

## TOEPLITZ AND HANKEL MATRICES AS HADAMARD–SCHUR MULTIPLIERS

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ABSTRACT. The *Hadamard product* of two matrices  $M = (m_{ij})$  and  $A = (a_{ij})$  is defined by  $M \circ A = (m_{ij}a_{ij})$ . A matrix  $M$  is a *Hadamard–Schur multiplier* (in short, HSM) if  $\|M\|_{\mathcal{H}} = \sup\{\|M \circ A\| \mid A : l^2 \rightarrow l^2, \|A\| \leq 1\} < \infty$ . Let  $\mu$  be a complex measure on the circle. An exact formula is found for the multiplier norm of the Toeplitz matrix  $T_\mu = (\hat{\mu}(i-j))_{i,j \geq 0}$ :  $\|T_\mu\|_{\mathcal{H}} = \|\mu\|_{\mathfrak{M}}$ . For the Hankel matrices  $\Gamma_\mu = (\hat{\mu}(i+j))_{i,j \geq 0}$ , we have  $\|\Gamma_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}/H^1_-}$ , and for more general “skew diagonal” matrices we have  $\|(\hat{\mu}(im+jl))_{i,j \geq 0}\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$ , where  $l, m \in \mathbb{Z}$ . Analogs of these results for matrix-valued measures and the corresponding block HSMs are established. Next, a necessary condition of Peller’s type for  $\|\Gamma\|_{\mathcal{H}} < \infty$  is given. It is also shown that, for  $\Lambda \subset \mathbb{Z}_+$ , the norm  $\|\Gamma\|_{\mathcal{H}}$  is equivalent to  $\sup_{k \geq 1} |\gamma_k|$  on the set of Hankel matrices  $\Gamma = (\gamma_{i+j})$  with  $\gamma_k = 0$  for  $k \in \mathbb{Z}_+ \setminus \Lambda$  if and only if  $\Lambda$  is a finite union of lacunary sequences.

### §1. INTRODUCTION

The *Hadamard product* of two matrices  $M = (m_{ij})$  and  $A = (a_{ij})$  is defined by  $M \circ A = (m_{ij}a_{ij})$ . A matrix  $M$  is a *Hadamard–Schur multiplier* (in short, HSM) if

$$\|M\|_{\mathcal{H}} = \sup\{\|M \circ A\| \mid A : l^2 \rightarrow l^2, \|A\| \leq 1\} < \infty.$$

This means that  $A \mapsto M \circ A$  is a bounded map on the space  $\mathcal{L}(l^2)$  of all bounded operators on the space  $l^2 = l^2(\mathbb{Z}_+) = \{x = (x_i)_{i \geq 0} : \sum_{i \geq 0} |x_i|^2 < \infty\}$ .

The most famous Hadamard–Schur multiplier for  $(n \times n)$ -matrices is the triangular truncation operator for which  $(i, j) \mapsto m_{ij}$  is the indicator function of the lower diagonal triangle  $\Delta_n = \{(i, j) : 0 \leq j \leq i \leq n-1\}$ ,  $m_{ij} = \chi_{\Delta_n}(i, j)$  for  $0 \leq i, j \leq n-1$ . The classical Matsaev theorem says that  $\|\chi_{\Delta_n}\|_{\mathcal{H}} \approx \log(n+1)$ , that is, there exists  $c > 0$  such that  $\frac{1}{c} \cdot \log(n+1) \leq \|\chi_{\Delta_n}\|_{\mathcal{H}} \leq c \cdot \log(n+1)$  for every  $n \geq 1$ . We generalize this result in several directions.

In particular, we show that an infinite Toeplitz matrix

$$T = (a_{i-j})_{i,j \geq 0}$$

is an HSM if and only if there exists  $\mu \in \mathfrak{M}$  ( $\mathfrak{M} = \mathfrak{M}(\mathbb{T})$  is the space of all complex measures on the circle  $\mathbb{T}$ ) such that  $a_k = \hat{\mu}(k)$ ,  $k \in \mathbb{Z}$ , where  $\hat{\mu}(k)$  stands for the  $k$ th Fourier coefficient of  $\mu$ . (In this case we write  $T = T_\mu$ .) Moreover, we have  $\|T_\mu\|_{\mathcal{H}} = \|\mu\|_{\mathfrak{M}}$ . For a Hankel matrix

$$\Gamma_\mu = (\hat{\mu}(i+j))_{i,j \geq 0}$$

we have  $\|\Gamma_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}/H^1_-}$ . Moreover, the inequality

$$\|M_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$$

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is proved for more general “skew diagonal” matrices

$$M_\mu = (\hat{\mu}(im + jl))_{i,j \geq 0},$$

where  $l, m \in \mathbb{Z}$ . For  $lm > 0$ ,  $M_\mu$  is a Hankel-like matrix (all diagonals  $im + jl = \text{const}$  are finite), and for  $lm \leq 0$  it is Toeplitz-like (the same diagonals are infinite).

It should be mentioned that, essentially, the above upper estimates for Toeplitz and Hankel multipliers follow from the basic result of A. Grothendieck [11], especially as it was stated by G. Pisier (see [22, Theorem 5.1]): a matrix  $M = (m_{ij})$  is an HSM with  $\|M\|_{\mathcal{H}} \leq C$  if and only if  $m_{ij} = (x_i, y_j)$ , where  $x_i, y_j$  are elements of a Hilbert space and  $\sup_i \|x_i\| \cdot \sup_j \|y_j\| \leq C$ . See also [3]. For Hankel matrices, the above inequality is implicitly contained in [22, Theorem 6.2]; see also [23] and [19] for different proofs. Below, we give another elementary proof of a more general result (for the matrices  $M_\mu$ ) including the cases of  $T_\mu$  and  $\Gamma_\mu$ . Hankel type Hadamard–Schur multipliers were also considered by V. Peller in the important paper [20], where it was proved that  $\Gamma_\mu$  is an HSM if

$$\sup_{j \geq 1} \left( \int_{\mathbb{T}} |P_j \mu|^2 \omega \, dm \right) < \infty,$$

where  $\omega > 0$ ,  $\frac{1}{\omega} \in L^1(\mathbb{T})$ , and

$$P_j \mu = \sum_{k \in \Delta(j)} \hat{\mu}(k) z^k \quad \text{with } \Delta(j) = [s_j, s_{j+1}), \quad \inf_j (s_{j+1}/s_j) > 1.$$

This sufficient condition does not work for all measures  $\mu \in \mathfrak{M}$ . For instance, for the Dirac type  $\delta$ -measures we have

$$\left( \int_{\mathbb{T}} |P_j \mu|^2 \omega \, dm \right) \left( \int_{\mathbb{T}} \omega^{-1} \, dm \right) \geq \left( \int_{\mathbb{T}} |P_j \mu| \, dm \right)^2 \longrightarrow \infty$$

as  $j \longrightarrow \infty$ . In §3 we give a necessary condition of Peller’s type for  $\|\Gamma\|_{\mathcal{H}} < \infty$ .

Another special class of Hankel type Hadamard–Schur multipliers is also considered in the paper. Namely, let  $\Lambda \subset \mathbb{Z}_+$ , and let  $\mathcal{M}(\Lambda)$  be the set of matrices for which only  $\Lambda$ -diagonals may differ from zero,

$$\mathcal{M}(\Lambda) = \{M = (m_{ij}) : m_{ij} = 0 \text{ for } i + j \in \mathbb{Z}_+ \setminus \Lambda\}.$$

It is proved that if  $\Lambda$  is a finite union of lacunary sequences, then a matrix  $M \in \mathcal{M}(\Lambda)$  is an HSM if and only if  $\sup_{ij} |m_{ij}| < \infty$ . In particular, there are many Hankel HSMs that are not of the form  $\Gamma_\mu$  for a measure  $\mu \in \mathfrak{M}$  (to see this, it suffices to take  $\Gamma = (\gamma_{i+j})$  with  $\gamma_k = 1$  for  $k \in \Lambda$  and  $\gamma_k = 0$  for  $k \in \mathbb{Z}_+ \setminus \Lambda$  and then apply Helson’s theorem saying that Fourier coefficients of the form  $k \longmapsto \hat{\mu}(k)$  that take finitely many values are eventually periodic; see [12]). In the case of the Hankel HSMs, a similar corollary can also be derived from Peller’s result quoted above.

However, the problem of description of the Hankel type Hadamard–Schur multipliers is still open. It was raised by G. Bennet in [2]. It is known that the norm of the restriction of a Hankel multiplier  $\Gamma_\varphi$  to bounded Hankel operators coincides with the norm of the symbol  $\varphi$  as a Fourier multiplier of the Hardy space  $H^1$  ( $f \longmapsto f * \varphi$ ); to see this, it suffices to use the Nehari theorem and the  $H^1$ -BMOA duality. The space of  $H^1$ -multipliers has not been described either. The analysis of the finite sections of a Hankel HSM allows us to suppose that the Bennett problem is related to that of finding a true expression for the norm of a finite Toeplitz or Hankel matrix. We answer the latter question as follows. It is clear that

$$\|(T_\varphi)_n\| \leq \|\varphi\|_{L^\infty/L^\infty_{-n,n}} \quad \text{and} \quad \|(\Gamma_\varphi)_n\| \leq \|\varphi\|_{L^\infty/L^\infty_{1,2n-1}},$$

where  $(T_\varphi)_n = (\hat{\mu}(i - j))_{0 \leq i, j < n}$ ,  $(\Gamma_\varphi)_n = (\hat{\mu}(i + j))_{0 \leq i, j < n}$ , and

$$L_{m,n}^\infty = \{f \in L^\infty(\mathbb{T}) : \hat{f}(k) = 0 \text{ for } k \in (m, n)\}.$$

We show that, in fact, these norms are equivalent, namely,

$$\|\varphi\|_{L^\infty/L_{n,n}^\infty} \leq 3\|(T_\varphi)_n\| \quad \text{and} \quad \|\varphi\|_{L^\infty/L_{1,2n-1}^\infty} \leq 3\|(\Gamma_\varphi)_n\|.$$

In particular, this means that, given a Toeplitz matrix  $T$  of size  $n \times n$ , we can find  $\varphi \in L^\infty(\mathbb{T})$  such that  $T = (T_\varphi)_n$  and  $\|\varphi\|_{L^\infty} \leq 3\|T\|$ . This result was obtained independently by N. Nikol’skiĭ and A. Vol’berg (private communication, 2002). See also [1] for another approach to the same problem. It should also be mentioned that the above inequalities can be regarded as an extension to nonanalytic polynomials of the classical Carathéodory–Schur interpolation theorem (see §2 for more details).

Analogs of these results are still true for the corresponding block matrix HSMs. In particular, if  $\sigma = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \varphi(j) \leq i\}$  is the ordinate set of a function  $\varphi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\infty\}$ , and  $n = \text{card}(\varphi(\mathbb{Z}_+))$  is the number of different values of  $\varphi$  ( $n = \infty$  if  $\varphi$  takes infinitely many values), then  $\|\chi_\sigma\|_{\mathcal{H}} \approx \log(n + 1)$ .

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§2. TOEPLITZ MULTIPLIERS

We begin with calculating the norm  $\|T_\mu\|_{\mathcal{H}}$ , where  $\mu \in \mathfrak{M}(\mathbb{T})$  and  $T_\mu = (\hat{\mu}(i - j))_{i, j \geq 0}$ . Let  $\zeta \in \mathbb{T}$ , and let  $R_\zeta$  be the unitary operator on  $l^2$  defined on the basis vectors  $e_k$ ,  $k \geq 0$ , by

$$R_\zeta e_k = \zeta^k e_k.$$

The following lemma gives examples of “skew diagonal” Hadamard–Schur multipliers. It is also related to Toeplitz and Hankel matrices as to special cases.

**Lemma 2.1.** *Suppose  $\mu \in \mathfrak{M}(\mathbb{T})$  and  $l, m \in \mathbb{Z}$ . Then the matrix*

$$M_\mu = (\hat{\mu}(im + jl))_{i, j \geq 0}$$

*is an HSM, and  $\|M_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$ . Moreover,*

$$(M_\mu \circ A)x = \int_{\mathbb{T}} R_{\zeta^{-m}} A R_{\zeta^{-l}} x \, d\mu(\zeta)$$

*for every  $A \in \mathcal{L}(l^2)$  and every  $x \in l^2$  (the integral is norm convergent).*

*Proof.* Clearly, the function  $\zeta \mapsto R_{\zeta^{-m}} A R_{\zeta^{-l}} x$  is norm continuous on  $\mathbb{T}$  for every  $x \in l^2$ . Hence, it is Riemann integrable, and we have

$$\left\| \int_{\mathbb{T}} R_{\zeta^{-m}} A R_{\zeta^{-l}} x \, d\mu(\zeta) \right\| \leq \int_{\mathbb{T}} \|R_{\zeta^{-m}} A R_{\zeta^{-l}} x\| \, d|\mu|(\zeta) \leq \|A\| \cdot \|x\| \cdot \|\mu\|_{\mathfrak{M}}.$$

This means that the integral determines a bounded operator on  $l^2$ , say,  $B$ . For its matrix entries, we have

$$\begin{aligned} (Be_j, e_i) &= \left( \int_{\mathbb{T}} R_{\zeta^{-m}} A R_{\zeta^{-l}} e_j \, d\mu(\zeta), e_i \right) = \int_{\mathbb{T}} (A\zeta^{-jl} e_j, \zeta^{im} e_i) \, d\mu(\zeta) \\ &= \int_{\mathbb{T}} \zeta^{-jl-im} (Ae_j, e_i) \, d\mu(\zeta) = \hat{\mu}(jl + im)(Ae_j, e_i) \end{aligned}$$

for all  $i, j \in \mathbb{Z}_+$ . Consequently,  $B = M_\mu \circ A$  and  $\|M_\mu \circ A\| \leq \|\mu\|_{\mathfrak{M}} \|A\|$ , whence  $\|M_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$ . □

**Theorem 2.2.** *Let  $T = (t_{i-j})_{i,j \geq 0}$  be a Toeplitz matrix. Then  $T \in \mathcal{H}$  if and only if  $T = T_\mu$  for a (unique)  $\mu \in \mathfrak{M}$ . Moreover,  $\|T_\mu\|_{\mathcal{H}} = \|\mu\|_{\mathfrak{M}}$ , so that the norm of  $T_\mu$  coincides with the norm of its restriction to the set of Toeplitz operators.*

*Proof.* The “if” part and the inequality  $\|T_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$  were proved in Lemma 2.1. To check the reverse inequality, we test the multiplier  $T_\mu$  on Toeplitz operators  $T_\varphi$ ,  $\varphi \in L^\infty(\mathbb{T})$  (it suffices to consider step functions  $\varphi$  or continuous  $\varphi \in C(\mathbb{T})$ ). As is well known,  $\|T_\varphi\| = \|\varphi\|_\infty$  for every  $\varphi \in L^\infty(\mathbb{T})$  (see, e.g., [16, Chapter B.4]). Hence, we have  $\|T_\mu \circ T_\varphi\| = \|T_{\varphi * \mu}\| = \|\varphi * \mu\|_\infty$ , where  $\varphi * \mu$  stands for the convolution on  $\mathbb{T}$ . Therefore,

$$\begin{aligned} \|T_\mu\|_{\mathcal{H}} &\geq \sup \{ \|T_\mu \circ T_\varphi\| = \|\mu * \varphi\|_\infty : \varphi \text{ is a step function, } \|\varphi\|_\infty \leq 1 \} \\ &\geq \sup \left\{ \left| \int_{\mathbb{T}} \varphi(\bar{\zeta}) d\mu(\zeta) \right| : \varphi \text{ is a step function, } \|\varphi\|_\infty \leq 1 \right\} = \|\mu\|_{\mathfrak{M}}. \end{aligned}$$

Conversely, if  $T$  is an HSM, then  $T \circ T_\varphi$  is a bounded Toeplitz operator (and hence is of the form  $T_\psi$ ) for every  $\varphi \in L^\infty$ . Consequently,  $\hat{\psi}(k) = t_k \hat{\varphi}(k)$  for every  $k \in \mathbb{Z}$ , which means that the mapping  $\varphi \mapsto \psi$  determined by  $(t_k)_{k \in \mathbb{Z}}$  is a Fourier multiplier  $L^\infty \rightarrow L^\infty$ . As is well known (see, e.g., [9]), this implies that  $t_k = \hat{\mu}(k)$ ,  $k \in \mathbb{Z}$ , for a measure  $\mu \in \mathfrak{M}$ . □

*Remark 2.3.* The above arguments are applicable in a more general framework. Namely, let  $X$  be a Banach space, and let  $(e_k)_{k \geq 0}$  be a complete sequence in  $X$  (i.e.,  $\text{span}(e_k : k \geq 0) = X$ ) such that the formula

$$R_\zeta e_k = \zeta^k e_k, \quad k \geq 0,$$

determines bounded operators  $R_\zeta$  continuously depending on  $\zeta \in \mathbb{T}$ . Then, by the same integration process, it is easy to show that  $(e_k)_{k \geq 0}$  is a minimal sequence having a bounded biorthogonal sequence  $(e_k^*)_{k \geq 0}$ , i.e.,  $e_k^* \in X^*$  with  $(e_k, e_j^*) = \delta_{kj}$ . With every bounded operator  $A : X \rightarrow X$ , we associate its matrix with the entries  $(Ae_j, e_i^*)$ . Then the above arguments imply that, for every  $\mu \in \mathfrak{M}$ , the matrix  $\hat{\mu}(i - j)(Ae_j, e_i^*)$  corresponds to a bounded operator on  $X$ , say  $T_\mu \circ A$ , and  $\|T_\mu \circ A\| \leq \|\mu\|_{\mathfrak{M}} \|A\|$ .

Now, we briefly consider block Hadamard–Schur multipliers. Let  $H$  be a Hilbert space, and let  $(P_i^k)_{i \geq 0, k = 1, 2, 3}$  be three families of orthoprojections in  $H$  such that  $P_i^k P_j^k = 0$  for every  $i \neq j$  and every  $k$ , and  $\sum_{i \geq 0} P_i^k = I$ . We say that a matrix  $M = (m_{ij})$ , where  $m_{ij} : P_i^1 H \rightarrow P_i^3 H$ , is a *block Hadamard–Schur multiplier* if for every operator  $A \in \mathcal{L}(H)$  there exists  $B \in \mathcal{L}(H)$  such that  $m_{ij} P_i^1 A P_j^2 = P_i^3 B P_j^2$  for all  $i, j \geq 0$ . As in the scalar case, we denote  $B = M \circ A$  and define the multiplier norm  $\|M\|_{\mathcal{H}}$  in the usual way.

First, we consider the “square case” where  $P_i^1 = P_i^2 = P_i^3 =: P_i$  and all blocks are “of the same size”, i.e.,  $\text{rank } P_i = n \leq \infty$  for every  $i$ . We can unitarily identify all  $P_i H$ , and identify  $H$  with the space  $l^2(\mathbb{Z}_+, E)$  of  $E$ -valued sequences, where  $\dim E = n$ . This identification leads to the replacement of a multiplier  $(m_{ij})$  by  $(u_i^{-1} m_{ij} u_j)$ , where the  $u_j$  are appropriate unitary operators. Let  $\mu$  be an  $\mathcal{L}(E)$ -valued Borel measures on  $\mathbb{T}$  having finite norm variation

$$\|\mu\|_{\mathfrak{M}} = \sup \left\{ \sum_i \|\mu(\tau_i)\| : \tau_i \cap \tau_j = \emptyset \ (i \neq j), \ \tau_i \subset \mathbb{T} \right\} < \infty.$$

We denote by  $\mathfrak{M}(\mathbb{T}, \mathcal{L}(E))$  the vector space of such measures endowed with the above norm. Every measure  $\mu$  having a density with respect to a scalar measure  $\lambda$  is in  $\mathfrak{M}(\mathbb{T}, \mathcal{L}(E))$ : if  $\mu = f \cdot \lambda$ , i.e.,  $\mu(\tau) = \int_\tau f(\zeta) d\lambda(\zeta)$  for a function  $f \in L^1(\mathcal{L}(E))$ , then  $\|\mu\|_{\mathfrak{M}} = \int_{\mathbb{T}} \|f(\zeta)\| d\lambda(\zeta) < \infty$ .

Every operator  $A \in \mathcal{L}(E)$  can be viewed as a diagonal operator acting on  $l^2(E)$ , i.e.,  $A(x_j)_{j \geq 0} = (Ax_j)_{j \geq 0}$ . Now, given a step function  $\sum x_j \chi_{\tau_j}$  with  $x_j \in l^2(E)$ , we set in the usual way

$$\int_{\mathbb{T}} (d\mu(\zeta)) \left( \sum x_j \chi_{\tau_j} \right) = \sum \mu(\tau_j) x_j,$$

and then, passing to the limit, we can consistently define the integral  $\int_{\mathbb{T}} (d\mu(\zeta)) h(\zeta)$  at least for separable-valued functions  $h \in L^\infty(l^2(E))$ . In the case of a density measure  $\mu = f \cdot \lambda$ , where  $f(\zeta) \in \mathcal{L}(E)$ , we have  $\int_{\mathbb{T}} (d\mu(\zeta)) h(\zeta) = \int_{\mathbb{T}} f(\zeta) h(\zeta) d\lambda(\zeta)$ . We refer to [13, 8] for the details concerning vector-valued integration.

As in the scalar case, for a measure  $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{L}(E))$ , the corresponding Toeplitz matrix is denoted by  $T_\mu = (\hat{\mu}(i - j))_{i, j \geq 0}$ . It is known that  $T_\mu$  is a bounded operator on  $l^2(E)$  if and only if  $d\mu = \varphi dm$  with  $\varphi \in L^\infty(\mathcal{L}(E))$ ; in this case  $\|T_\mu\| = \|\varphi\|_\infty$  (see [15] or [16]).

The following lemma is an analog of the ‘‘Toeplitz part’’ of Lemma 2.1. Of course, it remains true for the  $M_\mu$ -type operators (except the norm equality), but we do not need this. The rotation operators  $R_\zeta$  are defined on  $l^2(E)$  in the same way as on  $l^2$ :  $R_\zeta e_k = \zeta^k e_k$  for every  $e_k = (\delta_{kj} e)_{j \geq 0}$ ,  $e \in E$ .

**Lemma 2.4.** *Suppose  $A \in \mathcal{L}(l^2(E))$  and  $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{L}(E))$ . Then*

$$(T_\mu \circ A)x = \int_{\mathbb{T}} (d\mu(\zeta)) R_{\bar{\zeta}} A R_\zeta x$$

for every  $x \in l^2(E)$ . The integral is norm convergent, and we have  $\|T_\mu \circ A\| \leq \|\mu\|_{\mathfrak{M}} \|A\|$ . Moreover,  $\|T_\mu\|_{\mathcal{H}} = \|\mu\|_{\mathfrak{M}}$ .

*Proof.* Clearly, the function  $\zeta \mapsto R_{\bar{\zeta}} A R_\zeta x$  is norm continuous on  $\mathbb{T}$  for every  $x \in l^2(E)$ , whence

$$\left\| \int_{\mathbb{T}} (d\mu(\zeta)) R_{\bar{\zeta}} A R_\zeta x \right\| \leq \sup_{\zeta \in \mathbb{T}} \|R_{\bar{\zeta}} A R_\zeta x\| \cdot \|\mu\|_{\mathfrak{M}} \leq \|A\| \cdot \|x\| \cdot \|\mu\|_{\mathfrak{M}}.$$

This means that the integral determines a bounded operator on  $l^2(E)$ , say,  $B$ . Since the  $\mu(\sigma)$  are diagonal operators on  $l^2(E)$ , we have

$$\begin{aligned} (Be_j, e_i)_{l^2(E)} &= \left( \int_{\mathbb{T}} (d\mu(\zeta)) R_{\bar{\zeta}} A R_\zeta e_j, e_i \right) = \int_{\mathbb{T}} ((d\mu(\zeta)) A \zeta^j e_j, \zeta^i e_i) \\ &= \int_{\mathbb{T}} \zeta^{j-i} ((d\mu(\zeta)) A e_j, e_i)_{l^2(E)} = (\hat{\mu}(i - j) A e_j, e_i)_{l^2(E)} = (\hat{\mu}(i - j) a_{ij} e_j, e_i)_E \end{aligned}$$

for every  $i, j \in \mathbb{Z}_+$ . Consequently,  $B = T_\mu \circ A$ ,  $\|T_\mu \circ A\| \leq \|\mu\|_{\mathfrak{M}} \|A\|$ , whence  $\|T_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$ .

As in Lemma 2.1, to check the reverse inequality, we test the multiplier  $T_\mu$  at Toeplitz operators  $T_\varphi$ ,  $\varphi \in L^\infty(\mathbb{T}, \mathcal{L}(E))$  (it suffices to consider step functions  $\varphi$ ). We have  $\|T_\mu \circ T_\varphi\| = \|T_{\mu * \varphi}\| = \|\mu * \varphi\|_\infty$ , where  $\mu * \varphi$  stands for convolution on  $\mathbb{T}$ . Therefore,

$$\begin{aligned} \|T_\mu\|_{\mathcal{H}} &\geq \sup \{ \|T_\mu \circ T_\varphi\| = \|\mu * \varphi\|_\infty : \varphi \text{ is a step function, } \|\varphi\|_\infty \leq 1 \} \\ &\geq \sup \left\{ \left| \int_{\mathbb{T}} (d\mu(\zeta)) \varphi(\bar{\zeta}) \right| : \varphi \text{ is a step function, } \|\varphi\|_\infty \leq 1 \right\} = \|\mu\|_{\mathfrak{M}}, \end{aligned}$$

and the result follows. □

**Lemma 2.5.** *Let  $M = (m_{ij}) \in \mathcal{H}(\mathcal{L}(l^2(E)))$  be an HSM multiplier, i.e., a bounded mapping taking  $A = (a_{ij}) \in \mathcal{L}(l^2(E))$  to  $M \circ A = (m_{ij} a_{ij}) \in \mathcal{L}(l^2(E))$ , where  $A = (a_{ij})$  is the standard matrix representation of an operator  $A : l^2(E) \rightarrow l^2(E)$ . Next, let  $P_j$ ,*

$Q_j$  be orthoprojections on  $E$ , and let  $P(x_j)_{j \geq 0} = (P_j x_j)_{j \geq 0}$ ,  $Q(x_j)_{j \geq 0} = (Q_j x_j)_{j \geq 0}$  for  $x = (x_j)_{j \geq 0} \in l^2(E)$ . Assume that  $m_{ij} Q_i E \subset Q_j E$  for every  $i, j \geq 0$ .

Then the block matrix  $QMQ$  is an HSM for the operators acting from  $Pl^2(E)$  into  $Ql^2(E)$ , and  $\|QMQ\|_{\mathcal{H}} \leq \|M\|_{\mathcal{H}}$ .

*Proof.* For every operator  $A : Pl^2(E) \rightarrow Ql^2(E)$ , the composition  $AP$  is its extension to  $l^2(E)$ , and conversely,  $QAP$  is bounded from  $Pl^2(E)$  to  $Ql^2(E)$  for every  $A \in \mathcal{L}(E)$ . Obviously,  $\|QAP\| \leq \|A\|$ . Since for an arbitrary operator  $A : Pl^2(E) \rightarrow Ql^2(E)$  we have  $Q_i m_{ij} a_{ij} P_j = m_{ij} a_{ij} P_j$  for every  $i, j$ , the desired inequality follows.  $\square$

**Corollary 2.6.** Let  $P_j, Q_j, P$ , and  $Q$  be orthoprojections as in Lemma 2.5, and let  $\mu \in \mathfrak{M}(\mathbb{T}, \mathcal{L}(E))$  be a measure satisfying  $\mu(\tau) Q_i E \subset Q_i E$  for every  $i \geq 0$  and  $\tau \subset \mathbb{T}$ .

Then the block Toeplitz matrix  $QT_{\mu}Q$  is an HSM for the operators acting from  $Pl^2(E)$  into  $Ql^2(E)$ , and  $\|QT_{\mu}Q\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$ . If  $\mu$  is a scalar measure (i.e., if  $\mu(\sigma) = \lambda(\sigma) \cdot I$  for every  $\sigma \subset \mathbb{T}$  with a scalar measure  $\lambda$ ) and  $Q_j \neq 0, P_j \neq 0$  for every  $j \geq 0$ , we have  $\|QT_{\mu}Q\|_{\mathcal{H}} = \|\mu\|_{\mathfrak{M}} = \|\lambda\|_{\mathfrak{M}}$ .

The first claim follows from Lemmas 2.4 and 2.5, because  $\hat{\mu}(i-j) Q_i E \subset Q_j E$  for every  $i, j$ .

For a scalar measure  $\mu$ , we can pass to smaller rank one projections  $0 \leq P'_j \leq P_j, 0 \leq Q'_j \leq Q_j$ , in which case the operator  $Q'T_{\mu}Q'$  is equivalent to an HSM on the scalar space  $l^2$ . This implies that  $\|QT_{\mu}Q\|_{\mathcal{H}} \geq \|Q'T_{\mu}Q'\|_{\mathcal{H}} = \|\lambda\|_{\mathfrak{M}}$ , and the formula  $\|QT_{\mu}Q\|_{\mathcal{H}} = \|\lambda\|_{\mathfrak{M}}$  follows.  $\square$

Now, we apply these results to get the classical Matsaev theorem; see [15, 6].

**Corollary 2.7.** Let  $M_n = (m_{ij}) = \chi_{\Delta_n}$  be the  $(n \times n)$ -matrix that determines the triangular truncation operator  $A \mapsto M_n \circ A$ , i.e.,  $m_{ij} = \chi_{\Delta_n}(i, j)$  for  $0 \leq i, j \leq n-1$ , where  $\Delta_n$  is the lower diagonal triangle  $\Delta_n = \{(i, j) : 0 \leq j \leq i \leq n-1\}$ ,

$$M_n = \chi_{\Delta_n} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

Then  $\frac{2}{\pi^2} \cdot \log(n+1) \leq \|\chi_{\Delta_n}\|_{\mathcal{H}} \leq 1 + \log(n)$  for every  $n \geq 1$ .

The corresponding block truncation operator determined by the matrix  $M_n$  on  $\mathcal{L}(l^2_n(E))$ ,  $(a_{ij}) \mapsto (m_{ij} a_{ij})$  with  $a_{ij} : E \rightarrow E$ , has the same norm as  $M_n$ .

*Proof.* Let  $T_n = T_{\mathcal{D}_n}$  be the Toeplitz matrix determined by the Dirichlet kernel  $\mathcal{D}_n(\zeta) = \sum_{k=0}^{n-1} \zeta^k, \zeta \in \mathbb{T}$ . Then  $\|T_n\|_{\mathcal{H}} = \|\mathcal{D}_n\|_{L^1(\mathbb{T})}$  and, as is well known,  $\frac{4}{\pi^2} \cdot \log(n+1) \leq \|\mathcal{D}_n\|_{L^1(\mathbb{T})} \leq 1 + \log(n)$  for every  $n \geq 1$ .

Next, let  $P_0$  be the orthoprojection onto the subspace

$$l^2_n = \{x = (x_j)_{j \geq 0} \in l^2 : x_j = 0 \text{ for } j \geq n\}.$$

Then  $\|\chi_{\Delta_n}\|_{\mathcal{H}} = \|P_0 T_n P_0\|_{\mathcal{H}} \leq \|T_n\|_{\mathcal{H}} \leq 1 + \log(n)$ .

For the lower estimate of  $\|\chi_{\Delta_n}\|_{\mathcal{H}}$ , we represent  $T_n$  as a sum of two block Toeplitz matrices,  $T_n = T' + T''$ , where  $T' = \sum_{k \geq 0} P_k T_n P_k$  and  $P_k$  is the orthoprojection onto

$$l^2_{(k+1)n} \ominus l^2_{kn} = \{x = (x_j)_{j \geq 0} \in l^2 : x_j = 0 \text{ for } j < kn \text{ and } j \geq (k+1)n\}.$$

Identifying  $P_k l^2$  with  $\mathbb{C}^n$  and  $l^2$  with  $l^2(\mathbb{C}^n)$ , we observe that  $T'$  is a diagonal operator with constant diagonal entries,  $T' = \text{Diag}(M_n)$ , i.e.,

$$T' = \begin{pmatrix} M_n & 0 & 0 & \dots \\ 0 & M_n & 0 & \dots \\ 0 & 0 & M_n & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Similarly,  $T'' = S \text{Diag}(M_n^T)$ , where  $S$  is the shift operator on  $l^2(E)$ ,  $S(x_0, x_1, \dots) = (0, x_0, x_1, \dots)$ , and  $M_n^T$  is the transpose of  $M_n$ . It is clear that a diagonal HSM  $M = (\delta_{ij} m_{ii})$ , which acts on  $\mathcal{L}(l^2(E))$  by the rule  $M \circ A = (\delta_{ij} m_{ii} \circ a_{ij})$ , where  $A = (a_{ij}) \in \mathcal{L}(l^2(E))$  and  $m_{ii} \in \mathcal{L}(E)$ , has the norm  $\|M\|_{\mathcal{H}} = \sup_{i \geq 0} \|m_{ii}\|_{\mathcal{H}}$ . Therefore,  $\|T'\|_{\mathcal{H}} = \|T''\|_{\mathcal{H}} = \|M_n\|_{\mathcal{H}}$ , whence  $\|M_n\|_{\mathcal{H}} \geq \|T_n\|/2 \geq \frac{2}{\pi^2} \log(n+1)$ .  $\square$

*Remark 2.8.* A slightly weaker upper bound for  $\|M_n\|$  can be found in the following elementary way. First, let  $n = 2^k$ . Then  $M_n$  can be written as a  $(2 \times 2)$ -matrix of the form

$$M_n = \begin{pmatrix} M_{n/2} & 0 \\ 0 & M_{n/2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbf{I} & 0 \end{pmatrix} = M'_n + M''_n,$$

where  $\mathbf{I} = (\mathbf{I}_{ij})_{0 \leq i, j < n/2}$  with  $\mathbf{I}_{ij} = 1$  for all  $i, j$ . As above,  $\|M'_n\|_{\mathcal{H}} = \|M_{n/2}\|_{\mathcal{H}}$ , and  $\|M''_n\|_{\mathcal{H}} = 1$  because  $M''_n \circ A = P_0 A P_1$  for every  $A \in \mathcal{L}(\mathbb{C}^n)$ , where  $P_0$  and  $P_1$  are the corresponding orthoprojections. Therefore,  $\|M\|_{\mathcal{H}} \leq \|M_{n/2}\|_{\mathcal{H}} + 1$ , and by induction,  $\|M_n\|_{\mathcal{H}} \leq 1 + k$ .

In general, if  $2^{k-1} < n \leq 2^k$ , then

$$\|M_n\|_{\mathcal{H}} \leq \|M_{2^k}\|_{\mathcal{H}} \leq 1 + k \leq 2 + \log_2(n) = 2 + \log_2 e \cdot \log(n). \quad \square$$

**Corollary 2.9.** *Let  $\sigma = \sigma_\varphi = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : \varphi(j) \leq i\}$  be the ordinate set of a function  $\varphi : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+ \cup \{\infty\}$ , and let*

$$n = \text{card}(\varphi(\mathbb{Z}_+))$$

*be the number of different values taken by  $\varphi$  ( $n = \infty$  if  $\varphi(\mathbb{Z}_+)$  is infinite). Then the indicator function of the matrix*

$$M_\varphi = \chi_\sigma = \begin{pmatrix} 1 & & & & & & & & & & \dots \\ 1 & 1 & & & & & & & & & \dots \\ 1 & 1 & 1 & & & & & & & & \dots \\ 1 & 1 & 1 & 1 & & & & & & & \dots \\ 1 & 1 & 1 & 1 & 1 & & & & & & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & & & & & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & & \dots \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & & & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

*(the empty entries are filled with zeros), is an HSM if and only if  $n < \infty$ , and in this case*

$$c \cdot \log(n+1) \leq \|M_\varphi\|_{\mathcal{H}} \leq 1 + \log(n),$$

*where  $c > 0$  is an absolute constant.*

*Proof.* Indeed, let  $Z(k) = \{j \in \mathbb{Z}_+ : \varphi(j) = k\}$ ,  $k \in \varphi(\mathbb{Z}_+)$ , and let  $P_k$  be the orthoprojections onto  $l^2_{Z(k)}$ , where

$$l^2_{Z(k)} = \{x = (x_j)_{j \geq 0} \in l^2 : x_j = 0 \text{ for } j \in \mathbb{Z}_+ \setminus Z(k)\}.$$

Writing  $\varphi(\mathbb{Z}_+) = \{k(1) < k(2) < \dots\}$ , we get  $l^2 = \sum_{s \geq 1} \oplus P_{k(s)} l^2$ . Let  $Q_i$  be the orthoprojections onto  $l^2_{J(i)}$ , where  $J(i) = [k(i), k(i + 1))$ ,  $i \geq 0$ , and  $k(0) = 0$ . Then every operator  $A \in \mathcal{L}(l^2)$  can be represented in a block matrix form  $A = (a_{rs}^*)_{r,s \geq 0}$ ,  $a_{rs}^* x = Q_r A x$  for  $x \in l^2_{Z(k(s))}$ . For a matrix  $M = (m_{ij})$ , the Hadamard product  $M \circ A$  has the following entries (here  $x \in l^2_{Z(k(s))}$ ):

$$\begin{aligned} (M \circ A)_{rs}^* x &= Q_r (M \circ A) \sum_{j \in Z(k(s))} (x, e_j) e_j \\ &= \sum_{i \in J(r)} \sum_{j \in Z(k)} (x, e_j) (M \circ A e_j, e_i) e_i \\ &= \sum_{i \in J(r)} \sum_{j \in Z(k)} (x, e_j) m_{ij} (A e_j, e_i) e_i. \end{aligned}$$

If  $M = M_\varphi$  and  $r < s$ , then  $m_{ij} = 0$  for  $j \in Z(k(s))$  and  $i \in J(r)$ , whence  $(M_\varphi \circ A)_{rs}^* = 0$ . Similarly,  $(M_\varphi \circ A)_{rs}^* = a_{ik}^*$  for  $r \geq s$ . This means that  $M_\varphi$  is a block triangular truncation HSM from  $Pl^2(l^2)$  to  $Ql^2(l^2) = l^2$ . Now, Corollaries 2.6 and 2.7 yield the result.  $\square$

**Corollary 2.10.** *Let  $\varphi : \mathbb{Z}_+ \rightarrow \mathbb{R}$ , and let  $M_{n,\varphi}$  be the  $(n \times n)$ -matrix with the entries  $m_{ij} = 1$  for  $i \geq \varphi(j)$ ,  $0 \leq j < n$ , and  $m_{ij} = 0$  otherwise. Let  $N(\varphi, n) = \text{card}\{\varphi(j) : 0 \leq j < n\}$ , where  $[\cdot]$  stands for the integral part of a number. Then*

$$C_1 \cdot \log N(\varphi, n) \leq \|M_{n,\varphi}\|_{\mathcal{H}} \leq C_2 \cdot \log N(\varphi, n)$$

with some absolute constants  $C_1, C_2$ . In particular, for a piecewise affine function  $\varphi_{\alpha,\beta}(x) = \alpha x + \beta$ , we have  $N(\varphi, n) = \lceil \alpha n \rceil$  if both  $\varphi_{\alpha,\beta}(0)$  and  $\varphi_{\alpha,\beta}(n - 1)$  are in the interval  $I = [0, n - 1]$ ,  $N(\varphi, n) = \lfloor n/\alpha \rfloor$  if both  $\varphi_{\alpha,\beta}(0)$  and  $\varphi_{\alpha,\beta}(n - 1)$  are not in  $I$ , and  $N(\varphi, n) = \lceil \alpha n - \beta \rceil$  if one of  $\varphi_{\alpha,\beta}(0), \varphi_{\alpha,\beta}(n - 1)$  is in  $I$  and the other is not.  $\square$

Bounds for the norms of the multipliers  $M_{n,\varphi_{\alpha,\beta}}$  restricted to Toeplitz and/or Hankel matrices were found in [4] (of course, the results differ from ours).

Now, we pass to the question as to how to express the norm of a finite Toeplitz matrix in terms of its symbol. The result will be used in the next section to treat Hankel HSMs. Let  $T = (t_{i-j})_{0 \leq i,j < n}$  be a Toeplitz  $(n \times n)$ -matrix, and let  $\varphi_T(z) = \sum_{|k| < n} t_k z^k, z \in \mathbb{T}$ , be its symbol. For  $m, n \in \mathbb{Z}$  with  $m < n$ , we denote

$$L_{m,n}^\infty = \{f \in L^\infty(\mathbb{T}) : \hat{f}(k) = 0 \text{ for } k \in (m, n)\}.$$

Note that  $L_{m,n}^\infty$  contains the subspace  $\bar{z}^{m-1} H_-^\infty + z^n H^\infty$  but is different from it; in fact, the latter has an infinite codimension in the former (see D. Newman’s theorem in [16, A.5.7.3(m)]). Given a function  $\varphi$ , we put

$$D(m, n; \varphi) = \inf\{\|f\|_\infty : \hat{f}(k) = \hat{\varphi}(k) \text{ for } k \in (m, n)\}.$$

If  $\varphi \in L^\infty$ , we have  $D(m, n; \varphi) = \|\varphi\|_{L^\infty/L_{m,n}^\infty} = \text{dist}_{L^\infty}(\varphi, L_{m,n}^\infty)$ .

**Theorem 2.11.** *Let  $T = (t_{i-j})_{0 \leq i,j < n}$  be a Toeplitz  $(n \times n)$ -matrix, and let  $\varphi_T(z) = \sum_{|k| < n} t_k z^k, z \in \mathbb{T}$ , be its symbol. Then*

$$\frac{1}{3} D(-n, n; \varphi_T) \leq \|T\| \leq D(-n, n; \varphi_T).$$

In other words, for every extension of  $T$  up to an infinite Toeplitz matrix  $T_\varphi$  we have  $\|T\| \leq \|T_\varphi\|$ , and there exists an extension satisfying  $\|T_\varphi\| \leq 3\|T\|$ .

*Proof.* Since  $T = P_n T_\varphi P_n$  for every extension of  $T$ , we get  $\|T\| \leq \|T_\varphi\|$ . Consequently,

$$\begin{aligned} \|T\| &\leq \inf\{\|T_\varphi\| : \varphi \in L^\infty(\mathbb{T}), T = P_n T_\varphi P_n\} = \inf\{\|\varphi\|_\infty : \varphi \in L^\infty(\mathbb{T}), T = P_n T_\varphi P_n\} \\ &= \inf\{\|\varphi\|_\infty : \varphi_T - \varphi \in L^\infty_{-n,n}\}, \end{aligned}$$

and the right-hand side inequality follows.

To prove the left-hand side inequality, first we construct a tridiagonal extension  $\tilde{T}$  of  $T$ , which will be block Toeplitz. For this, we consider the block structure on  $l^2$  determined by the subspaces  $l^2_{(k+1)n} \ominus l^2_{kn}$ ,  $k \geq 0$ , the same as in the proof of Corollary 2.7. Therefore, we write  $l^2 = l^2(\mathbb{C}^n)$ .

Setting  $T_1 = \text{diag}(T) : l^2(\mathbb{C}^n) \rightarrow l^2(\mathbb{C}^n)$ , we define the subdiagonal part  $T_2 = S_n \cdot \text{diag}(A)$  and the superdiagonal part  $T_3 = S_n^* \cdot \text{diag}(B)$  of  $\tilde{T}$ , where  $S_n$  and  $S_n^*$  stand for the forward and backward shifts on  $l^2(\mathbb{C}^n)$ , respectively, and  $A$  and  $B$  are operators on  $\mathbb{C}^n$  to be defined below. Thus,  $\tilde{T} = T_1 + T_2 + T_3$ ,

$$\tilde{T} = \begin{pmatrix} T & B & 0 & 0 & \dots \\ A & T & B & 0 & \dots \\ 0 & A & T & B & \dots \\ 0 & 0 & A & T & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

Since we look for a Toeplitz extension of  $T$ , the upper triangular part of  $A$  is determined by the lower triangular part of  $T$ :

$$a_{ij} = t_{i+n-j},$$

where  $0 \leq i < j < n$ . We seek a matrix  $A$  of the smallest norm  $\|A\|$  whose (strictly) upper triangular part coincides with the matrix written above. It is known (H. Dym and I. Gohberg; see [10, Chapter XXXIII]) that

$$\min \|A\| = \max \|A_k\|,$$

where  $A_k$  runs through all  $(k \times (n - k))$ -matrices of the form  $A_k = (a_{ij})_{0 \leq i < k \leq j < n}$ . In our case,  $A_k$  coincides with a section of our initial Toeplitz matrix  $T$ , namely, denoting  $s = j - k$ ,  $r = i + n - k$ , we get

$$A_k = (t_{i+n-j})_{0 \leq i < k \leq j < n} = (t_{r-s})_{n-k \leq r < n, 0 \leq s < n-k}.$$

This implies that there exists a matrix  $A$  with the upper triangular part given above and such that  $\|A\| \leq \|T\|$ . We define  $B$  by a similar procedure but applied to the lower triangular part of  $B$ , which has the Toeplitz structure borrowed from the upper triangular part of  $T$ .

It is clear that  $\tilde{T}$  is an extension of  $T$  and that  $\|\tilde{T}\| \leq \|T_1\| + \|T_2\| + \|T_3\| \leq 3\|T\|$ . Now, we set  $T_\varphi = P(\tilde{T})$ , where  $P$  is a projection from  $\mathcal{L}(l^2)$  onto the subspace of all Toeplitz operators; specifically, we define  $T$  by a diagonal averaging process; see, e.g., [16, B.4.7.6]. (In fact, we do not need the Banach generalized limits used in [16] because in our case the arithmetic means

$$\frac{1}{m+1} \sum_{k=0}^m S^{*k} \tilde{T} S^k$$

converge weakly to  $P(\tilde{T})$  as  $m \rightarrow \infty$ .) Clearly,  $T$  is an  $(n \times n)$ -section of  $T_\varphi$  and  $\|T_\varphi\| \leq \|\tilde{T}\| \leq 3\|T\|$ . This completes the proof.  $\square$

**Corollary 2.12.** *Let  $p(z) = \sum_{|k| < n} a_k z^k$ ,  $z \in \mathbb{T}$ . Then*

$$\|(T_p)_n\| \leq D(-n, n; p) \leq 3\|(T_p)_n\|. \quad \square$$

*Remark 2.13.* Clearly, the last corollary is an extension of the classical Carathéodory–Schur interpolation theorem to the case of nonanalytic polynomials  $p$ . For an analytic polynomial  $p$ , the Carathéodory–Schur theorem says that  $D(-\infty, n; p) = \|(T_p)_n\|$  (see, e.g., [16, B.3.2]). As observed by A. Volberg (private communication), for general non-analytic data  $p$  the identity of the latter type fails. For example, let  $T$  be the  $(2 \times 2)$ -symplectic matrix,  $T = J_2$ ,

$$J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then  $\|T\| = 1$ , but for every function  $\varphi = \cdots + \bar{z} + z + \cdots$  we have  $\|\varphi\|_\infty \geq \|\varphi\|_2 \geq \sqrt{2}$ . Therefore,  $D(-2, 2; \varphi) \geq \sqrt{2}\|T\|$ . We do not know the best possible constant in the inequalities of Theorem 2.11 and Corollary 2.12.

A similar extension problem for Hankel operators is considered in the next section.

### §3. HANKEL AND HANKEL-TYPE MULTIPLIERS

Now we view Hankel operators as Hadamard–Schur multipliers. We begin with the following lemma linking Hankel and Toeplitz operators.

**Lemma 3.1.** *Let  $J_n$  be the  $(n \times n)$ -symplectic matrix,*

$$J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

*Then  $\|AJ_n\| = \|A\|$  and  $\|MJ_n\|_{\mathcal{H}} = \|M\|_{\mathcal{H}}$  for every  $A, M \in \mathcal{L}(\mathbb{C}^n)$ . Moreover,  $J_n^2 = I$ , and*

$$(T_\varphi)_n J_n = (\Gamma_{z^{n-1}\varphi})_n, \quad (\Gamma_\varphi)_n J_n = (T_{z^{-n+1}\varphi})_n.$$

*Proof.* Clearly,  $J_n$  is a unitary operator on  $\mathbb{C}^n$ , and  $(M \circ A)J_n = (MJ_n) \circ (AJ_n)$ .  $\square$

The following theorem can easily be derived from Lemma 2.1. However, we give yet another proof by using the link between finite Hankel and Toeplitz matrices established in Lemma 3.1.

**Theorem 3.2.** *Let  $\mu \in \mathfrak{M}(\mathbb{T})$ . Then the Hankel matrix  $\Gamma_\mu = (\hat{\mu}(i+j))_{i,j \geq 0}$  is an HSM, and we have  $\|\Gamma_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}/H_-^1}$ , where  $H_-^1 = \{f \in L^1(\mathbb{T}) : \hat{f}(k) = 0 \text{ for } k \geq 0\}$ .*

*Proof.* Let  $(T_\mu)_n = (\hat{\mu}(i-j))_{0 \leq i,j < n}$  be the  $n$ th finite section of the Toeplitz operator  $T_\mu$  with an odd  $n \geq 1$ . Let  $P_L$  be the orthoprojection onto the subspace  $L = \text{Lin}(e_k : \frac{n-1}{2} \leq k < n) \subset \mathbb{C}^n$ . Then, by Lemma 3.1, the operator  $(T_\mu)_n J_n$  has a Hankel matrix, and moreover, the section  $P_L((T_\mu)_n J_n)|L$  has the matrix  $(\Gamma_\mu)_{(n+1)/2}$ . Therefore,

$$\|(\Gamma_\mu)_{(n+1)/2}\|_{\mathcal{H}} = \|P_L((T_\mu)_n J_n)|L\|_{\mathcal{H}} \leq \|(T_\mu)_n\|_{\mathcal{H}} \leq \|T_\mu\|_{\mathcal{H}} = \|\mu\|_{\mathfrak{M}}.$$

Letting  $n \rightarrow \infty$ , we get  $\|\Gamma_\mu\|_{\mathcal{H}} \leq \|\mu\|_{\mathfrak{M}}$ . Moreover,  $\Gamma_\mu = \Gamma_{\mu+\nu}$  if and only if  $\hat{\nu}(k) = 0$  for  $k \geq 0$ , whence  $\nu = h \cdot m$ , where  $h$  is a function in the Hardy space  $H_-^1$ , and  $m$  stands for the normalized Lebesgue measure on  $\mathbb{T}$  (we refer to the Riesz brothers theorem; see, e.g., [16]). Consequently,  $\|\Gamma_\mu\|_{\mathcal{H}} \leq \inf\{\|\mu + h \cdot m\|_{\mathfrak{M}} : h \in H_-^1\}$ .  $\square$

Now, we apply Lemma 3.1 and Theorem 2.11 to Hankel operators. First, we obtain the following analog of Corollary 2.12.

**Corollary 3.3.** *Let  $p(z) = \sum_{0 \leq k < 2n-1} \gamma_k z^k$ ,  $z \in \mathbb{T}$ , be a polynomial, and let  $(\Gamma_p)_n = (\gamma_{i+j})_{0 \leq i, j < n}$  be the corresponding Hankel matrix. Then the smallest norm  $\min \|\Gamma\|$  of a Hankel extension  $\Gamma$  of  $(\Gamma_p)_n$  is  $D(-1, 2n - 1; p)$ , and*

$$\|(\Gamma_p)_n\| \leq D(-1, 2n - 1; p) \leq 3\|(\Gamma_p)_n\|.$$

*Proof.* A Hankel operator  $\Gamma_\varphi$  is an extension of  $(\Gamma_p)_n$  if and only if  $\hat{\varphi}(k) = a_k$  for  $0 \leq k < 2n - 1$ . Therefore, by the Nehari theorem, the smallest norm among all extensions is  $\inf \|\varphi + H^\infty\|$ , where  $\varphi$  runs over the  $L^\infty$ -functions described above. Since  $H^\infty \subset L^\infty_{-1, 2n-1}$ , the identity  $D(-1, 2n - 1; p) = \min \|\Gamma_\varphi\|$  follows.

Using the formula  $(\Gamma_\varphi)_n J_n = (T_{z^{-n+1}\varphi})_n$  of Lemma 3.1, and then applying Corollary 2.12, we get the rest of our claim.  $\square$

Next, we deduce a necessary condition for a Hankel matrix  $\Gamma$  to be an HSM.

**Theorem 3.4.** *Let  $\Gamma = (\gamma_{i+j})_{i, j \geq 0}$  be a Hadamard-Schur multiplier, and let  $\varphi = \sum_{k \geq 0} \gamma_k z^k$  be the corresponding symbol (a distribution on  $\mathbb{T}$ ). Let  $1/\alpha$  be a positive integer, and let  $K_{m, n; \alpha}$  be the Fejér type kernel for the interval  $[m, n]$ , that is, the trigonometric polynomial with  $\hat{K}_{m, n; \alpha}(s) = 1$  for  $m \leq s \leq n$ ,  $\hat{K}_{m, n; \alpha}(s) = 0$  for  $s \leq m - \alpha(n - m)$  and  $s \geq n + \alpha(n - m)$ , and with linear  $\hat{K}_{m, n; \alpha}$  on the intervals  $[m - \alpha(n - m), m]$  and  $[n, n + \alpha(n - m)]$ . Then*

$$\sup_{m \leq n \leq \beta m} \|K_{m, n; \alpha} * \varphi\|_1 \leq 3\beta \|\Gamma\|_{\mathcal{H}},$$

where  $\beta = \frac{1}{\alpha} + 1$ .

*Proof.* Using the formula  $(\Gamma_\varphi)_n J_n = (T_{z^{-n+1}\varphi})_n$  of Lemma 3.1 once again, we get

$$\|\Gamma\|_{\mathcal{H}} \geq \|(\Gamma_\varphi)_n\|_{\mathcal{H}} = \|T_n\|_{\mathcal{H}},$$

where  $T_n = (T_{z^{-n+1}\varphi})_n$ . Since

$$\sup\{\|T_n \circ (T_f)_n\| : \|(T_f)_n\| \leq 1\} \leq \|T_n\|_{\mathcal{H}},$$

we see that

$$\sup\{\|(T_{\psi * f})_n\| : \|f\|_\infty \leq 1\} \leq \|T_n\|_{\mathcal{H}} \leq \|\Gamma\|_{\mathcal{H}},$$

where  $\psi = z^{-n+1}\varphi$ . By Theorem 2.11, for every  $f \in L^\infty$  with  $\|f\|_\infty \leq 1$ , there exists  $g \in L^\infty$  such that  $\hat{g} = \widehat{(\psi * f)}$  on the interval  $(-n, n)$  and

$$\|g\|_\infty = \|T_g\| \leq 3\|(T_{\psi * f})_n\|.$$

Now, we take any kernel  $K_{-N, N; \alpha}$  such that  $[-N - 2\alpha N, N + 2\alpha N] \subset [-n, n]$ ; it is easily seen that  $K_{-N, N; \alpha}$  is a sum of at most  $\frac{1}{\alpha} + 1 = \beta$  (shifted) Fejér kernels, so that  $\|K_{-N, N; \alpha}\|_1 \leq \frac{1}{\alpha} + 1 = \beta$ . Moreover,  $K_{-N, N; \alpha} * g = K_{-N, N; \alpha} * (\psi * f)$ , whence

$$\|K_{-N, N; \alpha} * (\psi * f)\|_\infty = \|K_{-N, N; \alpha} * g\|_\infty \leq \beta \|g\|_\infty \leq 3\beta \|\Gamma\|_{\mathcal{H}}$$

for every  $f$  with  $\|f\|_\infty \leq 1$ . This implies  $\|K_{-N, N; \alpha} * \psi\|_\infty \leq 3\beta \|\Gamma\|_{\mathcal{H}}$ , which is equivalent to the claim.  $\square$

**Corollary 3.5.** *Let  $\Lambda \subset \mathbb{Z}_+$ , let  $k \mapsto \gamma_k$  be the indicator function of  $\Lambda$ , and let  $\Gamma = (\gamma_{i+j})_{i, j \geq 0}$  be the corresponding Hankel matrix. Assume that there exist  $m, n, \alpha > 0$  such that  $m \leq n \leq \beta m$ , where  $\beta = 1 + \frac{1}{\alpha}$  and  $[m, n] \cap \Lambda = [m - \alpha(n - m), n + \alpha(n - m)] \cap \Lambda$ . Then*

$$\|\Gamma\|_{\mathcal{H}} \geq a \frac{\alpha}{3(1 + \alpha)} \log(\text{card}([m, n] \cap \Lambda)),$$

where  $a > 0$  is an absolute constant.

In particular,  $\Gamma \notin \mathcal{H}$  if, for a fixed  $\alpha$ , there exist intervals  $[m, n]$  as above with  $\sup(\text{card}([m, n] \cap \Lambda)) = \infty$ .

*Proof.* Using Theorem 3.4 and its notation, we get  $\|K_{m,n,\alpha} * \varphi\|_1 \leq 3\beta\|\Gamma\|_{\mathcal{H}}$ . On the other hand,

$$K_{m,n,\alpha} * \varphi = \sum_{j \in [m,n] \cap \Lambda} z^j.$$

Now, the result follows from the Konyagin–Smith theorem (see, e.g., [7]).  $\square$

We finish this section by exhibiting a vector space of Hankel type matrices  $M = (m_{ij})$  for which  $\sup |m_{ij}| < \infty$  implies that  $M \in \mathcal{H}$ . In particular, we obtain many examples of Hankel HSMs whose symbols are not measures.

For  $\Lambda \subset \mathbb{Z}_+$ , we denote by  $HL(\Lambda)$  the set of all *Hankel-like matrices*  $M = (m_{ij})_{i,j \geq 0}$  supported on  $\Lambda$ , that is, matrices having  $m_{ij} = 0$  for every  $i, j$  with  $i + j \in \mathbb{Z}_+ \setminus \Lambda$  and  $\sup_{i,j} |m_{ij}| < \infty$ . Let  $H(\Lambda)$  be the subset of  $HL(\Lambda)$  consisting of Hankel matrices. We recall that a sequence  $(n(k))_{k \geq 1}$  of integers is (Hadamard) lacunary if  $q = \inf_{k \geq 1} \frac{n(k+1)}{n(k)} > 1$ . Clearly, a sequence  $\Lambda$  that is a finite union of lacunary sequences is also a finite union of sequences with  $q \geq 2$ .

**Lemma 3.6.** *Let  $M \in HL(\Lambda)$ . If  $\Lambda$  is a union of  $N$  lacunary sequences with  $q \geq 2$ , then*

$$\|M\|_{\mathcal{H}} \leq 3N \cdot \sup_{i,j} |m_{ij}|.$$

*Proof.* It suffices to analyze the case of  $N = 1$ . Let  $n(k+1) \geq 2n(k)$  for  $k \geq 1$ , and let  $A = (a_{ij})$  be a bounded operator on  $l^2$ . We write  $M \circ A = D + K + L$ , where  $D$  is a diagonal matrix,  $K$  is strictly upper triangular, and  $L$  is strictly lower triangular. Clearly,  $\|D\| \leq \|A\| \sup_{i,j} |m_{ij}|$ .

We show that  $KK^*$  is a diagonal matrix. For this, it suffices to check that, for every  $j \geq 1$ , we have  $m_{ij} \overline{m_{kj}} = 0$  for  $k > i$  and  $i < j$ ,  $k < j$ . Indeed, if we suppose that  $m_{ij} \overline{m_{kj}} \neq 0$ , then  $i + j = n(s)$  and  $k + j = n(t)$  for some  $t > s \geq 1$ , whence  $k = n(t) - j \geq n(t) - n(s) \geq n(s) \geq j$ , a contradiction. Thus,  $KK^*$  is the diagonal matrix with diagonal entries  $\sum_{j>i} |m_{ij} a_{ij}|^2$ , and we obtain  $\|K\| = \|KK^*\|^{1/2} \leq \|A\| \sup_{i,j} |m_{ij}|$ .

Similarly,  $\|L\| = \|L^*L\|^{1/2} \leq \|A\| \sup_{i,j} |m_{ij}|$ , and the result follows.  $\square$

**Corollary 3.7.** *Suppose  $\Lambda$  is a finite union of lacunary sequences. Then the only matrices  $\Gamma \in H(\Lambda)$  generated by a measure are matrices with finitely many nonzero diagonals. Consequently, the norm  $\mu \mapsto \|\Gamma_\mu\|_{\mathcal{H}}$  is strictly weaker than the norm  $\mu \mapsto \|\mu\|_{\mathfrak{M}/H^\perp}$ .*

*Proof.* The first assertion follows from Helson’s theorem (see the Introduction). The second assertion follows from the first one. Indeed, assume that there exists  $C > 0$  such that  $\|\mu\|_{\mathfrak{M}/H^\perp} \leq C\|\Gamma_\mu\|_{\mathcal{H}}$  for every measure  $\mu$ . Then, given a Hankel HSM  $\Gamma = (\gamma_{i+j})$ , we can approximate it by its Fejér means  $\Gamma_{\Phi_n} \circ \Gamma = \Gamma_{\varphi * \Phi_n}$ , where  $\Phi_n$  stands for the Fejér kernel and  $\varphi = \sum_{n \geq 0} \gamma_k z^k$ . This leads to  $\|\Phi_n * \varphi\|_{\mathfrak{M}/H^\perp} \leq C\|\Gamma\|_{\mathcal{H}}$  for every  $n \geq 0$ . Passing to the weak limit, we conclude that  $\Gamma$  is generated by a measure, a contradiction.  $\square$

**Theorem 3.8.** *Let  $\Lambda \subset \mathbb{Z}_+$ . The following assertions are equivalent.*

- 1)  $HL(\Lambda) \subset \mathcal{H}$ .
- 2)  $H(\Lambda) \subset \mathcal{H}$ .
- 3)  $\Lambda$  is a finite union of lacunary sequences.

*Proof.* The implication 1)  $\implies$  2) is obvious, and the implication 3)  $\implies$  1) was proved in Lemma 3.6. Now we show that the implication 2)  $\implies$  3) follows from Corollary 3.5. Indeed, if  $H(\Lambda) \subset \mathcal{H}$ , then the norms  $\|\Gamma\|_{\mathcal{H}}$  and  $\sup_k |\gamma_k|$  are equivalent on  $H(\Lambda)$ . Let  $\gamma_k = 1$  for  $k \in \Lambda \cap [2^N, 2^{N+1}]$  and  $\gamma_k = 0$  otherwise, where  $N$  is an integer. Applying

Corollary 3.5 to an appropriate Hankel operator, the interval  $[2^N, 2^{N+1}]$ , and  $\beta = 2$ , we see that  $\text{card}([2^N, 2^{N+1}] \cap \Lambda)$  is uniformly bounded for all  $N = 1, 2, \dots$ . This means that  $\Lambda$  is a finite union of lacunary sequences (we extract subsequences by taking a point from each interval  $[2^{2N}, 2^{2N+1})$  and then by doing the same with the intervals  $[2^{2N+1}, 2^{2N+2}]$ ,  $N = 0, 1, \dots$ ).  $\square$

*Remark 3.9.* An interesting open problem is to describe the subsets  $\sigma \subset \mathbb{Z}_+ \times \mathbb{Z}_+$  such that the set

$$M(\sigma) = \{(m_{ij}) : m_{ij} = 0 \text{ for } (i, j) \in (\mathbb{Z}_+ \times \mathbb{Z}_+) \setminus \sigma \text{ and } \sup_{ij} |m_{ij}| < \infty\}$$

is contained in  $\mathcal{H}$ . Theorem 3.8 characterizes such Hankel-like subsets. For Toeplitz-like subsets the result is completely different. Namely, let

$$\sigma = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : i - j \in \Lambda\},$$

where  $\Lambda \subset \mathbb{Z}$ , and let  $T(\sigma)$  be the subset of  $M(\sigma)$  consisting of Toeplitz matrices. Then the following statements are equivalent:

- (1)  $M(\Lambda) \subset \mathcal{H}$ .
- (2)  $T(\Lambda) \subset \mathcal{H}$ .
- (3)  $\Lambda$  is a finite set.

Indeed, the implications (1)  $\implies$  (2) and (3)  $\implies$  (1) are obvious. To prove the implication (2)  $\implies$  (3), assume that  $\Lambda$  is infinite. Taking an infinite aperiodic subset  $\Lambda_1 \subset \Lambda$  and considering the corresponding Toeplitz matrix (i.e.,  $t_{ij} = 1$  for  $i - j \in \Lambda_1$  and  $t_{ij} = 0$  otherwise), and then using Theorem 2.2, we obtain a contradiction with the Helson theorem (see the Introduction).  $\square$

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