Алгебра и анализ
Том. 15 (2003), вып. 5

St. Petersburg Math. J.
Vol. 15 (2004), No. 5, Pages 753–771
S 1061-0022(04)00830-1

INVERSION THEOREMS FOR THE LOCAL POMPEIU TRANSFORMATION IN THE QUATERNION HYPERBOLIC SPACE

VIT. V. VOLCHKOV AND N. P. VOLCHKOVA

Abstract. A construction for inversion of the local Pompeiu transformation is obtained for the family consisting of two geodesic balls on the quaternion hyperbolic space.

§1. Introduction

Notation. We denote by $\mathbb{R}^n$ the real Euclidean space of dimension $n \geq 2$ with Euclidean norm $|\cdot|$, by $\mathcal{M}(n)$ the group of isometries of $\mathbb{R}^n$, and by dx Lebesgue measure on $\mathbb{R}^n$. Consider a collection $\mathcal{F} = (E_1, \dots, E_k)$ of compact subsets in $\mathbb{R}^n$ of positive measure. The (global) Pompeiu transformation $\mathcal{P}_{\mathcal{F}} : C(\mathbb{R}^n) \to C(\mathcal{M}(n))^k$ is defined by the formula

$$\mathcal{P}_{\mathcal{F}} f = (f_1, \dots, f_k), \tag{1.1}$$

where

$$f_i(g) = \int_{gE_i} f(x)\, dx, \quad g \in \mathcal{M}(n), \ \ i = 1, \dots, k$$

(see, e.g., [1, formulas (2.1) and (2.2)]). Similarly, for an open set $U \subset \mathbb{R}^n$, formula (1.1) determines the *local* Pompeiu transformation

$$\mathcal{P}_{\mathcal{F},U} : C\left(U\right) \to C\left(\mathcal{G}_1\right) \times \dots \times C\left(\mathcal{G}_k\right),$$

where $\mathcal{G}_i = \{g \in \mathcal{M}(n) : gE_i \subset U\}$, $i = 1, \dots, k$.

For given $\mathcal{F}$ and U, the following problem arises:

Problem ([2])**.** 1. To find out whether the transformation $\mathcal{P}_{\mathcal{F},U}$ is injective, and if not, to describe its kernel.

2. If the mapping $\mathcal{P}_{\mathcal{F},U}$ is injective, then to find the inverse mapping.

Known results. For individual $\mathcal{F}$ and U, the injectivity of the Pompeiu transformation and related questions were studied in many papers (see the surveys [2]–[4] with extensive bibliography, and also [5]). The most complete results were obtained for the family $\mathcal{F} = \left(\overline{\mathcal{B}}_{r_1}, \overline{\mathcal{B}}_{r_2}\right)$, where

$$\mathcal{B}_{r_i} := \{x \in \mathbb{R}^n : |x| < r_i\}$$

and $\overline{\mathcal{B}}_{r_i}$ is the closure of $\mathcal{B}_{r_i}$, $i = 1, 2$. In this case, the transformation (1.1) is injective if and only if

$$\frac{r_1}{r_2} \notin \mathcal{E}_n := \left\{ \frac{\lambda_1}{\lambda_2} \;\middle|\; \lambda_i > 0,\ J_{n/2}(\lambda_i) = 0,\ i = 1, 2 \right\},$$

where $J_{n/2}$ is the Bessel function [6].

2000 *Mathematics Subject Classification.* Primary 44A15, 53C65.

Key words and phrases. Quaternionic hyperbolic space, Pompeiu transformation, theorem on two radii.

Partly supported by grant no. 01.07/00241 from the Foundation for Basic Research of Ukraine.

Furthermore, if elements of $\mathcal{E}_n$ badly approximate r_1/r_2, i.e., for some positive constants c_1 and c_2 we have the estimate

$$\left|\frac{r_1}{r_2}-\frac{\lambda_1}{\lambda_2}\right|\geq\frac{c_1}{(1+\lambda_2)^{c_2}},$$

then there exist compactly supported (and explicitly constructed) distributions ν_1 and ν_2 such that

$$f=(f*\chi_{r_1})*\nu_1+(f*\chi_{r_2})*\nu_2,$$

where χ_{r_i} is the indicator function of $\mathcal{B}_{r_i}$, $i=1,2$ (see [7] and also [1, §3]).

Thus, the function f can be recovered by the convolutions $f*\chi_{r_1}$ and $f*\chi_{r_2}$.

Local results. For $\mathcal{F}=\left(\overline{\mathcal{B}}_{r_1},\overline{\mathcal{B}}_{r_2}\right)$ and $U=\mathcal{B}_R$, where $R>\max(r_1,r_2)$, the above questions are considerably more difficult. This is related to the violation of the group structure acting on the set of solutions of the equation $\mathcal{P}_{\mathcal{F}}f=0$. Here, new effects arise: the answer to Problem 1 depends to a large extent not only on the nature of the numbers r_1 and r_2, but also on the size of the ball $\mathcal{B}_R$. Berenstein, Gay, and Yger [1, 8] and Volchkov [9] proved the injectivity of $\mathcal{P}_{\mathcal{F},U}$ by different methods under the condition that $r_1/r_2\notin\mathcal{E}_n$ and $r_1+r_2<R$. In this case, the construction of the inverse of $\mathcal{P}_{\mathcal{F},U}$ becomes much more complicated [1]. Also, it should be mentioned that the method suggested in [9] made it possible to study completely the question on the injectivity of $\mathcal{P}_{\mathcal{F},U}$ also for all $R\leq r_1+r_2$.

Hyperbolic spaces. The above results have interesting generalizations related to replacing the triple $(\mathbb{R}^n,\mathcal{M}(n),dx)$ with the triple $(X,\mathcal{G},d\mu)$, where X is a manifold, $\mathcal{G}$ is a Lie group acting on X transitively, and $d\mu$ is a $\mathcal{G}$-invariant measure on X. The largest progress in this direction was achieved in the case where X is a noncompact symmetric space of rank 1 (see [10, Chapter 1, §4, Section 3]). This means that X is isometric to one of the hyperbolic spaces $\mathbb{H}^n(\mathbb{R})$, $\mathbb{H}^{2n}(\mathbb{C})$, and $\mathbb{H}^{4n}(\mathbb{Q})$ or to the hyperbolic Cayley plane $\mathbb{H}^{16}(\mathbb{C}ay)$. (Here, the superscript indicates the real dimension.) Many authors considered questions related to the injectivity of the global Pompeiu transformation on these spaces (see [2, 3]).

First local results for hyperbolic spaces appeared relatively recently in [11]–[14]. In particular, in [12, 13] a construction was proposed for inversion of the local Pompeiu transformation in the case where the family $\mathcal{F}$ consists of two geodesic balls in $\mathbb{H}^n(\mathbb{R})$ or $\mathbb{H}^{2n}(\mathbb{C})$. For other symmetric spaces, no similar recovering of a function has been known up to now.

Quaternion case. In the present paper, we solve this problem for the quaternion hyperbolic space $\mathbb{H}^{4n}(\mathbb{Q})$. In contrast to the case of $\mathbb{H}^n(\mathbb{R})$ or $\mathbb{H}^{2n}(\mathbb{C})$, the techniques developed in [15] by the first author play a principal role here. In particular, we use substantially the realization of irreducible components of the quasiregular representation of the compact symplectic group $\mathrm{Sp}(n)$ that was found in [15, §§4 and 5]. Furthermore, the description of the intertwining operators for such components that was obtained in [15, §7] is of importance in our constructions.

Structure of the paper. We formulate our main result in precise terms in §2. In §3, we fix the main notation and present facts required subsequently. In §§4–6, we develop the techniques involved in the proof of our main result. In §4, we find a specific integral presentation for the eigenfunctions of the Laplace–Beltrami operator on $\mathbb{H}^{4n}(\mathbb{Q})$. In §5, we compute the Fourier transforms of some distributions on $\mathbb{H}^{4n}(\mathbb{Q})$. In §6, we prove auxiliary assertions related to the construction of the inverse of the local Pompeiu transformation for the family consisting of two geodesic balls in $\mathbb{H}^{4n}(\mathbb{Q})$. Our main result is proved in §7.

§2. STATEMENTS OF THE MAIN RESULTS

Let $n \geq 2$. We denote by $\mathbb{Q}^n$ the n-dimensional left quaternion Euclidean space with scalar product

$$\langle a, b\rangle := \sum_{k=1}^{n} a_k \overline{b}_k,$$

where $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n) \in \mathbb{Q}^n$ (see, e.g., [16, Chapter 1, §2]). We set

$$a_k = \alpha_k + \alpha_{n+k} j \quad \text{and} \quad b_k = \beta_k + \beta_{n+k} j,$$

where $\alpha_k, \alpha_{n+k}, \beta_k, \beta_{n+k} \in \mathbb{C}$, $1 \leq k \leq n$, and identify $\mathbb{Q}^n$ with the $2n$-dimensional complex Euclidean space $\mathbb{C}^{2n}$ via the correspondence

$$(a_1, \dots, a_n) \mapsto (\alpha_1, \dots, \alpha_{2n}).$$

We have

$$\langle a, b\rangle = \langle a, b\rangle_{\mathbb{C}} - (a, b)_{\mathbb{C}}\, j, \tag{2.1}$$

where

$$\langle a, b\rangle_{\mathbb{C}} = \sum_{k=1}^{2n} \alpha_k \overline{\beta}_k \quad \text{and} \quad (a, b)_{\mathbb{C}} = \sum_{k=1}^{n} \left(\alpha_k \beta_{n+k} - \alpha_{n+k} \beta_k\right).$$

We define

$$B^{4n} := \{a \in \mathbb{Q}^n : |a| < 1\} = \{z \in \mathbb{C}^{2n} : |z| < 1\},$$

where $|a|^2 = \langle a, a\rangle$. The matrix

$$q_{ik}(a) := \frac{\left(1 - |a|^2\right) \delta_{ik} + \overline{a}_i a_k}{\left(1 - |a|^2\right)^2}, \quad 1 \leq i, k \leq n, \tag{2.2}$$

where δ_{ik} is the Kronecker delta, is quaternion symmetric ($q_{ik} = \overline{q}_{ki}$) and induces the structure of a Riemannian manifold on B^{4n}. The distance $d\,(z, w)$ between two points $z, w \in B^{4n}$ in the metric (2.2) is given by the formula

$$d(z, w) := \frac{1}{2} \log \frac{|1 - \langle z, w\rangle| + \sqrt{|w - z|^2 + |\langle w, z\rangle|^2 - |z|^2 |w|^2}}{|1 - \langle z, w\rangle| - \sqrt{|w - z|^2 + |\langle w, z\rangle|^2 - |z|^2 |w|^2}}$$

(see [15, Lemma 6]). The Riemannian measure on B^{4n} has the form

$$d\mu(z) = \frac{dm(z)}{(1 - |z|^2)^{2n+2}},$$

where dm is the Lebesgue measure on $\mathbb{C}^{2n}$ (see [15, Lemma 2]). The space $\mathbb{H}^{4n}(\mathbb{Q})$ is isometric to the ball B^{4n} with the metric (2.2) (see [15, Theorem 2]).

Let L denote the Laplace–Beltrami operator on $\mathbb{H}^{4n}(\mathbb{Q})$,

$$(\mathcal{L} f)(z) = \frac{(Lf)(z)}{4(1 - |z|^2)}$$

(see [15, formula (22)]). For nonnegative integers p, q, and $m \leq \min(p, q)$, we define

$$\mathcal{H}(p, q, m) := \{f \in \mathcal{H}(p, q) : \mathcal{L} f = (m - 1)(m - s) f\},$$

where $s = p + q$ and $\mathcal{H}(p, q)$ is the space of homogeneous harmonic polynomials of bidegree (p, q) in $\mathbb{C}^{2n}$ (see [17, Chapter 12]). Next, we identify the space $\mathcal{H}(p, q)$ with the space of restrictions of its elements to the unit sphere $\mathbb{S}^{4n-1}$. The space $L^2(\mathbb{S}^{4n-1})$ is the orthogonal direct sum of the spaces $\mathcal{H}(p, q, m)$ for $p, q \geq 0$ and $0 \leq m \leq \min(p, q)$ (see [15, Theorem 3]).

Let ρ, σ be the polar coordinates in $\mathbb{C}^{2n}$ ($\rho = |z|$ for $z \in \mathbb{C}^{2n}$, and if $z \neq 0$, then $\sigma = z/|z|$), and let $\{S^k_{p,q,m}\}_{k=1}^{N(n,p,q,m)}$ be a fixed orthonormal basis in the space $\mathcal{H}(p, q, m)$.

We denote by $B_R = \{z \in B^{4n} : d(0,z) < R\}$ the open geodesic ball of radius $R > 0$ centered at the origin. Every function f locally integrable in B_R (notation: $f \in L_{\mathrm{loc}}(B_R)$) gives rise to a Fourier series,

$$f(\rho\sigma) \sim \sum_{p,q=0}^{\infty} \sum_{m=0}^{\min(p,q)} \sum_{k=1}^{N(n,p,q,m)} f^k_{p,q,m}(\rho) S^k_{p,q,m}(\sigma), \quad \rho \in (0, \tanh R),$$

where

$$f^k_{p,q,m}(\rho) = \int_{\mathbb{S}^{4n-1}} f(\rho\sigma)\overline{S^k_{p,q,m}(\sigma)}\, d\sigma, \tag{2.3}$$

and $d\sigma$ is the normalized surface measure on $\mathbb{S}^{4n-1}$. In order to recover f, it suffices to know the Fourier coefficients $f^k_{p,q,m}$.

Throughout this paper,

$$\lambda \in \mathbb{C}, \quad \nu = \nu(\lambda) = \frac{2n+1-i\lambda}{2}, \quad \text{and} \quad \nu' = \nu(-\lambda).$$

For $\alpha, \beta \geq 0$, we define

$$\varphi_\lambda^{(\alpha,\beta)}(t) := F\left(\frac{\alpha+\beta+1-i\lambda}{2}, \frac{\alpha+\beta+1+i\lambda}{2}; \alpha+1; -\sinh^2 t\right),$$

where F is the hypergeometric function. We also set

$$\Phi_\lambda^{s,m}(\rho) := \frac{\Gamma(\nu+s-m)\Gamma(2n)}{\Gamma(2n+s)\Gamma(\nu-m)} \rho^s (1-\rho^2)^\nu F(\nu+s-m, \nu-1+m; 2n+s; \rho^2), \tag{2.4}$$

where Γ is the gamma-function. The spherical functions on $\mathbb{H}^{4n}(\mathbb{Q})$ are of the form

$$\varphi_\lambda(z) = \varphi_\lambda^{(2n-1,1)}(d(0,z)) = \Phi_\lambda^{(0,0)}(|z|) = \int_{\mathbb{S}^{4n-1}} e_{\nu,\omega}(z)\, d\omega,$$

where

$$e_{\nu,\omega}(z) = \left(\frac{1-|z|^2}{|1-\langle z,\omega\rangle|^2}\right)^\nu, \quad \omega \in \mathbb{S}^{4n-1}$$

(see [18] and also [15, Lemma 26]). The Fourier transform of a compactly supported distribution T on $\mathbb{H}^{4n}(\mathbb{Q})$ is defined as follows:

$$\widetilde{T}(\lambda,\omega) := \langle T, e_{\nu,\omega}\rangle, \quad \lambda \in \mathbb{C}, \ \omega \in \mathbb{S}^{4n-1}.$$

If T is a *radial* distribution, i.e.,

$$\langle T, f\rangle = \langle T, f\circ\tau\rangle, \quad \tau \in \mathrm{Sp}(n), \ f \in C^\infty(B^{4n}),$$

then for each $\omega \in \mathbb{S}^{4n-1}$ the Fourier transform $\widetilde{T}(\lambda,\omega)$ coincides with the spherical transform $\widetilde{T}(\lambda) = \langle T, \varphi_\lambda\rangle$. In particular, for the indicator function χ_t of the ball B_t, we have

$$\widetilde{\chi}_t(\lambda) = \frac{\pi^{2n}}{\Gamma(2n+1)} (\sinh t)^{4n} (\cosh t)^4 \varphi_\lambda^{(2n,2)}(t)$$

(see [13, Lemma 3.3]). All zeros λ of the function $\varphi_\lambda^{(2n,2)}(t)$ are real and simple and lie symmetrically with respect to the point $\lambda = 0$ (see [15, §1] and also §6 below). We put

$$\mathcal{N}(t) := \{\lambda > 0 : \varphi_\lambda^{(2n,2)}(t) = 0\}.$$

As in [10, Chapter 2, §5], we denote by $T_1 \times T_2$ the convolution of two distributions on $\mathbb{H}^{4n}(\mathbb{Q})$ (one is assumed to have compact support). Our main result in the present paper is as follows.

Theorem 2.1. *Suppose that* $0 < r_1 < r_2$, $r_1 + r_2 < R$, *and* $\mathcal{N}(r_1) \cap \mathcal{N}(r_2) = \varnothing$. *For any* $p, q \geq 0$, $0 \leq m \leq \min(p, q)$, $1 \leq k \leq N(n, p, q, m)$, *and* $t \in (0, R)$, *there exist distributions* $\mathcal{U}_{l,i}$ $(l \in \mathbb{N}, i = 1, 2)$ *with the following properties*:

1) $\operatorname{supp}\mathcal{U}_{l,i} \subset B_{R-r_i}$ *for* $l \in \mathbb{N}$ *and* $i = 1, 2$;

2) *for each function* $f \in C^\infty(B_R)$, *we have*

$$f^k_{p,q,m}(\tanh t) = \lim_{l\to\infty} \left(\langle \mathcal{U}_{l,1}, f \times \chi_{r_1}\rangle + \langle \mathcal{U}_{l,2}, f \times \chi_{r_2}\rangle\right). \tag{2.5}$$

Several remarks are in order here. As has already been noticed, analogs of Theorem 2.1 for $\mathbb{R}^n$, $\mathbb{H}^n(\mathbb{R})$, and $\mathbb{H}^{2n}(\mathbb{C})$ were obtained in [1, 12, 13]. The distributions $\mathcal{U}_{l,i}$ are constructed explicitly in §7; the construction is based on the Paley–Wiener theorem for the spherical transformation on $\mathbb{H}^{4n}(\mathbb{Q})$ (see [10, Chapter 4, §7] and [13, 19]). Since

$$(f \times \chi_{r_i})(w) = \int_{d(z,w)<r_i} f(z)\, d\mu(z), \quad w \in B_{R-r_i}, \;\; i = 1, 2,$$

we see that Theorem 2.1 contains a construction for recovering a function by its globular mean values on the quaternion hyperbolic space. In particular, if $r_1 + r_2 < R$, $\mathcal{N}(r_1) \cap \mathcal{N}(r_2) = \varnothing$, and the function $f \in L_{\mathrm{loc}}(B_R)$ has zero integrals (with respect to the measure μ) over all closed geodesic balls of radii r_1 and r_2 and lying in B_R, then $f = 0$ in B_R (see [10, Chapter 1, Proof of Theorem 4.2]). We note that these conditions on r_1 and r_2 are sharp, and that the question on the injectivity of $\mathcal{P}_{(\mathcal{B}_{r_1}, \mathcal{B}_{r_2}), U}$ for hyperbolic spaces was studied completely in [14, 18], and [20]–[23] by different methods.

Our method of the proof of Theorem 2.1 allows us to obtain similar assertions for other families of distributions on $\mathbb{H}^{4n}(\mathbb{Q})$. We present a result in this direction.

Let σ_r denote the surface delta-function concentrated on the sphere $S_r = \{z \in B^{4n} : d(0, z) = r\}$.

Theorem 2.2. *Suppose* $r > 0$ *and* $R > 2r$. *Then for any* $p, q \geq 0$, $0 \leq m \leq \min(p, q)$, $1 \leq k \leq N(n, p, q, m)$, *and* $t \in (0, R)$, *there exist distributions* $\mathcal{V}_{l,i}$ $(l \in \mathbb{N}, i = 1, 2)$ *with the following properties*:

1) $\operatorname{supp}\mathcal{V}_{l,i} \subset B_{R-r}$ *for* $l \in \mathbb{N}$ *and* $i = 1, 2$;

2) *for each function* $f \in C^\infty(B_R)$, *we have*

$$f^k_{p,q,m}(\tanh t) = \lim_{l\to\infty} \left(\langle \mathcal{V}_{l,1}, f \times \chi_{r}\rangle + \langle \mathcal{V}_{l,2}, f \times \sigma_{r}\rangle\right).$$

As a consequence, we see that if a function $f \in C^\infty(B_R)$ has zero integrals over all closed geodesic balls and spheres of fixed radius $r < R/2$, then f vanishes in B_R.

The main results of the present paper were announced in [24]. Concerning other aspects of the Pompeiu problem on symmetric spaces, see [2, 3, 25, 26] and the references therein. The authors thank V. V. Volchkov and V. P. Zastavnyĭ for useful discussions and remarks.

§3. Notation and auxiliary constructions

As usual, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}_+$ denote the set of positive integers, integers, and nonnegative integers, respectively. We write

$$z = (z_1, \dots, z_{2n}) \in \mathbb{C}^{2n}, \quad z_k = x_k + iy_k, \;\; x_k, y_k \in \mathbb{R}^1, \;\; k = 1, \dots, 2n.$$

Next, we identify $\mathbb{C}^{2n}$ and $\mathbb{R}^{4n}$ via the correspondence

$$(z_1, \dots, z_{2n}) \mapsto (x_1, \dots, x_{2n}, y_1, \dots, y_{2n}).$$

We set $z^\eta = z_1^{\eta_1}\cdots z_{2n}^{\eta_{2n}}$, where $\eta = (\eta_1,\dots,\eta_{2n}) \in \mathbb{Z}_+^{2n}$ is a multiindex of *length* $|\eta| = \eta_1+\dots+\eta_{2n}$. For each polynomial

$$P(z) = \sum_{|\eta|\le p, |\varkappa|\le q} c_{\eta,\varkappa} z^\eta \overline{z}^\varkappa, \quad c_{\eta,\varkappa}\in\mathbb{C},$$

we denote by $P\left(\frac{\partial}{\partial \overline{z}}\right)$ the differential operator

$$P\left(\frac{\partial}{\partial \overline{z}}\right) := \sum_{|\eta|\le p, |\varkappa|\le q} c_{\eta,\varkappa}\frac{\partial^{|\eta|+|\varkappa|}}{\partial z^\varkappa \partial \overline{z}^\eta},$$

where

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right).$$

Since $\mathcal{H}(p,q)$ is the linear hull of functions of the form

$$(c_1z_1+\dots+c_{2n}z_{2n})^p\,(d_1\overline{z}_1+\dots+d_{2n}\overline{z}_{2n})^q,$$

where

$$c_1d_1+\dots+c_{2n}d_{2n}=0, \quad c_k,d_k\in\mathbb{C},\ \ k=1,\dots,2n$$

(see [13] and [17, Subsection 12.5]), it follows that for $H_{p,q}\in\mathcal{H}(p,q)$ and $f\in C^\infty(B^{4n})$ we have

$$D_{p,q}\left((1-|z|^2)^\lambda f(z)\right)(0) = D_{p,q}f(0) \tag{3.1}$$

and

$$D_{p,q}\left(|z|^{2k}f(z)\right)(0) = 0, \quad k\in\mathbb{N}, \tag{3.2}$$

where $D_{p,q} = H_{p,q}\left(\frac{\partial}{\partial\overline{z}}\right)$ (see [13, Lemma 4.2]). Furthermore,

$$D_{p,q}\left(\left(\frac{1-|z|^2}{|1-\langle z,\omega\rangle_{\mathbb{C}}|^2}\right)^{\frac{2n-i\lambda}{2}}\right)(0) = \frac{\Gamma\left(\frac{2p+2n-i\lambda}{2}\right)\Gamma\left(\frac{2q+2n-i\lambda}{2}\right)}{\Gamma^2\left(\frac{2n-i\lambda}{2}\right)}H_{p,q}(\omega), \quad \omega\in\mathbb{S}^{4n-1}$$

(see [13, Proof of Lemma 4.3]). Combining this with the identity

$$\int_{\mathbb{S}^{4n-1}}\left(\frac{1-|z|^2}{|1-\langle z,\omega\rangle_{\mathbb{C}}|^2}\right)^{\frac{2n-i\lambda}{2}}H_{p,q}(\omega)\,d\omega = \frac{\Gamma\left(\frac{2p+2n-i\lambda}{2}\right)\Gamma\left(\frac{2q+2n-i\lambda}{2}\right)\Gamma(2n)}{\Gamma^2\left(\frac{2n-i\lambda}{2}\right)\Gamma(2n+s)} \times\left(1-|z|^2\right)^{\frac{2n-i\lambda}{2}}F\left(\frac{2p+2n-i\lambda}{2},\frac{2q+2n-i\lambda}{2};2n+s;|z|^2\right)H_{p,q}(z)$$

(see [13, formulas (4.9) and (4.11)]), we obtain

$$D_{p_1,q_1}(k_1)(\overline{S^k_{p,q}})(0) = \begin{cases} 0 & \text{if } (p_1,q_1,k_1)\ne(p,q,k),\\ \frac{\Gamma(2n+p+q)}{\Gamma(2n)} & \text{if } (p_1,q_1,k_1)=(p,q,k),\end{cases} \tag{3.3}$$

where $\left\{S^k_{p,q}\right\}_{k=1}^{N(n,p,q)}$ is a fixed orthonormal basis in $\mathcal{H}(p,q)$ and $D_{p_1,q_1}(k_1) = S^{k_1}_{p_1,q_1}\left(\frac{\partial}{\partial\overline{z}}\right)$ (see (3.1) and (3.2)).

We define

$$\mathcal{Z}(p,q,m) := \{f\in\mathcal{H}(p,q,m) : f\circ\tau = f,\ \tau\in\mathrm{Sp}(n-1)\}$$

(hereinafter, we regard $\mathrm{Sp}(n-1)$ as the isotropy subgroup in $\mathrm{Sp}(n)$ of the point $e_1 = (1, 0, \ldots, 0) \in \mathbb{C}^{2n}$) and set

$$Z_{p,q,m}(z) := \sum_{l=0}^{m} P_l(z) F_{l,p,q,m}(z),$$

where

$$P_l(z) = \left(\sum_{k=2}^{n} \left(|z_k|^2 + |z_{n+k}|^2 \right) \right)^l,$$

$$F_{l,p,q,m}(z) = \sum_{k=p-m}^{p-l} a_{lk} z_1^k z_{n+1}^{p-l-k} \overline{z}_1^{k+m-p} \overline{z}_{n+1}^{q-l-k-m+p},$$

and the constants a_{lk} are determined by the following recurrence relations:

$$\begin{aligned}
&a_{m,p-m} = 1, \\
&(m-l)a_{l,p-m} = a_{l,p-m+1}, \quad 0 \le l \le m-1, \\
&a_{l,k+1}(k+1)(k+1+m-p) + a_{lk}(p-l-k)(q-l-k-m+p) \\
&\qquad = -(l+1)(2n+l-2)a_{l+1,k}, \quad 0 \le l \le m-1, \quad p-m \le k \le p-l-1.
\end{aligned}$$

The functions

$$\left\{ D^l \left(Z_{p,q,m} \right) \right\}_{l=0}^{s-2m}, \quad \text{where } D = z_{n+1} \frac{\partial}{\partial z_1} - \overline{z}_1 \frac{\partial}{\partial \overline{z}_{n+1}},$$

form an orthogonal basis of the space $\mathcal{Z}(p,q,m)$ (see [15, Lemma 14]). Also, we mention the relation

$$\begin{aligned}
&D^l \left(z_1^\alpha z_{n+1}^\beta \overline{z}_1^\gamma \overline{z}_{n+1}^\delta \right) \\
&\quad = \sum_{k=\max(0,l-\delta)}^{\min(l,\alpha)} (-1)^{l-k} \frac{l!}{k!(l-k)!} \frac{\alpha!}{(\alpha-k)!} \frac{\delta!}{(\delta-l+k)!} z_1^{\alpha-k} z_{n+1}^{\beta+k} \overline{z}_1^{\gamma+l-k} \overline{z}_{n+1}^{\delta-l+k},
\end{aligned} \tag{3.4}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}_+$.

We fix a point $z \in \mathbb{S}^{4n-1}$ and consider the linear functional on $\mathcal{H}(p,q,m)$ that takes a function $f \in \mathcal{H}(p,q,m)$ to the number $f(z)$. By the Riesz theorem, there exists a unique function $K_z \in \mathcal{H}(p,q,m)$ such that

$$f(z) = \int_{\mathbb{S}^{4n-1}} f(\sigma) \overline{K_z(\sigma)} \, d\sigma, \quad f \in \mathcal{H}(p,q,m). \tag{3.5}$$

The functions K_z possess the following properties (see [15, Lemmas 15 and 16]):

1) for each function $f \in L^2\left(\mathbb{S}^{4n-1}\right)$, we have

$$(\pi_{p,q,m} f)(z) = \int_{\mathbb{S}^{4n-1}} f(\sigma) \overline{K_z(\sigma)} \, d\sigma,$$

where $\pi_{p,q,m} : L^2\left(\mathbb{S}^{4n-1}\right) \to \mathcal{H}(p,q,m)$ is the orthogonal projection;

2) $K_z(w) = \overline{K_w(z)}$ for $z, w \in \mathbb{S}^{4n-1}$;

3) $K_{\tau z} = K_z \circ \tau^{-1}$ for $\tau \in \mathrm{Sp}(n)$;

4) $K_z = K_z \circ \tau$ for any $\tau \in \mathrm{Sp}(n)$ such that $\tau z = z$;

5) $K_z(z) = K_w(w)$ for $z, w \in \mathbb{S}^{4n-1}$;

6) $K_{e_1}(z) = \alpha_{p,q,m} D^{q-m} \left(Z_{p,q,m} \right)(z)$, where

$$\alpha_{p,q,m} = \overline{D^{q-m}(Z_{p,q,m})(e_1)} \Big/ \int_{\mathbb{S}^{4n-1}} \left| D^{q-m}(Z_{p,q,m})(\sigma) \right|^2 d\sigma.$$

We denote by $\mathfrak{T}(\tau)$ the quasiregular representation of the group $\mathrm{Sp}(n)$ in $\mathbb{S}^{4n-1}$: for $f \in L^2\left(\mathbb{S}^{4n-1}\right)$, we have

$$(\mathfrak{T}(\tau)f)(\sigma) = f\left(\tau^{-1}\sigma\right),$$

where $\sigma \in \mathbb{S}^{4n-1}$ and $\tau \in \mathrm{Sp}(n)$). Next, we denote by $\mathfrak{T}_{p,q,m}(\tau)$ the restriction of $\mathfrak{T}(\tau)$ to $\mathcal{H}(p,q,m)$, and let $t^{p,q,m}_{l,k}(\tau)$, $1 \le l,k \le N(n,p,q,m)$, be the matrix of the representation $\mathfrak{T}_{p,q,m}(\tau)$, i.e.,

$$\mathfrak{T}_{p,q,m}(\tau)S^k_{p,q,m} = \sum_{l=1}^{N(n,p,q,m)} t^{p,q,m}_{l,k}(\tau)S^l_{p,q,m}.$$

The representation $\mathfrak{T}(\tau)$ is an orthogonal direct sum of irreducible unitary representations $\mathfrak{T}_{p,q,m}(\tau)$, $p,q \in \mathbb{Z}_+$, $0 \le m \le \min(p,q)$, and two representations $\mathfrak{T}_{p_i,q_i,m_i}(\tau)$, $i = 1,2$, are equivalent if and only if $p_1 + q_1 = p_2 + q_2$ and $m_1 = m_2$ (see [15, Theorems 4 and 5]). It follows that for $f \in L_{\mathrm{loc}}(B_R)$, $\sigma \in \mathbb{S}^{4n-1}$, and almost all $\rho \in (0, \tanh R)$ we have

$$N(n,p,q,m)\int_{\mathrm{Sp}(n)} f(\rho\tau^{-1}\sigma)\overline{t^{p,q,m}_{l,k}(\tau)}\,d\tau = \sum_{\substack{p_1+q_1=s\\ p_1,q_1\ge m}} f^k_{p_1,q_1,m}(\rho)S^l_{p_1,q_1,m}(\sigma), \tag{3.6}$$

where $d\tau$ is the normalized Haar measure on $\mathrm{Sp}(n)$ (see [15, Lemma 20]).

We introduce the differential operators $D_1(s,m)$ and $D_2(s,m)$ that act on the functions $\varphi \in C^1(0,\tanh R)$ as follows:

$$(D_1(s,m)\varphi)(\rho) = \frac{(1-\rho^2)^{2n+s+1-m}}{\rho^{4n+s-2}}\frac{d}{d\rho}\left(\frac{\rho^{4n+s-2}}{(1-\rho^2)^{2n+s-m}}\varphi(\rho)\right),$$

$$(D_2(s,m)\varphi)(\rho) = \frac{\rho^{s-1}}{(1-\rho^2)^{s-m-2}}\frac{d}{d\rho}\left(\frac{(1-\rho^2)^{s-1-m}}{\rho^{s-1}}\varphi(\rho)\right).$$

If a function f in $C^2(B_R)$ has the form $f(z) = \varphi(\rho)H_{p,q,m}(\sigma)$, where $H_{p,q,m} \in \mathcal{H}(p,q,m)$, then

$$((L + 4(2n+s-m)(m-s+1)I)f)(z) = (D_2(s,m)D_1(s,m)\varphi)(\rho)H_{p,q,m}(\sigma), \tag{3.7}$$

where I is the identity operator (see [15, Proof of Lemma 26]). We also need the relations

$$L(e_{\nu,\omega}) = -(\lambda^2 + (2n+1)^2)e_{\nu,\omega} \tag{3.8}$$

and

$$L\left(\Phi^{s,m}_\lambda(\rho)H_{p,q,m}(\sigma)\right) = -\left(\lambda^2 + (2n+1)^2\right)\Phi^{s,m}_\lambda(\rho)H_{p,q,m}(\sigma) \tag{3.9}$$

(see [15, formulas (87) and (88)]).

§4. An integral presentation for the functions $\Phi^{s,m}_\lambda(\rho)H_{p,q,m}(\sigma)$

We denote by (x,y) the Euclidean scalar product of vectors $x,y \in \mathbb{R}^n$, and by $\mathcal{H}(k)$ the space of spherical harmonics of degree k on $\mathbb{S}^{n-1}$ (see [27, Chapter 4, §2]).

Lemma 4.1. *Suppose $H_k \in \mathcal{H}(k)$ and $x \in \mathbb{S}^{n-1}$. Then*

$$\int_{\mathbb{S}^{n-1}} (\xi,x)^k H_k(\xi)\,d\xi = \frac{k!}{2^k}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+k\right)}H_k(x), \tag{4.1}$$

where $d\xi$ is the normalized surface measure on $\mathbb{S}^{n-1}$.

Proof. By the Funk–Hecke theorem (see [28, Subsection §11.4]), we have

$$\int_{\mathbb{S}^{n-1}} (\xi,x)^k H_k(\xi)\,d\xi = \lambda_{k,n}H_k(x), \tag{4.2}$$

where

$$\lambda_{k,n}=\frac{2^{n-3}\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{n}{2}\right)\Gamma(k+1)}{\pi\Gamma(k+n-2)}\int_{-1}^{1}t^{k}C_{k}^{\frac{n-2}{2}}(t)(1-t^{2})^{\frac{n-3}{2}}\,dt,$$

and $C_k^{\frac{n-2}{2}}$ is the Gegenbauer polynomial of degree k and of order $\frac{n-2}{2}$ (see, e.g., [28, Subsection 11.1]). Since

$$\int_{-1}^{1}t^{k}C_{k}^{\frac{n-2}{2}}(t)(1-t^{2})^{\frac{n-3}{2}}\,dt=\frac{\sqrt{\pi}\Gamma(k+n-2)\Gamma\left(\frac{n-1}{2}\right)}{2^{k}\Gamma(n-2)\Gamma\left(k+\frac{n}{2}\right)}$$

(see [29, Chapter 9, Subsection 4.8, formula (8)]), relation (4.2) and the Legendre duplication formula for the Γ-function (see [30, Subsection 1.2, formula (15)]) imply (4.1). □

Lemma 4.2. *Suppose that* $l\in\mathbb{Z}_+$ *and* $l\neq q-m$. *Then for* $\rho\in[0,1)$ *we have*

$$\int_{\mathbb{S}^{4n-1}}\left(\frac{1-\rho^2}{|1-\rho\langle\sigma,e_1\rangle|^2}\right)^{\nu}D^{l}\left(Z_{p,q,m}\right)(\sigma)\,d\sigma=0.$$

Proof. If $\sigma=(\sigma_1,\dots,\sigma_{2n})\in\mathbb{S}^{4n-1}$, then

$$\langle\sigma,e_1\rangle=\sigma_1+\sigma_{n+1}j,\quad |1-\rho\langle\sigma,e_1\rangle|^2=1-u_\rho(\sigma),$$

where

$$u_\rho(\sigma)=2\rho\operatorname{Re}\sigma_1-\rho^2\left(|\sigma_1|^2+|\sigma_{n+1}|^2\right)$$

(see (2.1)). The definition of the functions u_ρ and $Z_{p,q,m}$ implies that for $\theta\in[-\pi,\pi]$ we have

$$u_\rho(\sigma_1,\dots,\sigma_n,e^{i\theta}\sigma_{n+1},\sigma_{n+2},\dots,\sigma_{2n})=u_\rho(\sigma)\tag{4.3}$$

and

$$\begin{aligned}&D^{l}\left(Z_{p,q,m}\right)(\sigma_1,\dots,\sigma_n,e^{i\theta}\sigma_{n+1},\sigma_{n+2},\dots,\sigma_{2n})\\&\qquad=e^{i\theta(l+m-q)}\sum_{k=0}^{m}D^{l}\left(F_{k,p,q,m}\right)(\sigma)P_k(\sigma)\end{aligned}\tag{4.4}$$

(see (3.4)). Applying the formula

$$\int_{\mathbb{S}^{4n-1}}f(\sigma)\,d\sigma=\frac{1}{2\pi}\int_{\mathbb{S}^{4n-1}}d\sigma\int_{-\pi}^{\pi}f(\sigma_1,\dots,\sigma_n,e^{i\theta}\sigma_{n+1},\sigma_{n+2},\dots,\sigma_{2n})\,d\theta$$

(see [17, Subsection 1.4.7]), we complete the proof. □

Lemma 4.3. *Suppose* $H_{p,q,m}\in\mathcal{H}(p,q,m)$ *and* $z\in B^{4n}$. *Then*

$$\int_{\mathbb{S}^{4n-1}}e_{\nu,\omega}(z)H_{p,q,m}(\omega)\,d\omega=\Phi_{\lambda}^{s,m}(|z|)H_{p,q,m}\left(\frac{z}{|z|}\right).$$

Proof. Consider the function

$$f(z):=\left(\frac{1-|z|^2}{|1-\langle z,e_1\rangle|^2}\right)^{\nu}.$$

Since $Lf=-\left(\lambda^2+(2n+1)^2\right)f$ (see (3.8)), formula (3.6) and the Sp(n)-invariance of the operator L (see [15, Lemma 1]) yield

$$\sum_{\substack{p_1+q_1=s\\p_1,q_1\geq m}}\left(L+\left(\lambda^2+(2n+1)^2\right)I\right)\left(f^{k}_{p_1,q_1,m}(\rho)S^{l}_{p_1,q_1,m}(\sigma)\right)=0.$$

By the orthogonality of the spaces $\mathcal{H}(p_1,q_1,m)$ on $\mathbb{S}^{4n-1}$, we obtain

$$\left(L+\left(\lambda^2+(2n+1)^2\right)I\right)\left(f^{k}_{p_1,q_1,m}(\rho)S^{l}_{p_1,q_1,m}(\sigma)\right)=0\tag{4.5}$$

(see (3.7)). Since the function $f^k_{p,q,m}(\rho)$ is smooth at the origin, from (3.9), (2.3), and (4.5) we deduce that

$$\int_{\mathbb{S}^{4n-1}} \left(\frac{1-\rho^2}{|1-\rho\langle\sigma, e_1\rangle|^2}\right)^{\nu} H_{p,q,m}(\sigma)\,d\sigma = \alpha_1(\lambda)\Phi^{s,m}_{\lambda}(\rho), \tag{4.6}$$

where $\alpha_1(\lambda)$ does not depend on ρ. Next, we have

$$\int_{\mathbb{S}^{4n-1}} \frac{H_{p,q,m}(\omega)}{|1-\rho\langle\omega,\sigma\rangle|^{2\nu}}\,d\omega = \int_{\mathbb{S}^{4n-1}} H_{p,q,m}(\xi)\left(\int_{\mathbb{S}^{4n-1}} \frac{\overline{K_\omega(\xi)}}{|1-\rho\langle\omega,\sigma\rangle|^{2\nu}}\,d\omega\right)d\xi \tag{4.7}$$

(see (3.5)). Suppose that $\tau \in \mathrm{Sp}(n)$ and $\tau\sigma = e_1$. Then

$$\int_{\mathbb{S}^{4n-1}} \frac{K_\omega(\xi)}{|1-\rho\langle\omega,\sigma\rangle|^{2\overline{\nu}}}\,d\omega = \int_{\mathbb{S}^{4n-1}} \frac{K_\omega(\tau\xi)}{|1-\rho\langle\omega,e_1\rangle|^{2\overline{\nu}}}\,d\omega = \int_{\mathbb{S}^{4n-1}} \frac{\Psi_\omega(\tau\xi)}{|1-\rho\langle\omega,e_1\rangle|^{2\overline{\nu}}}\,d\omega, \tag{4.8}$$

where

$$\Psi_\omega(\xi) = \int_{\mathrm{Sp}(n-1)} K_\omega(g\xi)\,dg \tag{4.9}$$

(see [16, Chapter 1, Subsection 2.3] and also properties 3) and 4) of the functions K_z). Since $\Psi_\omega \in \mathcal{Z}(p,q,m)$, from [15, Lemma 14] it follows that

$$\Psi_\omega(\xi) = \sum_{l=0}^{s-2m} c_l(\omega) D^l\left(Z_{p,q,m}\right)(\xi), \tag{4.10}$$

where

$$c_l(\omega) = \gamma_l \overline{D^l\left(Z_{p,q,m}\right)(\omega)}$$

and

$$\gamma_l = \left(\int_{\mathbb{S}^{4n-1}} |D^l(Z_{p,q,m})(\xi)|^2\,d\xi\right)^{-1}$$

(see (4.9) and (3.5)). By property 6) of the functions K_z, relations (4.8) and (4.10) and Lemma 4.2 imply the identity

$$\int_{\mathbb{S}^{4n-1}} \frac{\overline{K_\omega(\xi)}}{|1-\rho\langle\omega,\sigma\rangle|^{2\nu}}\,d\omega = c\overline{K_{e_1}(\tau\xi)} \int_{\mathbb{S}^{4n-1}} \frac{K_{e_1}(\omega)}{|1-\rho\langle\omega,e_1\rangle|^{2\nu}}\,d\omega,$$

where the constant c depends only on p, q, m, and n. Consequently (see (4.7), (4.6), and (3.5)), we have

$$\int_{\mathbb{S}^{4n-1}} \left(\frac{1-\rho^2}{|1-\rho\langle\omega,\sigma\rangle|^2}\right)^{\nu} H_{p,q,m}(\omega)\,d\omega = \alpha_2(\lambda)\Phi^{s,m}_{\lambda}(\rho) H_{p,q,m}(\sigma), \tag{4.11}$$

where $\alpha_2(\lambda)$ does not depend on ρ and σ. It remains to prove that $\alpha_2(\lambda) \equiv 1$. Differentiating (4.11) s times with respect to ρ, putting $\rho = 0$, and recalling (2.1) and (2.4), we obtain

$$\alpha(\nu) = s!\frac{\Gamma(2n)\Gamma(\nu-m+s)}{\Gamma(2n+s)\Gamma(\nu-m)}\alpha_2(\lambda) H_{p,q,m}(\sigma),$$

where $\alpha(\nu)$ is a polynomial in ν of degree s with the leading coefficient equal to

$$2^s \int_{\mathbb{S}^{4n-1}} (\sigma,\omega)^s H_{p,q,m}(\omega)\,d\omega.$$

Applying (2.4) and (4.1), we see that $\alpha_2(\lambda) \equiv 1$, and Lemma 4.3 is proved. ☐

§5. MAIN FORMULA FOR THE FOURIER COEFFICIENTS $f^k_{p,q,m}(\tanh t)$

We denote by δ_0 the delta distribution centered at the origin of the space $\mathbb{H}^{4n}(\mathbb{Q})$.

Lemma 5.1. *If*

$$H_{p,q,m} \in \mathcal{H}(p,q,m), \quad D_{p,q,m} = H_{p,q,m}\left(\frac{\partial}{\partial \bar{z}}\right), \quad \mathcal{W}_{p,q,m} = (-1)^s D_{p,q,m}\delta_0,$$

then

$$\widetilde{\mathcal{W}}_{p,q,m}(\lambda,\omega) = \frac{\Gamma(\nu+s-m)}{\Gamma(\nu-m)} H_{p,q,m}(\omega). \tag{5.1}$$

Proof. Since the functions $\left\{S^k_{p,q,m}\right\}$, $0 \le m \le \min(p,q)$, $1 \le k \le N(n,p,q,m)$, form an orthonormal basis of the space $\mathcal{H}(p,q)$ (see [15, Lemma 9]), from (3.3) and Lemma 4.3 it follows (see (3.1) and (3.2)) that

$$\int_{\mathbb{S}^{4n-1}} D_{p_1,q_1,m_1}(k_1)\left(e_{\nu,\omega}\right)(0)\overline{S^k_{p,q,m}(\omega)}\,d\omega$$
$$= \begin{cases} 0 & \text{if } (p_1,q_1,m_1,k_1) \ne (p,q,m,k), \\ \frac{\Gamma(\nu+s-m)}{\Gamma(\nu-m)} & \text{if } (p_1,q_1,m_1,k_1) = (p,q,m,k), \end{cases}$$

where $D_{p_1,q_1,m_1}(k_1) = S^{k_1}_{p_1,q_1,m_1}\left(\frac{\partial}{\partial \bar{z}}\right)$. Consequently,

$$D_{p,q,m}\left(e_{\nu,\omega}\right)(0) = \frac{\Gamma(\nu+s-m)}{\Gamma(\nu-m)} H_{p,q,m}(\omega).$$

Using the definition of $D_{p,q,m}\delta_0$ (see [13, §4]), we arrive at 5.1. □

We put $T_{r,0,0} = \sigma_r$ and, for $s \in \mathbb{N}$,

$$T_{r,s,m}(z) = F\left(s-m+1, m; s; \frac{\tanh^2 r - |z|^2}{1-|z|^2}\right)\left(\frac{\tanh^2 r - |z|^2}{1-|z|^2}\right)^{s-1} \chi_r(z).$$

It is easily seen that $\widetilde{T}_{r,0,0}(\lambda) = \varphi_\lambda^{(2n-1,1)}(r)$ (see [13, Proof of Lemma 3.4]).

Lemma 5.2. *Let $s \in \mathbb{N}$. Then*

$$\widetilde{T}_{r,s,m}(\lambda) = \pi^{2n}\Gamma(s)(\sinh r)^{4n+2s-2}(\cosh r)^{4-2m} G_{s,m}(\lambda,r), \tag{5.2}$$

where

$$G_{s,m}(\lambda,r) := \frac{\varphi_\lambda^{(2n-1+s,s-2m+1)}(r)}{\Gamma(2n+s)}.$$

Proof. 1) First, we assume that $m \in \mathbb{N}$. Setting

$$h(t) := t^{s-1}F(s-m+1, m; s; t),$$

we have

$$\begin{aligned} \widetilde{T}_{r,s,m}(\lambda) &= \int_{B_r} h\left(\frac{\tanh^2 r - |z|^2}{1-|z|^2}\right)\varphi_\lambda(z)\,d\mu(z) \\ &= \frac{2\pi^{2n}}{\Gamma(2n)}\int_0^r h\left(\frac{\tanh^2 r - \tanh^2 t}{1-\tanh^2 t}\right)\varphi_\lambda^{(2n-1,1)}(t)(\sinh t)^{4n-1}(\cosh t)^3\,dt \end{aligned} \tag{5.3}$$

(see [13, formula (3.7)]). Changing the variables by the formula $u = \sinh^2 t/\sinh^2 r$, we obtain

$$\widetilde{T}_{r,s,m}(\lambda) = \frac{\pi^{2n}}{\Gamma(2n)}(\sinh r)^{4n}\int_0^1 h\left((1-u)\tanh^2 r\right)u^{2n-1}(1-\varepsilon u)F\left(\nu,\nu';2n;\varepsilon u\right)du, \tag{5.4}$$

where $\varepsilon = -\sinh^2 r$. Since

$$F(\alpha,\beta;\gamma;\lambda) = (1-\lambda)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta;\gamma;\lambda) \tag{5.5}$$

(see [30, Subsection 2.1, formula (23)]), we obtain

$$\widetilde{T}_{r,s,m}(\lambda) = \frac{\pi^{2n}}{\Gamma(2n)} (\sinh r)^{4n} \int_0^1 h\left((1-u)\tanh^2 r\right) u^{2n-1} F(\nu-1,\nu'-1;2n;\varepsilon u)\,du. \tag{5.6}$$

Since

$$\frac{\Gamma(\gamma)}{\Gamma(\gamma-l)} \lambda^{\gamma-1-l} F(\alpha,\beta;\gamma-l;\lambda) = \frac{d^l}{d\lambda^l}\left(\lambda^{\gamma-1} F(\alpha,\beta;\gamma;\lambda)\right) \tag{5.7}$$

(see [30, Subsection 2.8, formula (22)]), we have

$$\widetilde{T}_{r,s,m}(\lambda) = \frac{\pi^{2n}(\sinh r)^{4n}}{\Gamma(2n+s-m)} \times \int_0^1 h((1-u)\tanh^2 r) \left(\frac{d}{du}\right)^{s-m} (u^{2n+s-m-1} F(\nu-1,\nu'-1;2n+s-m;\varepsilon u))\,du.$$

Integration by parts yields

$$\widetilde{T}_{r,s,m}(\lambda) = (-1)^{s-m} \frac{\pi^{2n}(\sinh r)^{4n}}{\Gamma(2n+s-m)} \times \int_0^1 \left(\frac{d}{du}\right)^{s-m} (h((1-u)\tanh^2 r)) u^{2n+s-m-1} F(\nu-1,\nu'-1;2n+s-m;\varepsilon u)\,du,$$

whence

$$\widetilde{T}_{r,s,m}(\lambda) = \frac{\pi^{2n}\Gamma(s)}{\Gamma(m)\Gamma(2n+s-m)} (\sinh r)^{4n+2s-2} (\cosh r)^{4-2m} \times \int_0^1 u^{2n+s-m-1}(1-u)^{m-1}(1-\varepsilon u)^{m-s-1} F(\nu-1,\nu'-1;2n+s-m;\varepsilon u)\,du \tag{5.8}$$

(see (5.7) and also [30, Subsection 2.8, formula (4)]). By (5.5), we have

$$\widetilde{T}_{r,s,m}(\lambda) = \frac{\pi^{2n}\Gamma(s)}{\Gamma(m)\Gamma(2n+s-m)} (\sinh r)^{4n+2s-2} (\cosh r)^{4-2m} \times \int_0^1 u^{2n+s-m-1}(1-u)^{m-1} F(\nu+s-m,\nu'+s-m;2n+s-m;\varepsilon u)\,du. \tag{5.9}$$

Using the formula

$$F(\alpha,\beta;\gamma;\lambda) = \frac{\Gamma(\gamma)}{\Gamma(\delta)\Gamma(\gamma-\delta)} \int_0^1 u^{\delta-1}(1-u)^{\gamma-\delta-1} F(\alpha,\beta;\delta;u\lambda)\,du,$$

where $\operatorname{Re}\gamma > \operatorname{Re}\delta > 0$, $\lambda \neq 1$, and $|\arg(1-\lambda)| < \pi$ (see [30, Subsection 2.4, formula (2)]), from (5.9) and the definition of $\varphi_\lambda^{(\alpha,\beta)}(t)$ we obtain the assertion of the lemma for $m \in \mathbb{N}$.

2) Now, suppose that $m = 0$. Then, as before,

$$\widetilde{T}_{r,s,0}(\lambda) = \frac{\pi^{2n}}{\Gamma(2n)} (\sinh r)^{4n+2s-2} (\cosh r)^{2-2s} \times \int_0^1 u^{2n-1}(1-u)^{s-1}(1-\varepsilon u) F(\nu,\nu';2n;\varepsilon u)\,du.$$

Thus,

$$\begin{aligned}\widetilde{T}_{r,s,0}(\lambda)& \\ &= \frac{\pi^{2n}}{\Gamma(2n+s)}(\sinh r)^{4n+2s-2}(\cosh r)^{2-2s} \\ &\qquad \times \int_0^1 (1-u)^{s-1}\left(\frac{d}{du}\right)^s (u^{2n+s-1}(1-\varepsilon u)^{s+1}F(\nu+s,\nu'+s;2n+s;\varepsilon u))\,du,\end{aligned}\tag{5.10}$$

because

$$\begin{aligned}\frac{\Gamma(\gamma)}{\Gamma(\gamma-l)}&\lambda^{\gamma-1-l}(1-\lambda)^{\alpha+\beta-\gamma-l}F\left(\alpha-l,\beta-l;\gamma-l;\lambda\right) \\ &= \frac{d^l}{d\lambda^l}\left(\lambda^{\gamma-1}(1-\lambda)^{\alpha+\beta-\gamma}F\left(\alpha,\beta;\gamma;\lambda\right)\right)\end{aligned}$$

(see [30, Subsection 2.8, formula (27)]). Integrating by parts, we obtain

$$\begin{aligned}\widetilde{T}_{r,s,0}(\lambda) &= \frac{\pi^{2n}\Gamma(s)}{\Gamma(2n+s)}(\sinh r)^{4n+2s-2}(\cosh r)^{2-2s} \\ &\qquad \times \int_0^1 \frac{d}{du}\left(u^{2n+s-1}(1-\varepsilon u)^{s+1}F\left(\nu+s,\nu'+s;2n+s;\varepsilon u\right)\right)\,du \\ &= \frac{\pi^{2n}\Gamma(s)}{\Gamma(2n+s)}(\sinh r)^{4n+2s-2}(\cosh r)^{4}F\left(\nu+s,\nu'+s;2n+s;-\sinh^2 r\right) \\ &= \pi^{2n}\Gamma(s)(\sinh r)^{4n+2s-2}(\cosh r)^4 G_{s,0}(\lambda,r),\end{aligned}$$

and Lemma 5.2 is proved. □

For $H_{p,q,m}\in\mathcal{H}(p,q,m)$, we denote by $H_{p,q,m}\sigma_r$ the simple layer on S_r with surface density $H_{p,q,m}$ (see, e.g., [31, Chapter 1, Subsection 1.7]).

Lemma 5.3. *For $s\in\mathbb{N}$, we have*

$$H_{p,q,m}\sigma_r = \frac{\Gamma(2n)}{\pi^{2n}\Gamma(s)(\sinh r)^{4n-2}(\cosh r)^4}\mathcal{W}_{p,q,m}\times T_{r,s,m}.$$

Proof. Since

$$\begin{aligned}F\left(\alpha,\beta;\gamma;\lambda\right) &= (1-\lambda)^{-\alpha}F\left(\alpha,\gamma-\beta;\gamma;\frac{\lambda}{\lambda-1}\right) \\ &= (1-\lambda)^{-\beta}F\left(\gamma-\alpha,\beta;\gamma;\frac{\lambda}{\lambda-1}\right)\end{aligned}\tag{5.11}$$

(see [30, Subsection 2.1, formula (22)]), Lemma 4.3 implies

$$\widetilde{H_{p,q,m}}\sigma_r(\lambda,\omega) = \frac{\Gamma(\nu+s-m)\Gamma(2n)}{\Gamma(\nu-m)}(\sinh r)^{2s}(\cosh r)^{-2m}G_{s,m}(\lambda,r)H_{p,q,m}(\omega).\tag{5.12}$$

By (5.1) and (5.2), we obtain

$$\widetilde{H_{p,q,m}}\sigma_r = \frac{\Gamma(2n)}{\pi^{2n}\Gamma(s)(\sinh r)^{4n-2}(\cosh r)^4}\widetilde{\mathcal{W}}_{p,q,m}\widetilde{T}_{r,s,m},$$

which proves the lemma. □

Corollary 5.1. *Suppose $f\in C^\infty(B_R)$ and $0<t<R$. Then for $s\in\mathbb{N}$ we have*

$$f^k_{p,q,m}(\tanh t) = \frac{\Gamma(2n)(\cosh t)^{s-4}}{\pi^{2n}\Gamma(s)(\sinh t)^{4n+s-2}}\langle\overline{\mathcal{W}^k_{p,q,m}}\times T_{t,s,m},f\rangle,\tag{5.13}$$

where $\mathcal{W}^k_{p,q,m}=(-1)^s D_{p,q,m}(k)\delta_0$.

Proof. This is deduced from (2.3) and Lemma 5.3 with the help of simple transformations. □

§6. AUXILIARY FACTS ABOUT THE FUNCTIONS $G_{s,m}(\lambda,t)$

For $\alpha,\beta\geq 0$ and $t>0$, we define

$$\mathcal{N}_{\alpha,\beta}(t):=\{\lambda\mid\varphi_\lambda^{(\alpha,\beta)}(t)=0\}.$$

Lemma 6.1. *The following assertions are true*:

1) *the set* $\mathcal{N}_{\alpha,\beta}(t)$ *is symmetric with respect to the point* $\lambda=0$;

2) *if* $\lambda_0\in\left(\mathcal{N}_{\alpha,\beta}(t)\cap\mathbb{R}^1\right)\setminus\{0\}$, *then* λ_0 *is a simple zero of* $\varphi_\lambda^{(\alpha,\beta)}(t)$;

3) $\mathcal{N}_{\alpha,\beta}(t)\cap\mathcal{N}_{\alpha+1,\beta+1}(t)=\varnothing$;

4) *if* $\alpha+1\geq\beta$, *then* $\varphi_{iu}^{(\alpha,\beta)}(t)>0$ *for* $u\in\mathbb{R}^1$ *and* $\mathcal{N}_{\alpha,\beta}(t)\subset\mathbb{R}^1$.

Proof. 1) The symmetry of $\mathcal{N}_{\alpha,\beta}(t)$ with respect to $\lambda=0$ follows from the relation

$$\varphi_\lambda^{(\alpha,\beta)}(t)=\varphi_{-\lambda}^{(\alpha,\beta)}(t). \tag{6.1}$$

2) Setting

$$A_{\alpha,\beta}(u)=(\sinh u)^{2\alpha+1}(\cosh u)^{2\beta+1},$$

we obtain

$$\frac{d}{du}\left(A_{\alpha,\beta}(u)\frac{d}{du}\varphi_\lambda^{(\alpha,\beta)}(u)\right)=-(\gamma^2+\lambda^2)A_{\alpha,\beta}(u)\varphi_\lambda^{(\alpha,\beta)}(u), \tag{6.2}$$

$$4(1+\alpha)\frac{d}{du}\varphi_\lambda^{(\alpha,\beta)}(u)=-(\gamma^2+\lambda^2)\sinh(2u)\varphi_\lambda^{(\alpha+1,\beta+1)}(u), \tag{6.3}$$

where $\gamma=\alpha+\beta+1$ (see (5.11) and [30, Subsection 2, Section 2.1, formulas (1) and (7)]). It follows that

$$\begin{aligned}&(\lambda^2-\lambda_0^2)A_{\alpha,\beta}(u)\varphi_{\lambda_0}^{(\alpha,\beta)}(u)\varphi_\lambda^{(\alpha,\beta)}(u)\\ &\quad=\frac{d}{du}\left(A_{\alpha,\beta}(u)\left(\varphi_\lambda^{(\alpha,\beta)}(u)\frac{d}{du}\varphi_{\lambda_0}^{(\alpha,\beta)}(u)-\varphi_{\lambda_0}^{(\alpha,\beta)}(u)\frac{d}{du}\varphi_\lambda^{(\alpha,\beta)}(u)\right)\right)\\ &\quad=\frac{d}{du}\Bigg(\frac{A_{\alpha,\beta}(u)\sinh(2u)}{4(1+\alpha)}\\ &\qquad\qquad\times\left((\gamma^2+\lambda^2)\varphi_\lambda^{(\alpha+1,\beta+1)}(u)\varphi_{\lambda_0}^{(\alpha,\beta)}(u)-(\gamma^2+\lambda_0^2)\varphi_{\lambda_0}^{(\alpha+1,\beta+1)}(u)\varphi_\lambda^{(\alpha,\beta)}(u)\right)\Bigg),\end{aligned}$$

whence

$$\begin{aligned}&4(1+\alpha)\left(\lambda^2-\lambda_0^2\right)\int_0^t A_{\alpha,\beta}(u)\varphi_\lambda^{(\alpha,\beta)}(u)\varphi_{\lambda_0}^{(\alpha,\beta)}(u)\,du\\ &\qquad=-\left(\gamma^2+\lambda_0^2\right)A_{\alpha,\beta}(t)\sinh(2t)\varphi_{\lambda_0}^{(\alpha+1,\beta+1)}(t)\varphi_\lambda^{(\alpha,\beta)}(t).\end{aligned} \tag{6.4}$$

Passing to the limit as $\lambda\to\lambda_0$, we obtain

$$\begin{aligned}&8\,(1+\alpha)\,\lambda_0\int_0^t A_{\alpha,\beta}(u)\left(\varphi_{\lambda_0}^{(\alpha,\beta)}(u)\right)^2du\\ &\qquad=-\left(\gamma^2+\lambda_0^2\right)A_{\alpha,\beta}(t)\sinh(2t)\varphi_{\lambda_0}^{(\alpha+1,\beta+1)}(t)\;\frac{d}{d\lambda}\left(\varphi_\lambda^{(\alpha,\beta)}(t)\right)\Big|_{\lambda=\lambda_0}.\end{aligned} \tag{6.5}$$

Since

$$\overline{\varphi_\lambda^{(\alpha,\beta)}(u)}=\varphi_{\overline{\lambda}}^{(\alpha,\beta)}(u), \tag{6.6}$$

formula (6.5) implies $\frac{d}{d\lambda}\left(\varphi_\lambda^{(\alpha,\beta)}(t)\right)\big|_{\lambda=\lambda_0}\neq 0$, i.e., the multiplicity of λ_0 is equal to 1.

3) This follows from (6.2), (6.3), and the uniqueness theorem for solutions of a linear differential equation of the second order.

4) Suppose that $\alpha + 1 \geq \beta$. Since

$$\varphi_\lambda^{(\alpha,\beta)}(u) = (\cosh u)^{-i\lambda-\gamma} F\left(\frac{\gamma + i\lambda}{2}, \frac{\gamma + i\lambda}{2} - \beta; \alpha + 1; \tanh^2 u\right) \tag{6.7}$$

(see (5.11)), the definition of the hypergeometric function and identity (6.1) imply that

$$\varphi_{iu}^{(\alpha,\beta)}(t) > 0, \quad u \in \mathbb{R}^1. \tag{6.8}$$

Next, suppose that $\lambda \in \mathcal{N}_{\alpha,\beta}(t)$. Then $\lambda^2 = \overline{\lambda}^2$ (see (6.6) and (6.4)). By (6.8), we conclude that $\lambda \in \mathbb{R}^1$. Lemma 6.1 is proved. ☐

Corollary 6.1. *Let $t > 0$. Then all zeros λ of the function $G_{s,m}(\lambda, t)$ are real and simple and lie symmetrically with respect to the point $\lambda = 0$.*

In what follows, we need estimates of certain functions related to $G_{s,m}(\lambda, t)$. We set

$$P_{s,m}(\nu) := \frac{\Gamma(\nu + s - m)}{\Gamma(\nu - m)}.$$

Lemma 6.2. *Suppose $t > 0$ and $\lambda \in \mathbb{C}$. Then*

$$|G_{s,m}(\lambda, t)| \leq \frac{e^{t(2n+1+|\operatorname{Im}\lambda|)}}{\Gamma(2n)(\sinh t)^s(\cosh t)^{s-2m}|P_{s,m}(\nu)|}. \tag{6.9}$$

Proof. By the definition of $e_{\nu,\omega}$, we have

$$|e_{\nu,\omega}(z)| \leq e^{(2n+1+|\operatorname{Im}\lambda|)\operatorname{arctanh}|z|}, \quad z \in B^{4n}.$$

Since

$$G_{s,m}(\lambda, t) = \frac{\Phi_\lambda^{s,m}(\tanh t)}{\Gamma(2n)(\sinh t)^s(\cosh t)^{s-2m}P_{s,m}(\nu)}$$

(see (2.4) and (6.7)) we obtain (6.9) by applying Lemma 4.3. ☐

Lemma 6.3. *Suppose that $a_1, a_2, a_3 > 0$, $p, q \in \mathbb{Z}_+$, and $m \in \{0, \dots, \min(p, q)\}$. Let*

$$\varphi(\lambda) = \varphi_\lambda^{(2n,2)}(a_1)\varphi_\lambda^{(2n,2)}(a_2)\varphi_\lambda^{(2n-1+s,s-2m+1)}(a_3).$$

Then there exist positive constants c_1 and c_2 independent of λ such that for each integer $l \geq c_2$ we can choose $\rho_l(a_1, a_2, a_3) \in (l, l+1)$ such that the following condition is satisfied: if $|\lambda| = \rho_l(a_1, a_2, a_3)$ or $|\operatorname{Im}\lambda| \geq 1$ and $|\lambda| \geq c_2$, then

$$|\varphi(\lambda)| \geq \frac{c_1}{|\lambda|^{6n+s+\frac{1}{2}}} e^{(a_1+a_2+a_3)|\operatorname{Im}\lambda|}.$$

Proof. Since the function $\varphi(\lambda)$ is even, we can assume that $\operatorname{Re}\lambda \geq 0$. The asymptotic expansion (as $|\lambda| \to +\infty$) of the hypergeometric function (see [30, Subsection 2.3, formula (17)]) yields

$$\begin{aligned} \varphi(\lambda) = {} & \frac{c}{\lambda^{6n+s+\frac{1}{2}}} \cos\left(a_1\lambda - \frac{\pi}{4}\right) \cos\left(a_2\lambda - \frac{\pi}{4}\right) \cos\left(a_3\lambda - \frac{\pi}{4}(2s - 1)\right) \\ & + O\left(\frac{e^{(a_1+a_2+a_3)|\operatorname{Im}\lambda|}}{|\lambda|^{6n+s+\frac{3}{2}}}\right), \end{aligned} \tag{6.10}$$

where the constant c depends on s, m, a_1, a_2, and a_3. By the Lojasiewicz inequality, we have

$$|\cos\lambda| \geq \frac{\min(1, \operatorname{dist}(\lambda, E))}{\pi e} e^{|\operatorname{Im}\lambda|},$$

where $E = \left\{(2l + 1)\frac{\pi}{2}, l \in \mathbb{Z}\right\}$. Applying (6.10), we complete the proof of the lemma (see [1, Proof of Lemma 6.1]). ☐

Lemma 6.4. *Suppose* $0<r_1<r_2$, $r_1+r_2<R$, *and* $\mathcal{N}(r_1)\cap\mathcal{N}(r_2)=\varnothing$. *Let* $\{\varepsilon_M\}_{M=1}^{\infty}$ *be a strictly monotone increasing sequence of positive numbers with limit* $\frac{R}{r_1+r_2}-1$, *let* $R_M=(r_1+r_2)(1+\varepsilon_M)$, *and let* $R_0=0$. *Then, for any* $p,q\in\mathbb{Z}_+$, $m\in\{0,\dots,\min(p,q)\}$, $M\in\mathbb{N}$, *and* $t\in[R_{M-1},R_M)$, *there exist two sequences of radial distributions* $\mu_{l,i}$ ($l\in\mathbb{N}$ *and* $i=1,2$) *with the following properties*:

1) $\operatorname{supp}\mu_{l,i}\subset\overline{B}_{R_M-r_i}$ *for* $i=1,2$ *and* $l\in\mathbb{N}$;

2) *there exist positive constants*

$$C_1=C_1(p,q,m,r_1,r_2,R,\varepsilon_1,n)\quad\textit{and}\quad C_2=C_2(r_1,r_2,R,\varepsilon_1,n)$$

depending on the parameters indicated and such that for all $l\geq C_1$ *we have the inequality*

$$\begin{aligned}&|G_{s,m}(\lambda,t)-G_{1,0}(\lambda,r_1)\widetilde{\mu}_{l,1}(\lambda)-G_{1,0}(\lambda,r_2)\widetilde{\mu}_{l,2}(\lambda)|\\ &\quad\leq\frac{C_2}{l}\frac{10^s}{(\sinh t)^s(\cosh t)^{s-2m}}\frac{(1+|\lambda|)^{6n+3}}{|\nu^2P_{s,m}(\nu)|}e^{R_M|\operatorname{Im}\lambda|},\quad\lambda\in\mathbb{C}.\end{aligned}\tag{6.11}$$

Proof. Setting $\beta_M=(r_1+r_2)\varepsilon_M$ and $\rho_l=\rho_l(r_1,r_2,\beta_M)$, we consider the even entire function

$$h_l(\lambda)=\frac{1}{2\pi i}\int_{|\zeta|=\rho_l}\frac{G_{s,m}(\zeta,t)}{\zeta^{6n+2}\theta(\zeta)}\frac{\zeta^{6n+2}\theta(\zeta)-\lambda^{6n+2}\theta(\lambda)}{\zeta-\lambda}\,d\zeta,\tag{6.12}$$

where

$$\theta(\lambda)=G_{1,0}(\lambda,r_1)G_{1,0}(\lambda,r_2)G_{s,m}(\lambda,\beta_M).$$

By the Cauchy formula,

$$h_l(\lambda)+\frac{1}{2\pi i}\lambda^{6n+2}\theta(\lambda)\int_{|\zeta|=\rho_l}\frac{G_{s,m}(\zeta,t)}{\zeta^{6n+2}\theta(\zeta)}\,\frac{d\zeta}{\zeta-\lambda}=\begin{cases}G_{s,m}(\lambda,t)&\text{if } |\lambda|<\rho_l,\\0&\text{if } |\lambda|>\rho_l.\end{cases}\tag{6.13}$$

Applying Lemmas 6.2 and 6.3, we find positive constants $C_1=C_1(p,q,m,r_1,r_2,R,\varepsilon_1,n)$ and $C_2=C_2(r_1,r_2,R,\varepsilon_1,n)$ such that

$$|h_l(\lambda)-G_{s,m}(\lambda,t)|\leq\frac{C_2}{l}\frac{10^s}{(\sinh t)^s(\cosh t)^{s-2m}}\frac{(1+|\lambda|)^{6n+3}}{|\nu^2P_{s,m}(\nu)|}e^{R_M|\operatorname{Im}\lambda|},\quad\lambda\in\mathbb{C},\tag{6.14}$$

for all $l\geq C_1$ (see Lemma 6.1 and also [1, Proof of Proposition 8]). Now, we prove that the function $h_l(\lambda)$ can be written as

$$h_l(\lambda)=G_{1,0}(\lambda,r_1)\widetilde{\mu}_{l,1}(\lambda)+G_{1,0}(\lambda,r_2)\widetilde{\mu}_{l,2}(\lambda),\tag{6.15}$$

where the $\mu_{l,i}$ are some radial distributions supported in the ball $\overline{B}_{R_M-r_i}$, $i=1,2$. We put

$$h_{l,1}(\lambda)=\sum_{\substack{\alpha\in\mathcal{A}_1,\\|\alpha|<\rho_l}}\frac{G_{s,m}(\lambda,\beta_M)G_{s,m}(\alpha,t)}{(\lambda-\alpha)\alpha^{6n+2}G_{1,0}(\alpha,r_1)G_{1,0}(\alpha,r_2)}\left(\frac{d}{d\zeta}\big(G_{s,m}(\zeta,\beta_M)\big)\Big|_{\zeta=\alpha}\right)^{-1},$$

$$h_{l,k}(\lambda)=\begin{cases}\sum_{\substack{\alpha\in\mathcal{A}_k\\|\alpha|<\rho_l}}\frac{G_{1,0}(\lambda,r_{k-1})G_{s,m}(\alpha,t)}{(\lambda-\alpha)\alpha^{6n+2}G_{1,0}(\alpha,r_{4-k})G_{s,m}(\alpha,\beta_M)}\frac{1}{\frac{d}{d\zeta}(G_{1,0}(\zeta,r_{k-1}))\big|_{\zeta=\alpha}}&\text{if } k=2,3,\\ \sum_{\substack{\alpha\in\mathcal{A}_k,\\|\alpha|<\rho_l}}G_{1,0}(\lambda,r_{k-3})G_{s,m}(\lambda,\beta_M)\frac{d}{d\zeta}\Big(\frac{(\zeta-\alpha)^2G_{s,m}(\zeta,t)}{\zeta^{6n+2}\theta(\zeta)(\lambda-\zeta)}\Big)\Big|_{\zeta=\alpha}&\text{if } k=4,5,\end{cases}$$

where

$$\mathcal{A}_1=\mathcal{N}_{2n-1+s,s-2m+1}(\beta_M)\backslash\big(\mathcal{N}_{2n,2}(r_1)\cup\mathcal{N}_{2n,2}(r_2)\big),$$

$$\mathcal{A}_k=\begin{cases}\mathcal{N}_{2n,2}(r_{k-1})\backslash\mathcal{N}_{2n-1+s,s-2m+1}(\beta_M)&\text{if } k=2,3,\\ \mathcal{N}_{2n,2}(r_{k-3})\cap\mathcal{N}_{2n-1+s,s-2m+1}(\beta_M)&\text{if } k=4,5.\end{cases}$$

Using Lemma 6.1 and the residue theorem, we obtain

$$h_l(\lambda) = G_{1,0}(\lambda, r_1) f_{l,1}(\lambda) + G_{1,0}(\lambda, r_2) f_{l,2}(\lambda), \tag{6.16}$$

where

$$f_{l,1}(\lambda) = \lambda^{6n+2}\left(G_{s,m}(\lambda, \beta_M) h_{l,3}(\lambda) + G_{1,0}(\lambda, r_2) h_{l,1}(\lambda) + h_{l,5}(\lambda)\right), \tag{6.17}$$

$$\begin{aligned} f_{l,2}(\lambda) = {} & \lambda^{6n+2}\left(G_{s,m}(\lambda, \beta_M) h_{l,2}(\lambda) + h_{l,4}(\lambda)\right) \\ & + G_{1,0}(\lambda, r_1) G_{s,m}(\lambda, \beta_M) \\ & \times \operatorname{res}_{\zeta=0}\left(\frac{G_{s,m}(\zeta, t)}{\zeta^{6n+2}\theta(\zeta)}\left(\lambda^{6n+1} + \lambda^{6n}\zeta + \cdots + \zeta^{6n+1}\right)\right). \end{aligned} \tag{6.18}$$

Since $\mathcal{N}(r_1) \cap \mathcal{N}(r_2) = \varnothing$ and the sets $\mathcal{A}_k$ $(k = 1, \ldots, 5)$ are symmetric with respect to the origin, relations (6.17), (6.18), and (6.9) imply that $f_{l,1}$ and $f_{l,2}$ are even entire functions with

$$|f_{l,i}(\lambda)| \le d_i (1 + |\lambda|)^{6n+3} e^{(R_M - r_i)|\operatorname{Im}\lambda|}, \quad \lambda \in \mathbb{C}, \tag{6.19}$$

where the positive constants d_1 and d_2 do not depend on λ. Applying the Paley–Wiener theorem for the spherical transformation on $\mathbb{H}^{4n}(\mathbb{Q})$ (see [10, Chapter 4, §7] and also [13, 19]) and using (6.16), we obtain (6.15). Thus (see (6.14)), Lemma 6.4 is proved. ☐

The following result is proved in a similar way.

Lemma 6.5. *Suppose* $r > 0$ *and* $R > 2r$. *Let* $\{\varepsilon_M\}_{M=1}^{\infty}$ *be a strictly monotone increasing sequence of positive numbers with limit* $\frac{R}{2r} - 1$, *let* $R_M = 2r(1 + \varepsilon_M)$, *and let* $R_0 = 0$. *Then, for any* $p, q \in \mathbb{Z}_+$, $m \in \{0, \ldots, \min(p, q)\}$, $M \in \mathbb{N}$, *and* $t \in [R_{M-1}, R_M)$, *there exist two sequences of radial distributions* $\nu_{l,i}$ $(l \in \mathbb{N}, i = 1, 2)$ *with the following properties*:

1) $\operatorname{supp}\nu_{l,i} \subset \overline{B}_{R_M - r}$, $i = 1, 2$, $l \in \mathbb{N}$;

2) *there exist positive constants* $C_1 = C_1(p, q, m, r, R, \varepsilon_1, n)$ *and* $C_2 = C_2(r, R, \varepsilon_1, n)$ *depending on the parameters indicated and such that for all* $l \ge C_1$ *we have the inequality*

$$\begin{aligned} & |G_{s,m}(\lambda, t) - G_{1,0}(\lambda, r)\widetilde{\nu}_{l,1}(\lambda) - G_{0,0}(\lambda, r)\widetilde{\nu}_{l,2}(\lambda)| \\ & \qquad \le \frac{C_2}{l} \frac{10^s}{(\sinh t)^s (\cosh t)^{s-2m}} \frac{(1 + |\lambda|)^{6n+2}}{|\nu P_{s,m}(\nu)|} e^{R_M|\operatorname{Im}\lambda|}, \quad \lambda \in \mathbb{C}. \end{aligned}$$

§7. Proof of Theorems 2.1 and 2.2

First, we prove Theorem 2.1. Let $\{R_M\}$ be the sequence described in Lemma 6.4, and let $t \in [R_{M-1}, R_M)$. We set

$$\mathcal{U}_{l,i} := \frac{\Gamma(2n)}{\pi^{2n}} \frac{(\sinh t)^s (\cosh t)^{s-2m}}{(\sinh r_i)^{4n} (\cosh r_i)^4} \overline{\mathcal{W}^k_{p,q,m}} \times \mu_{l,i}.$$

Then (5.13) yields

$$\begin{aligned} & f^k_{p,q,m}(\tanh t) - \langle \mathcal{U}_{l,1}, f \times \chi_{r_1}\rangle - \langle \mathcal{U}_{l,2}, f \times \chi_{r_2}\rangle \\ & \qquad = 10^s \Gamma(2n) \langle \overline{\mathcal{W}^k_{p,q,m}} \times \mathcal{T}_l, f\rangle, \end{aligned} \tag{7.1}$$

where

$$\begin{aligned} \mathcal{T}_l = {} & \frac{(\sinh t)^s (\cosh t)^{s-2m}}{10^s \pi^{2n}} \\ & \times \left(\frac{T_{t,s,m}}{\Gamma(s)(\sinh t)^{4n+2s-2}(\cosh t)^{4-2m}} - \frac{\mu_{l,1} \times \chi_{r_1}}{(\sinh r_1)^{4n}(\cosh r_1)^4} - \frac{\mu_{l,2} \times \chi_{r_2}}{(\sinh r_2)^{4n}(\cosh r_2)^4}\right). \end{aligned}$$

Since

$$\widetilde{\mathcal{T}}_l(\lambda) = \frac{(\sinh t)^s (\cosh t)^{s-2m}}{10^s}\left(G_{s,m}(\lambda, t) - G_{1,0}(\lambda, r_1)\widetilde{\mu}_{l,1}(\lambda) - G_{1,0}(\lambda, r_2)\widetilde{\mu}_{l,2}(\lambda)\right)$$

(see §2 and (5.2)), relations (6.11) and (7.1) imply that

$$\begin{aligned}&|f^k_{p,q,m}(\tanh t) - \langle \mathcal{U}_{l,1}, f \times \chi_{r_1}\rangle - \langle \mathcal{U}_{l,2}, f \times \chi_{r_2}\rangle| \\ &\qquad \le \frac{C}{l}\,\frac{10^s}{(R-R_M)^{10n+2}} \max_{\substack{|\eta|+|\varkappa|\le 10n+2,\\ z\in B_{R'_M}}} \left|\frac{\partial^{|\eta|+|\varkappa|} f}{\partial z^{\eta}\partial \overline{z}^{\varkappa}}(z)\right|, \quad l \ge C_1,\end{aligned}$$

where $R'_M = \frac{2}{3}R + \frac{1}{3}R_M$, C_1 is the constant occurring in Lemma 6.4, and the constant C depends on r_1, r_2, R, n, and ε_1 (see [13, Proof of Theorem 5.2]). Thus, (2.5) is fulfilled, which proves Theorem 2.1. $\square$

Similarly, using Lemma 6.5 and setting

$$\mathcal{V}_{l,1} := \frac{\Gamma(2n)}{\pi^{2n}}\,\frac{(\sinh t)^s(\cosh t)^{s-2m}}{(\sinh r)^{4n}(\cosh r)^4}\,\overline{\mathcal{W}^k_{p,q,m}} \times \nu_{l,1},$$

$$\mathcal{V}_{l,2} := \frac{\Gamma(2n)}{2\pi^{2n}}\,\frac{(\sinh t)^s(\cosh t)^{s-2m}}{(\sinh r)^{4n-1}(\cosh r)^3}\,\overline{\mathcal{W}^k_{p,q,m}} \times \nu_{l,2},$$

we obtain Theorem 2.2. $\square$

REFERENCES

[1] C. A. Berenstein, R. Gay, and A. Yger, *Inversion of the local Pompeiu transform*, J. Anal. Math. **54** (1990), 259–287. MR 1041185 (91g:44004)

[2] C. A. Berenstein and D. Struppa, *Complex analysis and convolution equations*, Several Complex Variables. V, Itogi Nauki i Tekhniki Sovrem. Probl. Mat. Fund. Naprav., t. 54, VINITI, Moscow, 1989, pp. 5–111; English transl., Encyclopaedia Math. Sci., vol. 54, Springer-Verlag, Berlin, 1993, pp. 1–108. MR 1039621 (91d:32001)

[3] L. Zalcman, *A bibliographic survey of the Pompeiu problem*, Approximation by Solutions of Partial Differential Equations (Hanstholm, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 365, Kluwer Acad. Publ., Dordrecht, 1992, pp. 185–194. MR 1168719 (93e:26001)

[4] I. Netuka and J. Veselý, *Mean value property and harmonic functions*, Classical and Modern Potential Theory and Applications (Chateau de Bonas, 1993), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 430, Kluwer Acad. Publ., Dordrecht, 1994, pp. 359–398. MR 1321628 (96c:31001)

[5] L. Zalcman, *Supplementary bibliography to* : "*A bibliographic survey of the Pompeiu problem*", Radon Transforms and Tomography (South Hadley, MA, 2000), Contemp. Math., vol. 278, Amer. Math. Soc., Providence, RI, 2001, pp. 69–74. MR 1851479

[6] ———, *Analyticity and the Pompeiu problem*, Arch. Rational Mech. Anal. **47** (1972), 237–254. MR 0348084 (50:582)

[7] C. A. Berenstein and A. Yger, *Le problème de la déconvolution*, J. Funct. Anal. **54** (1983), no. 2, 113–160. MR 0724701 (85i:46048)

[8] C. A. Berenstein and R. Gay, *A local version of the two-circles theorem*, Israel J. Math. **55** (1986), 267–288. MR 0876395 (88f:42044)

[9] V. V. Volchkov, *Definitive version of a local two-radius theorem*, Mat. Sb. **186** (1995), no. 6, 15–34; English transl., Sb. Math. **186** (1995), no. 6, 783–802. MR 1349012 (96i:43005)

[10] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions*, Pure Appl. Math., vol. 113, Academic Press, Inc., Orlando, FL, 1984. MR 0754767 (86c:22017)

[11] V. V. Volchkov, *On a problem of Zalcman and its generalizations*, Mat. Zametki **53** (1993), no. 2, 30–36; English transl., Math. Notes **53** (1993), no. 1–2, 134–138. MR 1220807 (94i:43008)

[12] M. El Harchaoui, *Inversion de la transformation de Pompeiu dans le disque hyperbolique* (*cas de deux disques*), Publ. Mat. **37** (1993), 133–164. MR 1240929 (94m:43019)

[13] ———, *Inversion de la transformation de Pompeiu locale dans les espaces hyperboliques réel et complexe*: *cas des deux boules*, J. Anal. Math. **67** (1995), 1–37. MR 1383489 (97e:43006)

[14] V. V. Volchkov, *Two-radii theorems on spaces of constant curvature*, Dokl. Akad. Nauk **347** (1996), no. 3, 300–302; English transl., Dokl. Math. **53** (1996), no. 2, 199–201. MR 1393056 (97g:43006)

[15] Vit. V. Volchkov, *On functions with zero spherical means on quaternion hyperbolic space*, Izv. Ross. Akad. Nauk Ser. Mat. **66** (2002), no. 5, 3–32; English transl., Izv. Math. **66** (2002), no. 5, 875–903. MR 1965935 (2004e:43014)

[16] A. T. Fomenko, *Symplectic geometry: methods and applications*, Moskov. Gos. Univ., Moscow, 1988; English transl., Stud. Contemp. Math., vol. 5, Gordon and Breach Science Publishers, New York, 1988. MR 0994805 (90k:58065)

[17] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Grundlehren Math. Wiss., vol. 241, Springer-Verlag, New York–Berlin, 1980. MR 0601594 (82i:32002)

[18] C. A. Berenstein and L. Zalcman, *Pompeiu's problem on symmetric spaces*, Comment. Math. Helv. **55** (1980), 593–621. MR 0604716 (83d:43012)

[19] M. Eguchi, M. Hashizume, and K. Okamoto, *The Paley–Wiener theorem for distributions on symmetric spaces*, Hiroshima Math. J. **3** (1973), 109–120. MR 0338261 (49:3027)

[20] V. V. Volchkov, *A definitive version of the local two-radius theorem on hyperbolic spaces*, Izv. Ross. Akad. Nauk Ser. Mat. **65** (2001), no. 2, 3–26; English transl., Izv. Math. **65** (2001), no. 2, 207–229. MR 1842839 (2002g:43010)

[21] Vit. V. Volchkov, *Theorems on spherical means on complex hyperbolic spaces*, Dopov. Nats. Akad. Nauk Ukr. Mat. Prirodozn. Nauki **2000**, no. 4, 7–10. (Russian) MR 1799203 (2001k:32011)

[22] ______, *A definitive version of a local two-radii theorem on a quarternion hyperbolic space*, Dokl. Akad. Nauk **384** (2002), no. 4, 449–451. (Russian) MR 1929569 (2003i:43011)

[23] V. V. Volchkov, *A local two-radius theorem on symmetric spaces*, Dokl. Akad. Nauk **381** (2001), no. 6, 727–731. (Russian) MR 1892518 (2003b:43007)

[24] Vit. V. Volchkov and N. P. Volchkova, *Inversion of the local Pompeiu transformation on a quaternionic hyperbolic space*, Dokl. Akad. Nauk **379** (2001), no. 5, 587–590. (Russian) MR 1864615 (2002h:43011)

[25] C. A. Berenstein and M. Shahshahani, *Harmonic analysis and the Pompeiu problem*, Amer. J. Math. **105** (1983), 1217–1229. MR 0714774 (85d:32061)

[26] M. Shahshahani and A. Sitaram, *The Pompeiu problem in exterior domains in symmetric spaces*, Integral Geometry (Brunswick, Maine, 1984), Contemp. Math., vol. 63, Amer. Math. Soc., Providence, RI, 1987, pp. 267–277. MR 0876324 (88e:43008)

[27] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Math. Ser., No. 32, Princeton Univ. Press, Princeton, NJ, 1971. MR 0304972 (46:4102)

[28] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions.* Vol. 2, Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. MR 0698780 (84h:33001b)

[29] N. Ya. Vilenkin, *Special functions and theory of group representations*, 2nd ed., "Nauka", Moscow, 1991; English transl. of 1st ed., Transl. Math. Monogr., vol. 22, Amer. Math. Soc., Providence, RI, 1968. MR 0229863 (37:5429)

[30] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher transcendental functions.* Vol. 1, Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. MR 0698779 (84h:33001a)

[31] V. S. Vladimirov, *Generalized functions in mathematical physics*, "Nauka", Moscow, 1976; English transl. of 2nd Russian ed., "Mir", Moscow, 1979. MR 0564116 (80j:46062b)

Donetsk National University, Department of Mathematical Analysis and Function Theory, Ulitsa A. Malyshko 3, Donetsk, 83053, Ukraine
E-mail address: `volchkov@univ.donetsk.ua`

Donetsk National University, Department of Mathematical Analysis and Function Theory, Ulitsa A. Malyshko 3, Donetsk, 83053, Ukraine
E-mail address: `volchkov@univ.donetsk.ua`

Received 28/OCT/2002

Translated by N. YU. NETSVETAEV