$\Theta(z) = e_a(z)\prod_n B_n(z) = \exp(iaz)\prod_n \frac{\overline{z}_n}{z_n}\cdot\frac{z-z_n}{z-\overline{z}_n},$$

we have $|\Theta(t)-\Theta(x)| \le |e_a(t)-e_a(x)| + \sum_n |B_n(t)-B_n(x)|$. The argument of the function B_n is of the form $2\psi_n(t) = 2\arctan\frac{t-x_n}{y_n} + c$, and

$$\varphi'(t) = \frac{a}{2} + \sum_n \frac{y_n}{(t-x_n)^2+y_n^2} = \frac{a}{2} + \sum_n \psi_n'(t).$$

By Lemma 4,

$$\begin{aligned}\int_{|t-x|\ge\varepsilon}\frac{|B_n(t)-B_n(x)|}{|t-x|^{2+\delta}}\,dt &= 2\int_{|t-x|\ge\varepsilon}\frac{|\sin(\psi_n(t)-\psi_n(x))|}{|t-x|^{2+\delta}}\,dt \\ &\le C\frac{\psi_n'(x)}{\delta}\left(\frac{1}{\varepsilon^\delta}+\frac{1}{y_n^\delta}\right) \le C\frac{\psi_n'(x)}{\delta}\left(\frac{1}{\varepsilon^\delta}+\frac{1}{M^\delta}\right).\end{aligned}$$

Also, it is easy to check that

$$\int_{|t-x|\ge\varepsilon}\frac{|e_a(t)-e_a(x)|}{|t-x|^{2+\delta}}\,dt \le \frac{a}{\delta\varepsilon^\delta}. \qquad ☐$$

§3. WEIGHTED BERNSTEIN-TYPE INEQUALITY IN THE CASE WHERE $\Theta' \in L^\infty$

In this section we prove Theorem 1; we use the general method described in §1 and the estimates obtained in §2. The symbols A_i have the same meaning as in Proposition 1.

Proof of Theorem 1. Let $\Theta' \in L^\infty$, and let $w(t) = |\Theta'(t)|^{-p+\frac{1}{2}+\sigma}$. First, we note that $M^{-1} \le \|\varphi'\|_\infty$ because $\|\varphi'\|_\infty \ge \varphi'(x_n) \ge y_n^{-1}$ for any n.

Put $\varepsilon = \|\varphi'\|_\infty^{-1}$. By Lemma 3, $\sup_{|t-x|\le\varepsilon} |k_x'(t,x)| \le C\|\varphi'\|_\infty \varphi'(x)$, and

$$A_1 \le C_1(p)\|\varphi'\|_\infty^{p+\frac{1}{2}+\sigma}$$

(the constants C_1, C_2, etc. depend only on p). Obviously, $A_2 = C_2\|\varphi'\|_\infty^{\frac{1}{2}+\sigma}$.

Now assume that $p > 1$. Put $\delta = \sigma/p$. Then

$$I(x) = \int_{|t-x|>\varepsilon} \frac{|\Theta(t)-\Theta(x)|^q}{|t-x|^{q+1-\delta q}}\,dt = \int_{|t-x|>\varepsilon} \frac{|\Theta(t)-\Theta(x)|}{|t-x|^{2+\delta q}} \cdot \frac{|\Theta(t)-\Theta(x)|^{q-1}}{|t-x|^{q-1-2\delta q}}\,dt.$$

Applying estimate (12) and Lemma 5 (observe that $q - 1 - 2\delta q \ge 0$), we obtain

$$C_3(\varphi'(x))^{\frac{q-1}{2}-\delta q}\|\varphi'\|_\infty^{\frac{q-1}{2}-\delta q} \cdot \frac{\varphi'(x)}{\delta q}\|\varphi'\|_\infty^{\delta q} = C_3\frac{(\varphi'(x))^{\frac{q+1}{2}-\delta q}\|\varphi'\|_\infty^{\frac{q-1}{2}}}{\delta q}.$$

Therefore,

$$x(I(x))^{p/q} \le C_4(\varphi'(x))^{p-\frac{1}{2}-p\delta}\|\varphi'\|_\infty^{\frac{1}{2}}\delta^{-p/q},$$

whence

$$A_3 \le C_4\|\varphi'\|_\infty^{\frac{1}{2}}\delta^{-p/q}.$$

Finally, Proposition 1 implies the Bernstein-type inequality with the constant

$$C_5(\varepsilon^p A_1 + A_2 + \varepsilon^{-\delta p}(\delta p)^{-1}A_3) \le C_6\|\varphi'\|_\infty^{\frac{1}{2}+\sigma}(1+\sigma^{-p}) \le C_7\frac{\|\varphi'\|_\infty^{\frac{1}{2}+\sigma}}{\sigma^p}.$$

For $p = 1$ the proof is similar. □

As an application of Theorem 1, we give a new and shorter proof of the sufficiency of the condition $\Theta' \in C_0(\mathbb{R})$ in Dyakonov's theorem on the compactness of the operator D.

Proof. It is well known that a set $S \subset L^p$, $1 \le p < \infty$, is relatively compact if and only if S is bounded and satisfies the conditions

$$\sup_{f\in S}\int_{|x|>A} |f(x)|^p\,dx \to 0, \quad A \to \infty, \tag{13}$$

and

$$\sup_{f\in S}\|f(\cdot+h) - f\|_p \to 0, \quad h \to 0. \tag{14}$$

Now, let $S = \{f' : f \in K_\Theta^p, \|f\|_p \le 1\}$. Note that

$$f'(x+h) - f'(x) = \int_x^{x+h} f''(t)\,dt.$$

If $\Theta' \in L^\infty$, then the operator $f \mapsto f''$ is bounded as an operator from K_Θ^p to L^p, and $\|f''\|_p \le C(p)\|\Theta'\|_\infty^2\|f\|_p$ (see [4]). Hence,

$$\|f'(\cdot+h) - f'\|_p \le hC(p)\|\Theta'\|_\infty^2\|f\|_p,$$

and (14) follows. We show that inequality (4) and the condition $\lim_{|t|\to\infty}|\Theta'(t)| = 0$ imply (13). Fix $\sigma \in (0, 1/2)$. Then

$$\begin{aligned}\int_{|x|>A} |f'(x)|^p\,dx &\le \sup_{|x|>A} |\Theta'(x)|^{p-\frac{1}{2}-\sigma} \int_{|x|>A} \frac{|f'(x)|^p}{|\Theta'(x)|^{p-\frac{1}{2}-\sigma}}\,dx \\ &\le C \sup_{|x|>A} |\Theta'(x)|^{p-\frac{1}{2}-\sigma} \to 0, \quad A\to\infty. \qquad \square\end{aligned}$$

§4. Bernstein-type inequality for the functions that satisfy the connected level set condition

For the proof of Theorem 2 we shall use the following lemma, which is a particular case of a theorem due to A. B. Alexandrov (see [13]) (the latter characterizes the inner functions satisfying the connected level set condition).

Lemma 6. *Let Θ be an inner function satisfying the connected level set condition and such that $\Theta' \in L^\infty$. If Θ is not a finite Blaschke product, then*

$$\left|\frac{\Theta(t)-\Theta(x)}{t-x}\right| \le C|\Theta'(x)| \tag{15}$$

for any $t, x \in \mathbb{R}$, where $C = C(\Theta)$ is a positive constant.

Proof. We recall that the boundary spectrum $\rho(\Theta)$ of the function Θ is the set of all $\zeta \in \mathbb{R}\cup\{\infty\}$ such that $\liminf_{\operatorname{Im} z>0, z\to\zeta} |\Theta(z)| = 0$. The following result was stated in [13] (see also [16]): an inner function Θ satisfies the connected level set condition and $\infty \in \rho(\Theta)$ if and only if

$$\left|\frac{\Theta(z)-\Theta(w)}{z-w}\right| \le C(\Theta)\frac{1-|\Theta(z)|^2}{\operatorname{Im} z} \tag{16}$$

for all $z, w \in \mathbb{C}^+$.

The assumptions of the lemma ensure that $\infty \in \rho(\Theta)$. Thus, estimate (15) follows immediately from (16) (it remains to observe that $\lim_{z\to x}(1-|\Theta(z)|^2)/\operatorname{Im} z = 4\varphi'(x)$). $\square$

Proof of Theorem 2. Let I_j, $j = 1, 2, 3$, be the same integral operators as above; we put $\varepsilon = \|B'\|_\infty^{-1}$. Clearly,

$$A_1 \le C_1\|B'\|_\infty^{p+\sigma}, \quad A_2 = C_2\|\Theta'\|_\infty^{\sigma}.$$

Now, we estimate A_3. Put $\delta = \sigma/2p$. Then, by Lemma 6,

$$\begin{aligned}I(x) &= \int_{|t-x|\ge\varepsilon} \frac{|B(t)-B(x)|^q}{|t-x|^{q+1-\delta q}}\,dt \\ &\le C_3|B'(x)|^{q-2\delta q}\int_{|t-x|\ge\varepsilon}\frac{dt}{|t-x|^{1+\delta q}} = C_4\frac{|B'(x)|^{q-2\delta q}}{\delta\varepsilon^{\delta q}}.\end{aligned}$$

Consequently,

$$A_3 \le C_5\delta^{-p/q}\varepsilon^{-\delta p},$$

and Proposition 1 yields a Bernstein-type inequality in the space K_B^p with the constant

$$C_6(\varepsilon^p\|B'\|_\infty^{p+\sigma} + \|B'\|_\infty^\sigma + \delta^{-1-p/q}\varepsilon^{-2p\delta}) \le C_7\|B'\|_\infty^\sigma\sigma^{-p}. \qquad \square$$

Remarks. 1. The assumptions of Theorem 2 are fulfilled, in particular, for the Blaschke products with zeros $z_n = iy_n$, $n \in \mathbb{N}$, where $y_n = n^\alpha$, $\alpha > 1$, or $y_n = \gamma^n$, $\gamma > 1$.

2. Theorems 1 and 2 can be viewed as weighted estimates for the first order Calderón commutator C^1_ψ,

$$C^1_\psi f(x) = \text{P.V.} \int f(t) \frac{\psi(t)-\psi(x)}{(t-x)^2}\, dt, \quad x \in \mathbb{R},$$

where ψ is a complex-valued function of the real variable. We were interested in the case where $\psi = \Theta$ is inner. But, actually, we have used only some special properties of the function Θ, and the same arguments can easily be applied in a more general situation. We give the statement.

Theorem 6. *Let ψ be a twice differentiable function such that $\psi' \in L^\infty$ and $\psi''/\psi' \in L^\infty$. Assume that for some $\alpha \in (0,1]$ we have*

$$\sup_{t\in\mathbb{R}} \left| \frac{\psi(t)-\psi(x)}{t-x} \right| \le C_1 |\psi'(x)|^\alpha, \quad x \in \mathbb{R},$$

and also that

$$\int_{|t-x|>\varepsilon} \frac{|\psi(t)-\psi(x)|^{1+2\delta}}{|t-x|^{2+\delta}}\, dt \le C_2 |\psi'(x)|, \quad x \in \mathbb{R},$$

for some $\varepsilon > 0$ and all sufficiently small $\delta > 0$. Then for $1 < p < \infty$ we have

$$\int_{\mathbb{R}} |C^1_\psi f(x)|^p |\psi'(x)|^{-\beta} dx \le C_3 \|f\|_p^p, \quad f \in L^p(\mathbb{R}),$$

for any $\beta < p + \alpha - 1$.

Remark. In Theorems 1 and 2 the exponent α is equal to $1/2$ (see Lemma 2) and to 1 (see Lemma 6), respectively.

§5. PROOF OF THEOREM 3

The main idea of the proof is to individually estimate the integrals over the set where the function $|\Theta'|$ is large and over the set where the function $|\Theta'|$ is sufficiently small (in fact, this was already done in Theorem 1).

Lemma 7. *Suppose Θ is a meromorphic inner function, and $M = \inf_n y_n > 0$, $1 < p < \infty$. Set $E = \{t \in \mathbb{R} : \varphi'(t) \ge M^{-1}\}$. There is a constant $C > 0$ depending on p (but not on Θ and M) such that*

$$\sup_{s\in\mathbb{R}} \int_E \left| \frac{\Theta(t)-\Theta(s)}{t-s} \right|^p \frac{dt}{(\varphi'(t))^{p-1}} \le C.$$

Proof. First, we note that

$$\int_{E\cap\{|t-s|>M\}} \left| \frac{\Theta(t)-\Theta(s)}{t-s} \right|^p \frac{dt}{(\varphi'(t))^{p-1}} \le 2^p M^{p-1} \int_{|t-s|>M} \frac{dt}{|t-s|^p} \le \frac{2^{p+1}}{p-1}.$$

If $|s-t| \le M$, $t \in E$, then $\varphi'(s) \ge \frac{\varphi'(t)}{4} \ge \frac{1}{4M}$ by Lemma 1. Consequently,

$$\int_{|t-s|<(4\varphi'(s))^{-1}} \left| \frac{\Theta(t)-\Theta(s)}{t-s} \right|^p \frac{dt}{(\varphi'(t))^{p-1}}$$
$$\le \frac{1}{2\varphi'(s)} \cdot \sup_{|t-s|\le M} \frac{1}{(\varphi'(t))^{p-1}} \cdot \sup_{|t-s|\le M} \left| \frac{\Theta(t)-\Theta(s)}{t-s} \right|^p \le 2^{3p-1}.$$

The last inequality follows from Lemma 1 and the estimate $\left|\frac{\Theta(t)-\Theta(s)}{t-s}\right| \le \sup_{\langle s,t\rangle} |\Theta'(u)|$.

Finally, applying Lemma 1 once again, we get

$$\int_{(4\varphi'(s))^{-1}\le|t-s|\le M}\left|\frac{\Theta(t)-\Theta(s)}{t-s}\right|^p\frac{dt}{(\varphi'(t))^{p-1}}$$
$$\le \sup_{|t-s|\le M}\frac{1}{(\varphi'(t))^{p-1}}\cdot\frac{2^{p+1}(4\varphi'(s))^{p-1}}{p-1}\le\frac{2^{5p-3}}{p-1}. \qquad \square$$

Proof of Theorem 3. Put

$$E_1=\{t\in\mathbb{R}:\varphi'(t)<M^{-1}\},\quad E_2=\{t\in\mathbb{R}:\varphi'(t)\ge M^{-1}\}.$$

Repetition of the arguments used in the proof of Theorem 1 readily shows that for any $\sigma\in(0,1/2]$ we have

$$\int_{E_1}\frac{|f'(t)|^p}{|\Theta'(t)|^{p-\frac12-\sigma}}\,dt\le C(p)M^{-\frac12-\sigma}\sigma^{-p}\|f\|_p^p,\quad f\in K_\Theta^p.$$

The only difference is that we must take $\varepsilon=M$ and use (11) in place of (12). We are going to show that

$$\int_{E_2}\frac{|f'(t)|^p}{|\Theta'(t)|^p}\,dt\le C(p)\|f\|_p^p,\quad f\in K_\Theta^p.$$

As in the proof of Proposition 1, we split the integral (9) (which represents the derivative f') into three parts:

$$2\pi i f'(x)=I_1g(x)+\Theta'(x)I_2g(x)+I_3g(x)$$

with $g=\overline{\Theta}f$, but this time the integration limits will depend on x:

$$I_1g(x)=\int_{|t-x|<1/\varphi'(x)}g(t)\Phi(t,x)\,dt,\quad I_2g(x)=\int_{|t-x|\ge1/\varphi'(x)}\frac{g(t)}{x-t}\,dt,$$

and

$$I_3g(x)=\int_{|t-x|\ge1/\varphi'(x)}g(t)\frac{\Theta(t)-\Theta(x)}{(t-x)^2}\,dt.$$

We verify that $I_1g/|\Theta'|\in L^p(E_2)$, $1<p<\infty$. By Lemma 3,

$$|\Phi(t,x)|=|k_x'(t,x)|\le C[(\varphi'(x))^2+\varphi'(x)/M]\le 2C(\varphi'(x))^2$$

whenever $|t-x|\ge M$ and $x\in E_2$. Consequently,

$$\left|\frac{I_1g(x)}{\Theta'(x)}\right|\le C\varphi'(x)\int_{|t-x|<1/\varphi'(x)}|g(t)|\,dt$$
$$\le C\sup_{h>0}\frac1h\int_{|t-x|<h}|g(t)|\,dt=Cm_g(x),\quad x\in E_2.$$

It is well known that the maximal transformation $g\mapsto m_g$ is a bounded operator in L^p for $1<p<\infty$ (see [1, 24]). Thus, $\|I_1g/\Theta'\|_{L^p(E_2)}\le C(p)\|g\|_p$. A similar estimate for $\Theta'I_2g$ is obvious because the maximal Hilbert transformation is bounded in L^p.

Now we check that $I_3g/|\Theta'|\in L^p(E_2)$. By the Hölder inequality,

$$\int_{E_2}\left|\frac{I_3g(x)}{\varphi'(x)}\right|^p dx=\int_{E_2}\left|\int_{|t-x|\ge1/\varphi'(x)}\frac{\Theta(t)-\Theta(x)}{\varphi'(x)(t-x)^2}g(t)\,dt\right|^p dx$$
$$\le\int_{E_2}\left(\int_{|t-x|\ge1/\varphi'(x)}\left|\frac{\Theta(t)-\Theta(x)}{t-x}g(t)\right|^p\frac{dt}{(\varphi'(x))^{p-1}}\right)$$
$$\times\left(\int_{|t-x|\ge1/\varphi'(x)}\frac{dt}{(\varphi'(x))^{q-1}|t-x|^q}\right)^{p/q}dx.$$

Thus,

$$\int_{E_2}\left|\frac{I_3g(x)}{\varphi'(x)}\right|^p dx \le \left(\frac{2}{q-1}\right)^{p/q}\int_{E_2}\int_{\mathbb{R}}\left|\frac{\Theta(t)-\Theta(x)}{t-x}\right|^p \frac{|g(t)|^p}{(\varphi'(x))^{p-1}}\,dt\,dx.$$

Changing the order of integration in the double integral and applying Lemma 7, we get the inequality $\|I_3g/\Theta'\|_p \le C(p)\|g\|_p$. □

Examples. 1. Generally speaking, the term $|\Theta'|^p$ in w^{-1} cannot be replaced by $|\Theta'|^\alpha$ with $\alpha < p$, and so the exponent p is sharp (to see this, it suffices to consider the integral of the function $|k(\cdot,s)'|^p$ over the set $[s, s+\varphi'(s)^{-1}]$).

2. The condition $\inf_n y_n > 0$ is also essential. Let $z_n = x_n + iy_n$, where $x_n \to \infty$ and $y_n \to 0$ so fast that

$$\varphi'(t) \asymp \frac{y_n}{|t-z_n|^2}, \quad |t-x_n| \le 1.$$

Then $|\Theta'(t)| \asymp 1$, $t \in [x_n+\sqrt{y_n}, x_n+2\sqrt{y_n}]$. Put $f_n(z) = y_n^{1-1/p}(z-\overline{z}_n)^{-1}$ and observe that

$$\int_{x_n+\sqrt{y_n}}^{x_n+2\sqrt{y_n}} |f_n'(t)|^p\,dt = \int_{x_n+\sqrt{y_n}}^{x_n+2\sqrt{y_n}} \frac{y_n^{p-1}dt}{|t-z_n|^{2p}} \asymp y_n^{-1/2} \to \infty.$$

Consequently, $\|f_n'/[|\Theta'|^\alpha + |\Theta'|^\beta]\|_p \to \infty$ for any α and β, whereas $\|f_n\|_p \asymp 1$.

§6. WEIGHTED BERNSTEIN-TYPE INEQUALITIES AND EMBEDDING THEOREMS

We show that each weighted Bernstein-type inequality yields an embedding theorem for the corresponding model subspace.

Proposition 2. *Let Θ be a meromorphic inner function, let $1 \le p < \infty$, and let w be a positive function such that*

$$\int_{\mathbb{R}} |f'(t)|^p w(t)\,dt \le A\int_{\mathbb{R}} |f(t)|^p\,dt \tag{17}$$

for any $f \in K_\Theta^p$. Suppose μ is a Borel measure on the real line $\mathbb{R}$, and $\mathbb{R} = \bigcup_k I_k$, where the intervals I_k are disjoint and satisfy the condition

$$\rho = \sup_k |I_k|\left(\int_{I_k} (w(t))^{-\frac{1}{p-1}}\,dt\right)^{p-1} < \infty$$

for $1 < p < \infty$, $\rho = \sup_k[|I_k| \sup_{t\in I_k} w^{-1}(t)] < \infty$ for $p = 1$. Then:

1) *if $\mu(I_k) \le C|I_k|$, then $\mu \in \mathcal{C}_p(\Theta)$;*

2) *if $\mu(I_k) = o(|I_k|)$, then K_Θ^p embeds in $L^p(\mu)$ compactly;*

3) *if $A\rho < 1$ and $\mu(I_k) \asymp |I_k|$ (i.e., $C_1|I_k| \le \mu(I_k) \le C_2|I_k|$), then $\|f\|_{L^p(\mu)} \asymp \|f\|_p$ for $f \in K_\Theta^p$.*

Proof. 1) Fix $f \in K_\Theta^p$. Since the function f is continuous, the mean value theorem implies the existence of points $t_k \in I_k$ such that

$$\int_{I_k} |f(t)|^p\,d\mu(t) = \mu(I_k)|f(t_k)|^p.$$

Let $g(t) = f(t_k)$, $t \in I_k$. Then

$$\|f\|_{L^p(\mu)}^p = \sum_k \mu(I_k)|f(t_k)|^p \le C\sum_k |I_k|\cdot|f(t_k)|^p = C\|g\|_p^p.$$

Hence,

$$\|f\|_{L^p(\mu)} \le C^{1/p}\|g\|_p \le C^{1/p}(\|g-f\|_p + \|f\|_p).$$

We estimate $\|g-f\|_p$:

$$\|g-f\|_p^p = \sum_k \int_{I_k} |f(t)-f(t_k)|^p\,dt = \sum_k \int_{I_k} \left| \int_{t_k}^t f'(s)\,ds \right|^p dt.$$

By the Hölder inequality (here q is the exponent conjugate to p),

$$\begin{aligned}\|g-f\|_p^p &\le \sum_k \int_{I_k} \left(\int_{t_k}^t |f'(s)|^p w(s)\,ds \right) \|w^{-1/p}\|^p_{L^q(\langle t_k,t\rangle)}\,dt \\ &\le \sum_k |I_k|\cdot \|w^{-1/p}\|^p_{L^q(I_k)} \left(\int_{I_k} |f'(s)|^p w(s)\,ds \right).\end{aligned}$$

Since $\rho = \sup_k |I_k|\cdot\|w^{-1/p}\|_{L^q(I_k)}$, we have

$$\|g-f\|_p^p \le \rho \sum_k \int_{I_k} |f'(s)|^p w(s)\,ds \le \rho A \|f\|_p^p.$$

The last inequality follows from the Bernstein-type inequality (17). Statement 1) is proved.

2) We show that the identity operator Id acting from K^p_Θ to $L^p(\mu)$ can be approximated by finite rank operators. Fixing $N\in\mathbb{N}$ and $\varepsilon>0$, we put $J_N = \bigcup_{|k|\le N} I_k$. Then there exists a finite set of disjoint intervals $\{\Delta_l\}_{l=1}^L$ such that $|\Delta_l|\le\varepsilon$ and $J_N = \bigcup_{l=1}^L \Delta_l$. We fix some points $t_l\in\Delta_l$ and consider the operator

$$T_N f = \sum_{l=1}^L f(t_l)\chi_{\Delta_l},$$

where χ_{Δ_l} is the characteristic function of Δ_l. We prove that $\|T_N f - f\|_{L^p(\mu)} = o(\|f\|_p)$, $f\in K^p_\Theta$, as $N\to\infty$, $\varepsilon\to+0$. First, we note that

$$\|T_N f - f\|^p_{L^p(\mu)} = \int_{J_N} |T_N f - f|^p\,d\mu + \int_{\mathbb{R}\setminus J_N} |f|^p\,d\mu. \tag{18}$$

Clearly,

$$\int_{J_N} |T_N f(t) - f(t)|^p\,d\mu(t) = \sum_l \int_{\Delta_l} \left| \int_{t_l}^t f'(s)\,ds \right|^p d\mu(t) \le \mu(J_N)\varepsilon^p \sup_{s\in J_N} |f'(s)|^p.$$

By formula (9), $|f'(s)| \le \|f\|_p \|\Phi(\cdot,s)\|_q$. Consequently,

$$\int_{J_N} |T_N f - f|^p\,d\mu \le \mu(J_N)\varepsilon^p \sup_{s\in J_N} \|\Phi(\cdot,s)\|_q^p \|f\|_p^p.$$

Since Θ is meromorphic, the kernel Φ is continuous on $\mathbb{R}^2$. It is easily seen that $|\Phi(t,s)| \le (|\Theta'(s)|+2)|t-s|^{-1}$ if $|t-s|\ge 1$. Therefore, $\sup_{s\in I} \|\Phi(\cdot,s)\|_q < \infty$ for any bounded set $I\subset\mathbb{R}$.

Now we estimate the second term in (18). By the mean value theorem,

$$\int_{\mathbb{R}\setminus J_N} |f|^p\,d\mu = \sum_{|k|>N} \mu(I_k)|f(s_k)|^p,$$

where $s_k\in I_k$. Applying the arguments used in the proof of statement 1), we obtain

$$\int_{\mathbb{R}\setminus J_N} |f|^p\,d\mu \le \sup_{|k|>N} \frac{\mu(I_k)}{|I_k|} \sum_{|k|>N} |I_k||f(s_k)|^p \le \sup_{|k|>N} \frac{\mu(I_k)}{|I_k|} (1+(A\rho)^{1/p})^p \|f\|_p^p.$$

Thus, choosing N sufficiently large and then reducing ε for N fixed, we can make the norm of the operator $T_N - Id : K^p_\Theta \to L^p(\mu)$ arbitrarily small.

3) If $\mu(I_k) \asymp |I_k|$ and $A\rho < 1$, then

$$\|f\|_{L^p(\mu)} \ge C_1^{1/p}\|g\|_p \ge C_1^{1/p}(\|f\|_p - \|g-f\|_p) \ge C_1^{1/p}(1-(\rho A)^{1/p})\|f\|_p.$$

Hence, $\|f\|_{L^p(\mu)} \asymp \|f\|_p$. □

Proposition 2 and Theorems 1 and 3 immediately yield Theorems 4 and 4′ (and also an analog of Theorem 4 for $p = 1$). It suffices to apply Proposition 2 to the weights $w(t) = |\Theta'(t)|^{-(p-1/2-\sigma)}$ and $w(t) = [|\Theta'(t)|^{p-1/2-\sigma} + |\Theta'(t)|^p]^{-1}$.

Remark. Combined with Theorems 1 and 3, Proposition 2 gives also sufficient conditions for the equivalence of the L^p-norm generated by a measure μ and the natural norm of the space K^p_Θ. Such measures were completely described by Volberg [19] in the case where $1 < p < \infty$ and the measures are of the form vm, where m is Lebesgue measure on the line and v is nonnegative and bounded: $\|f\|_{L^p(vm)} \asymp \|f\|_p$, $f \in K^p_\Theta$, if and only if

$$\inf_{z\in\mathbb{C}^+}(|\Theta(z)| + \tilde{v}(z)) > 0, \tag{19}$$

where $\tilde{v}$ denotes the harmonic extension of the function v to the upper half-plane. It is known that for $p = 1$ condition (19) is not sufficient for the equivalence of the corresponding norms; see [27]. The description of the general measures generating an equivalent norm is a very difficult problem. Its solution for the case of the classical Paley–Wiener space was found quite recently by Ortega-Cerda and Seip [28].

The case of $\mu = \chi_E m$, where χ_E is the characteristic function of a set $E \subset \mathbb{R}$, is of special interest. It is well known that the measure $\chi_E m$ determines an equivalent norm on the Paley–Wiener space PW^p_a if and only if the set E is relatively dense, that is, there exist numbers $L, \delta > 0$ such that $m(E \cap \Delta) \ge \delta$ for any interval Δ of length at most L (see [20]). In this case, the measure $\chi_E m$ determines an equivalent norm on each model subspace K^p_Θ with $\Theta' \in L^\infty$ (see [4]).

Now we state a condition ensuring the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_{L^1(\chi_E m)}$ on the space K^1_Θ; this condition follows from Theorem 1 and Proposition 2.

Corollary. *Suppose $\Theta' \in L^\infty$, $E \subset \mathbb{R}$, and I_k are disjoint intervals such that $\mathbb{R} = \bigcup_k I_k$ and*

$$\rho = \sup_k \Big[|I_k| \sup_{t\in I_k} |\Theta'(t)|^{\frac{1}{2}-\delta}\Big] < \infty$$

for some $\delta \in (0, 1/2)$. There exists an absolute constant ρ_0 such that if $\|\Theta'\|_\infty^{\frac{1}{2}+\delta}\delta^{-1}\rho < \rho_0$ and $m(E \cap I_k) \asymp |I_k|$, then

$$\int_E |f(t)|\,dt \asymp \|f\|_1, \quad f \in K^1_\Theta.$$

We conclude this section with examples of functions Θ and measures that satisfy the conditions of Theorem 4, but do not belong to the class $\mathcal{C}(\Theta)$; so, the Volberg–Treil theorem is not applicable to these measures. We shall consider Blaschke products with sufficiently sparse zeros, that is, essentially "multicomponent" functions. Let $\mathcal{M}_p(\Theta, \delta)$ denote the class of measures satisfying the conditions of Theorem 4. We show that there exist inner functions Θ such that $\mathcal{C}(\Theta)|_\mathbb{R} \subset \mathcal{M}_p(\Theta, \delta)$ for some $\delta > 0$ (and $\mathcal{C}(\Theta)|_\mathbb{R} \ne \mathcal{M}_p(\Theta, \delta)$). Thus, Theorem 4 provides nontrivial examples of embeddings.

Examples. 1. Let B be a Blaschke product with zeros $z_k = x_k + i$, $k \in \mathbb{Z}$, such that $\lim_{|k|\to\infty}(x_{k+1} - x_k) = \infty$. We denote by $x(t)$, $t \in \mathbb{R}$, the point of the form x_k that is nearest to the point t. We assume that, in a sense, the summand corresponding to the nearest zero brings the main contribution to $\varphi'(t)$. Namely, let

$$\varphi'(t) \le \frac{C}{|t - x(t) + i|^\alpha}, \quad t \in \mathbb{R}, \tag{20}$$

where $1 < \alpha \le 2$ (clearly, α does not exceed 2). For instance, if $x_k = |k|^\beta \operatorname{sgn} k$, $\beta > 1$, then estimate (20) is fulfilled for $\alpha \le 2$, and if $x_k = 2^{|k|} \operatorname{sgn} k$, then (20) is true for any $\alpha < 2$.

We prove the following statement: if $\alpha > \frac{2q}{q+1}$, then there is $\delta > 0$ such that $\mathcal{C}(B)|_{\mathbb{R}} \subset \mathcal{M}_p(B,\delta)$.

Let $\delta > 0$ be such that $\delta < \frac{q+1}{2} - \frac{q}{\alpha}$. We split the interval $[x_k, (x_k + x_{k+1})/2]$ into smaller intervals $I_n = [a_n, a_{n+1}]$ satisfying

$$\sup_n |I_n| \biggl(\int_{I_n} |B'(t)|^{\frac{q+1}{2} - \delta} \, dt \biggr)^{p/q} < \infty \tag{21}$$

and such that $a_{n+1} - a_n \asymp a_{n+1} - x_k$. Let $a_0 = x_k$, $a_1 = x_k + 2$. If the points $a_0, a_1, \dots, a_n$ have already been chosen, we put $a_{n+1} - a_n = (a_n - x_k)^{(\alpha \frac{q+1}{2} - \alpha\delta - 1)\frac{p}{q}}$. We repeat this procedure as long as $a_{n+1} \le (x_k + x_{k+1})/2$. Observe that, by the choice of δ, we have

$$\Bigl(\alpha \frac{q+1}{2} - \alpha\delta - 1\Bigr)\frac{p}{q} > 1,$$

whence $a_{n+1} - a_n \asymp a_{n+1} - x_k$. We verify condition (21). Indeed,

$$\begin{aligned} &(a_{n+1} - a_n) \biggl(\int_{a_n}^{a_{n+1}} |B'(t)|^{\frac{q+1}{2} - \delta} \, dt \biggr)^{p/q} \\ &\quad \le C (a_{n+1} - a_n) \biggl(\Bigl(\frac{1}{a_n - x_k} \Bigr)^{\alpha \frac{q+1}{2} - \alpha\delta - 1} - \Bigl(\frac{1}{a_{n+1} - x_k} \Bigr)^{\alpha \frac{q+1}{2} - \alpha\delta - 1} \biggr)^{p/q} \le C. \end{aligned}$$

Similarly, moving to the left from the point x_{k+1}, we construct a partition of the interval $[(x_k + x_{k+1})/2, x_{k+1}]$, and repeating this procedure for any k yields a partition of the entire real line into intervals I_n satisfying (21).

Now, suppose that $\mu \in \mathcal{C}(B)|_{\mathbb{R}}$. Then

$$\mu([a_n, a_{n+1}]) \le \mu([x_k, a_{n+1}]) \le C_1 (a_{n+1} - x_k) \le C_2 (a_{n+1} - a_n).$$

Thus, each measure μ of class $\mathcal{C}(B)|_{\mathbb{R}}$ satisfies the Carleson condition for the intervals I_n, that is, $\mu \in \mathcal{M}_p(B,\delta)$.

2. Let $z_k = x_k + i$ be the same as in Example 1, and assume that estimate (20) is true with $\alpha = 2$. We choose a sequence $t_k > 0$ such that $t_k \to \infty$, but "much more slowly" than $x_{k+1} - x_k$. Namely, suppose that $x_k + 2t_k^p < x_{k+1}$. Put $\mu = \sum_k a_k \delta_{x_k + t_k}$. Obviously, if $\mu \in \mathcal{C}(B)$, then $a_k \le C_1 t_k$. Note also that the relation $\mu \in \mathcal{C}_p(B)$ means that $a_k \le C_2 t_k^p \le C_3 (\varphi'(t_k))^{-p/2}$ (it suffices to consider $\|k(\cdot, x_k + t_k)\|_{L^p(\mu)}$).

We show that for any $\sigma > 0$ the measure $\mu = \sum_k t_k^{p-\sigma} \delta_{x_k + t_k}$ belongs to the class $\mathcal{M}_p(B,\delta)$. Indeed, let $I_k = [x_k + t_k, x_k + t_k^p]$. Then $\mu(I_k) = t_k^{p-\sigma} \le C_2 |I_k|$ and

$$\begin{aligned} \sup_k |I_k| \biggl(\int_{I_k} |\Theta'(t)|^{\frac{q+1}{2} - \frac{q\sigma}{2p}} \, dt \biggr)^{p/q} &\le C_4 \sup_k t_k^{p-\sigma} \biggl(\int_{t_k}^{t_k^p} t^{-q-1+\frac{q\sigma}{p}} \, dt \biggr)^{p/q} \\ &\le C_4 (q + 1 - 2q\sigma/p)^{-p/q} < \infty. \end{aligned}$$

Thus, the measure μ satisfies the conditions of Theorem 4, and consequently, $\mu \in \mathcal{C}_p(B)$, but $\mu \notin \mathcal{C}(B)$.

§7. PERTURBATIONS OF BASES OF REPRODUCING KERNELS

The assertion to be proved here is somewhat more general than Theorem 5 and concerns the case of meromorphic inner functions with zeros separated away from the real axis.

Theorem 5′. *Let* Θ *be a meromorphic inner function with zeros* $\{z_n\}$ *such that* $\inf_n y_n > 0$. *Assume that* $s_k, t_k \in \mathbb{R}$ *and* $\inf_k(\varphi(t_{k+1}) - \varphi(t_k)) > 0$. *If*

$$d(\{s_k\}) = \sup_k \int_{\langle s_k, t_k\rangle} (|\Theta'(t)|^\delta + |\Theta'(t)|)\,dt < \infty \tag{22}$$

for some $\delta < 1/2$, *then the measures* $\mu = \sum_k \delta_{t_k}/\varphi'(t_k)$ *and* $\nu = \sum_k \delta_{s_k}/\varphi'(s_k)$ *simultaneously belong or do not belong to the class* $\mathcal{C}_2(\Theta)$.

Furthermore, there is $\varepsilon = \varepsilon(\Theta, \delta)$ *such that the norms* $\|\cdot\|_{L^2(\nu)}$ *and* $\|\cdot\|_2$ *are equivalent on* K^2_Θ *whenever* $\{k(\cdot, t_k)\}$ *is an orthogonal basis in* K^2_Θ *and* $d(\{s_k\}) < \varepsilon$.

Proof. Without loss of generality, we assume that $d(\{s_k\}) \le 1$. First, we note that there exist constants C_1 and C_2 depending on Θ and δ such that $C_1 \le \varphi'(s_k)/\varphi'(t) \le C_2$, $t \in \langle\, s_k, t_k\rangle$. Indeed, by the mean value theorem,

$$\int_{\langle s_k, t_k\rangle} ((\varphi'(t))^\delta + \varphi'(t))\,dt = |s_k - t_k|[(\varphi'(x_k))^\delta + \varphi'(x_k)],$$

where $x_k \in \langle s_k, t_k\rangle$. Hence, $|s_k - t_k| \le C_3(\varphi'(x_k))^{-\delta}$. If $C_3(\varphi'(x_k))^{-\delta} \le \sqrt{M/4\varphi'(x_k)}$, then, by Lemma 1, $\varphi'(t) \asymp \varphi'(x_k)$ for $t \in \langle\, s_k, t_k\rangle$. Otherwise, we have $\varphi'(x_k) \ge C_4$, which yields $\varphi'(t) \asymp \varphi'(x_k)$ once again. Thus,

$$\|f\|^2_{L^2(\nu)} \asymp \sum_k \frac{|f(s_k)|^2}{\varphi'(t_k)}.$$

Furthermore,

$$\begin{aligned}\frac{|f(t_k) - f(s_k)|^2}{\varphi'(t_k)} &= \frac{1}{\varphi'(t_k)}\left|\int_{\langle s_k, t_k\rangle} f'(t)\,dt\right|^2 \\ &\le \int_{\langle s_k, t_k\rangle} [(\varphi'(t))^\delta + \varphi'(t)]\,dt \int_{\langle s_k, t_k\rangle} \frac{|f'(t)|^2}{\varphi'(t_k)[(\varphi'(t))^\delta + \varphi'(t)]}\,dt \\ &\le C_5 d(\{s_k\}) \int_{\langle s_k, t_k\rangle} \frac{|f'(t)|^2}{(\varphi'(t))^{\delta+1} + (\varphi'(t))^2}\,dt.\end{aligned}$$

Now, since $\delta + 1 < 3/2$, we can apply Theorem 3:

$$\int_{\mathbb{R}} \frac{|f'(t)|^2}{(\varphi'(t))^{\delta+1} + (\varphi'(t))^2}\,dt \le C_6\|f\|_2^2.$$

Since $\inf_k(\varphi(t_{k+1}) - \varphi(t_k)) > 0$, from (22) it follows that there is a positive integer N such that any point $x \in \mathbb{R}$ lies in at most N intervals $\langle s_k, t_k\rangle$. Therefore,

$$\begin{aligned}\sum_k \frac{|f(t_k) - f(s_k)|^2}{\varphi'(t_k)} &\le C_5 d(\{s_k\}) \sum_k \int_{\langle s_k, t_k\rangle} \frac{|f'(t)|^2}{(\varphi'(t))^{\delta+1} + (\varphi'(t)^2)}\,dt \\ &\le C_7 d(\{s_k\}) N \|f\|_2^2.\end{aligned}$$

Thus, $\nu \in \mathcal{C}_2(\Theta)$ if and only if $\mu \in \mathcal{C}_2(\Theta)$. Now, if $\{k(\cdot, t_k)\}$ is a basis in K^2_Θ (that is, $2\pi\|f\|^2_{L^2(\mu)} = \|f\|_2^2$), then

$$\|f\|^2_{L^2(\nu)} \ge C_8(1 - C_9 d(\{s_k\}))\|f\|_2^2, \quad f \in K^2_\Theta.$$

So, the norms $\|\cdot\|_{L^2(\nu)}$ and $\|\cdot\|_2$ are equivalent on K^2_Θ if $d(\{s_k\})$ is sufficiently small. □

Remark. In terms of the kernels $k(\cdot, s_k)$ the equivalence of the norms $\|\cdot\|_{L^2(\nu)}$ and $\|\cdot\|_2$ means that the system $\{k(\cdot, s_k)/\|k(\cdot, s_k)\|_2\}$ forms a frame in the space K^2_Θ (we recall that a system $\{f_n\}$ in a Hilbert space H is called a *frame* if $C_1\|f\|_H^2 \le \sum_n |\langle f, f_n\rangle_H|^2 \le C_2\|f\|_H^2$, $f \in H$, for some positive constants C_1 and C_2).

Example. We show that for $\delta > 1/2$ condition (22) in Theorem 5 does not imply that $\nu \in \mathcal{C}_2(\Theta)$.

Let Θ be a Blaschke product with the zeros $z_n = 2^n + i$, $n \in \mathbb{N}$. The sequence $\{z_n\}$ satisfies the Carleson interpolation condition. Consequently, by the Shapiro–Shields theorem (see [1, 2]), any function f in the space K^2_Θ can be represented as an unconditionally convergent series $f(z) = \sum_n \frac{c_n}{z - \overline{z}_n}$, and $\|f\|_2^2 \asymp \sum_n |c_n|^2$. So, to verify that $\nu \in \mathcal{C}_2(\Theta)$, we must study the boundedness of a certain operator in $\ell^2(\mathbb{N})$ defined by an infinite matrix.

Fix $\delta > 1/2$ and set $t_n = 2^n + 2^{\frac{n}{2\delta}}$. It is easy to show that $\mu = \sum_n \delta_{t_n}/\varphi'(t_n) \in \mathcal{C}_2(\Theta)$. It suffices to note that $\varphi'(t_n) \asymp 2^{-n/\delta}$ and to make sure that the matrix $a_{nk} = 2^{\frac{n}{2\delta}}(2^n + 2^{\frac{n}{2\delta}} - 2^k + i)^{-1}$ gives rise to a bounded operator in $\ell^2(\mathbb{N})$ (the sums over the rows and the columns are uniformly bounded).

Put $s_n = 2^n + 2^{n-1}$. Condition (22) is fulfilled because

$$\varphi'(t) \asymp \frac{1}{(t - 2^n)^2} + \frac{n}{2^{2n}}, \quad t \in [t_n, s_n].$$

At the same time, $\nu = \sum_n \delta_{s_n}/\varphi'(s_n) \notin \mathcal{C}_2(\Theta)$ (see [12, Example 6.1]).

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Received 6/MAR/2003

Translated by THE AUTHOR