

## BOOK REVIEW

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*Hankel operators and their applications*, by Vladimir Peller, Springer Monographs in Mathematics, Springer-Verlag, New York–Berlin–Heidelberg, 2003, viii+784 pp.

Hankel operators arise naturally in surprisingly many analytic questions and invoke an impressive combinations of ideas and methods of different nature—operator-theoretic and function-theoretic. It is somewhat curious that the same words would have been far less justified about 50 years ago, in spite of a much greater age of the notion of a *Hankel matrix* (the latter can be attributed to the XIX century). The author of the monograph under review has been playing a principal role in the events that were happening in the theory of Hankel operators through the last decades. The book is devoted to the results of investigations in which the author was involved, and it contains also a good deal of attendant material, starting with elementary facts about Hankel operators. Similar topics were touched upon in some earlier monographs (see, e.g., [3, 4, 5]), but with lesser thoroughness and completeness. Thus, the appearance of a careful and systematic presentation in one volume is useful and timely, especially if we keep in mind that some proofs are simplified compared to original publications, details are polished, and many quite recent developments are covered.

However, the book is not an all-inclusive encyclopedia on Hankel operators—one volume would not suffice for that. We quote a portion of the Introduction that explains what the reader cannot expect to find.

“For the last 20 years many interesting results have been obtained about various generalizations of Hankel operators (commutators of multiplications and Calderón–Zygmund operators, paracommutators, Hankel operators on Bergman spaces, Hankel operators on function spaces on a polydisk, on the unit ball in  $\mathbb{C}^n$ , on classical domains, etc.). However, it is physically impossible to cover such generalizations in one book, and we restrict ourselves to classical Hankel operators.

“Even under this constraint it is hardly possible to cover all aspects of Hankel operators and their applications (for example, this book does not include applications of Hankel operators in noncommutative geometry, perturbation theory, or asymptotics of Toeplitz determinants).”

So, what is included in the book? I would like to avoid a sequential chapter-by-chapter description of the content, indicating the main themes instead. At least three large topics can be distinguished: “spectral theory of functions” and approximation; control theory and the inverse spectral problem; vector-valued versions.

Usually, classical Hankel operators arise in two standard representations: either as the operators whose matrix is of the form  $\{a_{i+j}\}_{j,j=1}^{\infty}$  in some orthonormal basis, or as the operators

$$(1) \quad H_{\varphi} : H^2 \rightarrow H^2; \quad H_{\varphi}(f) = \mathbb{P}_-(f\varphi), \quad f \in H^2.$$

Here  $H^2 = \overline{\text{span}}_{L^2(\mathbb{T})}\{z^n\}_{n \geq 0}$  and  $H^2_- = \overline{\text{span}}_{L^2(\mathbb{T})}\{z^n\}_{n < 0}$  are the (boundary) Hilbertian Hardy classes on the circle  $\mathbb{T}$ ,  $\varphi$  is a function on  $\mathbb{T}$  (the *symbol* of the Hankel operator in question), and  $\mathbb{P}_-$  is the orthogonal projection of  $L^2(\mathbb{T})$  onto  $H^2_-$ . The relationship

between these representations is simple: if we fix the trigonometric bases  $\{z^n\}_{n \geq 0}$  and  $\{z^n\}_{n < 0}$  in the spaces  $H^2$  and  $H_-^2$ , respectively, then  $H_\varphi$  acquires the matrix

$$\begin{pmatrix} \hat{\varphi}(-1) & \hat{\varphi}(-2) & \hat{\varphi}(-3) \\ \hat{\varphi}(-2) & \hat{\varphi}(-3) & \\ \hat{\varphi}(-3) & & \end{pmatrix}.$$

It turns out that the properties of  $\varphi$  and  $H_\varphi$  are in intimate relationship, and that combination of operator and function-theoretic ideas is quite efficient in this situation (a manifestation of what is sometimes called “spectral theory of functions”). It should be noted that these aspects of the theory were also touched upon in the monographs [4, 5]. Here we only discuss approximation problems in brief, without dwelling on other topics.

The symbol  $\varphi$  is not uniquely determined by the operator  $H_\varphi$ : clearly, the latter depends only on the function  $\mathbb{P}_-\varphi$ . The operator  $H_\varphi$  is bounded if and only if the set of its symbols contains an  $L^\infty(\mathbb{T})$ -function, and it is compact if and only if the same set contains a continuous function (a different way to express these conditions is to write  $\mathbb{P}_-\varphi \in \text{BMO}$  and, respectively,  $\mathbb{P}_-\varphi \in \text{VMO}$ ). Moreover, we have

$$\|H_\varphi\| = \text{dist}(\varphi, H^\infty), \quad \|H_\varphi\|_e = \text{dist}(\varphi, H^\infty + C(\mathbb{T}))$$

( $\|\cdot\|_e$  stands for the essential norm, i.e., the distance to the set of compact operators). In fact, this dualism involving functions and operators extends very far: for instance, it turns out that rational approximation of functions in BMO and the behavior of the  $s$ -numbers of Hankel operators are one and the same subject in essence. Indeed, the distance of a Hankel operator to the set of rank  $m$  operators is attained at a Hankel operator of rank  $m$  (Adamyán–Arov–Kreĭn), and  $H_\varphi$  is of finite rank if and only if the function  $\mathbb{P}_-\varphi$  is rational (this result perhaps can be viewed as the starting point of the entire theory: it was proved by Kronecker in 1881).

Not entering in the details, note that, with the help of these ideas, the author of the book found at one time the first cases where “direct” and “inverse” theorems of rational approximation coincide; this was done on the basis of the results described in the next paragraph.

In the 1980s it became clear (the book’s author played a key role in that) that the Hankel operators belonging to the Schatten–von Neumann class  $S_p$  admit a transparent description in terms of the symbol:

$$H_\varphi \in S_p \iff \mathbb{P}_-\varphi \in B_p^{1/p}, \quad 0 < p < \infty.$$

Here  $B_p^{1/p}$  is a space among the Besov function classes. A detailed presentation of this result and related material is given in Chapter 6.

The natural development of the theme suggests that at this point one should look more closely at the (nonlinear) operator of best approximation by analytic functions in the metric of  $L^\infty(\mathbb{T})$ . This is dealt with in Chapter 7, along with the treatment of similar best approximation operators whose definition involves also rational functions. Among other things, it is shown that the best approximation operator takes many classical function spaces  $X$  on the circle to themselves. The “recovery problem” is solved: suppose  $u$  is a unimodular function; under what conditions is it true that  $\mathbb{P}u \in X$  whenever  $u \in X$ ?

The results of Chapter 7 are very beautiful and interesting in themselves; however, the theme culminates in Chapter 9, where it is shown that these results make it possible to explicitly describe various regularity conditions for stationary Gaussian processes in terms of the spectral function.

The second big theme of the book is related to the so-called inverse spectral problem for Hankel operators. (Incidentally, this problem is also related to stationary Gaussian processes; see §8.6.) A. V. Megretskiĭ, V. V. Peller, and S. R. Treil discovered in 1993

that the operators unitarily equivalent to selfadjoint Hankel operators admit quite a transparent description in intrinsic terms. The proof, which is a technical *tour de force*, is presented in Chapter 12. It is indirect and is based on the fact that Hankel operators arise in control theory in a natural way, so that, in place of the operator of the form (1) that is unitarily equivalent to a given operator, it is possible to look for the corresponding dynamical system, which is easier. The way in which Hankel operators are related to control theory is explained in Chapter 11, along with the analysis of control problems these operators help to resolve.

The third theme (permeating through the entire text) can be summarized as follows: most of the results presented in the book have vector-valued analogs, which are often useful. The word “vector-valued” means that the Hardy spaces in (1) are replaced with the corresponding classes of Hilbert-space-valued functions. Then  $\varphi$  becomes operator-valued (consequently, its Fourier coefficients  $\widehat{\varphi}(-n)$  are operators). An apex of the development is in Chapter 14, in which it is explained that if the values of  $\varphi$  are matrices, then, in place of the best approximation by analytic functions, it is natural to consider *superoptimal* approximation (i.e., the  $s$ -numbers are minimized successively), and many problems arising in this setting are discussed.

Surely, the content of the book is not exhausted by the above three themes. Spectral theory often enters in arguments in terms of the Toeplitz operators  $T_\varphi : H^2 \rightarrow H^2$ ,  $T_\varphi f = \varphi f - H_\varphi f$ , to which a separate chapter is devoted. In another chapter, among other things, the essential spectrum for Hankel operators is calculated, and a quasinilpotent Hankel operator is constructed. It goes without saying that classical topics (such as the Hilbert matrix, Wiener–Hopf factorization, or the Nevanlinna–Picck problem) are analyzed in due detail. Not aiming at a complete list, I only add to the above that in Chapter 15 Pisier’s counterexample (1996) to the Halmos problem is presented (i.e., an example of a polynomially bounded operator not similar to a contraction). This problem had been defying all attempts of solution for a long time, and in that story the role of Hankel operators (especially vectorial) also was prominent.

Given its size, the book contains a surprisingly small number of editorial errors, but some of them are unfortunate. For instance, Chapter 12 starts with the claim that a problem posed in §8.5 is solved in it, and the same claim is repeated later, on p. 492. However, §8.5 does not contain a statement of any problem, it is §8.6 that is meant. On p. 306 the author chooses to use the term “Köthe space” (there is a definition), and then suddenly (line 3 from below) switches to “ideal space” without any explanation (the two terms are nearly synonymous, but no mention of this can be found).

The most irritating among the editorial mistakes I was able to recognize is related to “tiresome trifles”, which present a problem for any author who describes an extensive theory. Tiresome trifles are statements that are *nearly* commonplaces. For this reason, it is natural not to discuss them in detail (to save space, not to digress, etc.); however such a discussion cannot be avoided totally because of the adverb “nearly” in the preceding sentence. In the book under review such “trifles” are mathematical facts of fairly high level. For instance, the standard information about the space BMO is among them, which presumes that the reader is well educated in analysis. However, practically nowhere is the reader left alone with these trifles: references and/or minimal explanations are included usually. But there is *one exception*. In the proof of Theorem 1.7 on p. 10 the following fact is used implicitly: if the essential norm of an operator on a Hilbert space is strictly smaller than the operator norm, then the latter is attained at some vector in the unit ball. This fact (known not universally, though widely) cannot be classified as difficult, but some concentration is required to prove it, so that an explicit statement and a reference would have been quite in order. An additional slight aggravation is contributed by the

fact that the symbol  $\|\cdot\|_e$  (occurring at the same place) is defined only 15 pages later (here, however, the standard remedy helps: consult the Appendix at the end of the book).

By the way, in the oldest book in the reference list to this review (i.e., in the monograph [1]), the role of a “tiresome trifle” was played not by the space BMO (surely!) but by ... the Lebesgue decomposition of monotone functions (this may evoke pleasant thoughts about progress in general). For a long time, the book [1] had been the main reference on the asymptotics of Toeplitz determinants, and perhaps it has not yet lost its topicality. More recent information on this subject can be found in [2].

I want to add some words about style. The presentation in the book under review is accurate and transparent, but it deviates slightly towards a bare listing of facts. Informal comments do occur, but the author clearly prefers that the results publicize themselves. Mostly, they do; but some evident possibilities to incorporate more vividness (by the way, this would not have been space-consuming) were overlooked. For instance, the results of Chapter 7 are very nice and do not need extensive motivations. However, it would have been still better if the reader would have been informed beforehand that, two chapters later, they would be employed in the theory of stationary Gaussian processes. In the introduction to Chapter 7 (otherwise, very good) no word is said about that; amazingly, nor is Chapter 7 (the only among all!) mentioned by a single word in the general introduction to the book.

Nevertheless, my criticism is entirely about possibilities to slightly improve an already good thing. Generally, scarce typos and editorial defects do not harden reading. Any mathematical library should find room for this monograph. Experts may use it as a reference-book, and professionals who want to learn the subject (including students, under due supervision) as a text-book.

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