SOME GEOMETRIC PROPERTIES OF CONVEX BODIES. II

V. V. MAKEEV

Abstract. Topological means are used for the study of approximation of 2-dimensional sections of a 3-dimensional convex body by affine-regular pentagons and approximation of a centrally symmetric convex body by a prism. Also, the problem of estimating the relative surface area of the sphere in a normed 3-space, the problem on universal covers for sets of unit diameter in Euclidean space, and some related questions are considered.

Throughout, by a convex body $K \subset \mathbb{R}^n$ (a figure for $n = 2$) we mean a compact convex subset of $\mathbb{R}^n$ with nonempty interior.

We denote by $G_k(\mathbb{R}^n)$ (respectively, $G_k^+(\mathbb{R}^n)$) the Grassmann manifold of nonoriented (respectively, oriented) $k$-planes in $\mathbb{R}^n$ passing through $O \in \mathbb{R}^n$. We let

$$\gamma_k^n : E_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n) \quad \text{and} \quad (\gamma_k^n)^+ : E_k^+(\mathbb{R}^n) \to G_k^+(\mathbb{R}^n)$$

be the tautological fiber bundles, where the fiber over an (oriented) $k$-plane $\alpha \in G_k(\mathbb{R}^n)$ is $\alpha$ itself regarded as a $k$-dimensional vector space.

We say that a field of convex bodies (or figures; a CB- or a CF-field) is given in a vector bundle $\gamma$ if in each fiber $\alpha$ of $\gamma$ we mark a convex body $K(\alpha)$ depending continuously on $\alpha$. A CB-field is pointed if for each $\alpha$ we also mark a point $x(\alpha) \in K(\alpha)$ depending continuously on $\alpha$. (In other words, $x(\alpha)$ is a section of $\gamma$.)

If $\lambda \in \mathbb{R}$ and $P$ is an affine-regular polygon (e.g., a pentagon or a parallelogram) with center $O(P)$, then $\lambda P$ denotes the polygon homothetic to $P$ with homothety ratio $\lambda$ and homothety center $O(P)$.

We denote by $S(K)$ the area of a figure $K \subset \mathbb{R}^2$.

§1. Fields of convex figures in $\gamma_2^3$ and $(\gamma_2^3)^+$, and 2-dimensional sections of convex bodies in $\mathbb{R}^3$

First, we prove two corollaries to the following known result.

Theorem [2]. Each CF-field in $\gamma_2^3$ contains a figure circumscribed about an affine-regular octagon.

Corollary 1. Suppose $C$ is the bounded component of a cubic surface in $\mathbb{R}^3$. Then each inner point $O$ of $C$ lies in a plane intersecting $C$ along an ellipse.

Proof. Indeed, $C$ is convex automatically, because otherwise $C$ intersects some line at 4 points. Consequently, the section of $C$ by some 2-plane through $O$ is circumscribed about an affine-regular octagon. Then this section is a component of a cubic, intersects an ellipse at 8 points, and, consequently, is an ellipse by the Bézout theorem. \(\square\)
Each pointed CF-field in $\gamma_2^3$ contains a figure $K$ containing a parallelogram $P$ such that

$$P \subset K \subset \left(1 + \frac{\sqrt{2}}{2}\right)P.$$  

Proof. Indeed, the figure $K$ circumscribed about an affine-regular octagon $\Omega = A_1 \ldots A_8$ possesses the required property. In this case, $K$ is contained in the octagonal star $\Sigma$ bounded by segments of the rays that extend the sides of $\Omega$. Letting $P$ be the parallelogram $A_2A_4A_6A_8$, we easily see that $P \subset K \subset \left(1 + \frac{\sqrt{2}}{2}\right)P$. □

Remark. Considering fields of disks, we see that $1 + \frac{\sqrt{2}}{2}$ cannot be replaced by a constant smaller than $\sqrt{2}$.

Theorem 1. Each pointed CF-field $(K(\alpha), x(\alpha))$ in $(\gamma_2^2)^+$ contains a figure $K(\alpha)$ circumscribed about an affine-regular pentagon with center at the marked point $x(\alpha)$.

Corollary 3. Each field of centrally symmetric convex figures in $(\gamma_2^2)^+$ or $\gamma_2^3$ contains a figure circumscribed about an affine-regular decagon. □

Corollary 4. If a field of centrally symmetric convex figures in $(\gamma_2^3)^+$ consists of bounded components of curves of degree at most 4, then the field contains an ellipse.

Indeed, by the Bézout theorem, the figure circumscribed about an affine-regular decagon is an ellipse. □

Corollary 5. Each convex bounded centrally symmetric quartic in $\mathbb{R}^3$ has a planar central section that is an ellipse. □

The proof of Theorem 1 uses two topological lemmas.

Lemma 1. Suppose $n \in \mathbb{N}$, $W$ is a compact oriented $2n$-manifold, the cyclic group $\mathbb{Z}_{2n+1}$ freely acts on $W$, and the boundary $\partial W$ of $W$ with standard orientation is the union of two closed $\mathbb{Z}_{2n+1}$-invariant manifolds $M$ and $M'$. We let $\mathbb{Z}_{2n+1}$ act on $\mathbb{R}^{2n+1}$ by cyclic permutations of coordinates of points and denote by $l \subset \mathbb{R}^{2n+1}$ the line determined by the equations $x_1 = x_2 = \ldots = x_{2n+1}$.

Suppose that $F : W \to \mathbb{R}^{2n+1}$ is a continuous $\mathbb{Z}_{2n+1}$-equivariant mapping such that $F(M)$ and $F(M')$ do not intersect $l$. Then

$$\deg(F) : M \to \mathbb{R}^{2n+1} \setminus l + \deg(F) : M' \to \mathbb{R}^{2n+1} \setminus l \equiv 0 \mod 2n + 1.$$  

Proof. After a slight perturbation, we can assume that $F$ is smooth and transversal to $l$. Then, $F^{-1}(l)$ consists of a finite number of orbits of points of $W$. We surround the points by small balls $D_1, \ldots, D_N$ such that they are mutually disjoint and disjoint from $M$ and $M'$ and, furthermore, the balls with centers in one $\mathbb{Z}_{2n+1}$-orbit are mapped to one another under the action of $\mathbb{Z}_{2n+1}$.

The mapping $F$ takes the $2n$-manifold $\bar{W} := W \setminus \bigcup_{i=1}^N \partial D_i$ with boundary to $\mathbb{R}^{2n+1} \setminus l \cong S^{2n-1}$. Consequently, $\deg(F) : \partial \bar{W} \to \mathbb{R}^{2n+1} \setminus l = 0$. We have

$$\partial \bar{W} = M \cup M' \cup \bigcup_{i=1}^N \partial D_i.$$  

By construction, the $\mathbb{Z}_{2n+1}$-equivariance of $F$ implies that $\deg(F) : \bigcup_{i=1}^N \partial D_i \to \mathbb{R}^{2n+1} \setminus l)$ is divisible by $2n + 1$, which completes the proof. □
Remark. For prime $2n+1$, Stiefel manifolds yield interesting examples in which the degree under consideration is nonzero (see [1] and below). The author does not know whether this is possible in the case where $2n+1$ is composite.

We denote by $SO(3)$ the group of orientation-preserving rotations of $\mathbb{R}^3$ about some point.

**Lemma 2.** Suppose $W$ is a compact 4-manifold, the cyclic group $\mathbb{Z}_5$ acts freely on $W$, and the boundary $\partial W$ of $W$ with standard orientation is the union of two closed $\mathbb{Z}_5$-invariant 3-manifolds $M$ and $M'$, where $M'$ is $SO(3)$ with standard action of $\mathbb{Z}_5$. We let $\mathbb{Z}_5$ act on $\mathbb{R}^5$ by cyclic permutations of the coordinates of points.

Suppose that $\mathcal{F} : W \to \mathbb{R}^5$ is a continuous $\mathbb{Z}_5$-equivariant mapping. Then $\mathcal{F}(M)$ contains a point of the form $(x, x, x, x, y)$, where $y \leq x$ (or, optionally, $y \geq x$).

**Proof.** Let $l \subset \mathbb{R}^5$ be the line determined by the equations $x_1 = x_2 = \cdots = x_5$. If $\mathcal{F}(M) \cap l \neq \emptyset$, then we are done. Otherwise, it suffices to prove the following assertion (see [1]).

**Assertion.** If $\mathcal{F}(M) \cap l = \emptyset$, then the degree of the mapping

$$F| : M \to \mathbb{R}^5 \setminus l \simeq S^3$$

is not divisible by 5.

**Proof.** After a slight perturbation, we can assume that the mapping $F' : M' \to \mathbb{R}^5$ is a $\mathbb{Z}_5$-equivariant mapping with image in $\mathbb{R}^5 \setminus l$. It is well known that in this case $\deg(F') : M' \to \mathbb{R}^5 \setminus l$ is not divisible by 5 (see [1]; actually, $\deg(F') = -1$). Now, the required result follows from Lemma 1. \hfill \Box

**Proof of Theorem 1.** We consider positively oriented affine-regular pentagons inscribed in the figures $K(\alpha)$. If $K(\alpha)$ is a generic smooth field of smooth convex figures in $(\mathbb{S}^2)^+$, then the pentagons constitute a compact oriented smooth 3-manifold $M$, on which the cyclic group $\mathbb{Z}_5$ acts by cyclic permutations of the vertices of the pentagons.

If all figures $K(\alpha)$ are ($C^1$-close to) disks, then, obviously, $M \cong SO(3)$.

We define a continuous mapping

$$F : M \to \mathbb{R}^5$$

as follows. If $P \subset \alpha$ is a pentagon $A_1 \ldots A_5$ inscribed in $K(\alpha)$ and with center $O(P)$, then the $i$th coordinate of $F(P)$ is the orthogonal projection of $x(\alpha)$ to the oriented axis $O(P)A_i$ with origin at $O(P)$, $i = 1, \ldots, 5$.

By construction, $F$ is $\mathbb{Z}_5$-equivariant if $\mathbb{Z}_5$ acts on $\mathbb{R}^5$ by cyclic permutations of the coordinates of points.

Using a smooth generic deformation $K_t(\alpha)$, $t \in [0, 1]$, we deform the initial field $K_0(\alpha) := K(\alpha)$ into a field of figures $K_1(\alpha)$ close to disks. Then the oriented affine-regular pentagons inscribed in the figures $K_t(\alpha)$ form a cobordism between $M$ and $M' \cong SO(3)$. By Lemma 2, $F(M)$ contains a point $F(P) = (x, x, x, x, y)$. Simple geometric arguments show that $x = 0$, whence $O(P) = x(\alpha)$.

**Theorem 2.** Each CF-field $K(\alpha)$ in $(\mathbb{S}^2)^+$ contains a convex figure $K$ circumscribed about an affine-regular pentagon $P$ and such that

$$(*) \quad P \subset K \subset \left(1 + \frac{1}{2 \sin 54^\circ}\right)P \subset 1.6181P.$$

This estimate is sharp for the field consisting of the sections of a tetrahedron $T$ that pass through an inner point of $T$. 


Proof. If \( P \) is a pentagon \( A_1 \ldots A_5 \) inscribed in \( K(\alpha) \), then we draw the support lines of \( K(\alpha) \) parallel to \( A_1A_2, A_2A_3, \ldots, A_5A_1 \) and denote by \( B_1, \ldots, B_5 \) the points where these support lines touch \( K(\alpha) \). We define
\[
F(P) := (S(\triangle A_1A_2B_1), \ldots, S(\triangle A_5A_1B_5)).
\]
As in the proof of Theorem 1, we see that there is a 2-plane \( \alpha \) such that for a certain pentagon \( P \) inscribed in \( K(\alpha) \) we have
\[
f(P) = (x, x, x, x, y),
\]
where \( y \leq x \). We show that \( P \) is the required pentagon.

After an affine transformation, we may assume that \( P = A_1 \ldots A_5 \) is a regular pentagon. Obviously, the altitude of the triangles \( \triangle A_1A_2B_1, \ldots, \triangle A_4A_5B_4 \) is maximal possible if \( S(\triangle A_5A_1B_5) = 0 \), and the figure \( K(\alpha) \) circumscribed about \( P \) is the isosceles trapezoid shown in Figure 1.

Simple calculations show that in this case we have \((*)\).

Remark. From [11] it follows that each CF-field in \((\gamma_2^3)^+\) contains a figure \( K \) circumscribed about a regular pentagon \( P \), in which case we have
\[
P \subset K \subset (1 + \sqrt{5})P.
\]
(Certainly, the same is true for any affine-regular pentagon inscribed in \( K \).) In the general case, the constant \( 1 + \sqrt{5} = 1 + \tan(\pi/5) \cot(\pi/10) \) here cannot be made smaller. Indeed, suppose that the point \( O \) lies near the apex of a regular triangular pyramid \( T \) the lateral edge of which is many times longer than the edge of the base. We consider the CF-field in \((\gamma_2^3)^+\) consisting of sections of \( T \) by planes through \( O \). In this case, the sections intersecting the base of \( T \) are very prolate, while for the other triangular sections the above constant, obviously, cannot be improved, because two sides of a pentagon inscribed into a triangle lie on the sides of the triangle.

If in Theorem 2 and in the situation considered above we lift the condition that the pentagon \( P \) is inscribed, then the sharp values of the constants are not known to the author.

§2. The relative surface area of the sphere in a normed 3-space

Definition. Suppose \( P \) is a polyhedron in a finite-dimensional normed space with unit ball \( K \). For each hyperface \( F \) of \( P \), we take the ratio of the area of \( F \) and the area of the central section of \( K \) parallel to \( F \). The sum of these ratios over all hyperfaces of \( P \) is the relative surface area of \( P \).

In [11], the author constructed a one-parameter family of affine images of a cube-octahedron which are inscribed in the unit ball of a normed 3-space and have relative surface area of at least \( 5/2 \).
To obtain an upper estimate, we approximate the ball of unit diameter in a normed 3-space by a circumscribed hexagonal prism.

We need the following result.

**Theorem 10.** Suppose \( \Pi = A_1 \ldots A_{12} \) is a regular hexagonal prism, and \( A_{13} \) and \( A_{14} \) are points that lie outside \( \Pi \) on the symmetry axis and are symmetric to each other with respect to the center. Each centrally symmetric convex body \( K \subset \mathbb{R}^3 \) centered at \( O \) is circumscribed about an affine image of the 14-tope \( P_{14} = A_1 \ldots A_{14} \) with the same center and such that the parallel support planes of \( K \) at the images of \( A_{13} \) and \( A_{14} \) are parallel to the images of the base planes of \( \Pi \).

**Corollary 6.** Each centrally symmetric convex body \( K \subset \mathbb{R}^3 \) is circumscribed about an affine-regular hexagonal bipyramid \( A_1 \ldots A_8 \) with the same center and such that the support planes of \( K \) at \( A_7 \) and \( A_8 \) are parallel to the plane of the base \( A_1 \ldots A_6 \), while the support planes of \( K \) at \( A_1, \ldots, A_6 \) are parallel to the axis \( A_7 A_8 \) of the bipyramid.

**Proof.** We apply the above theorem. As the lateral edge of \( \Pi \) in \( P_{14} \) tends to zero while the length of \( A_{13} A_{14} \) remains constant, in the limit we obtain the required bipyramid. \( \square \)

**Theorem 3.** The unit ball \( K \) in a normed 3-space is inscribed in a centrally symmetric hexagonal prism with relative surface area not exceeding \( \frac{32}{3} \).

**Corollary 7.** The relative surface area of \( K \) is at most \( \frac{32}{3} \).

**Remark.** If \( K \) is a Euclidean ball, then the regular prism is a hexagonal prism circumscribed about \( K \) with minimal relative surface area, which is equal to \( 12\sqrt{3}/\pi \).

The proof involves the following lemma.

**Lemma 2.** Suppose that the unit disk in a 2-dimensional normed space is an affine-regular hexagon \( H \). Then the area of any centrally symmetric convex figure \( K \) circumscribed about \( H \) is at most \( \frac{4}{3}S(H) \), and the perimeter of \( K \) (with respect to the norm) is at most 8. Both estimates are sharp.

**Proof.** Drawing the support lines of \( K \) at the vertices of \( H \), we reduce the proof to the case where \( K \) is a centrally symmetric hexagon.

1) Simple variational arguments show that the area of \( K \) is maximal if the vertices of \( H \) are the midpoints of the sides of \( K \), or \( K \) is a parallelogram. In both cases, we have \( S(K) = \frac{4}{3}S(H) \).

2) Let \( ED \) and \( E'D' \) be a pair of parallel sides of \( K \), and let \( CC' \) be the diameter of \( H \) parallel to them (see Figure 2). Continuing the opposite sides \( AB \) and \( A'B' \) that contain (respectively) \( C \) and \( C' \), to the intersection with the lines \( ED \) and \( E'D' \), we obtain a
Theorem 3. Each convex body $K \subset \mathbb{R}^3$ is circumscribed about an affine-regular hexagonal bipyramid $A_1 \ldots A_8$ such that the support planes of $K$ at $A_7$ and $A_8$ are parallel to the plane of the base $A_1 \ldots A_6$.

Proof. We prove this theorem for strictly convex smooth bodies $K$; in the general case, the theorem is obtained by passage to the limit.

For $\alpha \in G_2(\mathbb{R}^3)$, we draw the support planes $\alpha_1$ and $\alpha_2$ of $K$ parallel to $\alpha$, and also the secant plane $\alpha_3$ equidistant from $\alpha_1$ and $\alpha_2$ and parallel to them. We join the points of tangency of $\alpha_1$ and $\alpha_2$ with $K$ by a segment $I$ and denote by $A(\alpha)$ the orthogonal projection of the point $I \cap \alpha_3$ to the plane $\alpha$. Let $B(\alpha)$ be the orthogonal projection to $\alpha$ of the set of the centers of the affine-regular hexagons inscribed in $\alpha_3 \cap K$.

It suffices to prove that for some $\alpha \in G_2(\mathbb{R}^3)$ we have $A(\alpha) \subset B(\alpha)$. By construction,

$$C_1 := \{A(\alpha) \mid \alpha \in G_2(\mathbb{R}^3)\}$$

is the image of a section of $\gamma^3_2$ that realizes the generator of $H_2(E_2(\mathbb{R}^3); \mathbb{Z}_2)$.

We denote by $\Omega$ the 8-manifold of affine-regular hexagons lying in the planes $\alpha_3$. If $K$ is generic, then the affine-regular hexagons inscribed into all possible sections $\alpha_3 \cap K$ constitute a compact smooth 2-manifold $\mathcal{H}$ in $\Omega$. Then

$$C_2 := \bigcup \{B(\alpha) \mid \alpha \in G_2(\mathbb{R}^3)\}$$

is a continuous image of $\mathcal{H}$ that intersects a generic fiber $\alpha \in E_2(\mathbb{R}^3)$ at an odd number of points, because a generic convex figure is circumscribed about an odd number of affine-regular hexagons (see [2]). Thus, $C_2$ also realizes the generator of $H_2(E_2(\mathbb{R}^3); \mathbb{Z}_2)$. Consequently, the $\mathbb{Z}_2$ intersection number of $C_1$ and $C_2$ is equal to 1, whence $C_1 \cap C_2 \neq \emptyset$. \qed
Remark. It is easily seen that the volume of any inscribed affine-regular bipyramid in Theorem 4 is at least $\text{Vol}(K)/6$.

§3. Universal covers for sets of unit diameter in Euclidean space

Definitions and examples. A subset $A \subset \mathbb{R}^n$ is a universal cover for sets of unit diameter if each subset of $\mathbb{R}^n$ of diameter not exceeding 1 is contained in a congruent image of $A$.

A universal cover $A$ is rigid if the set of bodies of constant unit width and such that each of them is contained in finitely many congruent images of $A$ is dense in the Hausdorff metric.

Rigid covers are well known in dimensions 2 and 3. Each centrally symmetric hexagon of unit width is a rigid cover in dimension 2. Borsuk’s solution of the Borsuk problem in the plane involved the cover having the form of a regular hexagon of unit width (see [5]).

A regular rhombo-dodecahedron of unit width is a rigid cover in 3-space (see [4–7]). The one-parameter families of rigid covers constructed in [7, 8] consist of centrally symmetric dodecahedra circumscribed about a ball of unit diameter. Is it true that each universal cover contains a rigid universal cover?

A universal cover $A$ is an $s$-cover if there is a $C^1$ open and dense set of smooth bodies having constant unit width and such that each of them is contained in an odd number of congruent images of $A$.

By definition, all $s$-covers are rigid. All rigid covers mentioned above are $s$-covers. For $n \geq 3$, the author knows no examples of rigid covers that are not $s$-covers.

The following theorem yields infinite series of $s$-covers in Euclidean spaces.

Theorem 5. Suppose that $A_n \subset \mathbb{R}^n$ is a centrally and mirror-symmetric $s$-cover bounded by a finite number of regular hypersurfaces. Let $\Pi$ be the right prism in $\mathbb{R}^{n+1}$ with base $A_n$ and unit height, and let $A_{n+1}$ denote the intersection of $\Pi$ with two unit balls centered at the centers of the bases of $\Pi$. Then $A_{n+1}$ is an $s$-cover in $\mathbb{R}^{n+1}$.

Proof. Suppose that $K \subset \mathbb{R}^{n+1}$ is a generic smooth body of constant unit width. In the total space $E_n(\mathbb{R}^{n+1})$ of $\gamma_{n+1}$, we construct two $n$-dimensional cycles $C_1$ and $C_2$ intersecting the generic fiber at an odd number of points.

For $\alpha \in G_n(\mathbb{R}^{n+1})$, we denote by $s(\alpha)$ the point of intersection of $\alpha$ with the line containing the diameter of $K$ perpendicular to $\alpha$. Obviously, $s$ is a section of $\gamma_{n+1}$, and its image $C_1 = s(G_n(\mathbb{R}^{n+1}))$ intersects each fiber at a unique point.

Consider the fiber bundle $\xi : E(\xi) \to G_n(\mathbb{R}^{n+1})$ such that the fiber over a hyperplane $\alpha \in G_n(\mathbb{R}^{n+1})$ is the set of the congruent images of $A_n$ that lie in $\alpha$. (By the mirror symmetry of $A_n$, no problems with orientation arise.) For $\alpha \in G_n(\mathbb{R}^{n+1})$, let $B_\alpha$ be the set of the congruent images of $A_n$ that contain the orthogonal projection of $K$ to $\alpha$, and let $B = \bigcup_\alpha B_\alpha \subset E(\xi)$. If $K$ is a generic smooth body, then $B$ is a smooth compact $n$-manifold in $E(\xi)$ intersecting the generic fiber at an odd number of points.

We consider the fibrewise mapping $p : E(\xi) \to E_n(\mathbb{R}^{n+1})$ that takes the congruent image of $A$ lying in $\alpha$ to its center in $\alpha$. Then $C_2 = p(B)$ is an $n$-dimensional cycle in $E_n(\mathbb{R}^{n+1})$ intersecting the generic fiber at an odd number of points. Consequently, $C_2$ is $\mathbb{Z}_2$-homologous to $C_1$.

As before, the $\mathbb{Z}_2$ intersection number of $C_1$ and $C_2$ in $E_n(\mathbb{R}^{n+1})$ is nonzero, i.e., in the generic situation they intersect at an odd number of points, which precisely means that the initial body $K$ is contained in an odd number of congruent images of $A_{n+1}$. □

Corollary 8 (to the proof). Suppose $A_n \subset \mathbb{R}^n$ is a centrally and mirror-symmetric $s$-cover bounded by a finite number of regular hypersurfaces. Then each pointed CB-field $(K(\alpha), x(\alpha))$ of constant unit width in $\gamma_{n+1}$ contains a body $K(\alpha)$ that is contained
in a congruent image of $A_n$ centered at $x(\alpha)$. If $K(\alpha)$ is a generic field consisting of smooth convex bodies of constant unit width, then the number of such fibers (and covers) is odd. □

Remarks. 1. The $s$-cover $A_{n+1}$ itself satisfies the assumptions of Theorem 5, which allows us to use it for constructing an $s$-cover $A_{n+2} \subset \mathbb{R}^{n+2}$, etc.

2. If we take one of the 1-, 2-, or 3-dimensional covers mentioned above as a “basis” cover, then Theorem 5 yields an infinite series of $s$-covers that are intersections of half-spaces, cylinders, and spheres of unit diameter.

3. In the text above, we presented polygonal and polyhedral $s$-covers in dimensions $\leq 3$. The author does not know of any polyhedral $s$-covers in dimensions $4, 5, \ldots$.

In [9], it was proved that each centrally symmetric 14-hedron $P$ circumscribed about a ball of unit diameter is a universal cover. However, these covers are certainly not rigid, because for each $P$ and each body $K$ of constant unit width in $\mathbb{R}^4$ there is a 3-parametric family of congruent images of $P$ each of which is circumscribed about $K$.

References


Universitetski˘ı Pr. 27, St. Petersburg 190000, Russia

Received 25/DEC/2002

Translated by N. YU. NETSVETAEV