

CYCLIC DARBOUX q -CHAINS

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ABSTRACT. A discrete q -analog is constructed for the Veselov–Shabat dressing chain (the latter is a generalization of the classical harmonic oscillator). It is shown that, as in the continuous case, the corresponding operator relations make it possible to completely determine the discrete spectra of the operators in the chain: each spectrum consists of several q -arithmetic progressions. Any cyclic q -chain can be realized by bounded selfadjoint difference operators, the spectrum of each of them is discrete, and the eigenvectors form a complete family in the Hilbert space of square-integrable sequences. Moreover, an explicit general solution is given for chains of length 2, and it is proved that the q -oscillator constructed in the paper weakly converges to the usual harmonic oscillator as $q \rightarrow 1$.

§1. INTRODUCTION

It is well known that the algebraic relation satisfied by the harmonic oscillator $L = -\frac{d^2}{dx^2} + \frac{\alpha^2 x^2}{4}$, namely,

$$(1) \quad L + \frac{\alpha}{2} = AA^+ = A^+A + \alpha$$

(here $A = d/dx + \alpha x/2$, and $A^+ = -d/dx + \alpha x/2$ is the differential operator adjoint to A) makes it possible to find the discrete spectrum and the set of eigenfunctions of L by the following method. The ground state ψ_0 corresponding to the minimal eigenvalue $\frac{\alpha}{2}$ can be found from the equation

$$A\psi_0 = 0,$$

and the other states are obtained by applying the creation operator A^+ to the ground state,

$$\begin{aligned} \psi_k &= (A^+)^k \psi_0, \\ L\psi_k &= \frac{(2k+1)\alpha}{2} \psi_k. \end{aligned}$$

It is well known that the eigenfunctions obtained in this way constitute a complete system in the space $\mathcal{L}_2(\mathbb{R})$; after normalization, they have the form

$$\begin{aligned} \psi_0(x) &= \sqrt[4]{\frac{\alpha}{2\pi}} e^{-\frac{\alpha x^2}{4}}, \\ \psi_k(x) &= \left(\sqrt{\frac{\alpha}{2}}\right)^k \frac{1}{\sqrt{k!\alpha^k}} H_k\left(x\sqrt{\frac{\alpha}{2}}\right) \psi_0(x), \end{aligned}$$

where the H_k are the Hermite polynomials. However, completeness does not follow directly from relation (1) because L is not a bounded and everywhere defined operator in the Hilbert space.

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By a *Darboux chain* we mean a sequence of selfadjoint differential operators L_1, L_2, \dots on \mathbb{R} connected by the Darboux transformations

$$(2) \quad L_j = A_j A_j^+ - \alpha_j = A_{j-1}^+ A_{j-1},$$

where $A_j = d/dx + f_j(x)$ is a first order differential operator. A chain is called *cyclic* if $L_{j+r} = L_j$ for some r and all $j = 1, 2, \dots$. The number r is called the *period* (or the *length*) of the chain. If $r = 1$, the operator $L_1 + \alpha/2$ is a harmonic oscillator.

A cyclic Darboux chain yields the integrable system of differential equations

$$(3) \quad (f_j + f_{j+1})' = f_{j+1}^2 - f_j^2 + \alpha_j$$

(here $f_j = f_{j+r}$ and $\alpha_j = \alpha_{j+r}$) mentioned in [1] and studied in detail in [2]. If $\alpha = 0$, where $\alpha = \sum_{j=1}^r \alpha_j$, then the operators of a cyclic Darboux chain are finite-gap. If $\alpha \neq 0$, equations (3) reduce to the Painlevé equations ($r = 3, 4$) and their higher analogs [3]. If $r = 2$, then equations (3) can be solved explicitly,

$$(4) \quad f_j(x) = \frac{\alpha_j - \alpha_{j+1}}{2(\alpha_j + \alpha_{j+1})x} + \frac{(\alpha_j + \alpha_{j+1})x}{4}, \quad \text{where } \alpha_{j+2} = \alpha_j.$$

In [2] it was shown that if $\alpha \neq 0$, then, as in the case of a harmonic oscillator, the algebraic relations (2) make it possible to determine the discrete spectra of the operators in the chain. In the case in question, the spectrum of each L_j consists of r arithmetic progressions. For example, the spectrum of L_1 has the form

$$k\alpha, \alpha_1 + k\alpha, (\alpha_1 + \alpha_2) + k\alpha, \dots, (\alpha_1 + \alpha_2 + \dots + \alpha_{r-1}) + k\alpha, \quad k = 0, 1, 2, \dots$$

However, for $r > 1$, contrary to the case of a harmonic oscillator, it is not so easy to find out whether a system of eigenfunctions is complete, because we must find explicit solutions of (3) to determine the eigenfunctions. If all α_j are equal, then the chain admits a particular solution $f_j(x) = \alpha x/2r$ corresponding to the zero initial data. In [2] it was shown that, if r is odd, for a small perturbation of the initial data and the parameters α_j , the solution of the chain remains nonsingular on the entire line, and the potential $u_1 = f_1^2 + f_1' - \alpha_1$ of the Schrödinger operator L_1 has “oscillator-like” asymptotics

$$u_1(x) = \frac{\alpha^2 x^2}{4r^2} + O(x) \quad \text{as } x \rightarrow \pm\infty$$

(the Weiss conjecture).

In [4], the authors studied the possibility to realize the Heisenberg relations (1) by a first order differential operator $A = a + bT$ on the one-dimensional lattice \mathbb{Z} , where T is the shift operator, $(T\psi)(n) = \psi(n+1)$, and a and b are arbitrary nonvanishing functions on integers. As was shown in [4], such an operator cannot be constructed so as to act on functions defined at all integral points. However, it is possible to construct an operator acting on functions defined on the “half-line” \mathbb{N} . In this case, like in the continuous case, the operator L is unbounded, and its coefficients b_n grow as $n^{1/2}$. Again, the system of eigenvectors of L is complete in $\mathcal{L}_2(\mathbb{N})$, which can be proved by expressing the eigenvectors in terms of Charlier polynomials.

Various realizations of the *q-harmonic oscillator*, i.e., operators L that satisfy

$$(5) \quad L = AA^+ - \alpha = qA^+A$$

for some $0 < q < 1$ and $\alpha > 0$ (or $q > 1$ and $\alpha < 0$), where $A = a + bT$ is a first order difference operator, were considered in [4, 6, 7]. The discrete spectrum of the *q*-oscillator can be found easily with the help of (5). It consists of a “*q*-arithmetic sequence”

$$\lambda_k = \alpha[k]_q = \alpha \frac{1 - q^k}{1 - q}, \quad k = 0, 1, 2, \dots,$$

and, obviously, is located in the interval $[0, \frac{q\alpha}{1-q})$ (or in the interval $(0, \frac{q\alpha}{q-1})$ if $q > 1$). *A priori*, this does not rule out the possibility that the q -oscillator can be realized by bounded operators in some Hilbert space. As was noted in [4], for such a realization of the q -oscillator, the operator $L = (a + bT)(a + bT)^+ - \alpha$ is nevertheless unbounded and, apparently, has continuous spectrum in the region $(\frac{q\alpha}{1-q}, \infty)$. In [6] it was shown that the eigenstates of the q -oscillator can be expressed in terms of Hermite q -polynomials.

A fundamentally different version of the q -oscillator was constructed in the paper [8], where the difference operator $A = aT^{-1} + bT$ defined on the integral lattice and satisfying relation (5) was considered. It turned out that, for this choice of A , the q -oscillator L is a bounded selfadjoint operator in the Hilbert space $\mathcal{L}_2(\mathbb{Z})$, and L differs from a compact operator by a constant, which implies that its eigenvectors form a complete set. Moreover, the operator L has no continuous spectrum.

In the present paper, we construct a realization of a Darboux q -chain, i.e., of a q -analog of (2), by bounded difference operators in the case where r is even. We prove that the system of eigenvectors of the operators in the q -chain is complete in $\mathcal{L}_2(\mathbb{Z})$ and study the spectral properties of these operators. Also, for $r = 2$, we present explicit formulas for the coefficients of the operators in the q -chain. For some special choice of the parameters, these formulas reduce to the formulas obtained in [8].

The results of the present paper were announced in [9].

§2. DARBOUX q -CHAINS

By a *Darboux q -chain* we mean a sequence of selfadjoint difference operators L_1, L_2, \dots on the one-dimensional lattice that satisfy the relations

$$(6) \quad L_j = A_j A_j^+ - \alpha_j = q A_{j-1}^+ A_{j-1},$$

where $A_j = a_j + b_j T$ is a first order difference operator and $a_j(n), b_j(n) \in \mathbb{R} \setminus \{0\}$. Without loss of generality, we may assume that $b_j(n) > 0$ for all j and n . An important peculiarity of the difference operators, as compared to the differential ones, is that the notion of a cyclic chain can be defined in various ways. We say that a q -chain (6) is *cyclic* with *period* r and *shift* s if

$$(7) \quad x L_{j+r} = T^{-s} L_j T^s$$

for all $j \geq 1$. The case treated previously in the literature was mostly the case of $s = 0$. However, in [4] and also in [10], the possibility of the “inverse” factorization, i.e., replacement of the operators $A_j = a_j + b_j T$ with the operators $A_j = a_j T^+ + b_j$, was indicated; actually, this is equivalent to the “direct factorization” for $s = r$ (the operator L_j is replaced with $T^{-j} L_j T^j$).

The cyclic chains of the form (6), (7) admit two discrete symmetries, which we need in what follows. Suppose that difference operators L_1, \dots, L_r satisfy relations (6) and (7) for some $q > 0$, $q \neq 1$, and $\alpha_j \neq 0$, $j = 1, \dots, r$. Then the mapping

$$(8) \quad L_j \mapsto \tilde{L}_j = \tilde{A}_j \tilde{A}_j^+ - \tilde{\alpha}_j = \tilde{q} \tilde{A}_{j+1}^+ \tilde{A}_{j+1},$$

where $\tilde{q} = \frac{1}{q}$ and $\tilde{\alpha}_j = -\frac{\alpha_j}{q}$ for all $j = 1, \dots, r$, transforms the given q -chain to a new \tilde{q} -chain $\tilde{L}_1, \dots, \tilde{L}_r$ with the same shift (this symmetry is given by changing the direction in which the “circle” \mathbb{Z}_r is passed). Next, there is another symmetry given by changing the direction of the “axis” \mathbb{Z} ,

$$(9) \quad L_j \mapsto \tilde{L}_j = \tilde{A}_j \tilde{A}_j^+ - \alpha_j = q \tilde{A}_{j-1}^+ \tilde{A}_{j-1},$$

where $\tilde{A}_j = \tilde{a}_j + \tilde{b}_j T$, and the coefficients are determined by the formulas

$$(10) \quad \tilde{a}_j(n) = b_j(j-n), \quad \tilde{b}_j(n) = a_j(j-n) \quad \text{for all } j \text{ and } n.$$

Proposition 1. *The operators of a cyclic Darboux q -chain (6), (7) are unbounded if $s \leq 0$ or $s \geq r$.*

Proof. The operator relations (6) together with the cyclicity condition (7) lead to the following system of difference equations:

$$(11) \quad \begin{cases} a_2^2(n) + b_2^2(n) = q(a_1^2(n) + b_1^2(n-1)) + \alpha_2, \\ \vdots \\ a_r^2(n) + b_r^2(n) = q(a_{r-1}^2(n) + b_{r-1}^2(n-1)) + \alpha_r, \\ a_1^2(n-s) + b_1^2(n-s) = q(a_r^2(n) + b_r^2(n-1)) + \alpha_1, \\ a_2(n)b_2(n-1) = qa_1(n-1)b_1(n-1), \\ \vdots \\ a_r(n)b_r(n-1) = qa_{r-1}(n-1)b_{r-1}(n-1), \\ a_1(n-s)b_1(n-s-1) = qa_r(n-1)b_r(n-1). \end{cases}$$

First, we consider the case where $s \leq 0$. Multiplying all equations of the second group in system (11), we obtain

$$(12) \quad a_1(n-s)b_1(n-s-1) \prod_{j=1}^r a_j(n) = q^r a_1(n)b_1(n-1) \prod_{j=1}^r a_j(n-1).$$

If $s = 0$, then, after elimination of common factors, relation (12) takes the form

$$(13) \quad f(n) = q^r f(n-1),$$

where $f(n) = \prod_{j=1}^r a_j(n)$. Therefore, $f(n)$ grows exponentially as $n \rightarrow -\infty$ (or as $n \rightarrow +\infty$ if $q > 1$), and, consequently, the coefficients of the difference operators A_j are unbounded. If $s < 0$, then we multiply equation (12) by the product

$$a_1(n+1)a_1(n+2) \cdots a_1(n-s-1)b_1(n)b_1(n+1) \cdots b_1(n-s-2),$$

arriving at (13) again, with

$$f(n) = \prod_{i=1}^{-s} a_1(n+i)b_1(n+i-1) \prod_{j=1}^r a_j(n).$$

Now, let $s \geq r$. The symmetry (9) transforms a q -chain with shift s to a q -chain with the nonpositive shift $r-s$ and with unbounded operators (see above). Using formulas (10), we easily show that the operators of the initial q -chain are also unbounded. The proposition is proved. \square

Now, we restrict our consideration to cyclic Darboux q -chains of even length r with shift $s = r/2$ (this approach was suggested by I. A. Dynnikov). We consider another q -chain of length r and with zero shift,

$$(14) \quad M_j = B_j B_j^+ - \alpha_j = q B_{j-1}^+ B_{j-1}, \quad M_{j+r} = M_j,$$

where

$$(15) \quad B_j = u_j T^{-1} + v_j T.$$

It is easy to prove that the cyclic q -chain (14), (15) reduces to the following system of difference equations:

$$(16) \quad \begin{cases} u_1^2(n) + v_1^2(n) = q(u_r^2(n+1) + v_r^2(n-1)) + \alpha_1, \\ \vdots \\ u_r^2(n) + v_r^2(n) = q(u_{r-1}^2(n+1) + v_{r-1}^2(n-1)) + \alpha_r, \\ u_1(n+1)v_1(n-1) = qu_r(n)v_r(n), \\ \vdots \\ u_r(n+1)v_r(n-1) = qu_{r-1}(n)v_{r-1}(n). \end{cases}$$

Direct inspection shows that the substitution

$$\begin{aligned} u_{2i+1}(2k+1) &= a_{2i+1}(k+i), & v_{2i+1}(2k+1) &= b_{2i+1}(k+i), \\ u_{2i}(2k) &= a_{2i}(k+i-1), & v_{2i}(2k) &= b_{2i}(k+i-1) \end{aligned}$$

reshapes equation (11) to an equation of the form (16) ($u_j(n)$ and $v_j(n)$ are defined only if $j+n$ is even).

Thus, the study of the existence of a solution for a cyclic q -chain (6), (7) for $s = r/2$ reduces to the study of system (16).

§3. CHAINS OF LENGTH 2

The case of q -chains of length $r = 2s = 2$ is of particular interest because, in this case, the general solutions of the corresponding equations (11) can be written explicitly. The general solution of the problem for $r = 2$ was communicated to the author by I. A. Dynnikov.

To find the general solution, we need the following technical trick. Putting

$$(17) \quad \xi_n = \begin{cases} u_1^2(n+1) & \text{if } n \text{ is even,} \\ u_2^2(n+1) & \text{if } n \text{ is odd,} \end{cases} \quad \eta_n = \begin{cases} v_1^2(n) & \text{if } n \text{ is odd,} \\ v_2^2(n) & \text{if } n \text{ is even,} \end{cases}$$

we transform the underdetermined system (16) into the following completely determined system:

$$(18) \quad \begin{cases} \xi_{n-1} + \eta_n = q(\xi_n + \eta_{n-1}) + \frac{\alpha_1 + \alpha_2}{2} + (-1)^{n-1} \frac{\alpha_1 - \alpha_2}{2}, \\ \xi_n \eta_{n-1} = q^2 \xi_{n-1} \eta_n. \end{cases}$$

It is easily seen that equations (18) coincide precisely with equations (16) for $r = 1$ and for the “blinking” parameters $\frac{\alpha_1 + \alpha_2}{2} + (-1)^{n-1} \frac{\alpha_1 - \alpha_2}{2}$ (in place of α), i.e., that the change (17) reduces our chain of length $r = 2$ and with shift $s = 1$ to the “blinking” chain (14), (15). In the paper [8] a particular solution for the chain (14), (15) for $r = 1$ can be found (in our setting, this corresponds to the case where $s = 1$, $r = 2$, and $\alpha_1 = \alpha_2$), and this solution possesses an additional symmetry with respect to the origin.

For definiteness, we assume that $0 < q < 1$, $\alpha_1 > 0$, and $\alpha_2 > 0$. Writing the second equation in (18) in the form

$$\frac{\xi_n}{\eta_n} = q^2 \frac{\xi_{n-1}}{\eta_{n-1}},$$

we observe that the ξ_n/η_n form a geometric progression. Therefore,

$$(19) \quad \xi_n = q^{2n+2\varphi+1} \eta_n,$$

where $\varphi = \frac{1}{2} (\log_q(\xi_0/\eta_0) - 1)$ is a constant determined by the initial data. Substituting this expression for ξ_n in the first equation in (18) and multiplying the resulting equation

by $1 - q^{2(n+\varphi)}$, we obtain

$$(20) \quad \zeta_n = q\zeta_{n-1} + \left(\frac{\alpha_1 + \alpha_2}{2} + (-1)^{n-1} \frac{\alpha_1 - \alpha_2}{2} \right) (1 - q^{2(n+\varphi)}),$$

where $\zeta_n = \eta_n (1 - q^{2(n+\varphi-1)}) (1 - q^{2(n+\varphi)})$.

The general solution of the difference equation (20) is given by the formula

$$\zeta_n = -kq^n + \frac{(\alpha_1 + \alpha_2)(1 + q^{2n+2\varphi+1})}{2(1-q)} + (-1)^{n-1} \frac{(\alpha_1 - \alpha_2)(1 - q^{2n+2\varphi+1})}{2(1+q)},$$

where k is a constant. Putting

$$c_n = \frac{\alpha_1 + \alpha_2}{1-q} + (-1)^n \frac{\alpha_1 - \alpha_2}{1+q}, \quad 2\kappa = kq^{-(\varphi+\frac{1}{2})},$$

we obtain the following expression for η_n :

$$(21) \quad \eta_n = \frac{1}{2} \frac{q^{n+\varphi+\frac{1}{2}} (-2\kappa + c_{n+1}q^{-(n+\varphi+\frac{1}{2})} + c_n q^{n+\varphi+\frac{1}{2}})}{(1 - q^{2(n+\varphi+1)})(1 - q^{2(n+\varphi)})}.$$

To construct a solution for our q -chain, we need to select the solutions (ξ_n, η_n) of (18) that are positive at all points of the integral lattice; it is easily seen that, by (19), it suffices to ensure the positivity of η_n . We consider the cases where $\varphi \notin \mathbb{Z}$ and $\varphi \in \mathbb{Z}$ separately.

If $\varphi \notin \mathbb{Z}$, then exactly one integer n_0 lies in the interval $(-\varphi - 1, -\varphi)$. Obviously, the denominator in (21) is positive if $n \neq n_0$ and is negative if $n = n_0$. Therefore, we must choose the constant κ so that the numerator of (21) be positive for $n \neq n_0$ and negative at n_0 . We put

$$f(n) = c_{n+1}q^{-(n+\varphi+\frac{1}{2})} + c_n q^{n+\varphi+\frac{1}{2}}.$$

It is easy to check that this function has its minimum on \mathbb{Z} at n_0 . Therefore, the positivity condition for η_n can be written as follows:

$$f(n_0) < 2\kappa < \min\{f(n_0 - 1), f(n_0 + 1)\}.$$

After transformations, this inequality takes the form

$$(22) \quad c_{[\varphi]}q^{-\theta} + c_{[\varphi]-1}q^\theta < 2\kappa < \min\{c_{[\varphi]}q^{\theta+1} + c_{[\varphi]-1}q^{-\theta-1}, c_{[\varphi]}q^{\theta-1} + c_{[\varphi]-1}q^{-\theta+1}\},$$

where $\theta = \varphi - [\varphi] - \frac{1}{2}$ and $[\varphi]$ is the integral part of φ .

Thus, for any $\varphi \notin \mathbb{Z}$, formulas (21) and (19) give rise to a solution for the given chain if the parameter κ satisfies (22).

Now, let $\varphi \in \mathbb{Z}$. In this case, the numerator in (21) attains its minimum at the half-integer $-\varphi - 1/2$, and the denominator in (21) vanishes at the integral points $-\varphi$ and $-\varphi - 1$. Therefore, to guarantee the existence of a solution, we need to choose the parameter κ so that the numerator in question vanish at the points $-\varphi$ and $-\varphi - 1$. Since $f(-\varphi) = f(-\varphi - 1)$, this numerator will vanish at $-\varphi$ and $-\varphi - 1$ if the following condition is fulfilled:

$$2\kappa = f(-\varphi) = c_{\varphi-1}q^{-1/2} + c_\varphi q^{1/2};$$

consequently, in this case the solution is defined on the entire integral lattice. If $\varphi \in \mathbb{Z}$, then there are no integers in the interval $(-\varphi - 1, -\varphi)$. Therefore, it suffices to check that the chosen solution is positive at the points $-\varphi$ and $-\varphi - 1$, which can be done by direct calculation. Thus, we have proved the following statement.

Proposition 2. Let $r = 2 = 2s$, $\alpha_1, \alpha_2 > 0$, and $0 < q < 1$. Then the general solution of problem (6), (7) is of the form $a_1(n) = \varepsilon\sqrt{\xi_{2n}}$, $b_1(n) = \sqrt{\eta_{2n+1}}$, $a_2(n) = \varepsilon\sqrt{\xi_{2n-1}}$, $b_2(n) = \sqrt{\eta_{2n}}$, where $\varepsilon = \pm 1$,

$$(23) \quad \begin{aligned} \xi_n &= \frac{1}{2} \frac{c_n - 2\kappa q^{-n-\varphi-\frac{1}{2}} + c_{n+1}q^{-2n-2\varphi-1}}{(1-q^{-2(n+\varphi)})(1-q^{-2(n+\varphi+1)}), \\ \eta_n &= \frac{1}{2} \frac{c_{n+1} - 2\kappa q^{n+\varphi+\frac{1}{2}} + c_n q^{2n+2\varphi+1}}{(1-q^{2(n+\varphi)})(1-q^{2(n+\varphi+1)}), \end{aligned}$$

φ is an arbitrary parameter, and κ satisfies the restrictions

$$c_{[\varphi]}q^{-\theta} + c_{[\varphi]-1}q^\theta < 2\kappa < \min(c_{[\varphi]}q^{\theta+1} + c_{[\varphi]-1}q^{-\theta-1}, c_{[\varphi]}q^{\theta-1} + c_{[\varphi]-1}q^{-\theta+1}) \quad \text{if } \varphi \notin \mathbb{Z}$$

and $2\kappa = c_\varphi q^{\frac{1}{2}} + c_{\varphi-1}q^{-\frac{1}{2}}$ if $\varphi \in \mathbb{Z}$ (in the latter case the fractions (23) must be reduced by $(1-q^{2(n+\varphi)})$ if $n \equiv \varphi \pmod{2}$ and by $(1-q^{2(n+\varphi+1)})$ if $n \equiv \varphi+1 \pmod{2}$), where $\theta = \varphi - [\varphi] - \frac{1}{2}$.

Remark 1. If $\varphi \in \mathbb{Z}$, then, after cancellation of the common factors, formulas (23) take the form

$$\begin{cases} \xi_n = \frac{1}{2} \frac{c_\varphi - c_{\varphi+1}q^{-n-\varphi-1}}{(1-q^{-2(n+\varphi+1)})(1+q^{-n-\varphi})} & \text{if } n \equiv \varphi \pmod{2}, \\ \eta_n = \frac{1}{2} \frac{c_{\varphi+1} - c_\varphi q^{n+\varphi+1}}{(1-q^{2(n+\varphi+1)})(1+q^{n+\varphi})} \\ \xi_n = \frac{1}{2} \frac{c_{\varphi+1} - c_\varphi q^{-n-\varphi}}{(1-q^{-2(n+\varphi)})(1+q^{-n-\varphi-1})} & \text{if } n \equiv \varphi+1 \pmod{2}. \\ \eta_n = \frac{1}{2} \frac{c_\varphi - c_{\varphi+1}q^{n+\varphi}}{(1-q^{2(n+\varphi)})(1+q^{n+\varphi+1})} \end{cases}$$

Remark 2. In the specific case where $\alpha_1 = \alpha_2$, the quantity c_n is independent of n , so that formulas (23) can be rewritten as follows:

$$(24) \quad \begin{aligned} \xi_n &= \frac{\alpha_1}{1-q} \frac{-\nu q^{-n-\varphi-\frac{1}{2}} + 1 + q^{-2n-2\varphi-1}}{(1-q^{-2(n+\varphi)})(1-q^{-2(n+\varphi+1)}), \\ \eta_n &= \frac{\alpha_1}{1-q} \frac{-\nu q^{n+\varphi+\frac{1}{2}} + 1 + q^{2n+2\varphi+1}}{(1-q^{2(n+\varphi)})(1-q^{2(n+\varphi+1)})} \end{aligned}$$

for $\varphi \notin \mathbb{Z}$ with $\nu = \kappa \frac{1-q}{\alpha_1}$ satisfying the restrictions

$$(25) \quad q^{-\theta} + q^\theta < \nu < \min\{q^{1-\theta} + q^{\theta-1}, q^{-\theta-1} + q^{\theta+1}\},$$

and

$$(26) \quad \begin{aligned} \xi_n &= \frac{\alpha_1}{(1-q)(1+q^{-n-\varphi})(1+q^{-n-\varphi-1})}, \\ \eta_n &= \frac{\alpha_1}{(1-q)(1+q^{n+\varphi})(1+q^{n+\varphi+1})} \end{aligned}$$

for $\varphi \in \mathbb{Z}$. The particular solution (26) was presented in [8].

In what follows, it is convenient to put $\nu = q^\tau + q^{-\tau}$ and rewrite formulas (24) and (26) in the form

$$(27) \quad \begin{aligned} \xi_n &= \frac{\alpha_1(1-q^{-n-\varphi-\frac{1}{2}-\tau})(1-q^{-n-\varphi-\frac{1}{2}+\tau})}{(1-q)(1-q^{-2(n+\varphi)})(1-q^{-2(n+\varphi+1)}), \\ \eta_n &= \frac{\alpha_1(1-q^{n+\varphi+\frac{1}{2}-\tau})(1-q^{n+\varphi+\frac{1}{2}+\tau})}{(1-q)(1-q^{2(n+\varphi)})(1-q^{2(n+\varphi+1)}), \end{aligned}$$

where $|\theta| < \tau < \min\{|\theta-1|, |\theta+1|\}$ if $\varphi \notin \mathbb{Z}$ and $\tau = \frac{1}{2}$ if $\varphi \in \mathbb{Z}$.

Proposition 2 allows us to make the following conclusions concerning the asymptotic behavior of the coefficients of the operators in the q -chain (6), (7):

$$(28) \quad a_1(n), a_2(n) \rightarrow 0, \quad b_1(n) \rightarrow \sqrt{\frac{\alpha_1 + q\alpha_2}{1 - q^2}}, \quad b_2(n) \rightarrow \sqrt{\frac{\alpha_2 + q\alpha_1}{1 - q^2}} \quad \text{as } n \rightarrow +\infty,$$

$$(29) \quad a_1(n) \rightarrow \sqrt{\frac{\alpha_1 + q\alpha_2}{1 - q^2}}, \quad a_2(n) \rightarrow \sqrt{\frac{\alpha_2 + q\alpha_1}{1 - q^2}}, \quad b_1(n), b_2(n) \rightarrow 0 \quad \text{as } n \rightarrow -\infty.$$

Since

$$(30) \quad \begin{aligned} L_j &= A_j A_j^+ - \alpha_j = (a_j + b_j T)(a_j + T^{-1}(b_j)T^{-1}) - \alpha_j \\ &= (a_j T^{-1}(b_j)) T^{-1} + (a_j^2 + b_j^2 - \alpha_j) + (T(a_j)b_j) T, \end{aligned}$$

the coefficients of $T^{\pm 1}$ in the formula for L_j , $j = 1, 2$, tend to zero as $n \rightarrow \pm\infty$, i.e., at infinity the operator L_j is equivalent to the operator of multiplication by the constant $\frac{\alpha_j + q\alpha_{j+1}}{1 - q^2}$ (here $j \in \mathbb{Z}_2$).

Now, we investigate the convergence of the discrete model (6), (7) to the continuous model (2) as $q \rightarrow 1$ in the case where $r = 2$ and $s = 1$. Let h be the lattice spacing, $x = nh$, and $T = \exp(h \frac{d}{dx})$. We assume that n is real and that the L_j are difference operators on the entire line \mathbb{R} .

Proposition 3. *If $r = 2$, $s = 1$, and $\alpha_1 = \alpha_2$, then the operators $L_j + \frac{\alpha_j}{2}$ corresponding to the solutions (27) for $\varepsilon = -1$ and $q = \exp(-\frac{\alpha}{8}h^2)$ converge weakly to a harmonic oscillator:*

$$\left(L_{1,2} + \frac{\alpha_1}{2}\right) \psi(x) = \left(-\frac{d^2}{dx^2} + \frac{\alpha_1^2}{4}x^2\right) \psi(x) + o(h) \quad \text{for all } \psi \in C^2(\mathbb{R}).$$

Proof. By (30), we have

$$\begin{aligned} L_j &= (a_j T^{-1}(b_j)) T^{-1} + (a_j^2 + b_j^2) + (T(a_j)b_j) T \\ &= a_j(x)b_j(x-h) \exp\left(-h \frac{d}{dx}\right) + a_j^2(x) + b_j^2(x) \\ &\quad + a_j(x+h)b_j(x) \exp\left(h \frac{d}{dx}\right) + \dots \\ &= A(x, h) + hB(x, h) \frac{d}{dx} + h^2C(x, h) \frac{d^2}{dx^2} + \dots, \end{aligned}$$

where

$$\begin{aligned} A(x, h) &= a_j^2(x) + b_j^2(x) + a_j(x+h)b_j(x) - a_j(x)b_j(x-h), \\ B(x, h) &= a_j(x+h)b_j(x) - a_j(x)b_j(x-h), \\ C(x, h) &= \frac{1}{2}(a_j(x+h)b_j(x) + a_j(x)b_j(x-h)). \end{aligned}$$

Using formulas (27), we expand the functions a_j and b_j in a power h -series; this yields the following asymptotic formulas for the coefficients of L_j :

$$A(x, h) = \frac{\alpha_1^2}{4}x^2 + O(h), \quad B(x, h) = O(h), \quad C(x, h) = -\frac{1}{h^2} + O(1).$$

The proposition is proved. \square

In a similar way, it can be proved that if $\alpha_1 \neq \alpha_2$, then L_j converges to

$$-\frac{d}{dx^2} + \frac{(\alpha_j + \alpha_{j+1})^2}{16}x^2 - \frac{\alpha_j}{2} - \frac{(\alpha_j - \alpha_{j+1})(\alpha_j + 3\alpha_{j+1})}{4(\alpha_j + \alpha_{j+1})^2x^2},$$

where $\alpha_{j+2} = \alpha_j$, i.e., to the operator of the usual Darboux chain (2) for $r = 2$ (see (4)).

§4. CHAINS OF ARBITRARY LENGTH

Now we turn to the study of cyclic Darboux q -chains (6), (7) of arbitrary even length r and with shift $s = r/2$. For $r > 2$, no explicit solution of the corresponding system (11) of difference equations is known. Moreover, *a priori*, it is unclear whether such a solution exists. Our goal in this section is to prove the following statement.

Theorem 1. *For an arbitrary even r , arbitrary positive $\alpha_1, \dots, \alpha_r$, and arbitrary q with $0 < q < 1$, problem (6), (7) with $s = r/2$ has an r -parametric family of solutions. For all j , the operator L_j is bounded and has only a discrete spectrum $\{\lambda_{j,0}, \lambda_{j,1}, \dots\}$, which belongs to the interval $[-\alpha_j, \|L_j\|]$. The points of the spectrum can be calculated by the Darboux method:*

$$(31) \quad \lambda_{j,0} = 0, \quad \lambda_{j+1,k+1} = q(\lambda_{j,k} + \alpha_j), \quad \lambda_{j+r,k} = \lambda_{j,k}.$$

For each j , the eigenfunctions of L_j are also calculated by the Darboux method,

$$(32) \quad A_{j-1}\psi_{j,0} = 0, \quad \psi_{j+1,k+1} = A_j^+\psi_{j,k},$$

and constitute a complete family in $\mathcal{L}_2(\mathbb{Z})$.

For an arbitrary even r , arbitrary negative $\alpha_1, \dots, \alpha_r$, and arbitrary $q > 1$, problem (6), (7) with $s = r/2$ has an r -parametric family of solutions. For all j , the operator L_j is bounded and has only a discrete spectrum $\{\lambda_{j,0}, \lambda_{j,1}, \dots\}$ which belongs to the interval $[0, \|L_j\|]$. The points of the spectrum can be calculated by the Darboux method:

$$\lambda_{j,0} = -\alpha_j, \quad \lambda_{j-1,k+1} = \frac{\lambda_{j,k}}{q} - \alpha_{j-1}, \quad \lambda_{j-r,k} = \lambda_{j,k}.$$

For each j , the eigenfunctions of L_j are also calculated by the Darboux method,

$$A_j^+\psi_{j,0} = 0, \quad \psi_{j-1,k+1} = A_{j-1}\psi_{j,k},$$

and form a complete family in $\mathcal{L}_2(\mathbb{Z})$.

We split the proof of the theorem into several steps. First, we prove the existence of at least one solution of problem (6), (7), and then prove that this solution can be perturbed. As was shown in §2, the initial q -chain reduces to the system (16) of difference equations. Using the notation $\xi_j(n) = u_j^2(n + 1)$, $\eta_j(n) = v_j^2(n)$, we obtain the following system of equations:

$$(33) \quad \begin{cases} \xi_j(n - 1) + \eta_j(n) = q(\xi_{j-1}(n) + \eta_{j-1}(n - 1)) + \alpha_j, \\ \xi_j(n)\eta_j(n - 1) = q^2\xi_{j-1}(n - 1)\eta_{j-1}(n), \end{cases}$$

where j is a cyclic index, $j \in \mathbb{Z}_r$. Thus, it suffices to study system (33) (in which the $\xi_j(n)$ and $\eta_j(n)$ are defined for all n) and prove that such a system of an arbitrary length r has a $2r$ -parametric family of solutions.

Remark 3. As was shown in §2, the initial q -chain reduces to a system of difference equations with respect to the variables $u_j(n)$, $v_j(n)$, where $j+n$ is even; the corresponding operators B_j of the form (15) act on the lattice $2\mathbb{Z} + (j + 1) \pmod{2}$. However, in the chain (14), these operators act on the entire axis \mathbb{Z} ; therefore, this sequence is equivalent to *two independent* chains of the form (6), (7). This explains the fact that, by Theorem 1, the initial chain is determined by r parameters, while a solution of system (33) (if we assume that the variables $\xi_j(n)$ and $\eta_j(n)$ are defined at all integer points) is determined by $2r$ parameters (this will be shown later).

Remark 4. The study of system (33) for odd r (which is also presented below) is not directly related to the proof of Theorem 1. However, this study is of independent interest because an arbitrary chain of the form (6), (7) and of length $r = 4t + 2$ can be transformed

into a “blinking” chain of the form (14), (15), with half-length $2t + 1$, in the same way as a given chain of length 2 was represented in §3 in the form of a “blinking” chain of length 1. Therefore, if $\alpha_j = \alpha_{j+2t+1}$ for all j , then the given chain is equivalent to a chain of the form (14), (15), i.e., to a system (33) of odd length. Such a choice of parameters is of special interest for us because we expect to prove that, in this case, the solutions of the discrete problem converge to solutions of a continuous problem (in the same sense as the difference operators converge to continuous ones for $r = 2$; see Proposition 3).

Proposition 4. *For all r , the system (33) of difference equations with $j \in \mathbb{Z}_r, 0 < q < 1$, and $\alpha_j > 0$ has a $2r$ -parametric family of solutions that are positive at all points of the integral lattice.*

Proof. First, for each j , we obtain explicit expressions for $\xi_j(n)$ and $\eta_j(n)$ in terms of $\xi_i(n - 1)$ and $\eta_i(n - 1), i = 1, \dots, r$. Eliminating $\xi_j(n)$ from the equations of the second type in (33) and substituting the resulting expressions in the equations of the first type, we obtain the following system of linear equations:

$$(34) \quad c_j \eta_j(n) = g_j, \quad \text{where } j \in \mathbb{Z}_r,$$

$$(35) \quad \begin{aligned} c_j &= \eta_{j-1}(n - 1), \\ d_j &= q^3 \xi_{j-2}(n - 1), \\ g_j &= \eta_{j-1}(n - 1)(q\eta_{j-1}(n - 1) - \xi_j(n - 1) + \alpha_j). \end{aligned}$$

Thus, for an arbitrary choice of the initial data $(\xi_j(0), \eta_j(0))$, where $j = 1, \dots, r$, the corresponding solution of system (33) can be extended to the positive “semiaxis” provided that, for all positive integers n , the determinant of the matrix of the linear system (34) is different from zero. Applying the Cramer formulas and representing the result in a compact form, we obtain

$$(36) \quad \eta_j(n) = \frac{\Delta_j}{\Delta}, \quad \xi_j(n) = q^2 \frac{\xi_{j-1}(n - 1)\eta_{j-1}(n)}{\eta_j(n - 1)} = q^2 \frac{\xi_{j-1}(n - 1)}{\eta_j(n - 1)} \frac{\Delta_{j-1}}{\Delta},$$

where

$$(37) \quad \Delta_j = \sum_{k=0}^{r-1} \frac{g_{j-2k}}{c_{j-2k}} \left(\prod_{i=0}^{k-1} \frac{d_{j-2i}}{c_{j-2i}} \right) \left(\prod_{i=1}^r c_i \right), \quad \Delta = \prod_{i=1}^r c_i - \prod_{i=1}^r d_i$$

if r is odd, and

$$(38) \quad \begin{aligned} \Delta_j &= \sum_{k=0}^{\frac{r}{2}-1} \frac{g_{j-2k}}{c_{j-2k}} \left(\prod_{i=0}^{k-1} \frac{d_{j-2i}}{c_{j-2i}} \right) \left(\prod_{i=1}^r c_i \right) - \sum_{k=1}^{\frac{r}{2}} \frac{g_{j+2k}}{d_{j+2k}} \left(\prod_{i=1}^{k-1} \frac{c_{j+2i}}{d_{j+2i}} \right) \left(\prod_{i=1}^r d_i \right), \\ \Delta &= \left(\prod_{i=0}^{\frac{r}{2}-1} c_{2i+1} - \prod_{i=0}^{\frac{r}{2}-1} d_{2i+1} \right) \left(\prod_{i=1}^{\frac{r}{2}} c_{2i} - \prod_{i=1}^{\frac{r}{2}} d_{2i} \right) \end{aligned}$$

if r is even. It is assumed that all indices in (36)–(38) are cyclic, i.e., they are added and subtracted modulo r , and that the products \prod are equal to 1 if the lower limit is greater than the upper limit. Observe that if r is even, then Δ_j can be factored as follows:

$$\Delta_j = \left(\prod_{i=1}^{\frac{r}{2}} c_{j+2i+1} - \prod_{i=1}^{\frac{r}{2}} d_{j+2i+1} \right) \left(\sum_{k=0}^{\frac{r}{2}-1} \frac{g_{j-2k}}{c_{j-2k}} \left(\prod_{i=0}^{k-1} \frac{d_{j-2k}}{c_{j-2k}} \right) \left(\prod_{i=0}^{\frac{r}{2}-1} c_{j+2i} \right) \right),$$

so that the expressions (38) have a common factor, which can be cancelled in (36).

Multiplying all equations of the second group in system (33), we obtain the following integral of this system:

$$(39) \quad q^{-2rn} \frac{\xi_1(n)\xi_2(n)\cdots\xi_r(n)}{\eta_1(n)\eta_2(n)\cdots\eta_r(n)} = \kappa,$$

where κ is a constant determined by the initial data,

$$\kappa = \frac{\xi_1(0)\xi_2(0)\cdots\xi_r(0)}{\eta_1(0)\eta_2(0)\cdots\eta_r(0)}.$$

Moreover, equations (33) have the symmetry $\xi_j(n) \longleftrightarrow \eta_j(-n)$. Therefore, if we seek a symmetric solution, $\xi_j(0) = \eta_j(0)$ for all $j = 1, \dots, r$, it suffices to prove only that it exists and is positive on the “semiaxis” \mathbb{N} . Moreover, by (36), it suffices to check that $\eta_j(n)$ is positive for all j (if we already know that $\xi_j(n-1)$ and $\eta_j(n-1)$ are positive).

Now, we construct a solution of (33) that is positive on the integral lattice and satisfies the condition $\xi_j(0) = \eta_j(0) = \rho > 0$ for all $j = 1, \dots, r$ and some ρ . Let

$$A_j(n) = q\eta_{j-1}(n-1) - \xi_j(n-1) + \alpha_j.$$

Consider the case where r is odd. Using (35) and (39), we can represent formulas (36), (37) as follows:

$$(40) \quad \eta_j(n) = \frac{1}{1 - q^{2nr+r}} \sum_{k=0}^{r-1} q^{3k} h_{j,j-2k}(n) A_{j-2k}(n),$$

where $h_{j,k}(n)$ is a certain product of fractions of the form $\frac{\xi_{i-1}(n-1)}{\eta_i(n-1)}$ (it is easy to find exact expressions for these products but we do not need this). We only note that

$$(41) \quad h_{j,j}(n) = 1, \quad h_{j,j-2} = \frac{\xi_{j-2}(n-1)}{\eta_{j-1}(n-1)},$$

and that $h_{j,k}$ involves more than one factor if $k \neq j, j-2$. Thus, it suffices to show that all $A_j(n)$ are positive.

It is easily seen that, for our choice of the initial data, the quantities

$$A_j(1) = \rho(q-1) + \alpha_j$$

are positive if we require that the parameter ρ satisfy the inequality

$$(42) \quad 0 < \rho < \frac{\hat{\alpha}}{1-q}$$

with $\hat{\alpha} = \min_j \alpha_j$. Therefore, $\xi_j(1) > 0$ and $\eta_j(1) > 0$ for all $j = 1, \dots, r$. Next, we use (36) to obtain

$$\begin{aligned} A_j(2) &= q\eta_{j-1}(1) - \xi_j(1) + \alpha_j = q\eta_{j-1}(1) - q^2\eta_{j-1}(1) + \alpha_j = q(1-q)\eta_{j-1}(1) + \alpha_j \\ &> \alpha_j > 0, \end{aligned}$$

i.e., the $\xi_j(2)$ and $\eta_j(2)$ are also positive for all j .

Now, we prove that the following relation is valid for all $j = 1, \dots, r$ and all $n \geq 3$:

$$q\eta_{j-1}(n-1) - \xi_j(n-1) = q\eta_{j-1}(n-1) \frac{A_j(n-2)}{\eta_j(n-2)}.$$

Indeed, considering the corresponding difference and using (33), we obtain

$$\begin{aligned}
& q\eta_{j-1}(n-1) - \xi_j(n-1) - q \frac{\eta_{j-1}(n-1)}{\eta_j(n-2)} (q\eta_{j-1}(n-3) - \xi_j(n-3) + \alpha_j) \\
&= q\eta_{j-1}(n-1) - \xi_j(n-1) - q \frac{\eta_{j-1}(n-1)}{\eta_j(n-2)} (\eta_j(n-2) - q\xi_{j-1}(n-2)) \\
&= -\xi_j(n-1) + q^2 \frac{\xi_{j-1}(n-2)\eta_{j-1}(n-1)}{\eta_j(n-2)} \\
&= 0.
\end{aligned}$$

Therefore,

$$A_j(n) = q\eta_{j-1}(n-1) - \xi_j(n-1) + \alpha_j = q\eta_{j-1}(n-1) \frac{A_j(n-2)}{\eta_j(n-2)} + \alpha_j > \alpha_j > 0,$$

provided that we have already proved that $\eta_{j-1}(n-1)$, $\eta_j(n-2)$, and $A_j(n-1)$ are positive. Thus, by induction, we see that if the initial data satisfy condition (42), then the corresponding symmetric solution is positive on the entire integral lattice.

In the case where r is even, the ideas of the proof are the same, but some essential details should be indicated. After reducing the fractions in formulas (36), we cannot use the integral (39). Therefore, (40) is replaced with the following formula:

$$(43) \quad \eta_j(n) = \frac{\sum_{k=0}^{\frac{r}{2}-1} q^{3k} h_{j,j-2k}(n) A_{j-2k}(n)}{1 - q^{\frac{3r}{2}} \prod_{i=1}^{\frac{r}{2}} \frac{\xi_{j+2i+1}(n-1)}{\eta_{j+2i}(n-1)}}.$$

Thus, compared to the case of an odd r , the only difference is that at each step we must check additionally that the denominator of the fraction (43) is positive. For $n = 1, 2$ (we use induction) this condition is obviously fulfilled; if $n > 2$, then, by the induction hypothesis, we have $A_j(n) > \alpha_j$, i.e., $q\eta_{j-1}(n-1) > \xi_j(n-1)$ for all $j = 1, \dots, r$. Therefore,

$$\prod_{i=1}^{\frac{r}{2}} \frac{\xi_{j+2i}(n-1)}{\eta_{j+2i-1}(n-1)} < \prod_{i=1}^{\frac{r}{2}} \frac{\xi_{j+2i}(n-1)}{q \xi_{j+2i}(n-1)} = q^{\frac{r}{2}},$$

which implies that the denominator in (43) is positive.

Thus, we have proved that, for all positive α_j , system (33) has a solution positive on the entire integral lattice and satisfying the condition $\xi_j(0) = \eta_j(0) = \rho$, $j = 1, \dots, r$, where the parameter ρ obeys (42), i.e., we have constructed a one-parameter family of solutions of system (33).

Now, we prove that a small perturbation of a solution in this family leads to a function that is positive at all points of the integral lattice. Indeed, in the proof of positivity the induction step is independent of the initial data. Therefore, it suffices to choose initial data such that the inequalities $A_j(1) > 0$ and $A_j(2) > 0$ are fulfilled for all $j = 1, \dots, r$. These inequalities determine an open set in \mathbb{R}^{2r} , and we have already proved that this set is nonempty. Thus, sufficiently small perturbations of the solution constructed above remain positive for $n > 0$. By the symmetry of the system, the solution is positive at all integers under sufficiently small perturbations. The proposition is proved. \square

We note that the existence of a $2r$ -parametric family of positive solutions of (33) implies the existence of an r -parametric solution of problem (6), (7).

On the set

$$\left\{ (\xi_1(n-1), \dots, \xi_r(n-1), \eta_1(n-1), \dots, \eta_r(n-1)) \mid \prod_{j=1}^r \eta_j(n-1) - q^{3r} \prod_{j=1}^r \xi_j(n-1) \neq 0 \right\}$$

we consider the shift F_n along the “trajectories” of system (33):

$$(44) \quad F_n : (\xi_1(n-1), \dots, \xi_r(n-1), \eta_1(n-1), \dots, \eta_r(n-1)) \\ \mapsto (\xi_1(n), \dots, \xi_r(n), \eta_1(n), \dots, \eta_r(n)).$$

The mapping F_n does not depend on n (see the explicit formulas (36)–(38)). Therefore, we drop the dependence of F_n and of the coordinates on n , namely, we write ξ_j and η_j instead of $\xi_j(n-1)$ and $\eta_j(n-1)$, respectively.

Proposition 5. *In the domain $\mathcal{D} = \{\eta_j \neq 0 \mid j = 1, \dots, r\}$, the mapping F has a unique fixed point*

$$N = \left(\underbrace{0, 0, \dots, 0}_r, \frac{\alpha_1 + q\alpha_r + \dots + q^{r-1}\alpha_2}{1 - q^r}, \frac{\alpha_2 + q\alpha_1 + \dots + q^{r-1}\alpha_3}{1 - q^r}, \dots, \frac{\alpha_r + q\alpha_{r-1} + \dots + q^{r-1}\alpha_1}{1 - q^r} \right),$$

and this point is attracting.

In the domain $\mathcal{E} = \{\xi_j \neq 0 \mid j = 1, \dots, r\}$, the mapping F has a unique fixed point

$$S = \left(\frac{\alpha_1 + q\alpha_r + \dots + q^{r-1}\alpha_1}{1 - q^r}, \frac{\alpha_2 + q\alpha_1 + \dots + q^{r-1}\alpha_r}{1 - q^r}, \dots, \frac{\alpha_r + q\alpha_{r-1} + \dots + q^{r-1}\alpha_1}{1 - q^r}, \underbrace{0, 0, \dots, 0}_r \right),$$

and this point is repelling.

Proof. We assume that $(\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_r)$ is a fixed point of F in \mathcal{D} and choose it as the initial point. Then the function (39) does not change under the iterations of F , which is possible only if $\kappa = 0$. This implies that at least one of the ξ_j is zero. There is no loss of generality in assuming that $\xi_1 = 0$. Considering the equations of the second group in (33), we immediately conclude that $\xi_2 = \xi_3 = \dots = \xi_r = 0$. Substituting this in the remaining equations in (33), we obtain the linear system

$$\begin{pmatrix} 1 & 0 & \dots & 0 & -q \\ -q & 1 & \dots & 0 & 0 \\ 0 & -q & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -q & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \vdots \\ \eta_r \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_r \end{pmatrix}.$$

Solving this system, we obtain the required formulas for the coordinates of the fixed point N .

Now, we prove that this point is attracting. For this, we need to calculate the Jacobian of F at N . First, we assume that r is odd; in this case all terms in (40) except for the first two do not contribute into the partial derivatives at the point N , because they involve

ξ_j with exponent greater than 1. Now, using (41), we can easily find the Jacobi matrix of F at N and prove that its characteristic polynomial has the form

$$(45) \quad \chi(\lambda) = (\lambda^r - q^{2r})(\lambda^r - q^r).$$

Thus, the Jacobi matrix $dF|_N$ has the following eigenvalues:

$$q\varepsilon_1, q\varepsilon_2, \dots, q\varepsilon_r, q^2\varepsilon_1, q^2\varepsilon_2, \dots, q^2\varepsilon_r,$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ are the r th roots of unity. The case where r is even is analyzed similarly. Although the explicit formulas (38) and (43) differ from (37) and (40), we arrive at the same result as in the case of where r is even if we drop the “inessential” terms (i.e., those not contributing into the entries of the Jacobian at N). Therefore, again, the characteristic polynomial has the form (45).

As the norm of a finite-dimensional linear operator, we can take the maximum of the absolute values of its eigenvalues if the matrix of it is diagonalizable. Since the norm is continuous in the vicinity of N , we have $\|dF\| \leq p < 1$. Thus, for all points X sufficiently close to N we have the estimate

$$\|F(X) - F(N)\| = \|dF|_Y\| \cdot \|X - N\| \leq p\|X - N\| < \|X - N\|,$$

where Y is a point in the segment connecting X and N . Thus, the point N is attracting.

The domains \mathcal{E} and \mathcal{D} correspond to each other under the symmetry $\xi_j(n) \longleftrightarrow \eta_j(-n)$. Therefore, \mathcal{E} contains a unique fixed point S symmetric to N . The point S is repelling because N is attracting. The proposition is proved. \square

Now, suppose that

$$(46) \quad \alpha_1 = \dots = \alpha_r.$$

It is easy to check that the solution of system (33) corresponding to the initial data

$$(47) \quad \xi_1(0) = \dots = \xi_r(0), \quad \eta_1(0) = \dots = \eta_r(0)$$

has the following property:

$$\xi(n) = \xi_1(n) = \dots = \xi_r(n), \quad \eta(n) = \eta_1(n) = \dots = \eta_r(n) \quad \text{for all } n \in \mathbb{Z}.$$

Since the $\xi(n)$ and $\eta(n)$ satisfy (18) (for $\alpha_1 = \alpha_2$), they have the asymptotics (28), (29). Thus, for all initial data of the form (47) such that the corresponding solution (see §3) is positive at all points of the integral lattice, this solution tends to the fixed point N as $n \rightarrow +\infty$ and to the fixed point S as $n \rightarrow -\infty$. Since these points are attracting and repelling, respectively, and the mapping (44) is continuous together with its inverse inside each curvilinear triangle given by relations (25), and depends continuously on the parameters α_j , a small perturbation of the initial data (47) and the parameters (46) do not change the asymptotics of a solution, i.e., the solution tends to the fixed points corresponding to α_j as $n \rightarrow \pm\infty$. By continuity, the solution remains positive after a sufficiently small perturbation of the initial data (47) and the parameters (46). Thus, for each even r , we can perturb the parameters (46) so as to construct operators L_1, \dots, L_r satisfying (6) and (7) and such that the coefficients of $T^{\pm 1}$ tend to zero as $n \rightarrow \infty$ and the operators themselves become equivalent to the operators of multiplication by a constant as $n \rightarrow \infty$.

Our goal is to prove that this asymptotic behavior of the coefficients of the operators in a chain remains valid not only after small perturbations of the parameters α_j , but also for arbitrary positive values of these parameters, i.e., we must prove that, for an arbitrary collection of positive numbers α_j , there exist initial data such that the corresponding solution is positive at all integers and provides a solution for the q -chain (6), (7) with a “proper” asymptotic behavior of coefficients.

We fix some starting parameters $\alpha_1 = \dots = \alpha_r$ and some initial data $\xi_1(0) = \dots = \xi_r(0) = \eta_1(0) = \dots = \eta_r(0)$ satisfying condition (42); the corresponding solution is positive and has the required asymptotics. Now, we shall change the values of the numbers α_j in such a way that none of them decrease; the solution will remain positive. It is easily seen that the points in the space of parameters corresponding to the positive solutions with “proper” asymptotics constitute an open set. Let \mathcal{A} be a connected component of this set containing the starting point $\alpha_1 = \dots = \alpha_r$. Suppose that, by increasing the parameters α_j as described above, we have reached the boundary of \mathcal{A} , i.e., we have found a limiting collection $\alpha_1, \dots, \alpha_r$ for which our solution with fixed initial data fails to have a “proper” asymptotics. We shall show that this situation is impossible.

Proposition 6. *In the interval $[0, \frac{\omega_j}{1-q^r})$, the discrete spectrum of the operator L_j of the cyclic q -chain (33) looks like this:*

$$(48) \quad \omega_j \frac{1-q^{kr}}{1-q^r}, \quad q^{kr} q\alpha_{j-1} + \omega_j \frac{1-q^{kr}}{1-q^r}, \quad \dots, \quad q^{kr} \sum_{i=1}^{r-1} q^i \alpha_{j-i} + \omega_j \frac{1-q^{kr}}{1-q^r},$$

$$k = 0, 1, \dots,$$

where

$$\omega_j = q^r \alpha_j + q^{r-1} \alpha_{j+1} + \dots + q \alpha_{j-1}$$

and j is a cyclic index. If the operators of the q -chain are bounded, then (48) is the entire discrete spectrum of L_j .

Proof. Let $\psi_{j,0}$ be the ground state of L_j , $A_{j-1}\psi_{j,0} = 0$. Then relations (33) imply that $L_j\psi_{j,0} = 0$ for all j , i.e., 0 belongs to the spectrum of each operator in the q -chain.

It is easily seen that if $L_j\psi = \lambda\psi$, then $A_j^+\psi$ is an eigenvector of L_{j+1} corresponding to the eigenvalue $q(\lambda + \alpha_j)$. Therefore, acting as in (32), we obtain the set of eigenvalues (31). Representing these expressions in the explicit form, we obtain (48).

Now, we show that the spectrum of L_j in the interval $[0, \frac{\omega_j}{1-q^r})$ is exhausted by the collection (48) of q -arithmetic progressions. Let $L_j\psi = \lambda\psi$ for some λ and j . Then the q -chain relations imply that $q\|A_{j-1}\psi\|^2 = \lambda\|\psi\|^2$, i.e., λ is nonnegative, and only the ground state corresponds to the eigenvalue $\lambda = 0$. Moreover, it is easy to check that either $\lambda = 0$, or $A_{j-1}\psi$ is an eigenvector of L_{j-1} corresponding to the eigenvalue $\frac{\lambda}{q} - \alpha_{j-1}$. Therefore,

$$L_j(A_j \dots A_{j-2} A_{j-1} \psi) = \frac{\lambda - \omega_j}{q^r} (A_j \dots A_{j-2} A_{j-1} \psi).$$

Repeated application of this operator gives the following sequence of eigenvalues:

$$(49) \quad \frac{\lambda - \omega_j \frac{1-q^{kr}}{1-q^r}}{q^{kr}}.$$

If $\lambda \in [0, \frac{\omega_j}{1-q^r})$, then this sequence tends to $-\infty$, which contradicts the fact that all eigenvalues are nonnegative. Thus, this sequence must break, i.e., for some j the application of the operator $A_j \dots A_{j-2} A_{j-1}$ must give the zero vector, which is possible only if we have reached the ground state. It follows that λ belongs to the collection (48).

Let $\lambda > \frac{\omega_j}{1-q^r}$. Then the sequence (49) increases and tends to $+\infty$. Therefore, it cannot break, and the application of $A_j \dots A_{j-2} A_{j-1}$ gives a sequence of eigenvalues that tends to $+\infty$. Thus, if the operators in the chain (6), (7) are bounded, then the collection (48) represents the entire discrete spectrum of L_j . \square

Proposition 7. *If $(\alpha_1, \dots, \alpha_r) \in \mathcal{A}$, then the corresponding operators $L_j - \frac{\omega_j}{1-q^r}$ are compact.*

Proof. The fact that a solution of system (33) has “proper” asymptotics means that the coefficients of $T^{\pm 1}$ in the expressions for the corresponding operators L_j tend to zero as $n \rightarrow \pm\infty$. Therefore, the coefficients of the “shifted” operator $L_j - \frac{\omega_j}{1-q^r}$ tend to zero as $n \rightarrow \pm\infty$. It follows that the operator $L_j - \frac{\omega_j}{1-q^r}$ can be approximated by a sequence of finite-dimensional difference operators of the form $c_{j,m}T^{-1} + d_{j,m} + f_{j,m}T$, where $c_{j,m}(n) = d_{j,m}(n) = f_{j,m}(n) = 0$ for $|n| > m$, which means that the operators $L_j - \frac{\omega_j}{1-q^r}$ are compact. The proposition is proved. \square

Corollary 1. *Under the assumptions of Proposition 7, the eigenvectors of each operator L_j form a complete family in $\mathcal{L}_2(\mathbb{Z})$.*

Corollary 2. *Under the assumptions of Proposition 7, the norm of L_j is equal to $\frac{\omega_j}{1-q^r}$.*

Corollary 1 is obtained by application of the Hilbert–Schmidt theorem to the self-adjoint compact operator $L_j - \frac{\omega_j}{1-q^r}$. Corollary 2 follows directly from Corollary 1, because the operator L_j is diagonalizable in the basis of eigenvectors, so that its norm is equal to the supremum of the absolute values of its eigenvalues.

Let $\mathbf{a} = (\alpha_1, \dots, \alpha_r)$ be a point on the boundary of \mathcal{A} . We choose a sequence $\{\mathbf{a}_m\}$ of points in \mathcal{A} that converges to \mathbf{a} and is such that all the coordinates of \mathbf{a}_m increase with m . Let $L_j(\mathbf{a})$ and $L_j(\mathbf{a}_m)$ denote the operators in the q -chain that correspond to the parameters \mathbf{a} and \mathbf{a}_m , respectively. Obviously, $\|L_j(\mathbf{a}_m)\| < \frac{\omega_j}{1-q^r}$ for all j and m . Therefore, the norm of the limit operator also does not exceed $\frac{\omega_j}{1-q^r}$. However, Proposition 6 yields the reverse inequality. Thus, we have $\|L_j(\mathbf{a})\| = \frac{\omega_j}{1-q^r}$ for all $j = 1, \dots, r$.

It is easy to show that the operator sequence $L_j(\mathbf{a}_m)$ converges weakly to the operator $L_j(\mathbf{a})$ for all j (we already know that the operators $L_j(\mathbf{a})$ are bounded). It follows that the spectrum of the limit operator $L_j(\mathbf{a})$ is discrete, and, by Proposition 6, it is of the form (48). Therefore, the limit “shifted” operator $L_j(\mathbf{a}) - \frac{\omega_j(\mathbf{a})}{1-q^r}$ is also compact.

Clearly, the coefficients of a compact difference operator must tend to zero as $n \rightarrow \pm\infty$. Therefore, the existence of the integral (39) implies that $a_j(n) \rightarrow 0$ as $n \rightarrow +\infty$ and $b_j(n) \rightarrow 0$ as $n \rightarrow -\infty$. This shows that the coefficients of the limit operator have a “proper” asymptotics, i.e., $\mathbf{a} \in \mathcal{A}$. We conclude that the set \mathcal{A} is open and closed simultaneously on the positive part \mathbb{R}_+^r of the parameter space. Thus, $\mathcal{A} = \mathbb{R}_+^r$.

We have proved that, for an arbitrary set $(\alpha_1, \dots, \alpha_r)$ of positive parameters, there exists a solution of system (33) that is positive at all points of the integer lattice and has a “proper” asymptotics as $n \rightarrow \pm\infty$. By Proposition 7, the corresponding “shifted” operators $L_j - \frac{\omega_j}{1-q^r}$ are compact. Therefore, their spectra are discrete and the eigenvalues constitute complete families in $\mathcal{L}_2(\mathbb{Z})$.

Applying the symmetry (8), we immediately reduce the case where $q > 1$ and $\alpha_j < 0$ to the preceding one. This completes the proof of the theorem.

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