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$C_{\cdot 0}$-CONTRACTIONS: A JORDAN MODEL AND LATTICES OF INVARIANT SUBSPACES

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Dedicated to the memory of Professor Yuri A. Abramovich

Abstract. A subclass $C_{\cdot 0}(C_0(P), \mathrm{fin})$ of the class of $C_{\cdot 0}$-contractions is introduced and studied. This subclass is a generalization of the subclass of $C_{\cdot 0}$-contractions with finite defect indices, and it includes the $C_{\cdot 0}$-contractions T for which $\dim \operatorname{Ker} T^* < \infty$ and the defect operator $(I - T^*T)^{1/2}$ belongs to the Hilbert–Schmidt class. For an operator of class $C_{\cdot 0}(C_0(P), \mathrm{fin})$, a Jordan model is constructed, and it is proved that the lattices of invariant subspaces remain isomorphic under the quasiaffine transformations.

Introduction

In this paper we consider operators acting on *separable* Hilbert spaces. By an operator we always mean a bounded linear map, and a subspace is always a closed linear subset of a Hilbert space.

Let T be an operator acting on a space $\mathcal{H}$. By $\operatorname{Lat} T$ we denote the *lattice of invariant subspaces* of T: $\operatorname{Lat} T = \{E \subset \mathcal{H} : TE \subset E\}$. If $E \in \operatorname{Lat} T$, then T has an upper triangular form relative to the decomposition $\mathcal{H} = E \oplus E^\perp$:

$$T = \begin{pmatrix} T|_E & P_E T|_{E^\perp} \\ 0 & P_{E^\perp} T|_{E^\perp} \end{pmatrix},$$

where P_E denotes the orthogonal projection onto the subspace E. Such a representation of T is called its *triangulation*.

Let $\mu_T = \mu(T)$ denote the *multiplicity* of the operator T, i.e., the minimum dimension of a reproducing subspace: $\mu_T = \min\{\dim E : \bigvee_{n=0}^{\infty} T^n E = \mathcal{H}\}$.

An operator T is called a *contraction* if $\|T\| \le 1$. A contraction T is said to be *absolutely continuous* if its unitary part is absolutely continuous or acts on the space $\{0\}$. For any absolutely continuous contraction T, a function calculus is defined on the Hardy class H^∞ in the unit disk $\mathbb{D}$ (see [1, III.2]). An absolutely continuous contraction T belongs to the *class* C_0 (T is a C_0-contraction, $T \in C_0$) if there exists a function $\varphi \in H^\infty$, $\varphi \not\equiv 0$, such that $\varphi(T) = \mathbb{O}$. The C_0-contractions of a certain special form (see [2, III.4.1] and also §1) are called *Jordan operators* of class C_0. Every C_0-contraction T is quasisimilar (see the definition below) to a unique Jordan operator of class C_0, which is called the Jordan model of T; see [2, III.5]. A certain property (P) for C_0-contractions will be used in what follows. Among several equivalent formulations of this property (see [2, VII.1]) we choose the following one. A C_0-contraction T acting on a space $\mathcal{H}$

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has *property* (P) if, whenever $E \in \operatorname{Lat} T$ and T is quasisimilar to the restriction $T|_E$, we must have $E = \mathcal{H}$ (see [2, VII.1.15]).

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, and let $T : \mathcal{H} \to \mathcal{H}$, $R : \mathcal{K} \to \mathcal{K}$, and $X : \mathcal{H} \to \mathcal{K}$ be operators. Suppose that X *intertwines* the operators T and R, i.e., $XT = RX$. Then the map $\mathcal{J}_X : \operatorname{Lat} T \to \operatorname{Lat} R$ given by $\mathcal{J}_X E = \operatorname{clos} XE$, $E \in \operatorname{Lat} T$, is well defined.

If X is an invertible operator, then T and R are *similar*. It is well known that similarity preserves many characteristics of the operators T and R; in particular, $\mu_T = \mu_R$, and $\mathcal{J}_X$ is an isomorphism of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} R$. We have $\mathcal{J}_X^{-1} = \mathcal{J}_{X^{-1}}$.

The property of pseudosimilarity is weaker than similarity. Operators T and R on spaces $\mathcal{H}$ and $\mathcal{K}$ are said to be *pseudosimilar* if there exist operators $X : \mathcal{H} \to \mathcal{K}$ and $Y : \mathcal{K} \to \mathcal{H}$ that intertwine T and R ($XT = RX$, $YR = TY$) and "agree" in the following sense: XY and YX belong to the algebras closed in the weak operator topology and generated by T and R, respectively, and in these algebras, the minimal ideals containing XY and YX coincide with the entire algebras. Under these conditions, X and Y are *quasiaffinities*, i.e., they have zero kernels and dense ranges. Also, $\mu_T = \mu_R$, and the maps $\mathcal{J}_X$ and $\mathcal{J}_Y$ are mutually inverse isomorphisms of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} R$. Concerning pseudosimilarity, we refer the reader to [3, 4]. If T and R are absolutely continuous contractions, ρ is an outer function of class H^∞, and X, Y satisfy $XT = RX$, $YR = TY$, $XY = \rho(R)$, $YX = \rho(T)$, then X and Y realize the pseudosimilarity of T and R (see [3, 4]).

Next, the property of quasisimilarity of operators (see [1, II] and [2, I]) is weaker than that of pseudosimilarity. Operators T and R are said to be *quasisimilar* if there exist quasiaffinities X and Y intertwining T and R: $XT = RX$, $YR = TY$; notation: $T \sim R$. In contrast to pseudosimilarity, the quasiaffinities X and Y may happen to be not related to each other. If $T \sim R$, then $\mu_T = \mu_R$. If T and R are quasisimilar weak contractions (see [5]) or contractions of class C_0 with property (P) (see [2, VII.1.21]), then $\operatorname{Lat} T$ and $\operatorname{Lat} R$ are isomorphic. However, quasisimilarity may fail to preserve lattices of invariant subspaces. Namely, there exist quasisimilar contractions of class C_0 whose lattices of invariant subspaces are not isomorphic. Unfortunately, the author could not find a reference to this fact in the literature. An example of such contractions is presented in §3. But even if quasisimilarity happens to preserve lattices of invariant subspaces, among the quasiaffinities intertwining T and R there may be no X and Y for which $\mathcal{J}_X$ and $\mathcal{J}_Y$ are mutually inverse (see [5, §4] and §7 of this paper).

A still weaker relation is established by quasiaffine transformation (see [1, II] and [2, I]): $T : \mathcal{H} \to \mathcal{H}$ is a *quasiaffine transform* of $R : \mathcal{K} \to \mathcal{K}$ if there exists a quasiaffinity $X : \mathcal{H} \to \mathcal{K}$ such that $XT = RX$; notation: $T \prec R$. If T and R are absolutely continuous contractions, $T \prec R$, and one of the contractions T and R belongs to the class C_0, then the other also belongs to C_0 and $T \sim R$ [1, III.4.6]; [2, III.2.1]. If T and R are weak contractions, then the condition $T \prec R$ also implies $T \sim R$ (see [3, 4, 5]). If $T \prec R$, then $\mu_R \le \mu_T$, but the inequality may be strict, and then T and R are not quasisimilar (see [6]). However, in this case the map $\mathcal{J}_X$ may still be an isomorphism of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} R$ (see [7]).

The notation $T \overset{ci}{\prec} R$ is used if there exists a family of operators $\{X_n\}_n$, $X_n : \mathcal{H} \to \mathcal{K}$, such that $X_n T = R X_n$, $\operatorname{Ker} X_n = \{0\}$ for all n, and $\bigvee_n X_n \mathcal{H} = \mathcal{K}$. If T is a contraction, S_ν is the unilateral shift of multiplicity ν, $1 \le \nu \le \infty$, and $T \prec S_\nu$, then $S_\nu \overset{ci}{\prec} T$ (see [8]). Suppose $\nu < \infty$ and the relation $T \prec S_\nu$ is realized by a quasiaffinity X. Then there exists a family of operators $\{Y_n\}_n$ that realizes the property $S_\nu \overset{ci}{\prec} T$ and is such that $XY_n = \delta_n(S_\nu)$ and $Y_n X = \delta_n(T)$, where $\delta_n \in H^\infty$, and the inner parts of the functions

δ_n have no common inner divisors (see [7]). Therefore, the map $\mathcal{J}_X$ is an isomorphism of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} S_\nu$; the inverse of it is the map $E \mapsto \bigvee_n Y_n E$, $E \in \operatorname{Lat} S_\nu$, $\operatorname{Lat} S_\nu \to \operatorname{Lat} T$ (see [7]). At the same time, as was shown in [6], a contraction T may satisfy $\mu_{S_\nu} < \mu_T$, and then T and S_ν will not be quasisimilar.

Finally, if for operators $T : \mathcal{H} \to \mathcal{H}$, $R : \mathcal{K} \to \mathcal{K}$, and $X : \mathcal{H} \to \mathcal{K}$ we have $XT = RX$ and $\operatorname{Ker} X = \{0\}$, then we write $T \overset{i}{\prec} R$.

The classes of contractions $C_{\alpha\beta}$, $\alpha, \beta = \cdot, 0, 1$, were introduced by Sz.-Nagy and Foiaş (see [1] and the references therein). A contraction T on a Hilbert space $\mathcal{H}$ belongs to the *class* $C_{\cdot 0}$ (T is a $C_{\cdot 0}$-contraction, $T \in C_{\cdot 0}$) if $T^{*n}x \xrightarrow[n\to\infty]{} 0$ for any $x \in \mathcal{H}$. T belongs to the *class* C_{00} (T is a C_{00}-contraction, $T \in C_{00}$) if $T \in C_{\cdot 0}$ and $T^* \in C_{\cdot 0}$. T belongs to the *class* C_{10} if $T \in C_{\cdot 0}$ and for any $x \in \mathcal{H}$, $x \neq 0$, the sequence $\{T^n x\}_{n=1}^\infty$ does not tend to zero. The *defect numbers* of T are $d_T = \dim(I - T^*T)\mathcal{H}$ and $d_{T^*} = \dim(I - TT^*)\mathcal{H}$. If $T \in C_{\cdot 0}$, then $d_T \le d_{T^*}$.

Every contraction T of class $C_{\cdot 0}$ admits a triangulation of the form

$$T = \begin{pmatrix} T_0 & * \\ 0 & T_1 \end{pmatrix}, \tag{0.1}$$

where T_0 is of class C_{00} and T_1 is of class C_{10}; see [1, II, §4].

For the contractions T of class $C_{\cdot 0}$ with $d_T < \infty$, a Jordan model J is constructed, $J = J_0 \oplus S_\nu$, where J_0 is a Jordan operator of class C_0, $\nu = d_{T^*} - d_T$, and S_ν is the unilateral shift of multiplicity ν. For the contraction T and its Jordan model J we have $J \overset{ci}{\prec} T \prec J$, where the relation $\overset{ci}{\prec}$ can be realized by at most two operators (see [6, 9, 10]). The operator J_0 is constructed in terms of the "invariant factors" of the characteristic function of T. Next, in [11] it was shown that J_0 is the Jordan model of the C_0-contraction T_0, and $T_1 \prec S_\nu$, where T_0 and T_1 are as in (0.1).

We introduce the following notation:

$C_{\cdot 0}(C_0, \mathrm{fin}) = \{T : T$ is a contraction of class $C_{\cdot 0}$, $T_0 \in C_0$, and $T_1 \prec S_k$, $k < \infty$, where T_0 and T_1 are as in (0.1)$\}$,

$C_{\cdot 0}(C_0(P), \mathrm{fin}) = \{T \in C_{\cdot 0}(C_0, \mathrm{fin})$: T_0 has property $(P)\}$.

A characterization of the C_{10}-contractions T such that $T \prec S_k$, $k < \infty$, can be found in [8].

Largely, in this paper we consider the class $C_{\cdot 0}(C_0(P), \mathrm{fin})$, which is an extension of the class of $C_{\cdot 0}$-contractions with $d_{T^*} < \infty$. For the contractions of class $C_{\cdot 0}(C_0(P), \mathrm{fin})$ we shall construct a Jordan model.

Theorem 0.1. *Suppose $T \in C_{\cdot 0}(C_0, \mathrm{fin})$, T_0 and T_1 are as in* (0.1), *J_0 is the Jordan model of the C_0-contraction T_0, $T_1 \prec S_k$, $k < \infty$, $J = J_0 \oplus S_k$. Then $J \overset{ci}{\prec} T \overset{ci}{\prec} J$, where both relations $\overset{ci}{\prec}$ can be realized by at most two operators. Also, the following assertions are true*: 1) *if $T \in C_{\cdot 0}(C_0(P), \mathrm{fin})$, then $J \overset{ci}{\prec} T \prec J$*; 2) *if J_0' is a Jordan operator of class C_0, $k' < \infty$, $J' = J_0' \oplus S_{k'}$, and $T \prec J'$, then $J = J'$.*

Our construction of the Jordan model for a $C_{\cdot 0}$-contraction T is based upon its triangulation of the form (0.1). This allows us to extend the class of $C_{\cdot 0}$-contractions T for which the Jordan model exists; in particular, the Jordan model is well defined for the $C_{\cdot 0}$-contractions T such that the operator $I - T^*T$ is of trace class and $\dim \operatorname{Ker} T^* < \infty$ (see Lemma 0.3 below and also the remarks in [12]). However, the relationship with the "invariant factors" of the characteristic function of the contraction T is not clearly seen under this approach even in the case where $d_{T^*} < \infty$, as was the case in [6, 9, 10].

Also for the contractions of class $C_{\cdot 0}(C_0(P), \mathrm{fin})$, we shall prove that the lattices of invariant subspaces are preserved under the quasiaffine transformations.

Theorem 0.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, $T : \mathcal{H} \to \mathcal{H}$ and $R : \mathcal{K} \to \mathcal{K}$ contractions of class $C_{\cdot 0}(C_0(P), \mathrm{fin})$, and $\mathcal{X} : \mathcal{H} \to \mathcal{K}$ a quasiaffinity with $\mathcal{X}T = R\mathcal{X}$. Then the map $\mathcal{J}_{\mathcal{X}} : \operatorname{Lat} T \to \operatorname{Lat} R$ is a lattice isomorphism.*

In the next lemma, simple sufficient conditions for a contraction to belong to the class $C_{\cdot 0}(C_0(P), \mathrm{fin})$ are established.

Lemma 0.3. *Let T be a contraction of class $C_{\cdot 0}$, and let $I - T^*T$ be a trace class operator. The following statements are equivalent*: 1) $T \in C_{\cdot 0}(C_0(P), \mathrm{fin})$; 2) $\dim \operatorname{Ker} T^* < \infty$.

Proof. Let $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$ be a triangulation of T of the form (0.1). By [13, Lemma 1.2], the operators $I - T_0^*T_0$ and $I - T_1^*T_1$ are of trace class. By [2, VI.3.12, VII.1.7], the contraction T_0 belongs to the class C_0 and has property (P), and $\dim \operatorname{Ker} T_0^* < \infty$. Therefore, condition 2) of the lemma is equivalent to $\dim \operatorname{Ker} T_1^* < \infty$. Suppose that condition 1) of the lemma is fulfilled; then $T_1 \prec S_k$, $k < \infty$, whence $\dim \operatorname{Ker} T_1^* = k$ (see [14] or [15]). Conversely, if 2) is true, then the conditions that $I - T_1^*T_1$ is of trace class and $k = \dim \operatorname{Ker} T_1^* < \infty$ imply $T_1 \prec S_k$ (see [14] or [15]). □

The paper is organized as follows. In §1 we collect the definitions, notation and auxiliary facts that are not presented in the Introduction. In §2 we prove a theorem about outer functions, which will be used in the construction of the Jordan model. In §3 we give an example of a contraction T such that $\operatorname{Lat} T = \operatorname{Lat} S_1$, $S_1 \prec T$, but $\mathcal{J}_X$ is not an isomorphism of the lattices $\operatorname{Lat} S_1$ and $\operatorname{Lat} T$ for any quasiaffinity X intertwining S_1 and T. On the basis of this example, we construct a quasiaffinity $\mathcal{X}$ commuting with the shift of infinite (countable) multiplicity and such that $\mathcal{J}_{\mathcal{X}}$ is not an automorphism of its lattice. In §4 we prove Theorem 0.1. In §5 we check that the parts and compressions of the contractions of classes $C_{\cdot 0}(C_0, \mathrm{fin})$ and $C_{\cdot 0}(C_0(P), \mathrm{fin})$ belong to the same classes. §6 is devoted to the proof of Theorem 0.2. In §7 we present an example of two quasisimilar $C_{\cdot 0}$-contractions with finite defect indices and such that no two lattice isomorphisms induced by intertwining quasiaffinities are inverse to each other.

§1. Definitions and preliminaries

All Hilbert spaces to be considered are assumed to be separable. By $I = I_{\mathcal{H}}$ we denote the identity operator on $\mathcal{H}$. The unitary equivalence of operators is denoted by the symbol $\cong$. The maximal common inner divisor of a family $\{f_i\}_i$ of functions, $f_i \in H^\infty$, is denoted by $\bigwedge_i f_i$.

Let $\mathcal{D}$, $\mathcal{D}'$ be Hilbert spaces; we denote by $H^2(\mathcal{D})$ the Hardy space of functions defined on $\mathbb{D}$ and taking values in $\mathcal{D}$; $H^\infty(\mathcal{D}' \to \mathcal{D})$ denotes the space of bounded analytic functions defined on $\mathbb{D}$ and taking values in the space of operators from $\mathcal{D}'$ to $\mathcal{D}$. If $\mathcal{D} = \mathbb{C}^\nu$, $1 \le \nu \le \infty$, then $H^2_\nu = H^2(\mathbb{C}^\nu) = \sum_{n=1}^{\nu} \oplus H^2$. The *unilateral shift S_ν of multiplicity ν* is the operator of multiplication by the independent variable z in the space H^2_ν, $\mu_{S_\nu} = \nu$.

Let $\theta \in H^\infty(\mathcal{D}' \to \mathcal{D})$. By $\widetilde{\theta}$ we denote the *associated function*: $\widetilde{\theta} \in H^\infty(\mathcal{D} \to \mathcal{D}')$, $\widetilde{\theta}(z) = \theta^*(\overline{z})$, $z \in \mathbb{D}$. We shall use the following terminology. The function θ is 1) *inner* if $\theta^*(\zeta)\theta(\zeta) = I_{\mathcal{D}'}$ for almost all $\zeta \in \partial\mathbb{D}$; 2) *outer* if $\operatorname{clos} \theta H^2(\mathcal{D}') = H^2(\mathcal{D})$; 3) *$*$-inner* if $\widetilde{\theta}$ is inner; 4) *$*$-outer* if $\widetilde{\theta}$ is outer; and 5) *two-sided inner* if θ is inner and $*$-inner.

Suppose that $\theta \in H^\infty(\mathcal{D}' \to \mathcal{D})$, θ is an inner function. The operator T_θ acts on the space $K_\theta = H^2(\mathcal{D}) \ominus \theta H^2(\mathcal{D}')$ by the formula $T_\theta f = P_\theta z f$, $f \in K_\theta$, where $P_\theta = P_{K_\theta}$. The operator T_θ is a $C_{\cdot 0}$-contraction, and the pure part of θ is the characteristic function of T_θ. Suppose T is a contraction of class $C_{\cdot 0}$ on a space $\mathcal{H}$, $\mathcal{D}_T = \operatorname{clos}(I - T^*T)\mathcal{H}$, $\mathcal{D}_{T^*} = \operatorname{clos}(I - TT^*)\mathcal{H}$, and $\theta_T \in H^\infty(\mathcal{D}_T \to \mathcal{D}_{T^*})$ is the characteristic function of T. Then the function θ_T is inner and $T \cong T_{\theta_T}$ (see [1, V, VI] and [2, V, 1]).

A *Jordan operator of class* C_0 is an operator of the form $J_0 = \sum_{n=1}^{\infty} \oplus T_{\theta_n}$, where the functions θ_n belong to H^∞ and are inner, and θ_{n+1} is a divisor of θ_n for all $n \geq 1$; it is possible that $\theta_n \equiv 1$ for n greater than some n_0. As was mentioned in the Introduction, any C_0-contraction is quasisimilar to its *Jordan model* J_0. For a C_0-contraction T, property (P) admits an equivalent reformulation in terms of the Jordan model $J_0 = \sum_{n=1}^{\infty} \oplus T_{\theta_n}$: T possesses property (P) if $\bigwedge_{n=1}^{\infty} \theta_n = 1$ (see [2, VII.1.9]).

The symbol $\varkappa(T)$ denotes the shift index of an operator T:

$$\varkappa(T) = \sup\{\nu : S_\nu \overset{i}{\prec} T\}$$

(see [6, §4]). Let T be a contraction of class $C_{\cdot 0}$ such that $T_0 \in C_0$ and $T_1 \prec S_\nu$, $1 \leq \nu \leq \infty$, where T_0 and T_1 are as in (0.1). Then

$$\varkappa(T) = \nu \tag{1.1}$$

(see [16, Proposition 4]). The following implication is true:

$$\text{if } T \overset{i}{\prec} R, \text{ then } \varkappa(T) \leq \varkappa(R). \tag{1.2}$$

Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, and let $T : \mathcal{H} \to \mathcal{H}$, $R : \mathcal{K} \to \mathcal{K}$, $X : \mathcal{H} \to \mathcal{K}$ be operators such that $XT = RX$. Then the map

$$\mathcal{J}_X : \operatorname{Lat} T \to \operatorname{Lat} R, \quad \mathcal{J}_X E = \operatorname{clos} XE, \ \ E \in \operatorname{Lat} T,$$

has the following properties (see [2, VII.1.19, VII.1.20]):

1) $\mathcal{J}_X$ is a lattice isomorphism if and only if $\mathcal{J}_X$ is a bijection;
2) $\mathcal{J}_X \operatorname{Lat} T = \operatorname{Lat} R$ if and only if $\mathcal{J}_{X^*}$ is injective;
3) $\mathcal{J}_X$ is injective if $E_1 = E_2$ whenever $E_1, E_2 \in \operatorname{Lat} T$, $E_1 \subset E_2$, and $\mathcal{J}_X E_1 = \mathcal{J}_X E_2$.

§2. Quasiaffine transforms of the shift S_k, $k < \infty$

Theorem 2.1. *Suppose that* $1 \leq k < \infty$, S_k *is the unilateral shift of multiplicity* k, $\mathcal{H}$ *and* $\mathcal{K}$ *are separable Hilbert spaces*, T *and* R *are contractions on* $\mathcal{H}$ *and* $\mathcal{K}$, *respectively, and* $T \prec S_k$, $R \prec S_k$. *Let* $\mathcal{X} : \mathcal{H} \to \mathcal{K}$ *be an operator with* $\operatorname{clos} \mathcal{X}\mathcal{H} = \mathcal{K}$ *and* $\mathcal{X}T = R\mathcal{X}$. *Then* $\mathcal{J}_{\mathcal{X}} : \operatorname{Lat} T \to \operatorname{Lat} R$ *is a lattice isomorphism*; *in particular*, $\operatorname{Ker} \mathcal{X} = \{0\}$.

Proof. First, we consider the case where $R = S_k$. Set $E = \operatorname{Ker} \mathcal{X}$; clearly, $E \in \operatorname{Lat} T$. Let $T = \left(\begin{smallmatrix} T' & * \\ 0 & T_1 \end{smallmatrix}\right)$ be the triangulation of T relative to the decomposition $\mathcal{H} = E \oplus E^\perp$, and let $X = \mathcal{X}|_{E^\perp}$. Obviously, X is a quasiaffinity and $XT_1 = S_k X$. By [8, Proposition 2], we have $S_k \overset{i}{\prec} T_1$, and $\left(\begin{smallmatrix} T' & 0 \\ 0 & S_k \end{smallmatrix}\right) \overset{i}{\prec} T$ by [16, Lemma 1]. Suppose that $E \neq \{0\}$. Then, since $T \prec S_k$, there exists a number l, $1 \leq l \leq k$, such that $T' \prec S_l$. By [8, Proposition 2], $S_l \overset{i}{\prec} T'$, whence $S_{l+k} = \left(\begin{smallmatrix} S_l & 0 \\ 0 & S_k \end{smallmatrix}\right) \overset{i}{\prec} T$ and $\varkappa(T) \geq l + k$. But this contradicts (1.1). Therefore, $\operatorname{Ker} \mathcal{X} = \{0\}$, and $\mathcal{X}$ is a quasiaffinity. In [7] it was proved that under these conditions $\mathcal{J}_{\mathcal{X}}$ is a lattice isomorphism.

Now, suppose that R is an arbitrary contraction satisfying the assumptions of the theorem, and that $X : \mathcal{K} \to H^2_k$ is a quasiaffinity satisfying $XR = S_k X$. Applied to $X\mathcal{X}$, the part of the theorem that we have already proved shows that $\mathcal{J}_{X\mathcal{X}}$ is an isomorphism of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} S_k$. Since $\mathcal{J}_{X\mathcal{X}} = \mathcal{J}_X \mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}_X$ is an isomorphism of the lattices $\operatorname{Lat} R$ and $\operatorname{Lat} S_k$ (see [7]), $\mathcal{J}_{\mathcal{X}}$ is an isomorphism of $\operatorname{Lat} T$ and $\operatorname{Lat} R$; in particular, $\operatorname{Ker} \mathcal{X} = \{0\}$. $\square$

Remark 2.2. In the case where the operators $I - T^*T$ and $I - R^*R$ are of finite rank or of trace class, the part of the theorem saying that $\operatorname{Ker} \mathcal{X} = \{0\}$ can be found in [14, 17].

Remark 2.3. In the case of the shift S_∞ of infinite (countable) multiplicity, the theorem is not true. An example of an operator $\mathcal{X}$ having dense range and nonzero kernel and commuting with S_∞ can be found in [17]. In §3 of the present paper we give an example of a quasiaffinity $\mathcal{X}$ commuting with S_∞ and such that the map $\mathcal{J}_\mathcal{X} : \operatorname{Lat} S_\infty \to \operatorname{Lat} S_\infty$ is not an isomorphism.

Let $1 \le k < \infty$, and let T be a contraction on a Hilbert space $\mathcal{H}$. Suppose that $X : \mathcal{H} \to H^2_k$ is a quasiaffinity for which $XT = S_k X$. Then T belongs to the class C_{10}, so that $T \cong T_\theta$, where $\theta = \theta_T$ is a function that is inner and $*$-outer, $\theta \in H^\infty(\mathcal{D}_T \to \mathcal{D}_{T^*})$. Without loss of generality, we assume that $T = T_\theta$. Set $\mathcal{D} = \mathcal{D}_{T^*}$; then there exists an outer function $\Phi \in H^\infty(\mathcal{D} \to \mathbb{C}^k)$ such that the quasiaffinity $X : K_\theta \to H^2_k$ acts by the formula $Xf = \Phi f$, $f \in K_\theta$. If $\mathcal{F}$ is a subspace of $\mathcal{D}$, $\dim \mathcal{F} = k$, then $\Phi|_\mathcal{F} \in H^\infty(\mathcal{F} \to \mathbb{C}^k)$. Let δ denote the determinant of the operator-valued function $\Phi|_\mathcal{F}$ (relative to a certain pair of orthonormal bases in the finite-dimensional spaces $\mathcal{F}$ and $\mathbb{C}^k$); it is obvious that $\delta \in H^\infty$. There exists an operator $Y : H^2_k \to K_\theta$ such that $YS_k = TY$, $XY = \delta(S_k)$, and $YX = \delta(T)$ (see [8, Lemma 3] or [7]). If $\delta \not\equiv 0$, then, obviously, $\operatorname{Ker} Y = \{0\}$.

Theorem 2.4. *Suppose that $\mathcal{D}$ is a separable Hilbert space, $1 \le k < \infty$, $\omega \in H^\infty$, ω is an inner function, $\Phi \in H^\infty(\mathcal{D} \to \mathbb{C}^k)$, and Φ is outer. Then there exists a subspace $\mathcal{F}$ of $\mathcal{D}$ such that $\dim \mathcal{F} = k$ and $\omega \wedge \delta = 1$, where $\delta = \det \Phi|_\mathcal{F}$.*

Corollary 2.5. *If $1 \le k < \infty$, T is a contraction on $\mathcal{H}$, and $T \prec S_k$, then there exist operators $Y_1, Y_2 : H^2_k \to \mathcal{H}$ such that $Y_i S_k = TY_i$, $\operatorname{Ker} Y_i = \{0\}$ for $i = 1, 2$, and $Y_1 H^2_k \vee Y_2 H^2_k = \mathcal{H}$.*

Proof. We choose a space $\mathcal{F}_1$ so that $\delta_1 = \det \Phi|_{\mathcal{F}_1} \not\equiv 0$, denote by ω the inner part of the function δ_1, and apply Theorem 2.4 to ω and Φ. As a result, we get a space $\mathcal{F}_2$ and a function $\delta_2 = \det \Phi|_{\mathcal{F}_2}$; by Theorem 2.4 we have $\delta_1 \wedge \delta_2 = 1$. Now, let Y_1, Y_2 be operators that correspond to δ_1, δ_2 in the sense explained before Theorem 2.4. Then $Y_1 H^2_k \vee Y_2 H^2_k = Y_1 X\mathcal{H} \vee Y_2 X\mathcal{H} = \delta_1(T)\mathcal{H} \vee \delta_2(T)\mathcal{H} = \mathcal{H}$, because $\delta_1 \wedge \delta_2 = 1$. □

Remark 2.6. Suppose $1 \le k < \infty$, T is a contraction, and $T \prec S_k$. Corollary 2.5 and the relation $\mu_{S_k} = k$ imply the estimate $k \le \mu_T \le 2k$. Invoking the characteristic 'disc', which is related to the lattice of invariant subspaces of an operator (see [18, 19]), we can refine the above estimate for the multiplicity of T:

$$k \le \mu_T \le k + 1. \tag{2.1}$$

Indeed, $\mu_T \le \operatorname{disc}(T)$ for any operator T (see [18, 1.5]), and $\operatorname{disc}(S_k) = k + 1$ (see [19, Corollary 18]). The following assertion follows immediately from the definition of $\operatorname{disc}(T)$. Let T and R be operators on spaces $\mathcal{H}$ and $\mathcal{K}$. Suppose that $X : \mathcal{H} \to \mathcal{K}$ is an operator such that $XT = RX$, $\mathcal{J}_X : \operatorname{Lat} T \to \operatorname{Lat} R$ is a lattice isomorphism, and $\mu_T < \infty$. Then $\operatorname{disc}(T) \le \operatorname{disc}(R)$. For $R = S_k$ we obtain (2.1). For the case where $d_{T^*} < \infty$, estimate (2.1) can be found in [20].

It is easily seen that if $1 \le k < \infty$, T is a contraction, $T \prec S_k$, and $\mu_T = k$, then $T \sim S_k$. Indeed, let $\Theta \in H^\infty(\mathbb{C}^{\nu-k} \to \mathbb{C}^\nu)$, where $k \le \nu \le \infty$, be the characteristic function of T; then $T \cong T_\Theta$, and there exist functions $f_1, \dots, f_k \in K_\Theta$ such that $\bigvee_{n \ge 0, j=1,\dots,k} T^n_\Theta f_j = K_\Theta$. Suppose that the relation $T_\Theta \prec S_k$ is realized by a quasiaffinity X. We set $\mathcal{E} = \bigvee_{n \ge 0, j=1,\dots,k} S^n_\nu f_j$ and $Y = P_\Theta : \mathcal{E} \to K_\Theta$. Obviously, the operator Y intertwines $S_\nu|_\mathcal{E}$ with T_Θ and has dense range. We have $S_\nu|_\mathcal{E} \cong S_l$ for some l with $l \le k$; let $\mathcal{U} : H^2_l \to \mathcal{E}$ be a unitary operator such that $\mathcal{U}S_l = S_\nu|_\mathcal{E}\mathcal{U}$. Put $\mathcal{X} = XY\mathcal{U}$; then $\mathcal{X}S_l = S_k\mathcal{X}$ and $\operatorname{clos} \mathcal{X}H^2_l = H^2_k$. This yields $\mu_{S_l} \ge \mu_{S_k}$, whence $l = k$, and Theorem 2.1 shows that

$\operatorname{Ker} Y\mathcal{U} = \{0\}$. Therefore, $S_k \prec T$. For the case where $d_{T^*} < \infty$, this assertion can be found in [20].

Remark 2.7. For $\dim \mathcal{D} < \infty$, Theorem 2.4 can be found in [20, Lemma 5.3], and, in a more general form, in [19, Lemma 15]. However, apparently, the proof does not admit a simple generalization to the case of $\dim \mathcal{D} = \infty$.

Remark 2.8. As will be seen from the proof of Theorem 2.4, this theorem is in fact a consequence of the lemma in [10, §2] (we state it as Lemma 2.9), because Theorem 2.4 is deduced from Lemma 2.9 with the help of simple computations.

For the proof of Theorem 2.4 we shall need the following lemmas.

Lemma 2.9 ([10, §2])**.** *Suppose* $\mathbf{I}$ *is an at most countable set,* ω *is an inner function,* $\omega \in H^\infty$, $\{f_{\mathbf{i}n} : \mathbf{i} \in \mathbf{I}, n = 1, 2, \dots\} \subset H^\infty$, $\sup\{\|f_{\mathbf{i}n}\|_\infty : \mathbf{i} \in \mathbf{I}, n = 1, 2, \dots\} < \infty$, *and* $\omega \wedge \bigwedge_{n=1,2,\dots} f_{\mathbf{i}n} = 1$ *for all* $\mathbf{i} \in \mathbf{I}$. *Then there exists a sequence* $\{x_n\}_n \subset \mathbb{C}$ *such that* $\sum_{n=1}^\infty |x_n| < \infty$ *and* $\omega \wedge \sum_{n=1}^\infty x_n f_{\mathbf{i}n} = 1$ *for all* $\mathbf{i} \in \mathbf{I}$.

In the sequel we shall use the following notation. Let $1 \le M, N < \infty$, and let I_N denote the unit matrix of size $N \times N$. Suppose $A = \{a_{ij}\}_{\substack{i=1,\dots,M\\ j=1,\dots,N}}$ is a matrix of size $M \times N$ and $1 \le K \le \min(M, N)$. For $\mathbf{i} = \{i_1, \dots, i_K\}$ and $\mathbf{j} = \{j_1, \dots, j_K\}$, where $1 \le i_1 < \dots < i_K \le M$, $1 \le j_1 < \dots < j_K \le N$, we denote by $\det\left(A|_{\mathbf{i}}^{\mathbf{j}}\right) = \det\{a_{i_u j_l}\}_{u,l=1,\dots,K}$ the corresponding minor of A, i.e., the determinant of the submatrix of A composed of the elements on the intersections of the rows with numbers $i_1, \dots, i_K$ and the columns with numbers $j_1, \dots, j_K$. If $K = N$ or $K = M$, we omit the upper or lower index, which means that in the calculation of the minor all columns or rows of A are involved. Let $\mathbf{i} = \{i_1, \dots, i_K\}$, $1 \le i_1 < \dots < i_K \le M$. We set $\mathbf{i}' = \{i'_1, \dots, i'_{M-K}\}$, where $1 \le i'_1 < \dots < i'_{M-K} \le M$, $\mathbf{i} \cup \mathbf{i}' = \{1, \dots, M\}$. Suppose $1 \le K, M, N < \infty$, and A and B are matrices of size $K \times M$ and $K \times N$, respectively. By $(A|B)$ we denote the block matrix of size $K \times (M + N)$, that is, the matrix whose first M columns coincide with the columns of A, and the remaining N columns coincide with the columns of B.

Lemma 2.10. *Let* $2 \le M < \infty$, $1 \le N \le M - 1$. *Suppose that* $B = \{b_{ul}\}_{\substack{u=1,\dots,M-1\\ l=M,\dots,M+N}}$ *is an* $(M - 1) \times (N + 1)$ *matrix, and* $(I_{M-1}|B)$ *is a block* $(M - 1) \times (M + N)$ *matrix. Let* $\mathbf{i} = \{i_1, \dots, i_{N+1}\}$, $1 \le i_1 < \dots < i_{N+1} \le M + N$, *and let the number* m, $1 \le m \le N + 1$, *be determined by the condition* $i_m \le M - 1 < i_{m+1}$. *Set* $s(M, m) = (-1)^{((M-1)M+(M-m-1)(M-m))/2}$. *Then*

$$\det((I_{M-1}|B)|^{\mathbf{i}'}) = (-1)^{i_1+\dots+i_m} s(M, m) \det\left(B|_{i_1,\dots,i_m}^{i'_{M-m},\dots,i'_{M-1}}\right).$$

The proof of Lemma 2.10 consists in standard computations and is omitted.

Lemma 2.11. *Suppose* $2 \le M < \infty$, $1 \le N \le M - 1$, *and* $\Psi = \{\varphi_{ul}\}_{\substack{u=1,\dots,M\\ l=1,\dots,M+N}}$ *is a matrix of size* $M \times (M + N)$, *where* $\varphi_{ul} \in H^\infty$ *for all* u *and* l. *Put* $g_l = \det(\Psi|^{1,\dots,M-1,l})$, $f = \det(\Psi|^{1+N,\dots,M+N})$, *and* $w = \det\left(\Psi|_{2,\dots,M}^{1,\dots,M-1}\right)$. *Then there exist functions* $h_l \in H^\infty$ *such that* $wf = \sum_{l=M}^{M+N} h_l g_l$.

Proof. Clearly, $w, f, g_l \in H^\infty$. If $w = 0$, the claim is obvious; let $w \ne 0$. We set $F = \{\varphi_{ul}\}_{\substack{u=2,\dots,M\\ l=1,\dots,M+N}}$ and $W = \{\varphi_{ul}\}_{\substack{u=2,\dots,M\\ l=1,\dots,M-1}}$. By assumption, $\det W = w \ne 0$; consequently, there exists a matrix $B = \{b_{ul}\}_{\substack{u=2,\dots,M\\ l=M,\dots,M+N}}$ of size $(M - 1) \times (N + 1)$ such that $F = W(I_{M-1}|B)$ (the elements of B are functions b_{ul} defined on the set $\{z \in \operatorname{clos} \mathbb{D} : w(z) \ne 0\}$). Applying Lemma 2.10 with $m = 1$, we easily see that $wb_{ul} \in H^\infty$ for all u, l.

The matrix B^T can be represented in the form $B^T = (A|C)$, where A is a matrix of size $(N+1)\times N$, and, accordingly, C is of size $(N+1)\times(M-N-1)$. Expanding the minors g_l of the matrix Ψ by its first row and applying Lemma 2.10, we obtain

$$\mathbf{g} = \begin{pmatrix} g_M \\ \vdots \\ g_{M+N} \end{pmatrix} = (-1)^M w\left(A \begin{pmatrix} \varphi_{11} \\ \vdots \\ \varphi_{1N} \end{pmatrix} + C \begin{pmatrix} \varphi_{1N+1} \\ \vdots \\ \varphi_{1M-1} \end{pmatrix} - \begin{pmatrix} \varphi_{1M} \\ \vdots \\ \varphi_{1M+N} \end{pmatrix} \right). \tag{2.2}$$

We put $a_l = (-1)^{N-M+l}\det(A|_{\{M,\dots,M+N\}\setminus\{l\}})$, $l = M,\dots,M+N$; then $\mathbf{a} = (a_M,\dots,a_{M+N})$ is a row of length $N+1$, and

$$\mathbf{a}A = \mathbb{O}. \tag{2.3}$$

Next, $a_l = (-1)^{N-M+l}\det\left(B|_{\{1,\dots,N\}}^{\{M,\dots,M+N\}\setminus\{l\}}\right)$, and we have

$$\det(F|^{\{1+N,\dots,M+N\}\setminus\{l\}}) = (-1)^{N(M-N)} w \det\left(B|_{\{1,\dots,N\}}^{\{M,\dots,M+N\}\setminus\{l\}}\right),$$

whence $wa_l \in H^\infty$ for $l = M,\dots,M+N$. Using formulas (2.2) and (2.3), Lemma 2.10, and the decomposition of the minor f of the matrix Ψ by the first row, we conclude that

$$w\mathbf{a}\mathbf{g} = \sum_{l=M}^{N} (wa_l) g_l = s_{MN} w f,$$

where $s_{MN} = \pm 1$ depends only on M and N. □

Lemma 2.12. *Let $\mathcal{D}$ be a Hilbert space, let $2 \le M < \infty$, and let $\Psi \in H^\infty(\mathcal{D} \to \mathbb{C}^M)$ be an outer function. Fixing orthonormal bases in the spaces $\mathcal{D}$ and $\mathbb{C}^M$, we denote the matrix of the operator-valued function Ψ relative to these bases also by Ψ. Set $w_j = \det\left(\Psi|_{\{1,\dots,M\}\setminus\{j\}}^{1,\dots,M-1}\right)$, $j = 1,\dots,M$, and $g_l = \det(\Psi|^{1,\dots,M-1,l})$, $l = M, M+1,\dots$. Then $\bigwedge_{j=1}^M w_j = \bigwedge_{l=M,M+1,\dots} g_l$.*

Proof. Put $\psi_0 = \bigwedge_{j=1}^M w_j$, $\psi = \bigwedge_{l=M,M+1,\dots} g_l$. Expanding the minors g_l by the column of Ψ with index l, we see that ψ_0 is a divisor of ψ. Now we prove that ψ divides ψ_0. Let $\ell = \{l_1,\dots,l_M\}$, $1 \le l_1 < \dots < l_M$, and let N, $1 \le N \le M-1$, be the number determined by the condition $l_{M-N-1} \le M-1 < l_{M-N}$. We set $f_\ell = \det(\Psi|^\ell)$ and fix j, $1 \le j \le M$. We rearrange the rows and columns of Ψ so that the jth row become the first, and the columns with indices $l_1,\dots,l_M$ get indices $N+1,\dots,M+N$; then w_j, f_ℓ, g_l will remain as before up to signs. By Lemma 2.11, there exist functions $h_{l_{M-N}},\dots,h_{l_M} \in H^\infty$ such that $w_j f_\ell = \sum_{n=M-N}^M h_{l_n} g_{l_n}$. Consequently, ψ is a divisor of $w_j f_\ell$ for all $j = 1,\dots,M$, $\ell = \{l_1,\dots,l_M\}$. Let $\psi_j = \psi \wedge w_j$; then $\psi = \psi_j\psi_{0j}$, and ψ_{0j} is a divisor of f_ℓ for all ℓ. Also, by assumption, ψ_{0j} is a divisor of g_l for $l = M, M+1,\dots$. Therefore, ψ_{0j} is a divisor of the inner parts of all minors of the outer matrix Ψ, which implies that $\psi_{0j} = 1$ (this well-known fact follows from the results of [10] and [11, Lemma 2.7] applied to the inner part of the function $\widetilde{\Psi}$; see also [21, I.6]). Thus, ψ is a divisor of w_j for all $j = 1,\dots,M$, and we see that ψ is a divisor of ψ_0. □

Lemma 2.13. *Suppose $\mathcal{F}$, $\mathcal{D}$ are Hilbert spaces, $1 \le k < \infty$, $\dim\mathcal{F} = k$, and $\{e_n\}_{n=1}^\infty$ is an orthonormal basis in $\mathcal{D}$. Let $A : \mathcal{D} \to \mathbb{C}^k$, $B : \mathcal{F} \to \mathcal{D}$ be operators; their matrices relative to the basis $\{e_n\}_{n=1}^\infty$ and the fixed orthonormal bases in the spaces $\mathcal{F}$ and $\mathbb{C}^k$ will still be denoted by A, B. Then $\det AB = \sum_{\substack{\mathbf{i}=\{i_1,\dots,i_k\} \\ 1\le i_1<\dots<i_k<\infty}} \det(A|^{\mathbf{i}})\det(B|_{\mathbf{i}})$.*

Proof. Set $P_N = P_{\bigvee_{n=1}^N e_n}$. It is easily seen that $AP_NB \xrightarrow[N\to\infty]{} AB$ in the strong operator topology and hence (see [22, III.6.3]), in the trace norm. Therefore (see ([22, VI.1.1] or

[2, VI, §1]), $\det AP_N B \xrightarrow[N\to\infty]{} \det AB$. It remains to apply the Binet–Cauchy formula to $\det AP_N B$. $\square$

Proof of Theorem 2.4. Let $\{\varepsilon_i\}_{i=1}^k$ be an orthonormal basis in $\mathbb{C}^k$. For $1 \le M \le k$, we put $\mathbf{I}_M = \{\mathbf{i} = \{i_1, \dots, i_M\} : 1 \le i_1 < \dots < i_M \le k\}$, and $P_{\mathbf{i}} = P_{\bigvee_{n=1}^M \varepsilon_{i_n}}$. The following fact will be proved by induction on M: in the space $\mathcal{D}$ there exists a subspace $\mathcal{F}_M$ such that $\dim \mathcal{F}_M = M$ and $\omega \wedge \det(P_{\mathbf{i}}\Phi|_{\mathcal{F}_M}) = 1$ for all $\mathbf{i} \in \mathbf{I}_M$.

The base of induction: $M = 1$. Let $\{e_l\}_{l=1,2,\dots}$ be an orthonormal basis in $\mathcal{D}$, and let $\{\varphi_{il}\}_{\substack{i=1,\dots,k\\ l=1,2,\dots}}$ be the matrix of the operator-valued function Φ relative to the bases $\{\varepsilon_i\}_{i=1}^k$ and $\{e_l\}_{l=1,2,\dots}$. Since Φ is an outer function, $\bigwedge_{l=1,2,\dots} \varphi_{il} = 1$ for all $i = 1, \dots, k$, which allows us to apply Lemma 2.9 to ω and $\{\varphi_{il}\}_{\substack{i=1,\dots,k\\ l=1,2,\dots}}$. We obtain a sequence $\{x_l\}_{l=1,2,\dots}$ such that $\omega \wedge \big(\sum_{l=1,2,\dots} x_l\varphi_{il}\big) = 1$ for all $i = 1, \dots, k$. Set $t_l = x_l/\big(\sum_{n=1,2,\dots} |x_n|^2\big)^{1/2}$ and $\mathcal{F}_1 = \mathbb{C}\big(\sum_{l=1,2,\dots} t_l e_l\big)$. Then in the basis $\{e_l\}_{l=1,2,\dots}$ the matrix of the identity embedding operator $\mathrm{id} : \mathcal{F}_1 \to \mathcal{D}$ is the column $\{t_l\}_{l=1,2,\dots}$. Therefore, $\det P_i\Phi|_{\mathcal{F}_1} = \det((P_i\Phi)\,\mathrm{id}) = \sum_{l=1,2,\dots} \varphi_{il} t_l$, $i = 1, \dots, k$.

The induction step: suppose that the assertion is true for $M - 1$. We take an orthonormal basis $\{e_l\}_{l=1,2,\dots}$ of $\mathcal{D}$ such that $\mathcal{F}_{M-1} = \bigvee_{l=1}^{M-1} e_l$. If $\mathbf{i} \in \mathbf{I}_M$, then $\Phi_{\mathbf{i}} = P_{\mathbf{i}}\Phi \in H^\infty(\mathcal{D} \to \mathbb{C}^M)$ is an outer function; we set $g_{\mathbf{i}l} = \det(\Phi_{\mathbf{i}}|^{1,\dots,M-1,l}) = \det(P_{\mathbf{i}}\Phi|^{1,\dots,M-1,l})$, $l = M, M+1, \dots$. By Lemma 2.12 applied to $\Phi_{\mathbf{i}}$ and the induction assumption, $\omega \wedge \bigwedge_{l=M,M+1,\dots} g_{\mathbf{i}l} = 1$ for all $\mathbf{i} \in \mathbf{I}_M$. Applying Lemma 2.9 to ω and $\{g_{\mathbf{i}l}\}_{\substack{\mathbf{i}\in\mathbf{I}_M\\ l=M,M+1,\dots}}$, we obtain a sequence $\{x_l\}_{l=M,M+1,\dots}$ such that $\omega \wedge \big(\sum_{l=M,M+1,\dots} x_l g_{\mathbf{i}l}\big) = 1$ for all $\mathbf{i} \in \mathbf{I}_M$. Put $t_l = x_l/\big(\sum_{n=M,M+1,\dots} |x_n|^2\big)^{1/2}$, $l = M,\ M+1, \dots$. The vectors $f_1, \dots, f_M \in \mathcal{D}$ are determined by the relation

$$\begin{pmatrix} f_1 \\ \vdots \\ f_M \end{pmatrix} \left(\begin{array}{ccc|ccc} 1 & & \mathbb{O} & & & \\ & \ddots & & & \mathbb{O} & \\ \mathbb{O} & & 1 & & & \\ \hline 0 & \dots & 0 & t_M & t_{M+1} & \dots \\ 1 & & M-1 & & & \end{array}\right) \begin{pmatrix} e_1 \\ \vdots \\ e_{M-1} \\ e_M \\ e_{M+1} \\ \vdots \end{pmatrix}, \tag{2.4}$$

$\mathcal{F}_M = \bigvee_{n=1}^M f_n$. Let $\mathrm{id} : \mathcal{F}_M \to \mathcal{D}$ denote the identical embedding, and let V be the matrix in (2.4). Then V^T is the matrix of the operator id relative to the bases $\{f_n\}_{n=1}^M$ and $\{e_l\}_{l=1,2,\dots}$. By Lemma 2.13, we have

$$\det(P_{\mathbf{i}}\Phi|_{\mathcal{F}_M}) = \det((P_{\mathbf{i}}\Phi)\,\mathrm{id}) = \sum_{\substack{\ell=\{l_1,\dots,l_M\}\\ 1\le l_1<\dots<l_M}} \det(P_{\mathbf{i}}\Phi|^{\ell}) \det(V^T|_{\ell}).$$

Next, $\det(V^T|_\ell) = \det(V|^\ell) \ne 0$ only if $\ell = \{1, \dots, M-1, l\}$, $l = M, M+1, \dots$. Consequently,

$$\det(P_{\mathbf{i}}\Phi|_{\mathcal{F}_M}) = \sum_{l=M,M+1,\dots} \det(P_{\mathbf{i}}\Phi|^{1,\dots,M-1,l}) \cdot \det(V|^{1,\dots,M-1,l}) = \sum_{l=M,M+1,\dots} g_{\mathbf{i}l} t_l.$$

By construction, $\omega \wedge \det(P_{\mathbf{i}}\Phi|_{\mathcal{F}_M}) = 1$ for all $\mathbf{i} \in \mathbf{I}_M$. $\square$

§3. Examples

Example 3.1. For $\lambda \in \mathbb{D}$, let $b_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}$, $z \in \mathbb{D}$. Suppose that $\mathcal{K} = \sum_{n=1}^\infty \oplus K_{b_\lambda^2}$ and $\mathcal{H} = K_{b_\lambda} \oplus \mathcal{K}$ are Hilbert spaces, and $J = \sum_{n=1}^\infty \oplus T_{b_\lambda^2}$ and $T = T_{b_\lambda} \oplus J$ are operators on

$\mathcal{K}$ and $\mathcal{H}$, respectively. It is not difficult to show (e.g., with the help of [2, VI.5.8]) that J is the Jordan model of the C_0-contraction T, so that T and J are quasisimilar and fail to have property (P).

Set $E = K_{b_\lambda} \oplus \{0\}$ and $\mathcal{E} = \{0\} \oplus \mathcal{K}$; then $E, \mathcal{E} \in \operatorname{Lat} T$ and the following is true: $E \cap \mathcal{E} = \{0\}$, $E \vee \mathcal{E} = \mathcal{H}$, and if $E' \in \operatorname{Lat} T$ and $E' \subset E$, then either $E' = \{0\}$ or $E' = E$. Suppose that there exist nontrivial subspaces $F, \mathcal{F} \in \operatorname{Lat} J$ possessing the same properties as E, $\mathcal{E}$. Then the C_0-contraction $J|_F$ has no nontrivial invariant subspaces, which yields (see [1, III.6.3] and [2, II.4.5]) $\dim F = 1$, that is, $F = \mathbb{C}h$, where $h \in \mathcal{K}$, $h \neq 0$, and $Jh = \lambda h$. It is easily seen that there exists a vector $g \in \mathcal{K}$ such that $Jg = h + \lambda g$. Next, we have $\mathcal{K} = F \dotplus \mathcal{F}$, whence $g = ch + f$ with $c \in \mathbb{C}$, $f \in \mathcal{F}$. Therefore, $Jg = c\lambda h + Jf = h + \lambda ch + \lambda f$, whence $h = \lambda f - Jf \in \mathcal{F}$; we arrive at a contradiction with the assumption.

Thus, the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} J$ are not isomorphic.

Example 3.2. Let $S = S_1$ be the unilateral shift (of multiplicity 1) acting on H^2, and let $b_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}$, $z \in \mathbb{D}$, $\lambda \in \mathbb{D}$. Suppose that $\varphi \in H^\infty$, $\|\varphi\|_\infty \leq 1$, φ is a generator of the algebra H^∞ in the weak* topology (see [23, 24]), and $\varphi \neq s b_\lambda$ for any $s \in \partial\mathbb{D}$, $\lambda \in \mathbb{D}$. Let $T = \varphi(S)$, $T : H^2 \to H^2$, i.e., $Th = \varphi h$, $h \in H^2$. Then

$$\operatorname{Lat} T = \operatorname{Lat} S = \{\theta H^2, \theta \in H^\infty, \theta \text{ is inner}\} \tag{3.1}$$

(see [23]), in particular, $\mu_T = 1$, i.e., T is a cyclic operator. In [25] it was shown that T is a contraction of class $C_{\cdot 0}$.

Let $\alpha = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$, and let m be normalized Lebesgue measure on $\partial\mathbb{D}$. Then $m(\alpha) < 1$, because otherwise the function φ, $\varphi \neq s b_\lambda$, would be inner and (see (3.1)) could not be a generator of H^∞.

If $m(\alpha) = 0$, then T is a contraction of class C_{00} (see [25]). If $m(\alpha) > 0$, then T is a contraction of class C_{10}, and its minimal isometric extension U satisfying $T \prec U$ is a unitary operator (namely, U is the operator of multiplication by φ in the space $L^2(\alpha)$). Since the minimal isometric extension of a C_{10}-contraction is uniquely determined, T is not a quasiaffine transform of the unilateral shift (of any multiplicity); see [25].

Since T is a cyclic absolutely continuous contraction not belonging to the class C_0, we have $S \prec T$, i.e., there exists a quasiaffinity $X : H^2 \to H^2$ such that $XS = TX$ (see, e.g., [16, §1]). However, the map $\mathcal{J}_X : \operatorname{Lat} S \to \operatorname{Lat} T = \operatorname{Lat} S$ is not a lattice isomorphism for any quasiaffinity X intertwining S and T.

To check this, set $X1 = g$; then $g \in H^2$ and g is the image of the cyclic vector 1 for S. Hence, g is a cyclic vector for T, and, by (3.1), it is an outer function. Since $XS = TX$, we have $XS^n = T^n X$ for all $n \geq 1$, whence $Xz^n = XS^n 1 = T^n X 1 = \varphi^n g$ for all $n \geq 1$. Let

$$f \in H^2, \quad f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} |a_n| < \infty. \tag{3.2}$$

It is easily seen that

$$Xf = (f \circ \varphi) \cdot g. \tag{3.3}$$

The conditions imposed on φ imply that $\operatorname{clos} \varphi(\mathbb{D}) \neq \operatorname{clos} \mathbb{D}$. Indeed, by [24, Corollary 3], we have $\varphi(\mathbb{D}) = \operatorname{int} \operatorname{clos} \varphi(\mathbb{D})$. Consequently, if $\operatorname{clos} \varphi(\mathbb{D}) = \operatorname{clos} \mathbb{D}$, then $\varphi(\mathbb{D}) = \mathbb{D}$, i.e., φ is a conformal mapping of the disk $\mathbb{D}$ onto itself. Therefore, $\varphi = s b_\lambda$ for some $s \in \partial\mathbb{D}$, $\lambda \in \mathbb{D}$, which contradicts the assumption.

We take $\lambda \in \mathbb{D} \setminus \operatorname{clos} \varphi(\mathbb{D})$ and set $E = b_\lambda H^2$. Then $E \in \operatorname{Lat} S$ and $E = \bigvee_{n=0}^{\infty} b_\lambda z^n$. Obviously, the functions $b_\lambda z^n$ satisfy (3.2) and, consequently, (3.3). Therefore, $\mathcal{J}_X E = \operatorname{clos} XE = \bigvee_{n=0}^{\infty} X b_\lambda z^n = \bigvee_{n=0}^{\infty} (\varphi \circ b_\lambda) \varphi^n g = \bigvee_{n=0}^{\infty} T^n((\varphi \circ b_\lambda) g)$. Since $\lambda \notin \operatorname{clos} \varphi(\mathbb{D})$,

we have $\inf_{\mathbb{D}} |\varphi \circ b_\lambda| > 0$, which implies that $\varphi \circ b_\lambda$ is an outer function. Therefore, $(\varphi \circ b_\lambda) \cdot g$ is an outer function, and by (3.1) it is a cyclic vector for T. Thus, $\mathcal{J}_X(b_\lambda H^2) = \bigvee_{n=0}^{\infty} T^n((\varphi \circ b_\lambda)g) = H^2$.

Example 3.3. Consider the contraction $T : H^2 \to H^2$, $Th = \varphi h$, $h \in H^2$, that occurred in Example 3.2. Since $(I - T^*T)h = P_+(1 - |\varphi|^2)h$, $h \in H^2$, we have $\operatorname{clos}(I - T^*T)H^2 = H^2$, and therefore $d_T = \infty$. Since $T \in C_{\cdot 0}$, we have $d_{T^*} \geq d_T$. Let $\theta \in H^\infty(\mathbb{C}^\infty \to \mathbb{C}^\infty)$ be the characteristic function of the contraction T; then θ is inner and $T \cong T_\theta$, where the contraction T_θ acts on the space $K_\theta = H^2_\infty \ominus \theta H^2_\infty$. Clearly, with respect to the decomposition $H^2_\infty = \theta H^2_\infty \oplus K_\theta$, the shift S_∞ has the form

$$S_\infty = \begin{pmatrix} S_\infty|_{\theta H^2_\infty} & * \\ 0 & T_\theta \end{pmatrix}.$$

As was shown in Example 3.2, there exists a quasiaffinity $X : H^2 \to K_\theta$ and a subspace $E \in \operatorname{Lat} S_1$ such that $XS_1 = T_\theta X$, $\mathcal{J}_X E = \operatorname{clos} XE = K_\theta$, and $E \neq H^2$.

It is easily seen (see [16, Lemma 1] and also the construction of the operators $\mathcal{Y}_i$ in §4 of the present paper) that there exists a quasiaffinity $\mathcal{X}_1 : \theta H^2_\infty \oplus H^2 \to H^2_\infty$ such that $\mathcal{X}_1(S_\infty|_{\theta H^2_\infty} \oplus S_1) = S_\infty \mathcal{X}_1$, and relative to the decompositions of the spaces $\theta H^2_\infty \oplus H^2$ and $H^2_\infty = \theta H^2_\infty \oplus K_\theta$, the map $\mathcal{X}_1$ has the form

$$\mathcal{X}_1 = \begin{pmatrix} I_{\theta H^2_\infty} & * \\ 0 & X \end{pmatrix}.$$

It follows that $\mathcal{J}_{\mathcal{X}_1}(\theta H^2_\infty \oplus E) = \theta H^2_\infty \oplus K_\theta = H^2_\infty$. Consequently, $\mathcal{J}_{\mathcal{X}_1}$ is not an isomorphism of the lattices $\operatorname{Lat}(S_\infty|_{\theta H^2_\infty} \oplus S_1)$ and $\operatorname{Lat} S_\infty$.

Clearly, $S_\infty|_{\theta H^2_\infty} \oplus S_1 \cong S_\infty$. Let $\mathcal{U} : \theta H^2_\infty \oplus H^2 \to H^2_\infty$ be a unitary operator such that $\mathcal{U} S_\infty = (S_\infty|_{\theta H^2_\infty} \oplus S_1)\mathcal{U}$. If $\mathcal{X} = \mathcal{X}_1 \mathcal{U}$, then $\mathcal{X} S_\infty = S_\infty \mathcal{X}$, and $\mathcal{J}_{\mathcal{X}} : \operatorname{Lat} S_\infty \to \operatorname{Lat} S_\infty$ is not an isomorphism.

§4. Construction of a Jordan model

In this section Theorem 0.1 will be proved, i.e., a Jordan model will be constructed for the contractions that belong to the class $C_{\cdot 0}(C_0(P), \mathrm{fin})$ defined in the Introduction. Although, formally, the way of constructing the model presented here differs from the method of constructing the Jordan model for the contractions T of class $C_{\cdot 0}$ with $d_T < \infty$ (see [6, 9, 10]), the basic principle is the same: the possibility to choose appropriate relatively prime functions. The construction presented here is not independent of [6, 9, 10]; moreover, it is a different interpretation of the results of [10]. Namely, the operator-valued functions acting from a (possibly) infinite-dimensional space to a finite-dimensional one are not an object (as in [10]) but a method of study. We note also that for the construction of the Jordan model presented here we use the same method as in the proof of Theorem 2 in [26]; see also [3, Lemma 1.5] and [4, §4.6].

Let T be a contraction of class $C_{\cdot 0}(C_0, \mathrm{fin})$, and let $\theta \in H^\infty(\mathcal{D}_T \to \mathcal{D}_{T^*})$ be its characteristic function; then $T \cong T_\theta$. Without loss of generality we assume that $T = T_\theta$. Suppose $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$ is a triangulation of T of the form (0.1), $\theta = \Theta\Omega$ is the $*$-canonical factorization of θ, $\Omega \in H^\infty(\mathcal{D}_T \to \mathcal{D}_T)$ is a two-sided inner function, and $\Theta \in H^\infty(\mathcal{D}_T \to \mathcal{D}_{T^*})$ is an inner and $*$-outer function (see [1, VII]). Then

$$K_\theta = \Theta K_\Omega \oplus K_\Theta \tag{4.1}$$

is the decomposition of K_θ corresponding to the triangulation $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$.

By assumption, T_0 is a C_0-contraction. Let ω denote its minimal annihilator; then ω is an inner function in H^∞. By assumption, there exists a quasiaffinity $X : K_\Theta \to H^2_k$ with $XT_1 = S_k X$, $k < \infty$. Acting in the same way as in Corollary 2.5, we construct operators

$Y_1,\ Y_2 : H^2_k \to K_\Theta$ and functions $\delta_1, \delta_2 \in H^\infty$ such that $Y_iS_k = T_1Y_i$, $XY_i = \delta_i(S_k)$, $Y_iX = \delta_i(T)$, $\omega \wedge \delta_i = 1$, $i = 1, 2$, and $\delta_1 \wedge \delta_2 = 1$ (namely, Theorem 2.4 allows us to choose δ_1 so that $\omega \wedge \delta_1 = 1$, and δ_2 so that $(\omega\delta_1) \wedge \delta_2 = 1$). Next, there exist functions $\Psi_i \in H^\infty(\mathbb{C}^k \to \mathcal{D}_{T^*})$ such that $Y_ih = P_\Theta\Psi_ih$, $h \in H^2_k$, $i = 1, 2$ (see [1, II.2.3] and [2, V.1.24]). We define operators $\widetilde{Y}_i : H^2_k \to K_\theta$ by the formulas $\widetilde{Y}_ih = P_\theta\Psi_ih$, $h \in H^2_k$. Set $\Delta_i = P_{\Theta K_\Omega}\delta_i(T)|_{K_\Theta}$; then the triangulations of the operators $\delta_i(T)$ relative to the decomposition (4.1) are of the form $\delta_i(T) = \left(\begin{smallmatrix} \delta_i(T_0) & \Delta_i \\ 0 & \delta_i(T_1) \end{smallmatrix}\right)$, $i = 1, 2$. The operators that intertwine the contractions T and $T_0 \oplus S_k = \left(\begin{smallmatrix} T_0 & 0 \\ 0 & S_k \end{smallmatrix}\right)$ relative to the decomposition (4.1) are defined by the formulas

$$\begin{gathered} \mathcal{Y}_i = \begin{pmatrix} I_{\Theta K_\Omega} & P_{\Theta K_\Omega}\widetilde{Y}_i \\ 0 & Y_i \end{pmatrix}, \\ \mathcal{X}_i = \begin{pmatrix} \delta_i(T_0) & \Delta_i - P_{\Theta K_\Omega}\widetilde{Y}_iX \\ 0 & X \end{pmatrix}, \\ \mathcal{Y}_i : \mathcal{K} \to K_\theta, \quad \mathcal{X}_i : K_\theta \to \mathcal{K}, \\ \mathcal{K} = \begin{pmatrix} \Theta K_\Omega \\ H^2_k \end{pmatrix} = \Theta K_\Omega \oplus H^2_k. \end{gathered} \tag{4.2}$$

A straightforward calculation shows that $\mathcal{X}_iT = (T_0 \oplus S_k)\mathcal{X}_i$, $\mathcal{Y}_i(T_0 \oplus S_k) = T\mathcal{Y}_i$, $\mathcal{X}_i\mathcal{Y}_i = \delta_i(T_0 \oplus S_k)$, $\mathcal{Y}_i\mathcal{X}_i = \delta_i(T)$, $\operatorname{Ker}\mathcal{Y}_i = \{0\}$, $i = 1, 2$. Next, we have $\mathcal{Y}_i\mathcal{X}_iK_\theta = \delta_i(T)K_\theta$, $i = 1, 2$, so that $\mathcal{Y}_1\mathcal{K} \vee \mathcal{Y}_2\mathcal{K} \supset \delta_1(T)K_\theta \vee \delta_2(T)K_\theta = K_\theta$ (the last-written identity follows from the relation $\delta_1 \wedge \delta_2 = 1$). Similarly, $\mathcal{X}_1K_\theta \vee \mathcal{X}_2K_\theta = \mathcal{K}$. Now we show that $\operatorname{Ker}\mathcal{X}_i = \{0\}$, $i = 1, 2$. Let $h = \left(\begin{smallmatrix} h_0 \\ h_1 \end{smallmatrix}\right) \in K_\theta = \left(\begin{smallmatrix} \Theta K_\Omega \\ K_\Theta \end{smallmatrix}\right)$, and let $\mathcal{X}_ih = 0$. Then $Xh_1 = 0$, and since $\operatorname{Ker} X = \{0\}$, we see that $h_1 = 0$ and $h = \left(\begin{smallmatrix} h_0 \\ 0 \end{smallmatrix}\right)$. Therefore, $\mathcal{X}_ih = \delta_i(T_0)h_0 = 0$. Since $\delta_i \wedge \omega = 1$, we obtain $\operatorname{Ker}\delta_i(T_0) = \{0\}$ (see [1, III.4.2] and [2, II.4.10]). Thus, $h_0 = 0$ and $\operatorname{Ker}\mathcal{X}_i = \{0\}$, $i = 1, 2$.

Remark 4.1. If $T_1 \sim S_k$, then there exists an operator $Y : H^2_k \to K_\Theta$ having the same properties as Y_i, $i = 1, 2$, and such that $\rho = \delta_1 = \delta_2$ is an outer function (see, e.g., [26, 27, 7]). Formulas (4.2) yield operators $\mathcal{X}$ and $\mathcal{Y}$ intertwining T and $T_0 \oplus S_k$ and such that $\mathcal{X}\mathcal{Y} = \rho(T_0 \oplus S_k)$, $\mathcal{Y}\mathcal{X} = \rho(T)$; this means that T and $T_0 \oplus S_k$ are pseudosimilar.

It has already been noticed in this paper that (see [2, III.5.1]) every C_0-contraction T_0 is quasisimilar to a certain Jordan operator J_0 called the Jordan model of the C_0-contraction T_0.

Thus, in the relation $T_0 \oplus S_k \overset{ci}{\prec} T \overset{ci}{\prec} T_0 \oplus S_k$ obtained above, we can replace T_0 with its Jordan model J_0 by using the quasiaffinities that realize the quasisimilarity $T_0 \sim J_0$. We conclude that $J \overset{ci}{\prec} T \overset{ci}{\prec} J$, where $J = J_0 \oplus S_k$, and each of the relations $\overset{ci}{\prec}$ can be realized by at most two operators. The principal part of Theorem 0.1 is proved.

In the original construction in [6, 10], a $C_{\cdot 0}$-contraction T with $d_T < \infty$ and its Jordan model J were connected by the relation $J \overset{ci}{\prec} T \overset{ci}{\prec} J$; in [9, 10] the property $T \overset{ci}{\prec} J$ was replaced by $T \prec J$. If T_0 is a C_0-contraction in (0.1), then $d_T = d_{T_0}$. If $d_T < \infty$ (as was the case in [6, 9, 10]), then $d_{T_0} < \infty$, and T_0 has property (P). In this paper, in order to replace $T \overset{ci}{\prec} J$ by $T \prec J$, we add property (P) to the assumptions. So, statement 1) of Theorem 0.1 is a consequence of the following lemma.

Lemma 4.2. *Suppose R_0 is a contraction of class C_0 with property (P), $1 \le \nu \le \infty$, T is a contraction of class $C_{\cdot 0}$, $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$ is a triangulation of T of the form* (0.1), *$T_0 \sim R_0$, $T_1 \prec S_\nu$, and $T \overset{i}{\prec} R_0 \oplus S_\nu$. Then $T \prec R_0 \oplus S_\nu$.*

Proof. Suppose that R_0 acts on a space $\mathcal{K}_0$ and T on a space $\mathcal{H}$. Let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ be the decomposition of $\mathcal{H}$ corresponding to the triangulation $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$, and let $\mathcal{X} : \mathcal{H} \to \mathcal{K}_0 \oplus H^2_\nu$ be an operator that realizes the relation $T \overset{i}{\prec} R_0 \oplus S_\nu$. Then $T_0 \prec (R_0 \oplus S_\nu)|_{\operatorname{clos} \mathcal{X}\mathcal{H}_0}$, and since $T_0 \in C_{00}$, we have $(R_0 \oplus S_\nu)|_{\operatorname{clos} \mathcal{X}\mathcal{H}_0} \in C_{00}$. It follows that $\operatorname{clos} \mathcal{X}\mathcal{H}_0 \subset \mathcal{K}_0$. Since T_0 and $R_0|_{\operatorname{clos} \mathcal{X}\mathcal{H}_0}$ belong to the class C_0 and $T_0 \prec R_0|_{\operatorname{clos} \mathcal{X}\mathcal{H}_0}$, we have $T_0 \sim R_0|_{\operatorname{clos} \mathcal{X}\mathcal{H}_0}$. By assumption, $T_0 \sim R_0$, whence $R_0 \sim R_0|_{\operatorname{clos} \mathcal{X}\mathcal{H}_0}$, and since R_0 has property (P), we have $\operatorname{clos} \mathcal{X}\mathcal{H}_0 = \mathcal{K}_0$ by [2, VII.1.15].

We set $\mathcal{K} = \operatorname{clos} \mathcal{X}\mathcal{H}$; then, obviously, $\mathcal{K}_0 \subset \mathcal{K}$ and $\mathcal{K} \in \operatorname{Lat}(R_0 \oplus S_\nu)$. Thus, $\mathcal{K} = \mathcal{K}_0 \oplus E$, where $E \in \operatorname{Lat} S_\nu$. Since $S_\nu|_E \cong S_{\nu_1}$ for some $\nu_1 \le \nu$, we have $R|_{\mathcal{K}} = R_0 \oplus S_\nu|_E \cong R_0 \oplus S_{\nu_1}$. The relation $T \prec R|_{\mathcal{K}}$ yields $T \prec R_0 \oplus S_{\nu_1}$. From (1.1) and (1.2) it follows that $\nu = \varkappa(T) \le \varkappa(R_0 \oplus S_{\nu_1}) = \nu_1$. Consequently, $\nu = \nu_1$. $\square$

Thus, a contraction T of class $C_{\cdot 0}(C_0(P), \mathrm{fin})$ and its Jordan model $J = J_0 \oplus S_k$, where J_0 is a Jordan operator of class C_0 with property (P), $k < \infty$, are connected by the relation

$$J \overset{ci}{\prec} T \prec J,$$

where the property $J \overset{ci}{\prec} T$ may be realized by at most two operators.

The uniqueness of a Jordan model for the contractions of class $C_{\cdot 0}(C_0(P), \mathrm{fin})$ and statement 2) of Theorem 0.1 are implied by the following fact (see [11] for the case of finite defect indices.)

Proposition 4.3. *Let T be a contraction, let $R \in C_{\cdot 0}(C_0, \mathrm{fin})$, and let $T \prec R$. Then $T \in C_{\cdot 0}(C_0, \mathrm{fin})$. If $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} R_0 & * \\ 0 & R_1 \end{smallmatrix}\right)$ are triangulations of T and R of the form* (0.1)*, and $R_1 \prec S_k$, $k < \infty$, then $T_0 \sim R_0$ and $T_1 \prec S_k$.*

Corollary 4.4. *Under the conditions of Proposition* 4.3*, the following statements are true*: 1) *the contractions T and R have the same Jordan model*; 2) *T and R belong or do not belong to the class $C_{\cdot 0}(C_0(P), \mathrm{fin})$ simultaneously.*

Corollary 4.5. *Let $J = J_0 \oplus S_k$, where J_0 is a Jordan operator of class C_0 with property (P), and $k < \infty$. If T is a contraction with $T \prec J$, then $T \in C_{\cdot 0}(C_0(P), \mathrm{fin})$, and J is the Jordan model for T.*

For the proof of Proposition 4.3 we need the following lemma.

Lemma 4.6. *Suppose T is a contraction on a space $\mathcal{H}$, R is a contraction of class $C_{\cdot 0}$ on a space $\mathcal{K}$, and $T \prec R$. Then $T \in C_{\cdot 0}$. Let $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} R_0 & * \\ 0 & R_1 \end{smallmatrix}\right)$ be triangulations of T and R of the form* (0.1)*, let $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ be the corresponding decompositions of the spaces $\mathcal{H}$ and $\mathcal{K}$, and let $\mathcal{X} : \mathcal{H} \to \mathcal{K}$ be a quasiaffinity with $\mathcal{X}T = R\mathcal{X}$. Set $K_0 = \operatorname{clos} \mathcal{X}\mathcal{H}_0$; then $K_0 \subset \mathcal{K}_0$. If $R_0 \in C_0$, $\mathcal{L} \in \operatorname{Lat} R_0$, $K_0 \subset \mathcal{L}$, $L = \mathcal{K}_0 \ominus \mathcal{L}$, and $X = P_{L \oplus \mathcal{K}_1} \mathcal{X}|_{\mathcal{H}_1}$, then $T_0 \in C_0$, X is a quasiaffinity, and $XT_1 = (P_{L \oplus \mathcal{K}_1} R|_{L \oplus \mathcal{K}_1})X$.*

Proof. The fact that T belongs to the class $C_{\cdot 0}$ follows directly from the property $T \prec R$ and the definition of the class $C_{\cdot 0}$. The inclusion $K_0 \subset \mathcal{K}_0$ can be checked in the same way as in Lemma 4.2. Next, suppose that $R_0 \in C_0$. Then $R_0|_{K_0} \in C_0$, and since $T_0 \prec R_0|_{K_0}$, we have $T_0 \in C_0$ (see [1, III.4.6] and [2, III.2.1]).

We set $R' = P_{L \oplus \mathcal{K}_1} R|_{L \oplus \mathcal{K}_1}$. The operators $\mathcal{X}$ and R have the following form relative to the decompositions $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\mathcal{K} = \mathcal{L} \oplus (L \oplus \mathcal{K}_1) : \mathcal{X} = \left(\begin{smallmatrix} * & * \\ 0 & X \end{smallmatrix}\right)$, $R = \left(\begin{smallmatrix} * & * \\ 0 & R' \end{smallmatrix}\right)$. The identities $\mathcal{X}T = R\mathcal{X}$ and $\operatorname{clos} \mathcal{X}\mathcal{H} = \mathcal{K}$ imply $XT_1 = R'X$ and $\operatorname{clos} X\mathcal{H}_1 = L \oplus \mathcal{K}_1$. Now we check that $\operatorname{Ker} X = \{0\}$. Put $E = \mathcal{H}_0 \oplus \operatorname{Ker} X$; it is easily seen that $E = \mathcal{X}^{-1}\mathcal{L} = \{h \in \mathcal{H} : \mathcal{X}h \in \mathcal{L}\}$. Therefore, $E \in \operatorname{Lat} T$, $\operatorname{clos} \mathcal{X}E \subset \mathcal{L}$, and $T|_E \prec R_0|_{\operatorname{clos} \mathcal{X}E}$. Since $R_0|_{\operatorname{clos} \mathcal{X}E} \in C_0$, we have $T|_E \in C_0$ (see [1, III.4.6] and [2, III.2.1]). Consequently, $E \subset \mathcal{H}_0$, whence $\operatorname{Ker} X = \{0\}$. $\square$

Proof of Proposition 4.3. We use the notation of Lemma 4.6. Applying Lemma 4.6 with $\mathcal{L} = \mathcal{K}_0$, we obtain $T_1 \prec R_1$, whence $T_1 \prec S_k$. Set $R^{(1)} = P_{\mathcal{K}\ominus K_0} R|_{\mathcal{K}\ominus K_0}$. Lemma 4.6 with $\mathcal{L} = K_0$ shows that $T_1 \prec R^{(1)}$. We denote $L_0 = \mathcal{K}_0 \ominus K_0$ and $R^{(0)} = R^{(1)}|_{L_0} = P_{L_0} R_0|_{L_0}$. Let ω be the minimal annihilator of the C_0-contraction $R^{(0)}$, and let $X = P_{L_0\oplus\mathcal{K}_1}\mathcal{X}|_{\mathcal{H}_1}$. Then the triangulation of $R^{(1)}$ relative to the decomposition of the space $\mathcal{K} \ominus K_0 = L_0 \oplus \mathcal{K}_1$ has the form $R^{(1)} = \left(\begin{smallmatrix} R^{(0)} & * \\ 0 & R_1 \end{smallmatrix}\right)$. Next, let $\mathfrak{X} : \mathcal{H}_1 \to H^2_k$ be a quasiaffinity such that $\mathfrak{X}T_1 = S_k\mathfrak{X}$. By Theorem 2.4, there exists a function $\delta \in H^\infty$ and an operator $\mathfrak{Y} : H^2_k \to \mathcal{H}_1$ such that $\omega \wedge \delta = 1$, $\mathfrak{Y}S_k = T_1\mathfrak{Y}$, $\mathfrak{X}\mathfrak{Y} = \delta(S_k)$, and $\mathfrak{Y}\mathfrak{X} = \delta(T_1)$. Putting $Z = X\mathfrak{Y}$, we see that $S_k \prec R^{(1)}|_{\operatorname{clos} ZH^2_k}$ and $\operatorname{clos} ZH^2_k = \operatorname{clos} X \operatorname{clos} \delta(T_1)\mathcal{H}_1 = \operatorname{clos} \delta(R^{(1)})(L_0 \oplus \mathcal{K}_1) = \operatorname{clos} \left(\begin{smallmatrix} \delta(R^{(0)}) & * \\ 0 & \delta(R_1) \end{smallmatrix}\right) \left(\begin{smallmatrix} L_0 \\ \mathcal{K}_1 \end{smallmatrix}\right)$. Since $\omega \wedge \delta = 1$, we have $\operatorname{clos} \delta(R^{(0)})L_0 = L_0$ (see [1, III.4.7] and [2, II.4.9]). Thus,

$$\operatorname{clos} ZH^2_k = \begin{pmatrix} L_0 \\ \mathcal{K}^{(1)} \end{pmatrix}$$

for some $\mathcal{K}^{(1)} \in \operatorname{Lat} R_1$ and $S_k \prec \left(\begin{smallmatrix} R^{(0)} & * \\ 0 & R_1^{(1)} \end{smallmatrix}\right)$, where $R_1^{(1)} = R_1|_{\mathcal{K}^{(1)}}$. Applying Lemma 4.6 to the contractions S_k and $\left(\begin{smallmatrix} R^{(0)} & * \\ 0 & R_1^{(1)} \end{smallmatrix}\right)$ with $\mathcal{L} = L_0$, we obtain $S_k \prec R_1^{(1)}$. Since $R_1 \prec S_k$ by assumption, we have $R_1^{(1)} \prec S_l$ for some $l \le k$. The relation $S_k \prec R_1^{(1)} \prec S_l$ implies that $l = k$ and $R_1^{(1)} \sim S_k$. By Remark 4.1, $\left(\begin{smallmatrix} R^{(0)} & * \\ 0 & R_1^{(1)} \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} R^{(0)} & 0 \\ 0 & S_k \end{smallmatrix}\right) = R^{(0)} \oplus S_k$. We have obtained the relation $S_k \prec R^{(0)} \oplus S_k$, which yields the inequality $\mu_{R^{(0)}\oplus S_k} \le \mu_{S_k} = k$. By [11, Lemma 1.2] we have $\mu_{R^{(0)}\oplus S_k} = \mu_{R^{(0)}} + k$, whence $\mu_{R^{(0)}} = 0$, i.e., $R^{(0)}$ acts on the zero space L_0. Thus, $\operatorname{clos} \mathcal{X}\mathcal{H}_0 = \mathcal{K}_0$, so that $T_0 \prec R_0$, and since T_0 and R_0 are C_0-contractions, we obtain $T_0 \sim R_0$. □

§5. Triangulations of contractions of classes $C_{\cdot 0}(C_0, \mathrm{fin})$ and $C_{\cdot 0}(C_0(P), \mathrm{fin})$

Let T be a contraction of class $C_{\cdot 0}$ with finite defect indices d_T and d_{T^*}, and let $E \in \operatorname{Lat} T$. From [1, VII.1.1, VII.3.3] and [2, V.1.21] it follows readily that $T|_E$ and $P_{E^\perp}T|_{E^\perp}$ are also contractions of class $C_{\cdot 0}$ with finite defect indices. In this section we prove an analog of this fact for the contractions in the classes $C_{\cdot 0}(C_0, \mathrm{fin})$ and $C_{\cdot 0}(C_0(P), \mathrm{fin})$.

Let $\mathcal{D}$ and $\mathcal{D}'$ be (separable) Hilbert spaces with $\dim \mathcal{D} \le \dim \mathcal{D}'$, let $\theta \in H^\infty(\mathcal{D} \to \mathcal{D}')$ be an inner function, and consider the $*$-*canonical* ($*$-*outer-inner*) factorization of θ:

(5.1) $\theta = \Theta\Omega$, where $\Omega \in H^\infty(\mathcal{D} \to \mathcal{D})$ is a two-sided inner function and $\Theta \in H^\infty(\mathcal{D} \to \mathcal{D}')$ is an inner and $*$-outer function.

It is well known (see [1, VII.3.4]) that the factorization $\theta = \Theta\Omega$ is related to the triangulation of T_θ of the form (0.1), and the corresponding decomposition of the space K_θ has the form $K_\theta = \Theta K_\Omega \oplus K_\Theta$.

For an invariant subspace E of T_θ, we consider the corresponding regular factorization of θ:

(5.2) $\theta = \theta_2\theta_1$, where $\theta_1 \in H^\infty(\mathcal{D} \to \mathcal{D}'')$ and $\theta_2 \in H^\infty(\mathcal{D}'' \to \mathcal{D}')$ are inner functions

($\mathcal{D}''$ is an auxiliary Hilbert space); see [1, VII.1.1, VII.3.3] and [2, V.1.21]. Then $E = \theta_2 K_{\theta_1}$. Let

(5.3) $\theta_1 = \Theta_1\Omega_1$, where $\Omega_1 \in H^\infty(\mathcal{D} \to \mathcal{D})$ is a two-sided inner function and $\Theta_1 \in H^\infty(\mathcal{D} \to \mathcal{D}'')$ is an inner and $*$-outer function,

be the $*$-canonical factorization of θ_1. Then

$$\Omega = \Omega_{22}\Omega_1, \text{ where } \Omega_{22} \in H^\infty(\mathcal{D} \to \mathcal{D}) \text{ is two-sided inner,} \tag{5.4}$$

$$\Theta\Omega_{22} = \theta_2\Theta_1, \tag{5.5}$$

and

$$\theta_2 K_{\theta_1} \cap \Theta K_\Omega = \Theta\Omega_{22} K_{\Omega_1}. \tag{5.6}$$

Indeed, Ω_1 is a right $*$-inner factor of θ, and, by definition, Ω is the maximal right $*$-inner factor of θ. Then Ω_1 is a right divisor of Ω. This proves (5.4) and (5.5). We set $\Xi H^2(\mathcal{D}_0) = \theta_2 H^2(\mathcal{D}'') \cap \Theta H^2(\mathcal{D})$, where $\mathcal{D}_0$ is a Hilbert space; $\Xi \in H^\infty(\mathcal{D}_0 \to \mathcal{D}')$ is an inner function. From (5.5) it follows immediately that $\Theta\Omega_{22}K_{\Omega_1} \subset \theta_2 K_{\theta_1} \cap \Theta K_\Omega$ and $(\theta_2 K_{\theta_1} \cap \Theta K_\Omega) \ominus \Theta\Omega_{22}K_{\Omega_1} = \Xi H^2(\mathcal{D}_0) \ominus \Theta\Omega_{22}H^2(\mathcal{D})$. The inclusion $\Theta\Omega_{22}H^2(\mathcal{D}) \subset \Xi H^2(\mathcal{D}_0)$ and the definition of Ξ imply that

$$\Theta\Omega_{22} = \Xi\Xi_0, \quad \Xi = \theta_2\Xi_2 = \Theta\Xi_1, \tag{5.7}$$

for some inner functions Ξ_0, Ξ_1, Ξ_2. By (5.5) and (5.7), we have $\Theta_1 = \Xi_2\Xi_0$ and $\Omega_{22} = \Xi_1\Xi_0$. Since Θ_1 is $*$-outer, Ξ_0 is $*$-outer; since Ω_{22} is $*$-inner, Ξ_0 is $*$-inner. Thus (see [1, V.23] and [2, V.1.17]), Ξ_0 is a constant function, whence $\Theta\Omega_{22}H^2(\mathcal{D}) = \theta_2 H^2(\mathcal{D}'') \cap \Theta H^2(\mathcal{D})$, and (5.6) is proved.

Now suppose that $T_\theta \in C_{\cdot 0}(C_0, \mathrm{fin})$. This means that T_Ω is a C_0-contraction and $T_\Theta \prec S_k$, $k < \infty$. This implies that Ω_{22} and Θ have scalar multiples, i.e., there exist functions $\omega, \delta \in H^\infty$, $\omega, \delta \not\equiv 0$, $A_2 \in H^\infty(\mathcal{D} \to \mathcal{D})$, and $A_0 \in H^\infty(\mathcal{D}' \to \mathcal{D})$ such that $A_2\Omega_{22} = \Omega_{22}A_2 = \omega I_{\mathcal{D}}$ and $A_0\Theta = \delta I_{\mathcal{D}}$ (see [1, III.6.1, VI.5.1, VII.2.1], [2, II.4.3, V.3.3] and [8, Proposition 1]). If $A_1 = A_2A_0\theta_2$, then $A_1 \in H^\infty(\mathcal{D}'' \to \mathcal{D})$ and $A_1\Theta_1 = A_2A_0\Theta\Omega_{22} = A_2\delta I_{\mathcal{D}}\Omega_{22} = \delta\omega I_{\mathcal{D}}$ (see (5.5)), whence $T_{\Theta_1} \prec S_\nu$, $\nu \le \infty$ [8, Proposition 1]. Since $T_{\Theta_1\Omega_1} = T_{\theta_1} \cong T_\theta|_E$, we have $T_{\theta_1} \overset{i}{\prec} T_\theta$. From (1.1) and (1.2) we deduce that $\nu = \varkappa(T_{\theta_1}) \le \varkappa(T_\theta) = k < \infty$.

Thus, we have proved the following assertion.

Proposition 5.1. *Let T be a $C_{\cdot 0}$-contraction on a space $\mathcal{H}$, and let $E \in \operatorname{Lat} T$. Suppose that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $E = E_0 \oplus E_1$ are the decompositions of $\mathcal{H}$ and E corresponding to triangulations of the $C_{\cdot 0}$-contractions T and $T|_E$ of the form* (0.1). *Then*: 1) $E_0 = E \cap \mathcal{H}_0$; 2) *if $T \in C_{\cdot 0}(C_0, \mathrm{fin})$, then $T|_E \in C_{\cdot 0}(C_0, \mathrm{fin})$*; 3) *if $T \in C_{\cdot 0}(C_0(P), \mathrm{fin})$, then $T|_E \in C_{\cdot 0}(C_0(P), \mathrm{fin})$.*

For the proof of a similar assertion about the contraction $P_{E^\perp}T|_{E^\perp}$, we need the following lemmas.

Lemma 5.2. *Suppose T is a contraction on a space $\mathcal{H}$, $1 \le k < \infty$, $T \prec S_k$, and $G \in \operatorname{Lat} T$. Let $P_{G^\perp}T|_{G^\perp} = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$ be a triangulation of the $C_{\cdot 0}$-contraction $P_{G^\perp}T|_{G^\perp}$ of the form* (0.1). *Then $T_0 \in C_0$, $\mu(T_0) \le k - l$, and $T_1 \prec S_l$, where $0 \le l \le k$.*

Proof. Let $X : \mathcal{H} \to H^2_k$ be a quasiaffinity, and let $XT = S_kX$. By Theorem 2.1 and property 2) of the map $\mathcal{J}_X$ (see §1), $\mathcal{J}_{X^*} : \operatorname{Lat} S_k^* \to \operatorname{Lat} T^*$ is a lattice isomorphism. Since $G^\perp \in \operatorname{Lat} T^*$, we have $G^\perp = \mathcal{J}_{X^*}F$ for some $F \in \operatorname{Lat} S_k^*$, and $S_k^*|_F \prec T^*|_{G^\perp}$. Therefore,

$$P_{G^\perp}T|_{G^\perp} = (T^*|_{G^\perp})^* \prec (S_k^*|_F)^* = P_F S_k|_F.$$

As is well known, there exists a number l, $0 \le l \le k$, and an inner function $\vartheta \in H^\infty(\mathbb{C}^{k-l} \to \mathbb{C}^k)$ such that $F = K_\vartheta$ and $P_F S_k|_F = T_\vartheta$. Applying Proposition 4.3 to the contractions $P_{G^\perp}T|_{G^\perp}$ and T_ϑ and using the known form of the Jordan model for T_ϑ (see [6, 9, 11]), we obtain the claim. $\square$

Lemma 5.3 ([16]). *Suppose $T \in C_{\cdot 0}(C_0, \mathrm{fin})$, $G \in \operatorname{Lat} T$, and $P_{G^\perp} T|_{G^\perp} \in C_{00}$. Then $P_{G^\perp} T|_{G^\perp} \in C_0$.*

Proof. By [16, Proposition 4] (see (1.1)), we have $\varkappa(T) < \infty$. If $P_{G^\perp} T|_{G^\perp}$ does not belong to the class C_0, then, by [16, Theorem 2], we have $S_n \overset{i}{\prec} P_{G^\perp} T|_{G^\perp}$ for any $n < \infty$. By [16, Lemma 1], we have $T|_G \oplus S_n \overset{i}{\prec} T$, whence $S_n \overset{i}{\prec} T|_G \oplus S_n \overset{i}{\prec} T$ and $\varkappa(T) \geq n$ for any n, a contradiction. □

Lemma 5.4. *Let $1 \leq k < \infty$, and suppose that $R_0 : \mathcal{H}_0 \to \mathcal{H}_0$ is a C_0-contraction with property (P), $R = R_0 \oplus S_k^*$, $\mathcal{E} \in \operatorname{Lat} R$, and $R|_{\mathcal{E}} \in C_0$. Then $R|_{\mathcal{E}}$ has property (P).*

Proof. Set $E_1 = \mathcal{E} \cap \mathcal{H}_0$, $L_1 = \mathcal{E} \cap H_k^2$, $E_2 = \operatorname{clos} P_{\mathcal{H}_0} \mathcal{E}$, $L_2 = \operatorname{clos} P_{H_k^2} \mathcal{E}$; then, obviously, $E_1, E_2 \in \operatorname{Lat} R_0$, $L_1, L_2 \in \operatorname{Lat} S_k^*$. Moreover, $E_1 \oplus L_1 \subset \mathcal{E}$ and $R|_{E_1 \oplus L_1} = R_0|_{E_1} \oplus S_k^*|_{L_1}$. Since $R|_{\mathcal{E}} \in C_0$ by assumption, we have $S_k^*|_{L_1} = R|_{L_1} \in C_0$, and $S_k^*|_{L_1}$ has property (P) because $k < \infty$. Since R_0 has property (P), so does $R_0|_{E_1}$. Thus, $R|_{E_1 \oplus L_1}$ has property (P) (see [2, VII.1.9, VII.1.17]).

Set $E = E_2 \ominus E_1$, $L = L_2 \ominus L_1$, and $\mathcal{F} = \mathcal{E} \ominus (E_1 \oplus L_1)$. By [2, VII.1.17] we see that $P_E R_0|_E$ is a C_0-contraction with property (P). By [5, Lemma 3], $P_L S_k^*|_L \in C_0$, and since $k < \infty$, $P_L S_k^*|_L$ has property (P) (see [2, VII.1.9]). Since $P_{E \oplus L} R|_{E \oplus L} = P_E R_0|_E \oplus P_L S_k^*|_L$, we conclude that $P_{E \oplus L} R|_{E \oplus L}$ has property (P) (see [2, VII.1.17]). Since $\mathcal{F} \in \operatorname{Lat} P_{E \oplus L} R|_{E \oplus L}$, we see that $(P_{E \oplus L} R|_{E \oplus L})|_{\mathcal{F}} = P_{\mathcal{F}} R|_{\mathcal{F}}$ has property (P) by [2, VII.1.17].

Application of [2, VII.1.17] to the triangulation of the C_0-contraction $R|_{\mathcal{E}}$ relative to the decomposition $\mathcal{E} = (E_1 \oplus L_1) \oplus \mathcal{F}$ shows that $R|_{\mathcal{E}}$ has property (P). □

Lemma 5.5. *Suppose $\mathcal{D}$, $\mathcal{D}'$ are Hilbert spaces, $1 \leq k < \infty$, $W \in H^\infty(\mathcal{D} \to \mathcal{D})$, T_W is a C_0-contraction with property (P), $Q \in H^\infty(\mathcal{D} \to \mathcal{D}')$, $T_Q \prec S_k$. Also, suppose $\Xi \in H^\infty(\mathcal{D} \to \mathcal{D}')$ is an inner function, $W_0 \in H^\infty(\mathcal{D}' \to \mathcal{D}')$ is two-sided inner, $QW = W_0 \Xi$, and $K_Q \cap K_{W_0} = \{0\}$. Then T_{W_0} is a C_0-contraction with property (P).*

Proof. We set $T = T_{QW}$ and apply Lemma 5.3 to T with $G^\perp = K_{W_0}$ to show that $T_{W_0} \in C_0$. Set $T_0 = T|_{QK_W}$, $R = T_0^* \oplus S_k^*$. Let $\mathcal{Y}_i : QK_W \oplus H_k^2 \to K_{QW}$ be the operator as in (4.2), $i = 1$ or $i = 2$, and let $\mathcal{E} = \operatorname{clos} \mathcal{Y}_i(QK_W \oplus H_k^2)$. Then $\mathcal{E} \in \operatorname{Lat} T$, $QK_W \subset \mathcal{E}$, and $T_0 \oplus S_k \prec T|_{\mathcal{E}}$, whence

$$P_{\mathcal{E}} T^*|_{\mathcal{E}} \prec R. \tag{5.8}$$

Set $\mathcal{F} = \operatorname{clos} P_{\mathcal{E}} K_{W_0}$; it is easily seen that $\mathcal{F} \in \operatorname{Lat} P_{\mathcal{E}} T^*|_{\mathcal{E}}$. Consider the operator $X : K_{W_0} \to \mathcal{F}$, $Xf = P_{\mathcal{E}} f$, $f \in K_{W_0}$. We shall show that $\operatorname{Ker} X = \{0\}$. If $f \in \operatorname{Ker} X$, then $f \in K_{W_0} \cap \mathcal{E}^\perp$, and since $\mathcal{E}^\perp \subset (QK_W)^\perp = K_Q$, we have $f \in K_{W_0} \cap K_Q = \{0\}$. Therefore, X is a quasiaffinity, and it is easy to check that $XT^*|_{K_{W_0}} = (P_{\mathcal{E}} T^*|_{\mathcal{E}})|_{\mathcal{F}} X$, so that $T^*|_{K_{W_0}} \prec (P_{\mathcal{E}} T^*|_{\mathcal{E}})|_{\mathcal{F}}$. From (5.8) it follows that $(P_{\mathcal{E}} T^*|_{\mathcal{E}})|_{\mathcal{F}} \prec R|_{\mathcal{L}}$ for some $\mathcal{L} \in \operatorname{Lat} R$. Hence, $T^*|_{K_{W_0}} \prec R|_{\mathcal{L}}$, and since $(T^*|_{K_{W_0}})^* = T_{W_0} \in C_0$, we have $R|_{\mathcal{L}} \in C_0$. By Lemma 5.4, $R|_{\mathcal{L}}$ has property (P). Consequently (see [2, III. 5, VII.1.9, VI.1.16]), T_{W_0} has property (P). □

Lemma 5.6. *If T is an absolutely continuous contraction, $E \in \operatorname{Lat} T$, and $P_{E^\perp} T|_{E^\perp} \in C_0$, then $\varkappa(T) = \varkappa(T|_E)$.*

Proof. Let $\mathcal{H}$ denote the space on which T acts, and let $\varphi \in H^\infty$ be the minimal annihilator of the C_0-contraction $P_{E^\perp} T|_{E^\perp}$. Let $S_n \overset{i}{\prec} T$. Then $S_n \cong S_n|_{\operatorname{clos} \varphi(S_n) H_n^2} \overset{i}{\prec} T|_{\operatorname{clos} \varphi(T)\mathcal{H}}$ by [11, Lemma 2.1]. It is easily seen that $\operatorname{clos} \varphi(T)\mathcal{H} \subset E$, hence $T|_{\operatorname{clos} \varphi(T)\mathcal{H}} = (T|_E)|_{\operatorname{clos} \varphi(T)\mathcal{H}}$, and $S_n \overset{i}{\prec} T|_E$. Thus, $\varkappa(T) \leq \varkappa(T|_E)$. The reverse inequality is obvious (see (1.2)). □

Lemma 5.7. *If T is a contraction on a space $\mathcal{H}$, $1 \le k < \infty$, $T \prec S_k$, $G \in \operatorname{Lat} T$, and $P_{G^\perp} T|_{G^\perp} \prec S_l$, where $1 \le l \le k$, then $T|_G \prec S_{k-l}$.*

Proof. We use the notation introduced in the proof of Lemma 5.2. Since $\mathcal{J}_X$ is a lattice isomorphism, it is not difficult to show that $\mathcal{J}_X G = F^\perp$. Consequently, $T|_G \prec S_k|_{\vartheta H^2_{k-l}} \cong S_{k-l}$. □

Now we return to the factorization $\theta = \theta_2\theta_1$ (see (5.2)). Let

$$\theta_2 = \Theta_2\Omega_2, \text{ where } \Omega_2 \in H^\infty(\mathcal{D}'' \to \mathcal{D}'') \text{ is a two-sided inner function and } \Theta_2 \in H^\infty(\mathcal{D}'' \to \mathcal{D}') \text{ is an inner and } *\text{-outer function,} \tag{5.9}$$

be the $*$-canonical factorization of θ_2. Then

$$\Theta = \Theta_2\Theta_{11} \tag{5.10}$$

and

$$\Omega_2\Theta_1 = \Theta_{11}\Omega_{22}, \text{ where } \Theta_{11} \in H^\infty(\mathcal{D} \to \mathcal{D}'') \text{ is inner and } *\text{-outer.} \tag{5.11}$$

Indeed, let $\Omega_2\Theta_1 = \Theta^{(1)}\Omega^{(2)}$ be the $*$-canonical factorization of $\Omega_2\Theta_1$, where $\Omega^{(2)} \in H^\infty(\mathcal{D} \to \mathcal{D})$ is two-sided inner and $\Theta^{(1)} \in H^\infty(\mathcal{D} \to \mathcal{D}'')$ is inner and $*$-outer. Then $\Theta\Omega_{22} = \theta_2\Theta_1 = \Theta_2\Omega_2\Theta_1 = \Theta_2\Theta^{(1)}\Omega^{(2)}$. Since Θ_2 and $\Theta^{(1)}$ are inner and $*$-outer, $\Theta_2\Theta^{(1)}$ is inner and $*$-outer. The uniqueness of the $*$-canonical factorization (see [1, V.3]) implies the existence of a unitary constant function $C : \mathcal{D} \to \mathcal{D}$ such that $\Theta = \Theta_2\Theta^{(1)}C^{-1}$, $\Omega_{22} = C\Omega^{(2)}$. Setting $\Theta_{11} = \Theta^{(1)}C^{-1}$, we obtain formulas (5.10) and (5.11).

Proposition 5.8. *If $T \in C_{\cdot 0}(C_0, \mathrm{fin})$ and $E \in \operatorname{Lat} T$, then $T|_E$, $P_{E^\perp}T|_{E^\perp} \in C_{\cdot 0}(C_0, \mathrm{fin})$ and*

$$\varkappa(T|_E) + \varkappa(P_{E^\perp}T|_{E^\perp}) = \varkappa(T). \tag{5.12}$$

If $T \in C_{\cdot 0}(C_0(P), \mathrm{fin})$, then $T|_E$, $P_{E^\perp}T|_{E^\perp} \in C_{\cdot 0}(C_0(P), \mathrm{fin})$.

Proof. Let θ be the characteristic function of T; then $T \cong T_\theta$. There is no loss of generality in assuming that $T = T_\theta$. The assertions about $T|_E$ have already been proved (see Proposition 5.1). Let $\theta = \theta_2\theta_1$ be the regular factorization of θ corresponding to the space E (see (5.1)–(5.6) and (5.9)–(5.11)). By assumption, $T_\Theta \prec S_k$, $k < \infty$. From (5.10) and Lemma 5.2 we obtain $T_{\Theta_2} \prec S_l$, $l \le k$. The relations (5.10) and $T_\Theta \prec S_k$ imply that $T_{\Theta_{11}} \prec S_n$, $n \le k$. We have $T_\Omega \in C_0$ by assumption. From (5.4) it follows that $T_{\Omega_{22}} \in C_0$ (see [1, III.6.1] and [2, II.4.3]). Applying Lemma 5.3 to the contraction $T_{\Theta_{11}\Omega_{22}}$ with $G^\perp = K_{\Omega_2}$ (see (5.11)), we see that $T_{\Omega_2} \in C_0$. Thus, $T_{\theta_2} = T_{\Theta_2\Omega_2} \in C_{\cdot 0}(C_0, \mathrm{fin})$.

Now, suppose that the C_0-contraction T_Ω has property (P). If $K_{\Omega_{00}} = K_{\Omega_2} \cap K_{\Theta_{11}}$, then

$$\Omega_2 = \Omega_{00}\Omega_{20} \quad \text{and} \quad \Theta_{11} = \Omega_{00}\Theta_{10}, \tag{5.13}$$

where $\Omega_{00}, \Omega_{20} \in H^\infty(\mathcal{D}'' \to \mathcal{D}'')$ are two-sided inner, and $\Theta_{10} \in H^\infty(\mathcal{D} \to \mathcal{D}'')$ is inner and $*$-outer. Clearly, $K_{\Theta_{10}} \cap K_{\Omega_{20}} = \{0\}$. Relation (5.10) implies $T_{\Theta_{11}} \prec S_m$, $m \le k$. Applying Lemma 5.2 with $G^\perp = K_{\Omega_{00}}$ to $T_{\Theta_{11}}$, we obtain $T_{\Omega_{00}} \in C_0$ and $\mu(T_{\Omega_{00}}) \le m \le k$. This implies that $T_{\Omega_{00}}$ has property (P) (see [2, VII.1.9]). Next, from (5.11) and (5.13) we obtain $\Theta_{20}\Theta_1 = \Theta_{10}\Omega_{22}$. Set $W = \Omega_{22}$, $W_0 = \Omega_{20}$, $Q = \Theta_{10}$, and $\Xi = \Theta_1$. It is easily seen that W, W_0, Q, and Ξ satisfy the assumptions of Lemma 5.5; hence, $T_{\Omega_{20}}$ has property (P). The first identity in (5.13) and [2, VII.1.17] show that T_{Ω_2} has property (P).

Now we check (5.12). From (1.1), (5.1), (5.3), and (5.9) we see that (5.12) is equivalent to the identity $\varkappa(T_{\Theta_1}) + \varkappa(T_{\Theta_2}) = \varkappa(T_\Theta)$. Relation (5.11) and Lemma 5.6 imply $\varkappa(T_{\Theta_1}) = \varkappa(T_{\Theta_{11}})$, and now (5.12) follows from (5.10) and Lemma 5.7. □

§6. Preservation of the lattice of invariant subspaces for contractions of class $C_{\cdot 0}(C_0(P),\mathrm{fin})$ under quasiaffine transformations

In this section we prove Theorem 0.2 (see the Introduction). The proof is based upon triangulations of $C_{\cdot 0}$-contractions of the form (0.1) and similar facts for C_0-contractions with property (P) (see [2, VII.1.21]) and for contractions that are quasiaffine transforms of the unilateral shift of finite multiplicity (see Theorem 2.1 of this paper).

In the following lemma we establish the surjectivity of the map $\mathcal{J}_{\mathcal{X}}$ (see properties 2) and 3) of the map $\mathcal{J}_{\mathcal{X}}$ in §1).

Lemma 6.1. *Under the assumptions of Theorem* 0.2, *suppose that* F, $\mathcal{F} \in \operatorname{Lat} R^*$, $F \subset \mathcal{F}$, *and* $\mathcal{J}_{\mathcal{X}^*}F = \mathcal{J}_{\mathcal{X}^*}\mathcal{F}$. *Then* $F = \mathcal{F}$.

Proof. Set $\mathcal{L} = \mathcal{J}_{\mathcal{X}^*}F = \mathcal{J}_{\mathcal{X}^*}\mathcal{F}$. Clearly, $\mathcal{L} \in \operatorname{Lat} T^*$, $R^*|_F \prec T^*|_{\mathcal{L}}$, and $R^*|_{\mathcal{F}} \prec T^*|_{\mathcal{L}}$. This implies that $P_{\mathcal{L}}T|_{\mathcal{L}} \prec P_F R|_F$ and $P_{\mathcal{L}}T|_{\mathcal{L}} \prec P_{\mathcal{F}}R|_{\mathcal{F}}$. By Proposition 5.8, $P_{\mathcal{L}}T|_{\mathcal{L}}$, $P_F R|_F$, $P_{\mathcal{F}}R|_{\mathcal{F}} \in C_{\cdot 0}(C_0(P),\mathrm{fin})$. By statement 1) of Corollary 4.4, the contractions $P_{\mathcal{L}}T|_{\mathcal{L}}$, $P_F R|_F$, and $P_{\mathcal{F}}R|_{\mathcal{F}}$ have the same Jordan model $J_0 \oplus S_k$, where J_0 is a Jordan operator of class C_0, $0 \le k < \infty$.

Let θ be the characteristic function of $P_{\mathcal{F}}R|_{\mathcal{F}}$, and $\theta = \theta_2\theta_1$ the regular factorization of θ corresponding to the invariant subspace $\mathcal{F} \ominus F$ of $P_{\mathcal{F}}R|_{\mathcal{F}}$ (see (5.2)–(5.5) and (5.9)–(5.11)). Since $P_{\mathcal{F}}R|_{\mathcal{F}} \cong T_\theta$ and $P_F R|_F \cong T_{\theta_2}$ (see [1, VII.1.1, VII.3.3] and [2, V.1.21]), we see that $J_0 \oplus S_k$ is a Jordan model of the contractions T_θ and T_{θ_2}. Therefore (see §4 and (5.1), (5.9)), $T_\Omega \sim J_0$, $T_{\Omega_2} \sim J_0$, $T_\Theta \prec S_k$, and $T_{\Theta_2} \prec S_k$. Applying (5.10) and (5.12) to T_Θ, we obtain $\varkappa(T_{\Theta_{11}}) = 0$. As was mentioned in the proof of Proposition 5.8, from (5.11) and Lemma 5.6 it follows that $\varkappa(T_{\Theta_1}) = \varkappa(T_{\Theta_{11}})$. This means that $T_{\Theta_1} \prec S_0$ and $T_{\Theta_{11}} \prec S_0$, that is, $K_{\Theta_1} = \{0\}$ and $K_{\Theta_{11}} = \{0\}$. Consequently, Θ_1 and Θ_{11} are unitary constant functions.

From (5.4) and (5.11) we deduce that $\Theta_{11}\Omega = \Theta_{11}\Omega_{22}\Omega_1 = \Omega_2\Theta_1\Omega_1$. Next, $T_{\Theta_{11}\Omega} \cong T_\Omega$, $T^*_{\Theta_{11}\Omega} \cong T_{\widetilde{\Omega}\widetilde{\Theta}_{11}}$, and $T^*_{\Omega_2} \cong T_{\widetilde{\Omega}_2}$ (see [1, VI.1, VI.3.3] and [2, V.1.20]); therefore, $T_{\widetilde{\Omega}_2} \sim T_{\widetilde{\Omega}\widetilde{\Theta}_{11}}$, and $T_{\widetilde{\Omega}\widetilde{\Theta}_{11}}$ is a C_0-contraction with property (P). The factorization $\widetilde{\Omega}\widetilde{\Theta}_{11} = (\widetilde{\Omega}_1\widetilde{\Theta}_1)\cdot\widetilde{\Omega}_2$ is regular; let E be the invariant subspace of $T_{\widetilde{\Omega}\widetilde{\Theta}_{11}}$ corresponding to it. Then $T_{\widetilde{\Omega}\widetilde{\Theta}_{11}}|_E \cong T_{\widetilde{\Omega}_2} \sim T_{\widetilde{\Omega}\widetilde{\Theta}_{11}}$, and since $T_{\widetilde{\Omega}\widetilde{\Theta}_{11}}$ has property (P), we get $E = K_{\widetilde{\Omega}\widetilde{\Theta}_{11}}$ (see [2, VII.1.15]). Thus, $\widetilde{\Omega}_1\widetilde{\Theta}_1$ is a unitary constant function, whence $\theta_1 = \Theta_1\Omega_1$ is a unitary constant function, $\mathcal{F} \ominus F = \{0\}$, and $F = \mathcal{F}$. □

For the proof of the injectivity of $\mathcal{J}_{\mathcal{X}}$ we need the following lemma.

Lemma 6.2. *Suppose* T *is a contraction on a space* $\mathcal{H}$, $T \in C_{\cdot 0}(C_0(P),\mathrm{fin})$, *and* $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ *is the decomposition of* $\mathcal{H}$ *corresponding to a triangulation of* T *of the form* (0.1). *Let* $E, \mathcal{E} \in \operatorname{Lat} T$, $E \subset \mathcal{E}$. *If the contractions* $T|_E$ *and* $T|_{\mathcal{E}}$ *have the same Jordan model, then* $E \cap \mathcal{H}_0 = \mathcal{E} \cap \mathcal{H}_0$.

Proof. By statement 3) in Proposition 5.1, $T|_E, T|_{\mathcal{E}} \in C_{\cdot 0}(C_0(P),\mathrm{fin})$. Let $J_0 \oplus S_k$ be the Jordan model of the contractions $T|_E$ and $T|_{\mathcal{E}}$. Suppose that $E = E_0 \oplus E_1$ and $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$ are those decompositions of the spaces E and $\mathcal{E}$ that correspond to the triangulations of $T|_E$ and $T|_{\mathcal{E}}$ of the form (0.1). By the definition of the Jordan model (see §4), we have $T|_{E_0} \sim J_0$ and $T|_{\mathcal{E}_0} \sim J_0$, whence $T|_{E_0} \sim T|_{\mathcal{E}_0}$. By statement 1) of Proposition 5.1, $E_0 = E \cap \mathcal{H}_0$ and $\mathcal{E}_0 = \mathcal{E} \cap \mathcal{H}_0$, so that $E_0 \subset \mathcal{E}_0$. Clearly, $E_0 \in \operatorname{Lat} T|_{\mathcal{E}_0}$. Since $T|_{\mathcal{E}_0}$ has property (P), we have $E_0 = \mathcal{E}_0$ (see [2, VII.1.15]). □

Proof of Theorem 0.2. Lemma 6.1 and properties 1)–3) of the map $\mathcal{J}_{\mathcal{X}}$ (see §1) show that it suffices to prove the following assertion: if $E, \mathcal{E} \in \operatorname{Lat} T$, $E \subset \mathcal{E}$, and $\mathcal{J}_{\mathcal{X}}E = \mathcal{J}_{\mathcal{X}}\mathcal{E}$, then $E = \mathcal{E}$.

For E and $\mathcal{E}$ as above, set $\mathcal{F} = \mathcal{J}_{\mathcal{X}} E = \mathcal{J}_{\mathcal{X}} \mathcal{E}$. Clearly, $T|_E \prec R|_{\mathcal{F}}$ and $T|_{\mathcal{E}} \prec R|_{\mathcal{F}}$. By statement 3) of Proposition 5.1, we have $T|_E, T|_{\mathcal{E}}, R|_{\mathcal{F}} \in C_{\cdot 0}(C_0(P), \mathrm{fin})$. Statement 1) of Corollary 4.4 shows that the contractions $T|_E$, $T|_{\mathcal{E}}$, and $R|_{\mathcal{F}}$ have one and the same Jordan model $J_0 \oplus S_k$, $k < \infty$.

Let $E = E_0 \oplus E_1$, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, and $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1$ be those decompositions of the spaces E, $\mathcal{E}$, and $\mathcal{F}$ that correspond to the triangulations of $T|_E$, $T|_{\mathcal{E}}$, and $R|_{\mathcal{F}}$ of the form (0.1). Applying Lemma 6.2 to $T|_{\mathcal{E}}$ and its invariant subspaces E and $\mathcal{E}$, we obtain $E_0 = \mathcal{E}_0$. Therefore, $E_1 \subset \mathcal{E}_1$; clearly, $E_1 \in \operatorname{Lat} P_{\mathcal{E}_1} T|_{\mathcal{E}_1}$.

Since $J_0 \oplus S_k$ is a Jordan model of the contractions $T|_{\mathcal{E}}$ and $R|_{\mathcal{F}}$, we have $P_{\mathcal{E}_1} T|_{\mathcal{E}_1} \prec S_k$ and $P_{\mathcal{F}_1} R|_{\mathcal{F}_1} \prec S_k$. Set $X = P_{\mathcal{F}_1} \mathcal{X}|_{\mathcal{E}_1} : \mathcal{E}_1 \to \mathcal{F}_1$. We apply Lemma 4.6 with $\mathcal{L} = \mathcal{F}_0$ to the contractions $T|_{\mathcal{E}}$, $R|_{\mathcal{F}}$ and the quasiaffinity $\mathcal{X}|_{\mathcal{E}} : \mathcal{E} \to \mathcal{F}$ intertwining these contractions. We see that X is a quasiaffinity intertwining the contractions $P_{\mathcal{E}_1} T|_{\mathcal{E}_1}$ and $P_{\mathcal{F}_1} R|_{\mathcal{F}_1}$. By Theorem 2.1, the map $\mathcal{J}_X : \operatorname{Lat} P_{\mathcal{E}_1} T|_{\mathcal{E}_1} \to \operatorname{Lat} P_{\mathcal{F}_1} R|_{\mathcal{F}_1}$ is a lattice isomorphism.

Now we apply Lemma 4.6 to the contractions $T|_E$, $R|_{\mathcal{F}}$ and the intertwining quasiaffinity $\mathcal{X}|_E : E \to \mathcal{F}$ with $\mathcal{L} = \mathcal{F}_0$. We see that $P_{\mathcal{F}_1} \mathcal{X}|_{E_1} : E_1 \to \mathcal{F}_1$ is a quasiaffinity; in particular, $\operatorname{clos} P_{\mathcal{F}_1} \mathcal{X}|_{E_1} E_1 = \mathcal{F}_1$. But $\operatorname{clos} P_{\mathcal{F}_1} \mathcal{X}|_{E_1} E_1 = \operatorname{clos} P_{\mathcal{F}_1} \mathcal{X}|_{\mathcal{E}_1} E_1 = \operatorname{clos} X E_1 = \mathcal{J}_X E_1$. Thus, $\mathcal{J}_X E_1 = \mathcal{F}_1 = \mathcal{J}_X \mathcal{E}_1$. Since $\mathcal{J}_X$ is a lattice isomorphism, we have $E_1 = \mathcal{E}_1$, whence $E = \mathcal{E}$. $\square$

§7. On pseudosimilarity of $C_{\cdot 0}$-contractions with finite defect indices

Let T and R be contractions of class $C_{\cdot 0}$, and let $d_{T^*}, d_{R^*} < \infty$ (recall that for a $C_{\cdot 0}$-contraction T we have $d_T \le d_{T^*}$). As was proved in §6, if $T \prec R$, then every quasiaffinity X such that $XT = RX$ realizes an isomorphism $\mathcal{J}_X$ of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} R$. It is well known that if $T \prec R$, then the reverse relation is not true in general (see [6]). Quasisimilarity of $C_{\cdot 0}$-contractions was studied in [26, 28, 20].

If $C_{\cdot 0}$-contractions T and R with $d_{T^*}, d_{R^*} < \infty$ are quasisimilar ($T \sim R$), then, by definition, there exist quasiaffinities X and Y such that $XT = RX$ and $YR = TY$. By Theorem 0.2, the maps $\mathcal{J}_X : \operatorname{Lat} T \to \operatorname{Lat} R$ and $\mathcal{J}_Y : \operatorname{Lat} R \to \operatorname{Lat} T$ are lattice isomorphisms. In this section we present an example of a $C_{\cdot 0}$-contraction T with $d_{T^*} < \infty$ and such that T is quasisimilar to its Jordan model J, and the isomorphisms $\mathcal{J}_X$ and $\mathcal{J}_Y$ of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat} J$ are not mutually inverse for any quasiaffinities X and Y intertwining T and J. This example is an appropriate modification of the example in [5, §4]. Note that the operator J below was treated in [26] from a slightly different point of view.

In [26, Theorem 2] it was proved that if $T = \left(\begin{smallmatrix} T_0 & * \\ 0 & T_1 \end{smallmatrix}\right)$ is the triangulation of a $C_{\cdot 0}$-contraction T with $d_{T^*} < \infty$ of the form (0.1) and $J = J_0 \oplus S_k$ is the Jordan model of T, then $T \sim J$ if and only if $T_1 \sim S_k$, and in this case there exists an outer function $\rho \in H^\infty$ and quasiaffinities $\mathfrak{X}$ and $\mathfrak{Y}$ such that $\mathfrak{X}T = (T_0 \oplus S_k)\mathfrak{X}$, $\mathfrak{Y}(T_0 \oplus S_k) = T\mathfrak{Y}$, $\mathfrak{X}\mathfrak{Y} = \rho(T_0 \oplus S_k)$, and $\mathfrak{Y}\mathfrak{X} = \rho(T)$ (see also Remark 4.1 in the present paper). Therefore, T and $T_0 \oplus S_k$ are pseudosimilar, and, in particular, the maps $\mathcal{J}_{\mathfrak{X}}$ and $\mathcal{J}_{\mathfrak{Y}}$ are mutually inverse isomorphisms of the lattices $\operatorname{Lat} T$ and $\operatorname{Lat}(T_0 \oplus S_k)$. The example below shows that, in general, in this situation T_0 cannot be replaced by J_0, i.e., T and J may fail to be pseudosimilar.

Let $\varphi, \psi \in H^\infty$. Suppose that φ and ψ are relatively prime inner functions, and that for arbitrary $h_1, h_2 \in H^2$ the function $\varphi h_1 + \psi h_2$ is not outer (see examples of such functions in [6, 26, 28]). We set $k = 1$, $T_0 = T_\varphi \oplus T_\psi$, and $J_0 = T_{\varphi\psi}$. It is well known that $T_0 \sim J_0$. Let $T = T_0 \oplus S_1$ and $J = J_0 \oplus S_1$; then T and J act on the spaces $\left(\begin{smallmatrix} K_\varphi \oplus K_\psi \\ H^2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} K_{\varphi\psi} \\ H^2 \end{smallmatrix}\right)$, respectively. Obviously, $T \sim J$.

Let $\mathcal{X} : \left(\begin{smallmatrix} K_{\varphi\psi} \\ H^2 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} K_\varphi \oplus K_\psi \\ H^2 \end{smallmatrix}\right)$ be a quasiaffinity, $\mathcal{X}J = T\mathcal{X}$. Then, by Theorem 0.2, $\mathcal{J}_{\mathcal{X}} : \operatorname{Lat} J \to \operatorname{Lat} T$ is a lattice isomorphism. Now we show that $\mathcal{J}_{\mathcal{X}}^{-1} \neq \mathcal{J}_{\mathcal{Y}}$ for any bounded operator

$$\mathcal{Y} : \begin{pmatrix} K_\varphi \oplus K_\psi \\ H^2 \end{pmatrix} \to \begin{pmatrix} K_{\varphi\psi} \\ H^2 \end{pmatrix}.$$

Suppose that there exists a bounded operator $\mathcal{Y} : \left(\begin{smallmatrix} K_\varphi \oplus K_\psi \\ H^2 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} K_{\varphi\psi} \\ H^2 \end{smallmatrix}\right)$ such that $\mathcal{J}_{\mathcal{X}} = \mathcal{J}_{\mathcal{Y}}^{-1}$. Then, obviously, $\operatorname{clos} \mathcal{Y}\mathcal{X}E = E$ for any $E \in \operatorname{Lat} J$. Applying [4, §8, Theorem 5] (see also the proof of Theorem 2.3 in [3]) to the operators J and $\mathcal{Y}\mathcal{X}$, we obtain $\mathcal{Y}\mathcal{X} = \rho(J)$, where $\rho \in H^\infty$. If $E = \left(\begin{smallmatrix} 0 \\ H^2 \end{smallmatrix}\right)$, then, clearly, $E \in \operatorname{Lat} J$, whence $\left(\begin{smallmatrix} 0 \\ H^2 \end{smallmatrix}\right) = E = \operatorname{clos} \mathcal{Y}\mathcal{X}E = \operatorname{clos} \rho(J)E = \left(\begin{smallmatrix} 0 \\ \operatorname{clos} \rho(S_1)H^2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ \operatorname{clos} \rho H^2 \end{smallmatrix}\right)$. Consequently, $H^2 = \operatorname{clos} \rho H^2$, and ρ is an outer function.

Next, we have $\mathcal{Y}T\mathcal{X} = \mathcal{Y}\mathcal{X}J = \rho(J)J = J\rho(J) = J\mathcal{Y}\mathcal{X}$, so that $\mathcal{Y}T = J\mathcal{Y}$ on $\mathcal{X}\left(\begin{smallmatrix} K_{\varphi\psi} \\ H^2 \end{smallmatrix}\right)$. Since $\mathcal{X}$ is a quasiaffinity, $\mathcal{Y}T = J\mathcal{Y}$ on the entire space $\left(\begin{smallmatrix} K_\varphi \oplus K_\psi \\ H^2 \end{smallmatrix}\right)$. By Lemma 4.6, $\mathcal{X}$ and $\mathcal{Y}$ have an upper triangular form: $\mathcal{X} = \left(\begin{smallmatrix} X & * \\ 0 & * \end{smallmatrix}\right)$, $\mathcal{Y} = \left(\begin{smallmatrix} Y & * \\ 0 & * \end{smallmatrix}\right)$. The identities $\mathcal{X}J = T\mathcal{X}$, $\mathcal{Y}T = J\mathcal{Y}$, and $\mathcal{Y}\mathcal{X} = \rho(J)$ imply the relations $XT_{\varphi\psi} = (T_\varphi \oplus T_\psi)X$, $Y(T_\varphi \oplus T_\psi) = T_{\varphi\psi}Y$, and $YX = \rho(T_{\varphi\psi})$. By [5, Lemma 6], $\rho = \varphi h_1 + \psi h_2$ for some $h_1, h_2 \in H^2$. By the assumption imposed on φ and ψ, the function ρ cannot be outer, a contradiction. Thus, $\mathcal{J}_{\mathcal{X}}^{-1} \neq \mathcal{J}_{\mathcal{Y}}$ for any bounded operator $\mathcal{Y}$, and $J_0 \oplus S_1$ and $T_0 \oplus S_1$ are not pseudosimilar.

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