

TOPOLOGICAL AND GEOMETRIC PROPERTIES OF GRAPH-MANIFOLDS

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ABSTRACT. This is a unified exposition of results (obtained by different authors) on the existence of π_1 -injective immersed and embedded surfaces in graph-manifolds, and also of nonpositively curved metrics on graph-manifolds. The basis for unification is provided by the notion of compatible cohomology classes and by a certain difference equation on the graph of a graph-manifold (the BKN-equation). Criteria for seven different properties of graph-manifolds are given at three levels: at the level of compatible cohomology classes; at the level of solutions of the BKN-equation; and in terms of spectral properties of operator invariants of a graph-manifold.

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§1. INTRODUCTION

Let M be a closed three-manifold. We say that M contains a π_1 -injectively immersed (embedded) surface if there is an immersion (embedding) $g : S \rightarrow M$ of a closed surface S with nonpositive Euler characteristic such that the induced homomorphism $g_* : \pi_1(S) \rightarrow \pi_1(M)$ of fundamental groups is injective. We say that M contains a virtually and π_1 -injectively embedded surface if some finite cover of the manifold M contains a π_1 -injectively embedded surface. We say that a manifold M is virtually fibered over the circle if some finite cover of it fibers over the circle with a fiber that is a closed surface with negative Euler characteristic. Finally, if M carries a Riemannian metric of nonpositive sectional curvature, we say that the manifold possesses an NPC-metric.

The properties listed above are significant for the theory of three-manifolds. One of the main conjectures of three-dimensional topology says that any closed irreducible manifold with infinite fundamental group contains a virtually and π_1 -injectively embedded surface (see [Sc]). In accordance with a conjecture by W. Thurston, any closed hyperbolic manifold (and even any hyperbolic manifold of finite volume) is virtually fibered over the circle (see [T, Conjecture 18]). From the geometric side, the classification of closed manifolds admitting NPC-metrics is highly nontrivial (see [L, BK2]) and by now is known up to the geometrization conjecture.

It is natural to ask about conditions ensuring that a closed three-manifold has a property among those listed above. This survey is devoted to solving this problem in a class \mathfrak{M} of orientable graph-manifolds to be described below. We restrict ourselves to that class for the following reasons. First, for the manifolds in \mathfrak{M} the properties listed above are of the same nature and are closely related to one another. Second, as a consequence of this, there is a complete solution of our problem in the class \mathfrak{M} . Finally, \mathfrak{M} is a sufficiently general and wide class of closed three-manifolds.

The manifolds in \mathfrak{M} can be described as follows. A manifold that is a trivial S^1 -fibration over a surface with negative Euler characteristic and nonempty boundary is called a *block*. A gluing of blocks M_1 and M_2 (possibly, $M_1 = M_2$) along boundary tori $T_1 \subset \partial M_1$ and $T_2 \subset \partial M_2$ is said to be *regular* if the S^1 -fibers coming from T_1 and T_2 are not homotopic on the gluing torus. In this case, the result of gluing is not a block. We define \mathfrak{M}_0 to be the class of connected, closed, and orientable manifolds glued from blocks in a regular way. Now, we define \mathfrak{M} as the class of orientable manifolds that admit a finite covering by a manifold of class \mathfrak{M}_0 . We also impose the restriction that

no manifold $M \in \mathfrak{M}$ contains a Klein bottle (this condition is purely technical and is introduced for simplicity of statements).

Every manifold in \mathfrak{M} also possesses a block structure. The blocks are Seifert fibered spaces, i.e., foliations by circles; in general, these foliations are not fibrations (for more detail, see Subsection 1.3). It is not difficult to understand that every manifold $M \in \mathfrak{M}_0$ is irreducible, i.e., every sphere (piecewise linearly) embedded in M bounds a ball. Thus, all manifolds in \mathfrak{M} are also irreducible. Another invariant description of the class \mathfrak{M} will be given in Subsection 1.3.2.

1.1. Properties. In fact, the solution of the characterization problem in the class \mathfrak{M} will be described for a wider spectrum of properties than those indicated at the beginning of the paper. In the study of π_1 -injectively immersed surfaces, we are interested in surfaces of negative Euler characteristic, i.e., we exclude tori (and Klein bottles) for the reason that their immersions and embeddings are easy to describe. It is well known that each π_1 -injectively immersed torus in an irreducible graph-manifold, in particular, in a manifold $M \in \mathfrak{M}$, is homotopic to a virtually embedded torus and, up to an isotopy, lies in some block of M and is parallel to the Seifert fibration of that block (see, e.g., [N3]).

Next, due to a block structure of $M \in \mathfrak{M}$, the horizontal surfaces can be distinguished among surfaces immersed in such a manifold. Namely, an immersion $g : S \rightarrow M$ is said to be *horizontal* if in every block it is transversal to the corresponding Seifert fibration. In that case, S has negative Euler characteristic. It is known that every horizontal immersion is π_1 -injective (see [RW]).

We are interested in the following properties of a manifold $M \in \mathfrak{M}$:

- (I) M contains a π_1 -injectively immersed surface of negative Euler characteristic;
- (HI) M contains an immersed horizontal surface;
- (E) M contains a π_1 -injectively embedded surface of negative Euler characteristic;
- (VE) M contains a virtually and π_1 -injectively embedded surface of negative Euler characteristic;
- (F) M is fibered over the circle (with a surface of negative Euler characteristic as a fiber);
- (VF) M is virtually fibered over the circle;
- (NPC) M admits an NPC-metric.

It is remarkable and not at all obvious that in \mathfrak{M} all properties listed above have actually the same nature. We explain this and give criteria for all the properties (I)–(NPC) at three levels:

- at the level of compatible cohomology classes (see Theorem 2.3);
- at the level of solutions of a difference equation (BKN-equation) on the graph of the manifold (see Theorem 3.1);
- in spectral terms of operators defined via numerical invariants of a manifold (see § 4);

The criteria in terms of compatible cohomology classes and in terms of solutions of the BKN-equation show a common nature of the properties and allow us to find various interrelations among them. On the other hand, except for some rare cases, these criteria give no method to judge on the basis of invariants of a manifold whether or not this manifold possesses a property from the list (I)–(NPC). On the contrary, the criteria in spectral terms are quite constructive and allow us to decide whether or not a given manifold has a given property. In particular, by using these criteria we show that no one-directed arrow in the next diagram (D) is invertible.

The general picture of relations among properties (I)–(NPC) can be described by the following implication diagram:

$$(D) \quad \begin{array}{ccccc} E & \implies & (VE = NPC \cup E) & \implies & I \\ \uparrow & & \uparrow & & \updownarrow \\ F & \implies & VF & \implies & HI, \\ & & \uparrow & & \\ & & NPC & & \end{array}$$

where an implication $A \implies B$ means that if a manifold in \mathfrak{M} possesses property A, then it also possesses property B. All implications in this diagram, except for $I \implies HI$ and $NPC \implies VF$, are almost obvious or trivial: if a manifold $M \in \mathfrak{M}$ has property (F), then any fiber of the corresponding fibration is an embedded horizontal surface (see [WSY]); the property to be horizontal is preserved under finite coverings. On the other hand, the coincidence $VE = NPC \cup E$ and the implications $I \implies HI$, $NPC \implies VF$ are nontrivial (see Subsection 5.5.4, Corollary 5.14, and Corollary 2.4), as well as the fact that no one-directed arrow in the diagram (D) is invertible (see Subsection 4.8).

The above list of properties may be extended, for example, to include the property of a graph-manifold to be the link of a singularity of a complex surface (see [N1, N2]) or to have a π_1 -injectively embedded surface not homological to a sum of tori.

1.2. Historical remarks. Our survey is not a literature guide. Rather, we tried to present a unified and possibly self-contained exposition of results obtained in this field by various authors. We use many ideas of original papers, referring to them in Historical remarks scattered over the paper. The survey also contains new results, fills some gaps and corrects mistakes that we have found in publications. The new results are the identity $(VE) = (NPC) \cup (E)$ and the criteria for all properties in terms of compatible cohomology classes and in terms of solutions of the BKN-equation (Theorems 2.3 and 3.1).

Properties (I)–(NPC) have been studied independently by a number of authors and by different methods. Comparison of the papers [BK2] and [N3] showed unexpected similarity in the description of properties (NPC) and (I). Analyzing this similarity, we came to the conclusion that these properties are of common nature. In its turn, this allowed us to develop a unified approach to the study of them and to discover new relations among them. In what follows we give a list of papers devoted to the properties (I)–(NPC) of graph-manifolds. We do not pretend that this list is complete, and apologize in advance to the authors whose papers in this field have escaped our attention.

Property (VF). In the paper [LW], a topological obstruction was found to this property. Namely, let M be a manifold of class \mathfrak{M} , and let V be the set of its blocks (here we use the notions and notation introduced in Subsection 1.3.2). For each vertex $v \in V$ of the graph Γ_M of M , a number t_v is defined by

$$t_v = |k_v| - \sum_{w \in \partial v} \frac{1}{|b_w|}.$$

One of the results of [LW] says that if all the numbers t_v , $v \in V$, are positive, then the manifold M has no finite cover fibered over the circle. In [N2], a criterion for (VF) was proved in terms of the so-called virtualizers. Unfortunately, that criterion makes it possible to decide whether or not a given graph-manifold has property (VF) only in some exceptional cases, because it is unclear how to find an appropriate virtualizer. A simple obstruction to (VF) and a criterion similar to that in [N2] were given in [WYY]. Closely related to (VF) is the paper [RW], where a criterion was given for a horizontal surface to be a virtual fiber of a fibration over the circle of some finite cover of M . That criterion

(Lemma 5.18 below) will be used in Subsection 5.5. The proof of the main results in [Sv1] (namely, the implication NPC \implies VF and a spectral criterion for (VF)) has a gap: that paper has no convincing argument showing the existence of rational solutions of the BKN-equation. We fill this gap in Subsections 5.3.3 and 5.8.

In [WSY] it was proved that each irreducible graph-manifold with nonempty boundary has a finite cover that fibers over the circle. This fact reflects a general principle saying that obstructions to the properties under study usually disappear for graph-manifolds with nonempty boundary.

Property (NPC). For the irreducible graph-manifolds, all standard obstructions to the existence of NPC-metrics disappear, namely, all solvable subgroups of the fundamental group are almost Abelian, and the centralizers virtually split (see, e.g., [CE, GW, LY]). So the first series of examples (see [L]) of graph-manifolds $M \in \mathfrak{M}_0$ without NPC-metrics came as even more of a surprise. In [BK1], a geometrization equation was obtained for a manifold $M \in \mathfrak{M}_0$; the solvability of it is equivalent to the existence of an NPC-metric on M . A criterion for (NPC) in terms of a quadratic form defined via numerical invariants of $M \in \mathfrak{M}_0$ was given in [BK2] (this criterion has a little flaw; see Subsection 6.6.1 at the end of the survey).

In [L] it was also shown that every irreducible graph-manifold with boundary carries an NPC-metric.

Properties (F) and (E). The obstruction constructed in [LW] and mentioned above is also an obstruction for a manifold $M \in \mathfrak{M}$ to fiber over the circle. Criteria for these properties were obtained in [N2] in terms of a reduced plumbing matrix for M . However, the criterion for (E) is inaccurate (see Subsection 4.9); rather, it is related to the property of a graph-manifold to contain a π_1 -injectively embedded surface not homological to a sum of tori. A spectral criterion for (E), which is close to Theorem 4.5 below, was obtained in the thesis [Sv2].

Property (VE). Criteria for this property were obtained in [N2] in terms of virtualizers and in much more efficient spectral terms close to Theorem 4.6 below.

Properties (I) and (HI). An equation on the graph of a manifold $M \in \mathfrak{M}$, whose solvability is equivalent to the existence of a π_1 -injective immersion $S \rightarrow M$ of a surface of negative Euler characteristic, was obtained in [N3]. This equation turned out to be the same as the equation in [BK1], so that we call it the BKN-equation. In the same paper [N3], a criterion of solvability of the BKN-equation in spectral terms was found. The implication I \implies HI was proved in [Sv2].

In [RW] it was shown that every irreducible graph-manifold with boundary contains a horizontal and properly immersed surface — the result preceding the above-mentioned theorem in [WSY] about the (VF) property for graph-manifolds with boundary.

A unified approach to the study of properties (I)–(NPC) was developed in the thesis [Sv2].

1.3. Preliminaries. Here we discuss the basic notions related to the Seifert fibered spaces and graph-manifolds. For more details on the Seifert fibered spaces, see [Sc].

1.3.1. *Seifert fibered spaces.* Consider the cylinder $D^2 \times [0, 1]$ fibered by the segments $\{y\} \times [0, 1]$, $y \in D^2$. Let $\varphi_{q,p} : D^2 \rightarrow D^2$ be the rotation of the disk by $2\pi q/p$, where q and p are coprime integers. The quotient space $T(q, p) = D^2 \times [0, 1] / \{(y, 0) \sim (\varphi_{q,p}(y), 1)\}$ with fiber structure induced by circles is called a fibered solid torus.

Now, let M be a compact orientable three-manifold fibered by circles (for simplicity we assume that M contains no Klein bottle). Then M is called a Seifert fibered space,

and every interior fiber $\lambda \subset M \setminus \partial M$ has a neighborhood saturated by fibers and fiberwise homeomorphic to a fibered solid torus $T(q, p)$, where the numbers $q = q(\lambda)$ and $p = p(\lambda)$ depend in general on the fiber λ . The boundary ∂M of a Seifert fibered space consists of tori fibered by parallel circles. An interior fiber λ is said to be *regular* if $p(\lambda) = 1$; otherwise it is called *singular*. The set Λ of singular fibers is finite, all regular fibers are freely homotopic to one another, and $M \setminus \bigcup_{\lambda \in \Lambda} \lambda$ is a fibration by circles.

The base of a Seifert fibered space, i.e., the quotient space $\mathcal{O}M$ of the fibers, is a 2-orbifold. Topologically, $\mathcal{O}M$ is a surface, which is denoted by FM and called the underlying surface. The Euler characteristic of the 2-orbifold $\mathcal{O}M$ is defined as follows:

$$\chi(\mathcal{O}M) = \chi(FM) - \sum_{\lambda \in \Lambda} \left(1 - \frac{1}{p(\lambda)}\right),$$

where $\chi(FM)$ is the Euler characteristic of the surface FM . If $\chi(\mathcal{O}M) < 0$, then the structure of Seifert foliation on the manifold M is unique up to isotopy (see [Sc]).

Waldhausen bases. Consider an oriented Seifert fibered space M with nonempty boundary and with (orientable) base orbifold of negative Euler characteristic. Let W be the set of boundary components. Choosing an orientation of fibers in M , we let $f_w \in H_1(T_w)$ be the homology class of an oriented fiber on the boundary torus $T_w \subset \partial M$, $w \in W$. Here $H_1(T_w) = H_1(T_w; \mathbb{Q})$ is the first homology group with rational coefficients.

In what follows, we omit the notation \mathbb{Q} in (co)homology groups for simplicity.

On the torus T_w , there is an orientation induced from M ; hence, the intersection form $\wedge_w : H_1(T_w) \times H_1(T_w) \rightarrow \mathbb{Q}$ is well defined. Every collection of elements $\{z_w, f_w\}_{w \in W}$, $z_w \in H_1(T_w)$, such that $z = \bigoplus_{w \in W} z_w$ lies in the kernel of the inclusion homomorphism $i_* : H_1(\partial M) \rightarrow H_1(M)$ and $z_w \wedge_w f_w = 1$ for all $w \in W$, will be called a Waldhausen basis of the Seifert fibered space M (the elements z_w, f_w form a basis of $H_1(T_w)$ for all boundary components $w \in W$). A Waldhausen basis always exists and is defined up to the transformation $z_w \mapsto z_w + n_w f_w$, where $n_w \in \mathbb{Q}$, $\sum_{w \in W} n_w = 0$. In the case where M is a trivial S^1 -fibration over a surface F , the elements z_w can be taken in $H_1(T_w; \mathbb{Z})$ ($f_w \in H_1(T_w; \mathbb{Z})$ always), and the choice of a Waldhausen basis is equivalent to the choice of a trivialization $M = F \times S^1$.

Framed Seifert fibered spaces. Let M be a Seifert fibered space such that for every boundary torus $T_w \subset \partial M$ an element $c_w \in H_1(T_w; \mathbb{Z})$ with $c_w \wedge_w f_w \neq 0$ is fixed. Then the space M is said to be *framed*, and the collection $C = \{c_w\}_{w \in W}$ is a *framing* of M . Using the following Lemma 1.1, in Subsection 1.3.2 we define *charges*, which are numerical invariants of a graph-manifold and play an important role in the criteria for properties (I)–(NPC).

Lemma 1.1. *Let $C = \{c_w\}$ be a framing of an oriented Seifert fibered space M . Then*

$$\sum_{w \in W} \frac{1}{c_w \wedge_w f_w} \cdot (i_w)_* c_w = e(M, C) \cdot f,$$

where $(i_w)_* : H_1(T_w) \rightarrow H_1(M)$ is the inclusion homomorphism induced by $i_w : T_w \hookrightarrow M$, $f \in H_1(M)$ is the homology class of a regular fiber, $(i_w)_* f_w = f$ for every $w \in W$, and $e(M, C)$ is a rational number.

Proof. Let $\{z_w, f_w\}_{w \in W}$ be a Waldhausen basis for M . Representing the class c_w as $c_w = b_w z_w + d_w f_w$, we see that $b_w = c_w \wedge_w f_w$ and

$$\sum_{w \in W} \frac{1}{c_w \wedge_w f_w} \cdot (i_w)_* c_w = \left(\sum_{w \in W} \frac{d_w}{b_w} \right) f.$$

The coefficient $e(M, C) = \sum_{w \in W} \frac{d_w}{b_w}$ is independent of the choice of a Waldhausen basis because so is the left-hand side, whence the claim. \square

The number $e(M, C)$ depends neither on the framing curves orientations, nor on the fiber orientation, and it changes the sign if the orientation of M is changed. All this immediately follows from the definitions. The number $e(M, C)$ will be called the *charge* of the framed Seifert fibered space (M, C) . It differs only by sign from the relative Euler number for (M, C) as introduced in [LW].

1.3.2. *Graph-manifolds.* At the beginning of the paper, we gave an informal description of the graph-manifold classes \mathfrak{M}_0 and \mathfrak{M} . From that description it does not immediately follow that a manifold cannot have different block decompositions. Here, we give an invariant description of the classes \mathfrak{M}_0 and \mathfrak{M} , which implies the uniqueness (up to isotopy) of a (maximal) block decomposition. Thus, various characteristics related to this decomposition are topological invariants of a graph-manifold.

Recall that the splitting of a three-manifold M along a proper surface $\mathcal{T} \subset M$ is the manifold $M|\mathcal{T}$ that is homeomorphic to the complement $M \setminus N(\mathcal{T})$ of a regular open neighborhood $N(\mathcal{T})$ of the surface \mathcal{T} . There is a continuous projection $\pi : M|\mathcal{T} \rightarrow M$, which is a homeomorphism $\pi^{-1}(M \setminus \mathcal{T}) \rightarrow M \setminus \mathcal{T}$ onto the complement of \mathcal{T} and also is a 2-fold covering $\pi^{-1}(\mathcal{T}) \rightarrow \mathcal{T}$ over \mathcal{T} ; furthermore, $\pi^{-1}(\mathcal{T}) \subset \partial(M|\mathcal{T})$.

The class \mathfrak{M} (respectively, its subclass \mathfrak{M}_0) consists of 3-dimensional, connected, closed, orientable manifolds that are not Seifert fibered spaces and have the following property. For any manifold $M \in \mathfrak{M}$ (respectively, $M \in \mathfrak{M}_0$) there is a finite collection \mathcal{T} of disjoint, embedded, incompressible tori in M such that every connected component of the splitting $M|\mathcal{T}$ is a compact Seifert fibered space with orientable base orbifold of negative Euler characteristic (respectively, a trivial S^1 -fibration over a surface of negative Euler characteristic). Furthermore, we impose the restriction that the manifolds in \mathfrak{M} contain no Klein bottle.

Let M be a manifold of class \mathfrak{M} . It is well known (see [JS, J]) that a minimal collection of tori $\mathcal{T} \subset M$ satisfying the above definition is unique up to isotopy. That collection is called the *JSJ-surface* in M , and the connected components of the splitting $M|\mathcal{T}$ are called the *maximal blocks* (or the *vertex manifolds*) of the manifold M . Therefore, the decomposition of M into the maximal blocks and the topological invariants of the maximal blocks are topological invariants of M . We note that every irreducible graph-manifold, in particular, a graph-manifold with NPC-metric, has a 2-fold cover of class \mathfrak{M} and a finite cover of class \mathfrak{M}_0 (see, e.g., [N2, RW]). Thus, the two approaches to the definition of \mathfrak{M} agree.

The graph of a graph-manifold. Associated with every $M \in \mathfrak{M}$ is its graph $\Gamma = \Gamma_M$, which is dual to the decomposition $M = \bigcup_v M_v$ of the manifold into maximal blocks. In other words, the vertex set V of Γ is the set of maximal blocks in M , and the set W of the oriented edges of Γ can be identified with the set of boundary components of all maximal blocks. Namely, an edge $w \in W$ is directed from a vertex v to a vertex v' if the boundary torus $T_w \subset \partial M_v$ is attached in M to the boundary torus $T_{-w} \subset \partial M_{v'}$, where the minus sign means the reverse edge orientation. The incompressible torus in M that results from gluing the boundary tori T_w and T_{-w} will be denoted by $T_{|w|}$. The set of the edges coming out from a vertex v is denoted by ∂v , $\partial v \subset W$, and if $w \in \partial v$, then we write $w^- = v$, $(-w)^+ = v$. The set of nonoriented edges is denoted by E . The elements $e \in E$ are identified with the pairs $(w, -w)$, $w \in W$.

Invariants of a graph-manifold. Here we describe some numerical invariants of a manifold $M \in \mathfrak{M}$ that play an important role in what follows. There are two types of them: intersection numbers and charges.

We fix an orientation of M . Then every boundary torus T_w , $w \in W$, receives the induced orientation, and the orientations of T_w and T_{-w} are opposite on $T_{|w|}$. Let $a \wedge_w b \in \mathbb{Z}$ be the intersection number of integral homology classes $a, b \in H_1(T_{|w|}; \mathbb{Z})$ with respect to the orientation coming from the side of T_w . Then $a \wedge_{-w} b = -a \wedge_w b$ because the orientations of the torus $T_{|w|}$ coming up from different sides are opposite. The operation $\wedge_w : H_1(T_{|w|}; \mathbb{Z}) \times H_1(T_{|w|}; \mathbb{Z}) \rightarrow \mathbb{Z}$ is extended by linearity to an operation $\wedge_w : H_1(T_{|w|}) \times H_1(T_{|w|}) \rightarrow \mathbb{Q}$.

The intersection number. For every maximal block, we fix one of the two possible orientations of its Seifert fibers. This distinguishes an element $f_w \in H_1(T_{|w|}; \mathbb{Z})$ corresponding to an oriented Seifert fiber for every boundary torus T_w . The integer

$$b_w = f_{-w} \wedge_w f_w$$

satisfies $b_w = b_{-w} \neq 0$ and is called the *intersection number* of fibers on the torus $T_{|w|}$. It changes the sign if the manifold orientation is changed, as well as if the orientation of one of the fibers f_w, f_{-w} is changed.

The form of intersection numbers. In general, it is impossible to choose orientations in such a way that all intersection numbers b_w , $w \in W$, have one and the same sign. The obstruction to this is the cohomology class $\rho \in H^1(\Gamma; \mathbb{Z}_2)$ (described in [W, §9.6]) of the cocycle $\text{sign } b : W \rightarrow \mathbb{Z}_2$, $\text{sign } b(w) = \text{sign } b_w$ (here and in what follows we use the multiplicative form of the group \mathbb{Z}_2). The class $\rho = [\text{sign } b]$ will be called the *form of intersection numbers*. Always, there is a 2-fold cover such that its form ρ is trivial.

In what follows, we always assume that the choice of a Seifert fiber orientation for a maximal block $M_v \subset M$ means also the choice of an element $f_v \in H_1(M_v; \mathbb{Z})$ corresponding to an oriented regular fiber, and of elements $f_w \in H_1(T_{|w|}; \mathbb{Z})$ representing this fiber on the boundary tori $T_{|w|}$, $w \in \partial v$. Then $(i_w)_* f_w = f_v$, where $(i_w)_* : H_1(T_{|w|}; \mathbb{Z}) \rightarrow H_1(M_v; \mathbb{Z})$ is the inclusion homomorphism.

The charge. If fiber orientations of all maximal blocks are fixed, then each maximal block $M_v \subset M$ has a natural framing C_v . Namely, as a distinguished class $c_w \in H_1(T_{|w|}; \mathbb{Z})$, $w \in \partial v$, we take the class of oriented Seifert fibers of the adjacent block, $c_w = f_{-w}$. The charge

$$k_v = e(M_v, C_v) \in \mathbb{Q}$$

of the framed Seifert fibered space (M_v, C_v) (see Subsection 1.3.1) is called the *charge of the maximal block* M_v . The properties of the charges of framed Seifert fibered spaces imply that the charges k_v , $v \in V$, are independent of the choice of Seifert fiber orientations of maximal blocks, change the sign if the manifold orientation is changed, and are topological invariants of the oriented manifold M .

The labeled graph. With every oriented graph-manifold M of class \mathfrak{M} , we associate its graph $\Gamma = \Gamma(V, W)$ together with the collection of absolute values of the intersection numbers $|B| = \{|b_w| \in \mathbb{N} : w \in W\}$, the collection of charges $K = \{k_v \in \mathbb{Q} : v \in V\}$, and the form of intersection numbers ρ . The quadruple $(\Gamma, |B|, K, \rho)$ is called the *labeled graph* of the manifold M . A graph-manifold cannot be recovered from its labeled graph. However, the information encoded in the labeled graph is sufficient to judge whether a given graph-manifold possesses any property on the list (I)–(NPC).

In many situations the form of intersection numbers ρ plays no role; then by the labeled graph we mean the triple $(\Gamma, |B|, K)$.

§2. COMPATIBLE COHOMOLOGY CLASSES

The key to a unified approach to properties (I)–(NPC) is the notion of a compatible collection of cohomology classes. Here we give the definition of this notion and formulate criteria for (I)–(NPC) in terms of it. These criteria (Theorem 2.3) do not look efficient if we want to decide whether or not a given manifold $M \in \mathfrak{M}$ possesses this or that property on the list (I)–(NPC). However, the criteria on the level of the BKN-equation (Theorem 3.1) and the criteria in spectral terms of operator invariants (Theorems 4.4–4.9), which are quite efficient, are based on the criteria discussed in this section.

2.1. Motivations and definitions. To motivate the definition of a compatible collection, we consider a π_1 -injective immersion $g : S \rightarrow M$ of a closed surface S of negative Euler characteristic in a manifold $M \in \mathfrak{M}$ (all properties (I)–(VF) imply the existence of such an immersion). Let \mathcal{T} be the JSJ-surface in M . Then g is homotopic to an immersion such that the preimage $g^{-1}(\mathcal{T})$ consists of a finite number of disjoint, simple, closed, noncontractible curves on the surface S , and the connected components of the preimage of every maximal block M_v are mapped either horizontally (horizontal components) or in parallel to the Seifert fibers of the block (vertical annuli); see [RW]. We assume in what follows that every π_1 -injective immersion under consideration has been put in this position.

Furthermore, we can assume that the vertical annuli of the immersion g that lie in one and the same block are split into parallel pairs. This can be achieved by taking the boundary of the collar of the surface $g(S)$. We fix an orientation of M and orientations of the Seifert fibers of its maximal blocks. This induces an orientation on the horizontal components of the intersection $g(S) \cap M_v$ for every maximal block $M_v \subset M$. We orient the vertical annuli so that the parallel annuli of every pair have opposite orientations. Therefore, the relative class $[g(S) \cap M_v] \in H_2(M_v, \partial M_v)$ of the oriented surface $g(S) \cap M_v$ is well defined. Let $l_v \in H^1(M_v)$ be the class dual to $[g(S) \cap M_v]$. The collection of classes $\{l_v : v \in V\}$ depends on the choice of a Seifert fiber orientation and is independent of the choice of an orientation for M . This collection satisfies certain conditions, and axiomatizing them we arrive at the notion of compatible cohomology classes.

For a class $l_v \in H^1(M_v)$ and an edge $w \in \partial v$, we introduce the notation $l_w := i_w^* l_v$, where the homomorphism $i_w^* : H^1(M_v) \rightarrow H^1(T_{|w|})$ is induced by the inclusion $i_w : T_{|w|} \hookrightarrow M_v$.

Lemma 2.1. *The collection of cohomology classes $\{l_v : v \in V\}$ obtained for a π_1 -injective immersion $g : S \rightarrow M$ satisfies the following conditions: $|l_{-w}(f_w)| \leq l_w(f_w)$, and if*

$$|l_{-w}(f_w)| = l_w(f_w), \quad |l_w(f_{-w})| = l_{-w}(f_{-w}),$$

then $l_{-w} = \pm l_w$ for every edge $w \in W$ of the graph Γ_M .

Proof. Let $c \subset g(S) \cap T_{|w|}$ be the image of a simple closed curve on the surface S . Then the curve c , being a boundary component of the surface $g(S) \cap M_{w^-}$ as well as of the surface $g(S) \cap M_{w^+}$, receives orientations from both, and these orientations do not necessarily coincide. The curve c is said to be consistent (inconsistent) if these two orientations agree (are opposite). We denote by c_w^+ (respectively, c_w^-) an element of $H_1(T_{|w|})$ that is the sum of the homology classes of consistent (respectively, inconsistent) curves lying in $g(S) \cap T_{|w|}$ and oriented as the boundary of the surface $g(S) \cap M_v$, where $v = w^-$ is the initial vertex of w . The above definition implies that $c_{-w}^+ = c_w^+$ and $c_{-w}^- = -c_w^-$. Computing the value of the cocycle l_w at some $x \in H_1(T_{|w|})$, we obtain $l_w(x) = (c_w^+ + c_w^-) \wedge_w x$. On the other hand, $l_{-w}(x) = (c_{-w}^+ + c_{-w}^-) \wedge_{-w} x = -(c_w^+ - c_w^-) \wedge_w x$. The orientation chosen for the surface $g(S) \cap M_v$ is such that the numbers $a_w^+ = c_w^+ \wedge_w f_w$ and $a_w^- = c_w^- \wedge_w f_w$ are

nonnegative. These formulas imply that $l_w(f_w) = a_w^+ + a_w^-$ and $l_{-w}(f_w) = -(a_w^+ - a_w^-)$. Therefore, for every edge $w \in W$ we have

$$|l_{-w}(f_w)| = |a_w^+ - a_w^-| \leq a_w^+ + a_w^- = l_w(f_w).$$

Now, suppose that $|l_{-w}(f_w)| = l_w(f_w)$ and $|l_w(f_{-w})| = l_{-w}(f_{-w})$ for some edge $w \in W$. Then $a_w^+ \cdot a_w^- = 0 = a_{-w}^+ \cdot a_{-w}^-$. Assume that $a_w^+ = 0$. This means that either the set of consistent curves on the torus $T_{|w|}$ is empty, or the consistent curves are vertical with respect to the block M_v . In the first case the class c_w^+ vanishes, whence $c_{-w}^+ = 0$ and $a_{-w}^+ = 0$, which easily implies that $l_{-w} = l_w$. In the second case, by the choice of orientations on the parallel pairs of vertical annuli, on the torus $T_{|w|}$ there are consistent as well as inconsistent curves that are vertical with respect to the block M_v , and, therefore, horizontal with respect to the adjacent block $M_{v'}$, $v' = w^+$. Thus, the numbers a_{-w}^+ , a_{-w}^- are both nonzero. This contradicts the aforesaid, and so the second case is excluded. Similarly, if $a_w^- = 0$, then $l_{-w} = -l_w$. \square

With each choice of orientations of the Seifert fibers of maximal blocks, an *oriented collection of cohomology classes* associates a collection $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ in such a way that the change of the fiber orientation of some block M_v changes the sign of the class l_v . Lemma 2.1 motivates the following definition. An oriented collection of cohomology classes $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ is said to be *compatible* if not all classes are zero and for every edge $w \in W$ the inequality $|l_{-w}(f_w)| \leq l_w(f_w)$ is true; furthermore, if $|l_{-w}(f_w)| = l_w(f_w)$ and $|l_w(f_{-w})| = l_{-w}(f_{-w})$, then $l_{-w} = \pm l_w$.

We emphasize that in this definition we talk about real cohomology classes, while Lemma 2.1 deals with rational ones.

By Lemma 2.1, the existence of compatible cohomology classes is a necessary condition for each of the properties (I)–(VF). In the next subsection we show that this condition is also necessary for property (NPC).

2.1.1. The case of NPC-metrics. It is well known (see, e.g., the papers [GW, LY, E, Sch, B1] and the book [CE]) that every NPC-metric g on a graph-manifold M has a rather special form. Namely, a JSJ-surface \mathcal{T} for M can be chosen in such a way that every torus $T_{|w|}$ is geodesic and flat, and the metric g locally splits along every maximal block M_v , i.e., every point $z \in M_v$ has a neighborhood $U_z \subset M_v$ isometric to the metric product $F_z \times (-\varepsilon, \varepsilon)$, where F_z is a surface of nonpositive Gaussian curvature. The splitting is naturally compatible with the Seifert fiber structure, the Seifert fibers are closed geodesics, and all regular fibers have one and the same length L_v .

We note that even if a block M_v is a trivial S^1 -bundle, i.e., $M_v \simeq F_v \times S^1$, the metric g may have no global splitting along M_v (see Remark 5.5 below). However, lifted to the universal cover \widetilde{M}_v , it splits globally, and \widetilde{M}_v is isometric to the metric product $A_v \times \mathbb{R}$, where A_v is a surface of nonpositive Gaussian curvature with geodesic boundary.

As usual, we fix an orientation of M and orientations of the Seifert fibers of every maximal block $M_v \subset M$. This determines an orientation of $\widetilde{M}_v = A_v \times \mathbb{R}$ and an orientation of the factor \mathbb{R} . The fundamental group $\pi_1(M_v)$ acts freely on \widetilde{M}_v by isometries leaving the splitting $\widetilde{M}_v = A_v \times \mathbb{R}$ invariant. The isometries leave the orientation of M invariant because the manifold M is orientable. Furthermore, they preserve the orientation of the factor \mathbb{R} , because M contains no Klein bottle. Therefore, we have a homomorphism $\varphi_v : \pi_1(M_v) \rightarrow \mathbb{R}$ such that to every isometry $\gamma \in \pi_1(M_v)$, $\gamma : A_v \times \mathbb{R} \rightarrow A_v \times \mathbb{R}$, it assigns the shift of the factor \mathbb{R} under γ . Since the group of homomorphisms $\pi_1(M_v) \rightarrow \mathbb{R}$ is canonically isomorphic to $H^1(M_v; \mathbb{R})$, we get a class $l_v \in H^1(M_v; \mathbb{R})$ corresponding to the homomorphism φ_v . By the choice of orientations, $l_v(f_v) = L_v > 0$ is the length of a regular fiber.

Lemma 2.2. *The collection $\{l_v : v \in V\}$ of cohomology classes described above is compatible. Moreover, for every edge $w \in W$ we have $|l_{-w}(f_w)| < l_w(f_w)$ and $l_w(f_w) \cdot l_w(f_{-w}) = l_{-w}(f_w) \cdot l_{-w}(f_{-w})$, where, we recall, $l_w = i_w^* l_v$.*

Proof. The induced flat metric on the torus $T_{|w|}$ gives rise to a scalar product g_w on $H_1(T_{|w|}; \mathbb{R}) \simeq \mathbb{R}^2$ such that $g_w(a, a)$ is the square of the length of a closed geodesic on $T_{|w|}$ representing an element $a \in H_1(T_{|w|}; \mathbb{Z})$. Then the functional $l_w : H_1(T_{|w|}; \mathbb{R}) \rightarrow \mathbb{R}$ is the projection onto the line $\mathbb{R} \cdot f_w$ with respect to this scalar product, $g_w(a, f_w) = l_w(a) \cdot l_w(f_w)$ for every $a \in H_1(T_{|w|}; \mathbb{R})$, because $g_w(f_w, f_w) = L_v^2 = (l_w(f_w))^2$ is the square of the length of a regular fiber of the block M_v . Since the scalar products g_w and g_{-w} on the torus $T_{|w|}$ coincide, we have

$$l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = g_{-w}(f_w, f_{-w}) = g_w(f_{-w}, f_w) = l_w(f_w) \cdot l_w(f_{-w}).$$

The vectors $f_{-w}, f_w \in H_1(T_{|w|}; \mathbb{R})$ are linearly independent; therefore, for their scalar product we have

$$|g_{-w}(f_w, f_{-w})| < \sqrt{g_w(f_w, f_w)} \sqrt{g_w(f_{-w}, f_{-w})} = l_w(f_w) \cdot l_{-w}(f_{-w}),$$

which implies $|l_{-w}(f_w)| < l_w(f_w)$. □

2.2. Criteria for properties (I)–(NPC). Now, we are ready to formulate criteria for properties (I)–(NPC) in terms of compatible cohomology classes.

Theorem 2.3. *Let M be a manifold of class \mathfrak{M} , and let $\Gamma = \Gamma(V, W)$ be its graph.*

- (I) *M has property (I) if and only if there is a compatible collection of cohomology classes for M ;*
- (HI) *M has property (HI) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomology classes for M such that $l_v(f_v) > 0$ for every vertex $v \in V$;*
- (E) *M has property (E) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomology classes for M such that for every edge $w \in W$ the following is true: if $l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0$, then $l_{-w} = \pm l_w$, where $l_w = i_w^* l_v$;*
- (VE) *M has property (VE) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomology classes for M such that $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$;*
- (F) *M has property (F) if and only if there is a function $\varepsilon : V \rightarrow \{\pm 1\}$ and a compatible collection $\{l_v : v \in V\}$ of cohomology classes for M such that $l_{-w}(f_w) = \varepsilon_v \varepsilon_{v'} l_w(f_w) \neq 0$ for every edge $w \in W$, where $v = w^-, v' = w^+$;*
- (VF) *M has property (VF) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomology classes for M such that $l_v(f_v) > 0$ for every vertex $v \in V$ and $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$;*
- (NPC) *M has property (NPC) if and only if there is a compatible collection $\{l_v : v \in V\}$ of cohomology classes for M such that $|l_{-w}(f_w)| < l_w(f_w)$ and $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$.*

The proof of Theorem 2.3 is given in § 5. The implication NPC \implies VF is an immediate consequence of Theorem 2.3.

Corollary 2.4. *If a closed graph-manifold M carries an NPC-metric, then M virtually fibers over the circle.*

Proof. As has already been mentioned, the manifold M has a 2-fold cover of class \mathfrak{M} . Thus, we may assume that $M \in \mathfrak{M}$. Comparing the criteria (VF) and (NPC) in Theorem 2.3, we see that M virtually fibers over the circle. □

§3. BKN-EQUATION

By Theorem 2.3, each of the properties (I)–(NPC) is characterized by the existence of the corresponding compatible cohomology classes. However, Theorem 2.3 gives no method to judge whether or not a given manifold $M \in \mathfrak{M}$ admits such classes. On the other hand, it turns out that, in a sense, these classes viewed as unknowns satisfy a difference equation on the graph of the manifold; this equation is called the BKN-equation. Its coefficients are charges and intersection numbers. This allows us to make one step further to the solution of the initial problem and to reformulate each property (I)–(NPC) in terms of solutions of the BKN-equation.

3.1. Deriving the BKN-equation. Let M be a manifold of class \mathfrak{M} , $\Gamma = \Gamma(V, W)$ its graph, and $X = (\Gamma, |B|, K)$ its labeled graph. Assume that there is a compatible collection $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ of cohomology classes on M (we assume that an orientation of M and orientations of the Seifert fibers of the maximal blocks are fixed). We define functions $a : V \rightarrow \mathbb{R}$ and $\gamma : W \rightarrow \mathbb{R}$ as follows: $a(v) = l_v(f_v)$ and

$$\gamma(w) = \begin{cases} \text{sign}(b_w) l_w(f_{-w})(l_{-w}(f_{-w}))^{-1} & \text{if } l_{-w}(f_{-w}) \neq 0, \\ 0 & \text{if } l_{-w}(f_{-w}) = 0, \end{cases}$$

where $l_w = i_w^* l_v$ and $(i_w)_* f_w = f_v \in H_1(M_v; \mathbb{Z})$ is the class representing a regular oriented fiber of M_v . For brevity, we put $a_v = a(v)$, $\gamma_w = \gamma(w)$. Observe that for every edge $w \in W$ we have $\gamma_w \cdot l_{-w}(f_{-w}) = \text{sign}(b_w) \cdot l_w(f_{-w})$. If $l_{-w}(f_{-w}) \neq 0$, then this identity follows from the definition of γ_w ; if $l_{-w}(f_{-w}) = 0$, then $l_w(f_{-w}) = 0$ by the compatibility of the collection $\{l_v : v \in V\}$. Thus, for every vertex $v \in V$ we have

$$\begin{aligned} \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} &= \sum_{w \in \partial v} \frac{\text{sign}(b_w) l_w(f_{-w})}{|b_w|} = \sum_{w \in \partial v} \frac{i_w^* l_v(f_{-w})}{f_{-w} \wedge_w f_w} \\ &= l_v \left(\sum_{w \in \partial v} \frac{(i_w)_* f_{-w}}{f_{-w} \wedge_w f_w} \right) = l_v(k_v f_v) = k_v a_v. \end{aligned}$$

Therefore, the functions a and γ satisfy the equation

$$k_v a_v = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+}, \quad v \in V,$$

which is called the BKN-equation over the labeled graph X . The coefficients of this equation are numerical invariants of the oriented manifold M : the charges k_v , $v \in V$, and the absolute values $|b_w|$ of the intersection numbers $w \in W$. The numbers γ_w , $w \in W$, and the charges are independent of the choice of fiber orientations of the maximal blocks, and they change the sign if the manifold orientation is changed.

Also, the compatibility condition for $\{l_v\}$ implies that the solution (a, γ) obtained above possesses the following property: $|\gamma_w| \leq 1$, and if $|\gamma_w| = |\gamma_{-w}| = 1$ for some edge $w \in W$, then $\gamma_{-w} = \gamma_w = \pm 1$.

For each of the properties (I)–(NPC), the unknown functions a and γ have a clear geometric meaning. For instance, if we talk about (NPC), then $a_v = L_v$ is the length of a regular fiber of the block M_v , and

$$\gamma_w = \text{sign}(b_w) \frac{g_w(f_w, f_{-w})}{\sqrt{g_w(f_w, f_w) \cdot g_w(f_{-w}, f_{-w})}}$$

is \pm cosine of the angle between the oriented fibers f_w and f_{-w} of adjacent fibrations on the common torus $T_{|w|}$. If we talk about property (HI), then a_v is the degree of the projection of the horizontal surface $g(S) \cap M_v$ onto the base orbifold $\mathcal{O}M_v$ of the maximal block M_v , where $g : S \rightarrow M$ is a horizontal immersion. In that case we have $|\gamma_w| \leq 1$ by

Lemma 2.1. A similar interpretation of the variables a and γ is possible for each of the remaining properties (I), (E)–(VF). In any case we have $a_v \geq 0$ for all vertices $v \in V$ and $|\gamma_w| \leq 1$ for all edges $w \in W$. Somewhat loosely, the function $a \in \mathbb{R}^V$ in a solution (a, γ) of the BKN-equation will be called the *length function*, and $\gamma \in \mathbb{R}^W$ will be called the *angle function*.

3.2. Criteria for properties (I)–(NPC). A solution (a, γ) of the BKN-equation is said to be *compatible* if

- the length function a is nonnegative and $a \neq 0$;
- the angle function γ satisfies the following conditions: $|\gamma_w| \leq 1$ for all $w \in W$, and if $|\gamma_w \cdot \gamma_{-w}| = 1$ for some edge $w \in W$, then $\gamma_{-w} = \gamma_w = \pm 1$;
- if $a_v = 0$ for some vertex $v \in V$, then $\gamma_w = \gamma_{-w} = 0$ for all edges $w \in \partial v$.

The last condition is introduced for convenience, and it is not an essential restriction, because if (a, γ) is a solution to the BKN-equation, then for

$$\gamma'_w = \begin{cases} \gamma_w & \text{if } a_{w^+} \cdot a_{w^-} \neq 0, \\ 0 & \text{if } a_{w^+} \cdot a_{w^-} = 0 \end{cases}$$

the collection (a, γ') solves the BKN-equation and satisfies the condition in question.

A solution (a, γ) of the BKN-equation is said to be *symmetric* if the angle function γ is symmetric, $\gamma_w = \gamma_{-w}$ for every edge $w \in W$. Now we are ready to formulate criteria for properties (I)–(NPC) in terms of solutions of the BKN-equation.

Theorem 3.1. *Let $X = (\Gamma, |B|, K, \rho)$ be the labeled graph of an oriented manifold $M \in \mathfrak{M}$.*

- (I) *M has property (I) if and only if the BKN-equation over the graph X has a compatible solution;*
- (HI) *M has property (HI) if and only if the BKN-equation has a compatible solution (a, γ) with positive length function a , $a_v > 0$ for every vertex $v \in V$;*
- (E) *M has property (E) if and only if the BKN-equation has a compatible and symmetric solution (a, γ) such that $\gamma_w = \gamma_{-w} = \pm 1$ for every edge $w \in W$ with $a_{w^-} \cdot a_{w^+} \neq 0$;*
- (VE) *M has property (VE) if and only if the BKN-equation has a compatible and symmetric solution (a, γ) ;*
- (F) *M has property (F) if and only if there is a compatible and symmetric solution (a, γ) of the BKN-equation with positive length function a and with angle function $\gamma : W \rightarrow \{\pm 1\}$ whose cohomology class is the form of intersection numbers, $[\gamma] = \rho \in H^1(\Gamma; \mathbb{Z}_2)$;*
- (VF) *M has property (VF) if and only if the BKN-equation has a compatible and symmetric solution (a, γ) with positive length function a ;*
- (NPC) *M has property (NPC) if and only if the BKN-equation has a compatible and symmetric solution (a, γ) with positive length function a and with angle function γ satisfying $\gamma_w = \gamma_{-w} \in (-1, 1)$ for every edge $w \in W$.*

The proof of Theorem 3.1 is given in § 5.

3.3. Historical remarks. The BKN-equation was obtained in [BK1, BK2] for the study of property (NPC), and independently in [N3] for the study of properties (I) and (HI). A relation equivalent to the BKN-equation for the particular case of $|\gamma| \equiv 1$ was obtained and used in [WSY]. This equation was used in [BK3] to derive a criterion for the property (NPC) with finite volume for infinite graph-manifolds. In the paper [BK4], it was generalized to the case of an arbitrary dimension $n \geq 4$ for manifolds glued from maximal blocks such that each of them is a trivial torus bundle over a compact surface

of negative Euler characteristic. In this case, in contrast to the case of $n = 3$, the charges are not numerical invariants, but some linear mappings. In [B2], the BKN-equation was interpreted as an analog of the Maxwell equations of classical electrodynamics. The criteria (I)–(VF) of Theorem 3.1 were obtained in the thesis [Sv2].

§4. SPECTRAL CRITERIA

Here we introduce operator invariants of a manifold $M \in \mathfrak{M}$, i.e., linear operators defined in terms of the labeled graph $(\Gamma, |B|, K, \rho)$ (see Subsection 1.3.2), and give criteria for properties (I)–(NPC) in spectral terms related to these operators. This completes the program of investigating properties (I)–(NPC) for the manifolds in \mathfrak{M} that was described in the Introduction.

4.1. Operator invariants. Let $\Gamma = \Gamma(V, W)$ be the graph of an oriented manifold $M \in \mathfrak{M}$, and let $(\Gamma, |B|, K, \rho)$ be its labeled graph. If not all charges of M are zero, then we always orient M so that at least one of the charges be positive.

All operator invariants we use are linear operators on \mathbb{R}^V symmetric with respect to the scalar product

$$(x, x') = \sum_{v \in V} x_v x'_v, \quad x = (x_v) \in \mathbb{R}^V,$$

so that we define them via the corresponding quadratic forms on \mathbb{R}^V . Note that the space \mathbb{R}^V has a canonical basis, which we identify with the vertex set V of the graph Γ .

We define symmetric operators D_M and D_M^+ with the help of the quadratic forms

$$(D_M x, x) = \sum_{v \in V} k_v x_v^2, \quad (D_M^+ x, x) = \sum_{v \in V} |k_v| x_v^2,$$

where $x = (x_v) \in \mathbb{R}^V$. For a symmetric function $\lambda : W \rightarrow \mathbb{Z}_2$ (a cocycle on the graph Γ), we define a symmetric operator J_M^λ as follows:

$$(J_M^\lambda x, x) = \sum_{w \in W} \frac{\lambda_w}{|b_w|} x_{w^-} x_{w^+}$$

(recall that we always use the multiplicative form of the group \mathbb{Z}_2).

4.1.1. *The operator A_M^+ .* We define a linear operator $A_M^+ : \mathbb{R}^V \rightarrow \mathbb{R}^V$ by

$$A_M^+ = D_M^+ - J_M,$$

where $J_M = J_M^\lambda$ for $\lambda \equiv 1$. In terms of this operator, we formulate criteria for properties (I) and (HI). As a motivation for the introduction of A_M^+ we prove the following lemma.

Lemma 4.1. *If a manifold $M \in \mathfrak{M}$ has one of the properties on the list (I)–(NPC), then the operator A_M^+ has a nonpositive eigenvalue.*

Proof. By Theorem 3.1, the BKN-equation has a compatible solution (a, γ) . Therefore,

$$\begin{aligned} (A_M^+ a, a) &= \sum_{v \in V} \left(|k_v| a_v^2 - a_v \sum_{w \in \partial v} \frac{1}{|b_w|} a_{w^+} \right) \\ &\leq \sum_{v \in V} a_v \operatorname{sign}(k_v) \left(k_v a_v - \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} \right) = 0, \end{aligned}$$

because $\operatorname{sign}(k_v) \gamma_w \leq 1$. Since the operator A_M^+ is symmetric, it has a nonpositive eigenvalue. \square

4.1.2. *The operators A_λ .* For a cocycle $\lambda : W \rightarrow \mathbb{Z}_2$, we put $A_\lambda = D_M - J_M^\lambda$. We are interested in the singularity properties of A_λ . Recall that a linear operator $A : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is called *singular* if it has a nontrivial kernel, and *supersingular* if its kernel contains an element $x \in \mathbb{R}^V$ with all coordinates x_v different from zero. We say that an operator $A : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is *weakly singular* if there is a nonzero vector $x \in \mathbb{R}^V$ such that $(Ax)_v = 0$ for all vertices v in the support of x , $x_v \neq 0$. (The properties of supersingularity and weak singularity depend on the choice of a basis. However, the space \mathbb{R}^V has a canonical basis V , so that these properties are manifold invariants for the corresponding operators defined below.)

Lemma 4.2. *Let $\lambda : W \rightarrow \mathbb{Z}_2$ be a symmetric function. The spectrum of the operator A_λ as well as its properties to be singular, supersingular, or weakly singular depend only on the cohomology class $[\lambda] \in H^1(\Gamma; \mathbb{Z}_2)$.*

Proof. Take $\lambda' = \lambda \cdot \sigma$, where σ is a coboundary, i.e., $\sigma_w = \varepsilon_w - \varepsilon_{w^+}$ for some function $\varepsilon : V \rightarrow \mathbb{Z}_2$ and every edge $w \in W$. Given a vector $x \in \mathbb{R}^V$, we consider the vector $x' \in \mathbb{R}^V$ such that $x'_v = \varepsilon_v x_v$. Then for every $\mu \in \mathbb{R}$ and every vertex $v \in V$ we have $(A_{\lambda'} x' - \mu x')_v = \varepsilon_v (A_\lambda x - \mu x)_v$, and the claim follows. \square

We formulate spectral criteria for properties (E) and (F) in terms of the singularity of the operators A_λ . By Lemma 4.2, we can assume that $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$.

4.1.3. *The operator H_M .* We formulate spectral criteria for properties (VE), (VF), and (NPC) in terms of a symmetric operator $H_M : \mathbb{R}^V \rightarrow \mathbb{R}^V$ to be defined below. We must take into account the distribution of the charge signs on the vertex set V of the graph Γ , which significantly complicates the definition of H_M .

The graph of sign components. Let E be the set of nonoriented edges of the graph Γ . Vertices $v, v' \in V$ of Γ lie in one and the same sign component if $v = v'$ or if there is a sequence of vertices $v_1 = v, \dots, v_n = v'$ such that for every $i = 1, \dots, n - 1$ the vertices $v_i, v_{i+1} \in V$ are connected by an edge with $k_{v_i} k_{v_{i+1}} > 0$. This gives an equivalence relation \sim on V . The quotient set $U = V / \sim$ is called the set of sign components, and it is the disjoint union $U = U_0 \cup U_+ \cup U_-$, where U_+ (U_-) is the set of sign components with positive (negative) charges, and U_0 is the set of vertices with zero charges. For $u \in U$, let Γ_u be the connected subgraph in Γ spanned by the vertices in the component u , i.e., the graph Γ_u contains all edges $e \in E$ between the vertices in u . Contracting every subgraph $\Gamma_u, u \in U$, to a point, we obtain a graph $G = G(U, E_0)$, which is called the graph of sign components of the labeled graph $(\Gamma, |B|, K)$. The vertex set of G is U , and its set of (nonoriented) edges E_0 consists of all edges $e \in E$ connecting vertices from different sign components. If $U \neq U_0$, then the set U_+ is nonempty by assumption. We denote by $p : \Gamma \rightarrow G$ the canonical projection.

Introducing the function $s : U \rightarrow \{0, \pm 1\}$. Recall that a graph G is said to be *bipartite* if its vertex set U can be represented as a disjoint union $U = P \cup N$ in such a way that every edge of the graph connects a vertex in P with a vertex in N . This property is equivalent to the condition that the graph G has no cycle with odd number of edges. If the graph G is connected, then the decomposition into the parts P and N is unique up to permutation.

We define an auxiliary function $s : U \rightarrow \{0, \pm 1\}$, which will be used in the definition of the operator H_M , as follows. If the graph $G(U, E_0)$ of sign components for the labeled graph $(\Gamma, |B|, K)$ is not bipartite, or if $U = U_0$, then we put $s(u) = 0$ for all $u \in U$. Otherwise, we choose a decomposition $U = P \cup N$ such that $P \cap U_+ \neq \emptyset$, and put $s(u) = 1$ if $u \in P$, and $s(u) = -1$ if $u \in N$.

Introducing the operator H_M . For every sign component $u \in U$ we put $W_u = W \cap p^{-1}(u)$, and define symmetric operators $D_u, J_u : \mathbb{R}^u \rightarrow \mathbb{R}^u$ via the quadratic forms

$$(D_u x, x) = s(u) \sum_{v \in u} k_v x_v^2, \quad (J_u x, x) = \sum_{w \in W_u} \frac{1}{|b_w|} x_{w^-} x_{w^+},$$

where $x = (x_v) \in \mathbb{R}^u$ (if the set W_u is empty, then $J_u = 0$).

Now, we define a symmetric operator $H_M : \mathbb{R}^V \rightarrow \mathbb{R}^V$ by

$$H_M = \bigoplus_{u \in U} (D_u - J_u).$$

Therefore, the edges of the graph Γ connecting different sign components give no contribution into the operator H_M . If all charges of $M \in \mathfrak{M}$ are nonzero and have one and the same sign, then the graph of sign components G reduces to a point. In this case, the operators A_M^+ and H_M coincide, $A_M^+ = H_M$.

As a motivation, we prove the following lemma.

Lemma 4.3. *If the BKN-equation over $M \in \mathfrak{M}$ has a compatible and symmetric solution (a, γ) , then $(H_M a, a) \leq 0$, in particular, the operator H_M has a nonpositive eigenvalue.*

Proof. Using the BKN-equation, we obtain

$$(H_M a, a) = \sum_{u \in U} \sum_{w \in W_u} (s(u) \gamma_w - 1) \frac{a_{w^-} a_{w^+}}{|b_w|} \leq 0.$$

The first equality in this chain follows from the fact that the sum of $s(u) \frac{\gamma_w}{|b_w|} a_{w^-} a_{w^+}$ over all edges $w \in W$ connecting vertices from different sign components is zero, because either $s(u) \equiv 0$, or each summand occurs in this sum twice with opposite signs $s(u) = 1, -1$, where $u \in U$ is the sign component from which the edge w goes out. Thus, the operator H_M has a nonpositive eigenvalue. \square

4.2. Spectral criterion for (I) and (HI). Recall that a symmetric operator $A : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is said to be *positive semidefinite* if $(Ax, x) \geq 0$ for every $x \in \mathbb{R}^V$. In the course of the proof of Theorems 2.3 and 3.1 we shall obtain Corollary 5.14, which says that the properties (I) and (HI) are equivalent.

Theorem 4.4. *A manifold $M \in \mathfrak{M}$ has properties (I)=(HI) if and only if one of the following conditions is fulfilled:*

- (1) *all charges of M have one and the same sign, and the operator A_M^+ is positive semidefinite and singular;*
- (2) *the operator A_M^+ has a negative eigenvalue.*

This theorem will be proved in Subsection 6.1.

4.3. Spectral criterion for (E).

Theorem 4.5. *A manifold $M \in \mathfrak{M}$ has property (E) if and only if the operator A_λ is weakly singular for some class $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$.*

This theorem will be proved in Subsection 6.2.

4.4. Spectral criterion for (VE).

Theorem 4.6. *A manifold $M \in \mathfrak{M}$ has the property (VE) if and only if the operator H_M has a nonpositive eigenvalue.*

This theorem will be proved in Subsection 6.3.

4.5. Spectral criterion for (F).

Theorem 4.7. *A manifold $M \in \mathfrak{M}$ has property (F) if and only if the operator A_ρ is supersingular, where $\rho \in H^1(\Gamma; \mathbb{Z}_2)$ is the form of intersection numbers.*

This theorem will be proved in Subsection 6.4.

4.6. Spectral criterion for (VF).

Theorem 4.8. *A manifold $M \in \mathfrak{M}$ has property (VF) if and only if one of the following conditions is fulfilled:*

- (1) *the operator H_M has a negative eigenvalue;*
- (2) *the operator H_M is positive semidefinite and supersingular.*

This theorem will be proved in Subsection 6.5.

4.7. Spectral criterion for (NPC).

Theorem 4.9. *A manifold $M \in \mathfrak{M}$ has property (NPC) if and only if the operator H_M has a negative eigenvalue, or the function $s : U \rightarrow \{0, \pm 1\}$ occurring in the definition of H_M is zero.*

This theorem will be proved in Subsection 6.6.

4.8. The implication diagram. Using the spectral criteria (Theorems 4.4–4.9), here we give examples which show that no one-directed arrow of the diagram (D) is invertible, and also that there are manifolds of class \mathfrak{M} possessing no property among those on the list (I)–(NPC).

4.8.1. *The graph-manifolds $M(\alpha)$.* All manifolds in the examples below belong to the class \mathfrak{M}_0 , and they all have one and the same graph Γ , which is a linear graph with three vertices v_1, v_2, v_3 and two (nonoriented) edges e_1, e_2 connecting the vertices v_1, v_2 and v_2, v_3 , respectively. The maximal blocks $M_i = M_{v_i}$, $i = 1, 2, 3$, have the following structure. The blocks M_1 and M_3 are the trivial bundles over the torus with a hole, and the block M_2 is the trivial bundle over the torus with two holes. The gluing between the blocks M_1 and M_2 is one and the same for all examples, and the gluings between M_2 and M_3 are parameterized by matrices $\alpha \in GL(2, \mathbb{Z})$ with determinant equal to -1 .

To describe these gluings, we fix trivializations $M_i = F_i \times S^1$ of the blocks, orientations of the S^1 -factors of all blocks, and orientations of the surfaces F_i , $i = 1, 2, 3$. For every boundary torus T_w , $w \in W$, this fixes a basis z_w, f_w of the group $H_1(M_v; \mathbb{Z})$, where $w \in \partial v$. Every element z_w represents the corresponding oriented boundary component of F_v , and f_w represents the oriented factor S^1 of the block M_v . We assume that $e_1 = (w_1, -w_1)$, $e_2 = (w_2, -w_2)$, where $w_1 \in \partial v_1$, $-w_1, w_2 \in \partial v_2$, and $-w_2 \in \partial v_3$. Then the gluing $g : T_{-w_1} \rightarrow T_{w_1}$ between the blocks M_1, M_2 is given by

$$\begin{aligned} g(z_{-w_1}) &= f_{w_1}, \\ g(f_{-w_1}) &= z_{w_1} + f_{w_1}. \end{aligned}$$

For

$$\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{Z})$$

with $ad - bc = -1$, the gluing $h_\alpha : T_{-w_2} \rightarrow T_{w_2}$ between the blocks M_2 and M_3 is given by α , i.e.,

$$\begin{aligned} h_\alpha(z_{-w_2}) &= az_{w_2} + cf_{w_2}, \\ h_\alpha(f_{-w_2}) &= bz_{w_2} + df_{w_2}. \end{aligned}$$

We denote by $M(\alpha)$ the graph-manifold resulting from these gluings.

The orientations of the S^1 -factors of the maximal blocks in $M(\alpha)$ and in the base surfaces determine orientations of the blocks themselves, and these orientations are compatible with the gluings, because the latter reverse the orientation of the corresponding boundary tori. This induces an orientation of the manifold $M(\alpha)$. Its intersection numbers are

$$b_1 := b_{w_1} = b_{-w_1} = 1, \quad b_2 := b_{w_2} = b_{-w_2} = b,$$

and the charges can easily be found by using Lemma 1.1:

$$k_1 := k_{v_1} = 1, \quad k_2 := k_{v_2} = \frac{d}{b}, \quad k_3 := k_{v_3} = -\frac{a}{b}.$$

4.8.2. *A graph-manifold without (I)-(NPC).* An example of such a manifold is $M(\alpha) \in \mathfrak{M}_0$ with the gluing

$$\alpha = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}.$$

The operator $A_\alpha^+ = A_{M(\alpha)}^+$ is given by the matrix

$$A_\alpha^+ = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

whose eigenvalues 1 and $2 \pm \sqrt{3}$ are positive. By Lemma 4.1, the manifold $M(\alpha)$ possesses no property on the list (I)-(NPC).

4.8.3. (I) $\not\Rightarrow$ (VE). Here we give an example of $M(\alpha)$ with property (I) but without (VE). This also shows that (HI) $\not\Rightarrow$ (VF). As α we take the matrix

$$\alpha = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}.$$

The operator $A_\alpha^+ = A_{M(\alpha)}^+$ is given by the matrix

$$A_\alpha^+ = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

and it has a negative eigenvalue, namely, $1 - \sqrt{2}$. By Theorem 4.4, $M(\alpha)$ has property (I). On the other hand, the operator $H_{M(\alpha)}$ is the identity, $H_{M(\alpha)} = \text{id}$. Thus, by Theorem 4.6, the manifold $M(\alpha)$ does not possess property (VE).

4.8.4. (VE) $\not\Rightarrow$ (E). Here we give an example of $M(\alpha)$ with property (VE) but without (E). As α we take the matrix

$$\alpha = \begin{bmatrix} -3 & 2 \\ -1 & 1 \end{bmatrix}.$$

The operator $H_\alpha = H_{M(\alpha)}$ is given by the matrix

$$H_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1/2 & -1/2 \\ 0 & -1/2 & 3/2 \end{bmatrix},$$

and it has a negative eigenvalue because $\det H_\alpha = -1$. By Theorem 4.6, the manifold $M(\alpha)$ has property (VE). The graph Γ of $M(\alpha)$ is simply connected, so that the group $H^1(\Gamma; \mathbb{Z}_2)$ is trivial, and the operator A_λ coincides with H_α for every $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$. Since the principal minors of the matrix H_α of every order are different from zero, the operator $A_\lambda = H_\alpha$ is not weakly singular. By Theorem 4.5, the manifold $M(\alpha)$ does not possess property (E).

4.8.5. (VE) $\not\Rightarrow$ (VF). Here we give an example of $M(\alpha)$ with property (VE) but without (VF). As α we take the matrix

$$\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

The operator $H_\alpha = H_{M(\alpha)}$ is given by the matrix

$$H_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and it is singular. By Theorem 4.6, the manifold $M(\alpha)$ has property (VE). On the other hand, the operator H_α is positive semidefinite and not supersingular, because $x_1 = x_2 = 0$ for every vector $x = (x_1, x_2, x_3)$ with $H_\alpha x = 0$. By Theorem 4.8, the manifold $M(\alpha)$ does not possess property (VF).

4.8.6. (E) $\not\Rightarrow$ (F), (VF) $\not\Rightarrow$ (F), (VF) $\not\Rightarrow$ (NPC). Here we give an example of $M(\alpha)$ with properties (E) and (VF) but without (F) and (NPC). As α we take the matrix

$$\alpha = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since the group $H^1(\Gamma; \mathbb{Z}_2)$ is trivial, the operator A_λ is given by the matrix

$$A_\lambda = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

for every $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$, and it is weakly singular, because $A_\lambda x = (0, 0, -1)$ for $x = (1, 1, 0)$. By Theorem 4.5, the manifold $M(\alpha)$ has property (E).

The operator $H_\alpha = H_{M(\alpha)}$ is given by the matrix

$$H_\alpha = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and it is positive semidefinite and supersingular, $H_\alpha x = 0$ for $x = (1, 1, 1)$. By Theorem 4.8, the manifold $M(\alpha)$ fibers virtually over the circle. On the other hand, the operator $A_\rho = A_\lambda$ is not singular, $\det A_\lambda = -1$. By Theorem 4.7, the manifold $M(\alpha)$ does not fiber over the circle. The operator H_α has no negative eigenvalue, and the function s occurring in the definition of H_α is nonzero. By Theorem 4.9, the manifold $M(\alpha)$ carries no NPC-metric.

4.9. Historical remarks. A spectral criterion for property (F) equivalent to Theorem 4.7, and a criterion for property (VE) close to Theorem 4.6 were obtained in [N2]. A spectral criterion for property (I) equivalent to Theorem 4.4 (but without the identity (I)=(HI)) was obtained in [N3]. In the same paper, an example of a graph-manifold of class \mathfrak{M} without property (I) was found, and it was shown that (I) $\not\Rightarrow$ (VE). A spectral criterion for property (E) was obtained in [Sv2], and for property (VF) such a criterion was formulated in [Sv1]. A spectral criterion for property (NPC) was obtained in [BK2] (see, however, Subsection 6.6.1). The example in Subsection 4.8.6 is a counterexample to [N2, Theorem D.1(3)], because the manifold $M(\alpha)$ in that example has property (E), and at the same time the matrix A_λ (the “decomposition matrix” in terms of [N2]) is not singular.

§5. PROOF OF THEOREMS 2.3 AND 3.1

We prove Theorems 2.3 and 3.1 simultaneously. The first step of the proof is the following simple but important extension lemma.

Lemma 5.1. *Assume that for a block M_v of an oriented manifold $M \in \mathfrak{M}$ an orientation of the Seifert fibers is fixed and a Waldhausen basis $\{z_w, f_w\}_{w \in \partial v}$ is chosen. For a given collection of cohomology classes $\{l_w \in H^1(T_{|w|}; \mathbb{R}) : w \in \partial v\}$, there is a class $l_v \in H^1(M_v; \mathbb{R})$ such that $i_w^* l_v = l_w$ for all $w \in \partial v$ if and only if the following conditions are fulfilled:*

- (1) $l_w(f_w) = a_v$ is independent of $w \in \partial v$;
- (2) $\sum_{w \in \partial v} l_w(z_w) = 0$.

Proof. Conditions (1) and (2) are necessary, because for every class $l_v \in H^1(M_v; \mathbb{R})$ with $i_w^* l_v = l_w$, $w \in \partial v$, the number $l_w(f_w) = l_v(f_v)$ is independent of $w \in \partial v$ and $\sum_{w \in \partial v} l_w(z_w) = l_v(\bigoplus_{w \in \partial v} (i_w)_* z_w) = 0$.

Conversely, assume that a collection of cohomology classes $\{l_w\}$ satisfies conditions (1) and (2). We show that the class $m_v = \bigoplus_{w \in \partial v} l_w \in H^1(\partial M_v; \mathbb{R})$ lies in the image of the homomorphism $i^* : H^1(M_v; \mathbb{R}) \rightarrow H^1(\partial M_v; \mathbb{R})$. For this, due to the exact cohomology sequence of the pair $(M_v, \partial M_v)$, it suffices to show that $m_v(\partial S) = 0$ for every relative class $S \in H_2(M_v, \partial M_v; \mathbb{R})$, where $\partial : H_2(M_v, \partial M_v; \mathbb{R}) \rightarrow H_1(\partial M_v; \mathbb{R})$ is the boundary homomorphism. Representing the class ∂S as $\partial S = \bigoplus_{w \in \partial v} c_w$ and decomposing the element c_w over the basis z_w, f_w , we obtain $c_w = \alpha z_w + \beta_w f_w$, where the coefficient α is independent of $w \in \partial v$, because the classes ∂S and $z = \bigoplus_{w \in \partial v} z_w$ in $H_1(\partial M_v; \mathbb{R})$ lie in the kernel of the homomorphism i_* . Thus,

$$m_v(\partial S) = \sum_{w \in \partial v} l_w(c_w) = \left(\sum_{w \in \partial v} \beta_w \right) \cdot a_v.$$

On the other hand,

$$0 = i_*(\partial S) = \sum_{w \in \partial v} (i_w)_*(c_w) = \left(\sum_{w \in \partial v} \beta_w \right) \cdot f_v.$$

Consequently, $m_v(\partial S) = 0$, and the class m_v lies in the image of i^* . Then any class $l_v \in H^1(M_v; \mathbb{R})$ with $i^* l_v = m_v$ is as required. □

5.1. Theorems 2.3 and 3.1 are equivalent. As a consequence of Lemma 5.1, we see that Theorems 2.3 and 3.1 are equivalent in the following sense.

Proposition 5.2. *Let $M \in \mathfrak{M}$ be an oriented manifold. For each property (A) on the list (I)–(NPC) we have the following equivalence: a compatible collection of cohomology classes on M satisfying condition (A) of Theorem 2.3 exists if and only if the BKN-equation over M has a compatible solution satisfying condition (A) of Theorem 3.1.*

Proof. Let $\{l_w \in H^1(M_v; \mathbb{R}) : v \in V\}$ be compatible cohomology classes satisfying condition (A) of Theorem 2.3 (we assume that fiber orientations of the maximal blocks are fixed). Then the functions $a : V \rightarrow \mathbb{R}$, $a(v) = l_v(f_v)$, and $\gamma : W \rightarrow \mathbb{R}$,

$$\gamma(w) = \begin{cases} \text{sign}(b_w) l_w(f_{-w}) (l_{-w}(f_{-w}))^{-1} & \text{if } l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0, \\ 0 & \text{if } l_w(f_w) \cdot l_{-w}(f_{-w}) = 0, \end{cases}$$

where $l_w = i_w^* l_v$, form a compatible solution of the BKN-equation (see Subsection 3.1). It is straightforward to check that this solution satisfies condition (A) of Theorem 3.1.

Conversely, let (a, γ) be a compatible solution of the BKN-equation satisfying condition (A) of Theorem 3.1. For every vertex $v \in V$, choose a Waldhausen basis $\{z_w, f_w\}_{w \in \partial v}$

of M_v . Then for the coefficients of the decomposition $f_{-w} = b_w z_w + d_w f_w$ we have the following: b_w is the intersection number for the edge $w \in \partial v$, and $k_v = \sum_{w \in \partial v} \frac{d_w}{b_w}$ is the charge of the vertex v (see Lemma 1.1 and Subsection 1.3.2). For an edge $w \in \partial v$, we define a class $l_w \in H^1(T_{|w|}; \mathbb{R})$ by $l_w(f_w) := a_v$ and $l_w(z_w) := \frac{\gamma_w}{|b_w|} a_{w^+} - \frac{d_w}{b_w} a_v$. Then

$$\sum_{w \in \partial v} l_w(z_w) = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} - k_v a_v = 0,$$

whence we see that the collection of classes $\{l_w : w \in \partial v\}$ satisfies the conditions of Lemma 5.1. By that lemma, there is a class $l_v \in H^1(M_v; \mathbb{R})$ such that $i_w^* l_v = l_w$ for all $w \in \partial v$. Since $l_w(f_{-w}) = \text{sign}(b_w) \gamma_w a_{w^+}$ and $|\gamma_w| \leq 1$, for the vertex $v' = w^+$ we have $|l_w(f_{-w})| \leq a_{v'} = l_{-w}(f_{-w})$. Assume that $|l_{-w}(f_w)| = l_w(f_w)$ and $|l_w(f_{-w})| = l_{-w}(f_{-w})$. If $l_w(f_w) = 0 = l_{-w}(f_{-w})$, then $l_w = l_{-w} = 0$. Otherwise, we have $l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0$, because the solution is compatible. Then $|\gamma_w| = |\gamma_{-w}| = 1$, whence $\gamma_{-w} = \gamma_w$, because, we recall, $\gamma_w \cdot \gamma_{-w} \neq -1$. It easily follows that $l_{-w} = \pm l_w$. Therefore, the cohomology classes $\{l_v : v \in V\}$ are compatible. It is straightforward to check that these classes satisfy condition (A) of Theorem 2.3. \square

Proposition 5.2 provides some additional flexibility for the proof of Theorems 2.3 and 3.1: it is convenient to use compatible cohomology classes in some cases, and compatible solutions of BKN-equations in other cases.

5.2. Local conditions of extension. Here we establish necessary and sufficient boundary conditions for every maximal block to have an extension to an NPC-metric on it (Lemma 5.3), and to have an extension to a horizontal immersion in it (Lemma 5.6). These conditions are an essential part of the proof of Theorems 2.3 and 3.1. Though Lemmas 5.3 and 5.6 are formulated differently and their proofs are technically different, there is an explicit analogy between them.

Lemma 5.3. *Assume that for a maximal block M_v of a manifold $M \in \mathfrak{M}$ an orientation of its Seifert fibers is fixed and a flat metric on the boundary ∂M_v is given, i.e., for every $w \in \partial v$ we have a positive definite quadratic form g_w on $H_1(T_{|w|}; \mathbb{R})$, and*

- (1) $g_w(f_w, f_w) = a_v^2 > 0$ is independent of $w \in \partial v$;
- (2) $\sum_{w \in \partial v} g(z_w, f_w) = 0$,

where $\{z_w, f_w\}_{w \in \partial v}$ is a Waldhausen basis of the block M_v . Then there is an NPC-metric g_v on M_v that extends the given one on the boundary ∂M_v , i.e., every boundary torus $T_{|w|}$, $w \in \partial v$, is flat and geodesic with respect to g_v , and on $T_{|w|}$ the metric g_v induces the metric g_w .

Remark 5.4. Condition (2) is independent of the choice of a Waldhausen basis; conditions (1) and (2) are also necessary for the existence of an NPC-metric on M_v with flat geodesic boundary (see Lemmas 2.2 and 5.1).

Sketch of the proof of Lemma 5.3. Given $w \in \partial v$, we define $l_w \in H^1(T_{|w|}; \mathbb{R})$ by $l_w(a) = \frac{g_w(a, f_w)}{\sqrt{g_w(f_w, f_w)}}$, $a \in H_1(T_{|w|}; \mathbb{R})$. In other words, the class l_w acts as the projection onto $\mathbb{R} \cdot f_w$ with respect to the scalar product g_w (cf. the proof of Lemma 2.2). From (1) and (2) it follows that the collection $\{l_w : w \in \partial v\}$ satisfies the conditions of Lemma 5.1 about extension; thus, there is a class $l_v \in H^1(M_v; \mathbb{R})$ that induces a collection $\{l_w\}$, $i_w^* l_v = l_w$, on the boundary ∂M_v . The class l_v gives rise to a homomorphism $\varphi_v : \pi_1(M_v) \rightarrow \mathbb{R}$; we use it to construct a representation $\psi_v : \pi_1(M_v) \rightarrow \text{Iso}(\mathbb{H}^2 \times \mathbb{R})$ of the fundamental group of the block M_v in the isometry group of the space $\mathbb{H}^2 \times \mathbb{R}$.

Now, we assume for simplicity that the block M_v is the trivial S^1 -bundle over a compact surface F_v of negative Euler characteristic. We fix a trivialization $M_v = F_v \times S^1$

and an orientation of the surface F_v , which is equivalent to the choice of a Waldhausen basis $\{z_w, f_w\}_{w \in \partial v}$. Then the elements z_w , $w \in \partial v$, represent the corresponding oriented components of the boundary ∂F_v .

Recall that, since the Euler characteristic of F_v is negative, for any positive numbers L_w , $w \in \partial v$, there is a metric of constant curvature -1 with geodesic boundary on the surface F_v and such that the length of the w -component equals L_w .

Now, for $w \in \partial v$, as L_w we take the length of z_w projected to the direction orthogonal to f_w with respect to the metric g_w . As before, this determines a hyperbolic metric on the surface F_v , and hence a representation $\eta_v : \pi_1(F_v) \rightarrow \text{Iso}(A_v)$ of the group $\pi_1(F_v)$ in the isometry group of the universal cover $A_v \subset \mathbb{H}^2$ of the surface F_v .

We define a representation $\psi_v : \pi_1(M_v) \rightarrow \text{Iso}(A_v \times \mathbb{R})$ by $\psi_v(\gamma) = (\eta_v \circ \pi_v(\gamma), \varphi_v(\gamma))$ for $\gamma \in \pi_1(M_v)$, where $\pi_v : \pi_1(M_v) \rightarrow \pi_1(F_v)$ is the projection homomorphism onto the first factor. It is easy to check that its image acts discretely and freely on the metric product $A_v \times \mathbb{R}$, and this gives the required NPC-metric on the block M_v .

In the general case the argument is similar: we use the fact that there is a hyperbolic structure on the orbifold $\mathcal{O}_v = \mathcal{O}M_v$ with the prescribed lengths L_w of the boundary components, which gives a representation $\eta_v : \pi_1(\mathcal{O}_v) \rightarrow \text{Iso}(A_v)$; then we take the homomorphism π_v from the exact sequence

$$1 \longrightarrow \mathbb{Z} \cdot f_v \longrightarrow \pi_1(M_v) \xrightarrow{\pi_v} \pi_1(\mathcal{O}_v) \longrightarrow 1$$

of the Seifert bundle for M_v . □

Remark 5.5. Typically, the NPC-metric constructed above on the trivial bundle $M_v \simeq F_v \times S^1$ is not a metric product: the holonomy of the circle S^1 along some noncontractible loops in F_v can be nontrivial. A metric is a product if it is possible to find a trivialization $z \in H_1(\partial M_v; \mathbb{Z})$, $z = \bigoplus_{w \in \partial v} z_w$, such that the element z_w is orthogonal to f_w with respect to the metric g_w for all $w \in \partial v$.

Lemma 5.6. *Assume that for an oriented block M_v of a manifold $M \in \mathfrak{M}$ an orientation of its Seifert fibers is fixed and for every $w \in \partial v$ two elements c_w^+ , $c_w^- \in H_1(T_{|w|}; \mathbb{Z})$ are given such that $\alpha_w^\pm = c_w^\pm \wedge f_w \geq 0$, the class $c_w = c_w^+ + c_w^-$ is even, i.e., $\frac{1}{2}c_w \in H_1(T_{|w|}; \mathbb{Z})$, and the following conditions are fulfilled:*

- (1) $c_w \wedge f_w = a_w > 0$ is independent of $w \in \partial v$;
- (2) $\sum_{w \in \partial v} z_w \wedge c_w = 0$,

where $\{z_w, f_w\}_{w \in \partial v}$ is a Waldhausen basis of the block M_v . Then there is an integer $d(M_v) \geq 1$ such that for any integer multiple $d \geq 1$ of $d(M_v)$ there is a horizontal immersion $g_v : S_v \rightarrow M_v$ of a compact surface S_v with boundary $\partial S_v = \bigcup_{w \in \partial v} (\gamma_w^+ \cup \gamma_w^-)$ (the union of the connected components) such that $[g_v(\gamma_w^\pm)] = d \cdot c_w^\pm$ for all $w \in \partial v$ with $\alpha_w^+ \cdot \alpha_w^- \neq 0$. If $\alpha_w^+ \cdot \alpha_w^- = 0$ for some $w \in \partial v$, then $[g_v(\gamma_w^+)] = [g_v(\gamma_w^-)] = \frac{d}{2} \cdot c_w$.

Remark 5.7. Condition (2) is independent of the choice of a Waldhausen basis; conditions (1) and (2) are also necessary for the existence of an immersion $g_v : S_v \rightarrow M_v$ with prescribed behavior on the boundary (see Lemmas 2.1 and 5.1).

The proof of Lemma 5.3 was based on the existence of a hyperbolic metric with prescribed lengths of the boundary components on a given surface (orbifold) of negative Euler characteristic. The proof of Lemma 5.6 is based on a similar existence property for coverings of a given surface of negative Euler characteristic with prescribed behavior on the boundary. Here is the precise statement.

Lemma 5.8. *Let F be a compact orientable surface of positive genus with nonempty boundary, and let W be the set of its boundary components. Assume that every boundary component $w \in W$ is covered by a collection S_w of $n_w \geq 1$ circles with multiplicities*

$a_w^s \geq 1$, $s \in S_w$, where the sum $\sum_{s \in S_w} a_w^s = a$ is independent of w and the number $\sum_w n_w$ of circles has the same parity as $a \cdot |W|$. Then the covering of the boundary ∂F can be extended to a covering of F of multiplicity a by a connected surface S .

Proof. It suffices to find a homomorphism of the group $\pi_1(F)$ into the permutation group of a fiber (which has cardinality a) with transitive image such that it maps the boundary components to permutations with the prescribed cycle structure. The parity condition implies that the product of the boundary cycles is an even permutation; by a result in [JLLP], this product is the commutator of an n -cycle and an involution. Now, since the genus of F is positive, we see that the required homomorphism does exist. \square

Proof of Lemma 5.6. First, we assume for simplicity that M_v is the trivial S^1 -bundle and that the trivialization $M_v = F_v \times S^1$ corresponds to the chosen Waldhausen basis. Taking if necessary a covering of F_v with multiplicity $d(M_v) := |\partial v|$ and with the same number of boundary components, we may assume that the genus of F_v is positive.

In the decompositions $c_w^\pm = \alpha_w^\pm z_w + \beta_w^\pm f_w$, the integers $\alpha_w^\pm = c_w^\pm \wedge_w f_w$ are nonnegative, $\alpha_w^+ + \alpha_w^- = a_v > 0$ is even for all $w \in \partial v$, and $\sum_{w \in \partial v} (\beta_w^+ + \beta_w^-) = 0$. We construct the required immersion $g_v : S_v \rightarrow F_v \times S^1$ as follows: $g_v = (g_v^{\text{hor}}, g_v^{\text{vert}})$. Here $g_v^{\text{hor}} : S_v \rightarrow F_v$ is a covering of degree a_v with the following property: on a boundary component $\gamma_w^\pm \subset \partial S_v$ it covers the corresponding component of ∂F_v with degree α_w^\pm if both α_w^+, α_w^- are nonzero, and both components γ_w^\pm cover the corresponding component of ∂F_v with degree $a_v/2$ if $\alpha_w^+ \cdot \alpha_w^- = 0$. Lemma 5.8 provides such a covering by a connected surface S_v . The parity condition is fulfilled because the number of the boundary components of S_v is twice the number of those of F_v , and a_v is even.

We require that the map $g_v^{\text{vert}} : S_v \rightarrow S^1$ have the degree β_w^\pm on the boundary component γ_w^\pm if $\alpha_w^+ \cdot \alpha_w^- \neq 0$, and the degree $\frac{1}{2}(\beta_w^+ + \beta_w^-)$ on both boundary components γ_w^+, γ_w^- if $\alpha_w^+ \cdot \alpha_w^- = 0$ ($\beta_w^+ + \beta_w^-$ is even because the class c_w is even). Since S_v is connected, the condition $\sum_{w \in \partial v} (\beta_w^+ + \beta_w^-) = 0$ implies that such a map does exist, because $[S_v, S^1] = H^1(S_v; \mathbb{Z})$. Now, the immersion g_v has the required properties by construction.

In the general case, we remove from M_v the interiors of the fibered solid tori (with disjoint closures) that are boundaries of the singular fibers, and denote by $\partial^s v$ the set of the new boundary components of the complement M'_v of the removed pieces. The manifold M'_v is a trivial S^1 -bundle over a compact orientable surface F'_v of negative Euler characteristic; furthermore, for every torus $T_w \subset \partial M'_v$, $w \in \partial^s v$, the class $c_w \in H_1(T_w; \mathbb{Z})$ of its meridian is fixed, i.e., a simple closed curve bounding a disk in the corresponding removed solid torus. We assume that the orientations of the meridians are chosen so that $\alpha_w := c_w \wedge_w f_w > 0$ for all $w \in \partial^s v$. We put $a'_v = a_v \cdot \prod_{w \in \partial^s v} \alpha_w$, $\partial^* v = \partial v \cup \partial^s v$, and for $w \in \partial^* v$ we denote $d_w = a'_v / \alpha_w \in \mathbb{Z}$, where $\alpha_w = a_v$ for $w \in \partial v$. Then all integral classes $c'_w = d_w c_w$, $w \in \partial^* v$, have one and the same intersection number with the regular fibers f_w , $c'_w \wedge_w f_w = a'_v$. Fix a trivialization $M'_v = F'_v \times S^1$. Then for the corresponding Waldhausen basis $\{z_w, f_w\}_{w \in \partial^* v}$ we have $\sum_{w \in \partial^* v} z_w \wedge_w c'_w = 0$.

For the collection of classes $c'_w = d \cdot c_w^\pm$, $w \in \partial v$, where $d = d(M_v) := a'_v / a_v$ (we multiply by $|\partial^* v|$ if necessary for the positivity of the genus), and c'_w , $w \in \partial^s v$, we construct an immersion $g'_v : S'_v \rightarrow M'_v$, $g'_v = (g_v^{\text{hor}}, g_v^{\text{vert}})$ as above with the only difference that, under the covering g_v^{hor} , every boundary component $w \in \partial^s v$ of the surface F'_v is covered by d_w boundary components of the surface S'_v with degree α_w on each. The parity condition reduces to the property that the number $\sum_{w \in \partial^s v} d_w$ must be even, and this is true because the above sum is an integer multiple of the even number a_v . The map $g_v^{\text{vert}} : S'_v \rightarrow S^1$ is subject to the additional condition that it must have the degree $z_w \wedge_w c'_w$ on each of the corresponding d_w boundary components of S'_v for every $w \in \partial^s v$.

We may assume that the image $g'_v(S'_v) \subset M'_v$ meets every boundary component $w \in \partial^s v$ over a collection of d_w curves parallel to the meridian. Attaching disks in the corresponding solid tori to these curves, we obtain the required immersion g_v for $d = d(M_v)$. Now, the case of an arbitrary multiple d of $d(M_v)$ follows easily. \square

5.2.1. *Historical remarks.* A statement close to Lemma 5.3 can be found in [L]. Lemma 5.6 is a particular case of [N3, Lemma 3.1]. Lemma 5.8 and its proof are taken from [N3].

5.3. Properties (imm), (HI), (E), (F), and (NPC).

5.3.1. *Existence of compatible cohomology classes.* Here we show that each of the properties (I), (HI), (E), (F), and (NPC) for a manifold $M \in \mathfrak{M}$ implies the existence of compatible cohomology classes satisfying the corresponding condition of Theorem 2.3. For this, we use the constructions of Subsection 2.1. For the virtual properties (VE) and (VF) this step of the proof is more difficult, and it is discussed in Subsection 5.4.

Proposition 5.9. *Each of the properties (I), (HI), (E), (F), and (NPC) of a manifold $M \in \mathfrak{M}$ implies the existence of a compatible collection of cohomology classes satisfying the corresponding condition of Theorem 2.3.*

Proof. For (I) and (NPC) this follows from Lemmas 2.1 and 2.2, respectively.

(HI) If an immersion $g : S \rightarrow M$ is horizontal, then for the corresponding compatible cohomology classes $\{l_v : v \in V\}$ we have $l_v(f_v) > 0$ by construction.

(E) Let $g : S \rightarrow M$ be a π_1 -injective embedding. Consider the corresponding compatible collection $\{l_v \in H^1(M_v; \mathbb{R}) : v \in V\}$ and assume that $l_w(f_w) \cdot l_{-w}(f_{-w}) \neq 0$ for some edge $w \in W$, where $l_w = i_w^* l_v$. Then the embedding g is horizontal in both blocks $M_v, M_{v'} \subset M$, $v = w^-, v' = w^+$. In this case, the intersection $g(S) \cap T_{|w|}$ is a collection of parallel circles, so that they all simultaneously are either consistent or not. Thus, $l_w = \pm l_{-w}$.

(F) Let $p : M \rightarrow S^1$ be a fibration. We fix a fiber $S = p^{-1}(t)$, $t \in S^1$. We may assume that the surface $S \subset M$ is horizontal. As usual, we employ this surface to define compatible cohomology classes $\{l_v : v \in V\}$. From horizontality it follows that $l_v(f_v) > 0$ for all $v \in V$. Fixing a generator α of the group $H^1(S^1; \mathbb{Z})$, we put $\varepsilon_v = \text{sign } \alpha(p_*(f_v)) \in \{\pm 1\}$. Then $l_v = \varepsilon_v p_v^* \alpha$, where p_v is the restriction of p to the maximal block M_v . Thus, for every edge $w \in W$ we have

$$l_{-w}(f_w) = i_{-w}^* l_{v'}(f_w) = i_{-w}^* \varepsilon_{v'} p_{v'}^* \alpha(f_v) = \varepsilon_{v'} p_v^* \alpha(f_v) = \varepsilon_{v'} \varepsilon_v l_w(f_w),$$

where v and v' are the vertices of w . \square

5.3.2. *Finding an NPC-metric on a graph-manifold.* To complete the proof of Theorems 2.3 and 3.1 for property (NPC), it remains to construct an NPC-metric on M in the case where a compatible collection of cohomology classes $\{l_v : v \in V\}$ satisfying condition (NPC) of Theorem 2.3 is given. We do this with the help of Lemma 5.3.

For $w \in \partial v$, we put $l_w = i_w^* l_v \in H^1(T_{|w|}; \mathbb{R})$ and define a symmetric bilinear form g_w on $H_1(T_{|w|}; \mathbb{R})$ by $g_w(f_w, f_w) = (l_w(f_w))^2$, $g_w(f_{-w}, f_{-w}) = (l_{-w}(f_{-w}))^2$, and $g_w(f_w, f_{-w}) = l_w(f_w) \cdot l_{-w}(f_{-w})$. The (NPC)-condition of compatibility $|l_{-w}(f_w)| < l_w(f_w)$ implies that the form g_w is positive definite. Furthermore, it obviously satisfies condition (1) in Lemma 5.3. Condition (2) in the same lemma is also fulfilled, because the forms g_w are defined via the classes l_v . By Lemma 5.3, we obtain an NPC-metric on every maximal block M_v . From the (NPC)-condition of symmetry $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ it follows that the metrics constructed coincide on the gluing tori (cf. the proof of Lemma 2.2). Therefore, we obtain an NPC-metric on M . (This metric is C^1 -smooth on M and real analytic inside every block M_v , being modeled on $\mathbb{H}^2 \times \mathbb{R}$. Its curvature

is nonpositive in Alexandrov’s sense. It should be noted, though, that this metric can easily be smoothed up to a C^∞ -smooth NPC-metric (see, e.g., [L]). \square

5.3.3. *Rational approximation.* To implement properties (I)–(VF), we need compatible collections of integral cohomology classes, while we only have real ones. Therefore, first we approximate real classes by rational ones, and then obtain integral classes from them. It is convenient to construct approximation in terms of solutions of the BKN-equation. The next proposition suffices for properties (I), (HI), (E), and (F). To implement the virtual properties (VE) and (VF), we need a symmetric approximating angle function γ' , and symmetric approximation is achieved by another method; see Subsection 5.8.

Proposition 5.10. *Let (a, γ) be a (real) compatible solution of the BKN-equation over $M \in \mathfrak{M}$. Then it can be approximated by rational compatible solutions (a', γ') either in such a way that*

- $\gamma' = \gamma$ and $a'_v = a_v$ for the vertices $v \in V$ with $a_v = 0$ if the angle function γ takes values in $\{0, \pm 1\}$;
or in such a way that
- $a'_v > 0$ for all $v \in V$ and $|\gamma'_w| < 1$ for all edges $w \in W$ if $|\gamma_{w_0}| < 1$ for some edge $w_0 \in W$.

In other words, if the angle function γ takes values in $\{0, \pm 1\}$ and $\gamma_w = 0$ for some edge $w \in W$, then we have two types of rational approximation to the solution (a, γ) . The approximation described in the first part will be used for properties (I), (HI), (E), and (F). The approximation described in the second part will be used for properties (I) and (HI).

The first part is easy to prove: when the (rational) angles γ are fixed, unknown lengths $a_v, v \in V$, satisfy a linear homogeneous system of $|V|$ equations with rational coefficients and zero determinant, which follows from the BKN-equation. By a classical theorem of linear algebra, the solutions of such a system can be parameterized by linear functions with rational coefficients. This easily implies the required approximation. We start the proof of the second part with construction of perturbations of the initial solution (within the class of all solutions), not caring about rationality first. This is achieved in the following three lemmas.

Lemma 5.11. *Let (a, γ) be a compatible solution of the BKN-equation. Then it can be approximated by compatible solutions (a', γ') such that $a'_v > 0$ for all $v \in V$.*

Proof. If the length function a has zeros on V , then, since the solution is nontrivial, there is a vertex $v \in V$ with $a_v > 0$ and $a_{v'} = 0$ for a neighboring vertex v' . Consider an edge $w \in \partial v$ coming to v' . Recall that then $\gamma_{w'} = \gamma_{-w'} = 0$ for all $w' \in \partial v'$ since the solution (a, γ) is compatible. We choose an arbitrarily small $a'_{v'} > 0$ and find γ'_{-w} with $k_{v'} a'_{v'} = \frac{\gamma'_{-w}}{|b_{-w}|} a_v$. Since the angle function γ vanishes at all edges going out of the vertex v' , this ensures that the BKN-equation is fulfilled over v' for the perturbed functions a', γ' . The BKN-equation for them is also fulfilled over the vertex v , because the function γ vanishes at all edges coming from v to v' . Obviously, this procedure leads to the required approximation. \square

Using this lemma, from now on we assume that the length function a is positive.

Lemma 5.12. *Let (a, γ) be a compatible solution of the BKN-equation. Assume that for a vertex $v \in V$ there is an edge $w_0 \in \partial v$ with $|\gamma_{w_0}| < 1$. Then (a, γ) can be approximated by compatible solutions (a', γ') such that $|\gamma'_w| < 1$ for all $w \in \partial v$.*

Proof. The unknown angles γ_w , $w \in \partial v$, are involved only in the BKN-equation over the vertex v :

$$k_v a_v = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w+}.$$

Thus, changing only them and keeping this identity, we obtain a new solution of the BKN-equation. We need to change only the γ_w , $w \in \partial v$, for which $|\gamma_w| = 1$. Since the length function a is positive, the absolute values of these γ_w can be reduced (to be as small as we wish) at the expense of γ_{w_0} . \square

Lemma 5.13. *Let (a, γ) be a compatible solution of the BKN-equation. Assume that $|\gamma_w| < 1$ for all $w \in \partial v$ for some vertex $v \in V$. Then (a, γ) can be approximated by compatible solutions (a', γ') such that $|\gamma'_{\pm w}| < 1$ for all $w \in \partial v$.*

Proof. We change the angle function γ on every edge $-w$ with $|\gamma_{-w}| = 1$ and $w \in \partial v$, and the length function a at the vertex v so that $|\gamma'_{-w}| \cdot a'_v = |\gamma_{-w}| \cdot a_v$, where the ratio $\frac{a'_v}{a_v} > 1$ is arbitrarily close to 1 (here $'$ means passage to an approximant). This yields the property $|\gamma'_{-w}| < 1$ for all edges $w \in \partial v$, with the BKN-equation satisfied for all final vertices of these edges (except the loops). To ensure the BKN-equation for the initial vertex v , we distribute the excess $k_v(a'_v - a_v)$ over the edges $w \in \partial v$, changing the values γ_w in an appropriate way. This can be done with preservation of the condition $|\gamma_w| < 1$, because the value $|k_v(a'_v - a_v)|$ can be chosen arbitrarily small. \square

Proof of Proposition 5.10. As has been explained before, for the proof of the first part of the proposition, a rational parameterization of the solutions of the BKN-equation can be employed, namely, the parametrization obtained when rational values of the angle function γ are fixed. The compatibility of the solution (a, γ) implies that the equations over the vertices $v \in V$ with $a_v = 0$ put no restriction on any approximant a' having the same zeros as a . Thus, it remains to prove the second part of the proposition.

It suffices to approximate the solution (a, γ) by rational solutions (a', γ') satisfying the assumptions of the proposition. Then the solutions (a', γ') sufficiently close to (a, γ) will be compatible automatically. Applying Lemmas 5.11, 5.12, and 5.13, we can assume that the initial solution (a, γ) of the BKN-equation satisfies the conditions $a_v > 0$ for all vertices $v \in V$ and $|\gamma_w| < 1$ for all edges $w \in W$.

Now, we approximate the solution (a, γ) by solutions (a', γ') , where the length function a' takes rational values. Fix $v \in V$ and approximate the number a_v by rational numbers $a'_v > 0$. We distribute the excess $k_v(a'_v - a_v)$ in the BKN-equation over the vertex v between the summands $\frac{\gamma_w}{|b_w|} a_{w+}$, $w \in \partial v$, changing only appropriate γ_w if necessary, keeping the condition $|\gamma_w| < 1$, and leaving the numbers a_{w+} invariant. This ensures the BKN-equation over the vertex v . For any neighboring vertex v' to v , only the summands on the right-hand side of the BKN-equation over v' that correspond to the edges $w \in \partial v'$ with $-w \in \partial v$ are changed in the course of the manipulations described. We compensate for these changes by the condition $\gamma'_w \cdot a'_v = \gamma_w \cdot a_v$ for every such edge w . We have $|\gamma'_w| < 1$ if the ratio $\frac{a_v}{a'_v}$ is sufficiently close to 1. Therefore, we may assume that the length function a in the initial compatible solution (a, γ) takes positive rational values at all vertices $v \in V$. Now, it is easy to approximate the numbers γ_w by rational numbers γ'_w for any edge $w \in W$, keeping the BKN-equation. \square

5.3.4. Finding a horizontal immersion. To complete the proof of Theorems 2.3 and 3.1 for properties (I) and (HI), we construct an immersed horizontal surface in M , provided there is a compatible collection of cohomology classes on M . Using Proposition 5.10, we find a compatible rational solution (a, γ) of the BKN-equation for which $a_v > 0$ for all $v \in V$, and if $|\gamma_{w_0}| = 1$ for some edge $w_0 \in W$, then $\gamma_{-w} = \gamma_w = \pm 1$ for all edges

$w \in W$. This gives a collection $\{l_v \in H^1(M_v; \mathbb{Z}) : v \in V\}$ of compatible integral classes satisfying the following stronger condition of compatibility: if $|l_{-w_0}(f_{w_0})| = l_{w_0}(f_{w_0})$ for some edge $w_0 \in W$, then $l_{-w} = \pm l_w$ for all edges $w \in W$.

For every edge $w \in W$ we put $l_w^+ = l_w + l_{-w}$, $l_w^- = l_w - l_{-w}$ and consider the elements $c_w^+, c_w^- \in H_1(T_{|w|}; \mathbb{Z})$ dual to l_w^+, l_w^- . These elements satisfy the conditions of Lemma 5.6, because

- $\alpha_w^\pm = c_w^\pm \wedge_w f_w = l_w(f_w) \pm l_{-w}(f_w) \geq 0$;
- the class $c_w = c_w^+ + c_w^-$ is even (being dual to the class $2l_w$);
- $c_w \wedge_w f_w = 2l_v(f_v) > 0$ for all $v \in V$ and $w \in \partial v$; condition (2) of Lemma 5.6 is also fulfilled because the classes $l_w, w \in \partial v$, are the boundary values for the class $l_v, i_w^* l_v = l_w$ (see Lemma 5.1).

Moreover, the condition of strong compatibility means that either $\alpha_w^+ \cdot \alpha_w^- \neq 0$ for all $w \in W$, or $\alpha_w^+ \cdot \alpha_w^- = 0$ for all $w \in W$. Since $c_{-w}^+ = c_w^+$, we have $c_{-w}^- = -c_w^-$, and if $\alpha_w^+ \cdot \alpha_w^- = 0$, then $c_{-w} = \pm c_w$, the horizontal immersions $g_v : S_v \rightarrow M_v$ given by Lemma 5.6 for each maximal block M_v can easily be pasted on every gluing torus $T_{|w|}$ at the expense of an appropriate multiplicity d universal for all blocks. This gives the required horizontal immersion $g : S \rightarrow M$. □

Corollary 5.14. *Property (I) implies property (HI).* □

5.3.5. *Finding a fibering over the circle.* To complete the proof of Theorems 2.3 and 3.1 for property (F), we construct a fibration $M \rightarrow S^1$ whose fiber is a closed surface of negative Euler characteristic, provided there are compatible cohomology classes $\{l_v : v \in V\}$ on M satisfying condition (F) in Theorem 2.3. This condition implies that $\varepsilon_v l_w = \varepsilon_{v'} l_{-w}$ for every edge $w \in W$ from v to v' . By the first part of Proposition 5.10, we may assume that the collection $\{l_v : v \in V\}$ consists of integral classes. Using the canonical isomorphism $H^1(M_v; \mathbb{Z}) = [M_v, S^1]$ and the fact that $l_v(f_v) \neq 0$, for every maximal block M_v we find a map $M_v \rightarrow S^1$ corresponding to the class $\varepsilon_v l_v$ of nonzero degree on the Seifert fibers for M_v . It is well known that such a map is homotopic to a fibration $p_v : M_v \rightarrow S^1$ (see the discussion of this in [N2]). Since $\varepsilon_v l_w = \varepsilon_{v'} l_{-w}$ for any edge $w \in W$ from v to v' , the fibrations p_v and $p_{v'}$ can be chosen to coincide on the gluing torus $T_{|w|}$. This yields the required fibration $p : M \rightarrow S^1$.

5.3.6. *Finding an embedded surface.* Now we implement property (E). The first part of Proposition 5.10 allows us to assume that all classes $l_v \in H^1(M_v; \mathbb{Z}), v \in V$, of the corresponding compatible collection are even. Given a block M_v with $l_v(f_v) \neq 0$, we find (as above) a fibration $M_v \rightarrow S^1$. Its fibers are horizontally embedded surfaces whose relative class corresponds to l_v under the isomorphism $H_2(M_v, \partial M_v; \mathbb{Z}) \simeq H^1(M_v; \mathbb{Z})$. If $l_v(f_v) \cdot l_{v'}(f_{v'}) \neq 0$ for neighboring vertices v and v' , then $l_w = \pm l_{-w}$ by condition (E) in Theorem 2.3, where the edge $w \in W$ goes from v to v' . Thus, the corresponding horizontal surfaces in the blocks $M_v, M_{v'}$ can be glued on the separating torus $T_{|w|}$ (after a slight deformation if necessary). This gives a surface S' horizontally embedded into the union of all maximal blocks M_v with $l_v(f_v) \neq 0$.

Now, assume that $l_v(f_v) = 0$ for some vertex $v \in V$. Moreover, we may assume that $l_{v'}(f_{v'}) \neq 0$ for a neighboring vertex v' . Consider an edge $w \in W$ from v to v' . The compatibility condition $|l_{-w}(f_w)| \leq l_w(f_w) = 0$ implies that the horizontal surface $S' \cap M_{v'}$ meets the torus $T_{|w|}$ by a collection of circles parallel to the Seifert fibers of $M_{v'}$. By the parity condition, the number of these circles is even. It is not difficult to close these circles by vertically embedded annuli not parallel to the block boundary in M_v that are pairwise disjoint (all edges $w \in \partial v$ are taken into account). We leave the details to the reader. Obviously, this procedure gives a closed surface embedded into M and such that the intersection of it with any block either is horizontal, or consists of

vertical annuli, or is empty. It is known (see [RW, Lemma 3.4]) that any such embedding is π_1 -injective. \square

This completes the proof of Theorems 2.3 and 3.1 for all properties except the virtual ones, (VE) and (VF).

5.4. Virtual properties: necessary conditions. We start the proof of Theorems 2.3 and 3.1 for the virtual properties (VE) and (VF). The proof of the “only if” part of the corresponding statements is based on the existence of characteristic coverings for graph-manifolds and on the Descent lemma (Lemma 5.16). This lemma also implies the existence of an NPC-metric on a graph-manifold if some finite cover of it carries an NPC-metric. This fact was proved earlier by another method in [KL] for a wider class of manifolds.

5.4.1. *Descent lemma.* Let $s \geq 1$ be an integer. A covering $p : \tilde{T} \rightarrow T$ of tori is said to be *s-characteristic* if the image of the induced homomorphism is $p_*(H_1(\tilde{T}; \mathbb{Z})) = s \cdot H_1(T; \mathbb{Z})$. We say that a covering $\tilde{M} \rightarrow M$ of manifolds of class \mathfrak{M} is *s-characteristic* if its restriction to each JSJ-torus in \tilde{M} is *s-characteristic*.

The existence of *s-characteristic* coverings of a manifold $M \in \mathfrak{M}$ with an arbitrary multiple s of some s_0 was proved in [LW] (by arguments in the spirit of the proof of Lemma 5.6); see also [N2]. Furthermore, this covering can be pushed through any given finite covering $\widehat{M} \rightarrow M$.

Characteristic coverings are good for the reason that they make it easy to trace the behavior of intersection numbers and charges. Namely, let $\tilde{M} \rightarrow M$ be an *s-characteristic* covering of a manifold of class \mathfrak{M} . Denote by $p : \tilde{\Gamma} \rightarrow \Gamma$ the induced map of the graphs $\tilde{\Gamma} = \Gamma_{\tilde{M}}$, $\Gamma = \Gamma_M$. The restriction $\tilde{M}_{\tilde{v}} \rightarrow M_v$ of the covering to any maximal block $\tilde{M}_{\tilde{v}} \subset \tilde{M}$ has the degree $d_{\tilde{v}} \cdot s^2$, and the number $s^2 \cdot \sum_{p(\tilde{v})=v} d_{\tilde{v}}$ is independent of the vertex $v \in V$ of the graph Γ and is equal to the degree of the covering $\tilde{M} \rightarrow M$.

We assume that an orientation of M is fixed, together with fiber orientations of its maximal blocks, and for \tilde{M} the orientations induced by the covering are chosen (this can be done because M contains no Klein bottles). The characteristic property implies that the intersection numbers for the edges $\tilde{w} \in \tilde{W}$ of $\tilde{\Gamma}$ and $w = p(\tilde{w}) \in W$ of Γ coincide, $b_{\tilde{w}} = b_w$. For the charges of $\tilde{v} \in \tilde{V}$ and $v = p(\tilde{v}) \in V$ we have

$$k_{\tilde{v}} = d_{\tilde{v}} \cdot k_v,$$

which is easy to deduce from the fact that every boundary torus T_w of the block M_v is *s-characteristically* covered by $d_{\tilde{v}}$ tori of the block $\tilde{M}_{\tilde{v}}$, so that their contribution to the charge $k_{\tilde{v}}$ is $d_{\tilde{v}}$ times the contribution of T_w to the charge k_v (for the details, see [LW, N2]). These two properties immediately imply the following lemma.

Lemma 5.15. *For the graphs $\tilde{\Gamma} = \Gamma_{\tilde{M}}$, $\Gamma = \Gamma_M$ of manifolds M , $\tilde{M} \in \mathfrak{M}$, consider the map $p : \tilde{\Gamma} \rightarrow \Gamma$ induced by a characteristic covering $\tilde{M} \rightarrow M$. Then for any solution (a, γ) of the BKN-equation over M the functions $\tilde{a} = a \circ p$, $\tilde{\gamma} = \gamma \circ p$ form a solution of the BKN-equation over \tilde{M} . \square*

Furthermore, only these two properties of characteristic coverings will be used in the proof of the following Descent lemma.

Lemma 5.16. *Let $\tilde{M} \rightarrow M$ be a characteristic covering of a manifold $M \in \mathfrak{M}$. If the BKN-equation over \tilde{M} has a compatible solution satisfying one of the conditions (VE), (VF), and (NPC) in Theorem 3.1, then the BKN-equation over M has a compatible solution satisfying the same condition.*

Proof. We use the notation introduced above for characteristic coverings. For every vertex $v \in V$ (every vertex of the graph of M) we define a quadratic form D_v on $\mathbb{R}^{\tilde{V}}$ by the formula

$$(D_v x, x) = \sum_{p(\tilde{v})=v} d_{\tilde{v}} \cdot x_{\tilde{v}}^2, \quad x \in \mathbb{R}^{\tilde{V}}.$$

Let $(\tilde{a}, \tilde{\gamma})$ be a compatible solution of the BKN-equation over the cover \tilde{M} . For every edge $w \in W$ (every edge of the graph of M) we define a quadratic form G_w on $\mathbb{R}^{\tilde{V}}$ by putting

$$(G_w x, x) = \sum_{p(\tilde{w})=w} \tilde{\gamma}_{\tilde{w}} x_{\tilde{w}^-} x_{\tilde{w}^+}, \quad x \in \mathbb{R}^{\tilde{V}}.$$

The condition $|\tilde{\gamma}_{\tilde{w}}| \leq 1$ implies that $(G_w x, x)^2 \leq (D_v x, x)(D_{v'} x, x)$ for every $x \in \mathbb{R}^{\tilde{V}}$, where $v = w^-$ is the initial vertex of the edge w and $v' = w^+$ its terminal vertex. To deduce this inequality, it suffices to apply the Cauchy–Schwartz inequality twice and to use the fact that the number of edges $\tilde{w} \in \partial\tilde{v}$ projected onto an edge w , $p(\tilde{w}) = w$, is equal to $d_{\tilde{v}}$ for every vertex $\tilde{v} \in \tilde{V}$ with $p(\tilde{v}) = v$.

Using the relations for the intersection numbers and the charges of M and of its cover \tilde{M} , and the BKN-equation

$$k_{\tilde{v}} \tilde{a}_{\tilde{v}} = \sum_{\tilde{w} \in \partial\tilde{v}} \frac{\tilde{\gamma}_{\tilde{w}}}{|b_{\tilde{w}}|} \tilde{a}_{\tilde{w}^+}, \quad \tilde{v} \in \tilde{V},$$

for $(\tilde{a}, \tilde{\gamma})$, we obtain

$$k_v (D_v \tilde{a}, \tilde{a}) = \sum_{w \in \partial v} \frac{1}{|b_w|} (G_w \tilde{a}, \tilde{a}), \quad v \in V.$$

Thus, the functions $a : V \rightarrow \mathbb{R}$ and $\gamma : W \rightarrow \mathbb{R}$ defined by $a_v = (D_v \tilde{a}, \tilde{a})^{1/2}$ (the form D_v is nonnegative) and

$$\gamma_w = \frac{(G_w \tilde{a}, \tilde{a})}{a_{w^-} \cdot a_{w^+}} \quad \text{if } a_{w^-} \cdot a_{w^+} \neq 0$$

and $\gamma_w = 0$ if $a_{w^-} \cdot a_{w^+} = 0$, make up a compatible solution of the BKN-equation for M : $|\gamma_w| \leq 1$ for all $w \in W$.

To complete the proof, we observe that the length function a is positive if so is \tilde{a} , and the angle function γ is symmetric if so is $\tilde{\gamma}$ ($\tilde{\gamma}_{-\tilde{w}} = \tilde{\gamma}_{\tilde{w}}$ for all $\tilde{w} \in \tilde{W}$). Furthermore, if $|\tilde{\gamma}| < 1$, then $|\gamma| < 1$. Consequently, each of the properties (VE), (VF), and (NPC) (see Theorem 3.1) of the BKN-equation over \tilde{M} implies the corresponding property of the BKN-equation over M . □

5.4.2. Solvability of the BKN-equation.

Proposition 5.17. *Each of the properties (VE) and (VF) of a manifold $M \in \mathfrak{M}$ implies that the BKN-equation over M has a compatible solution satisfying the corresponding condition of Theorem 3.1.*

Proof. Suppose a manifold $M \in \mathfrak{M}$ has one of the properties (VE), (VF). There is a finite covering $\tilde{M} \rightarrow M$ such that the manifold \tilde{M} has the corresponding property (E) or (F). We may assume that the covering is characteristic. Proposition 5.9 and the fact that Theorems 2.3 and 3.1 are equivalent show that the BKN-equation over \tilde{M} has a compatible solution $(\tilde{a}, \tilde{\gamma})$ that satisfies the corresponding property (E) of (F) in Theorem 3.1. In any case, the angle function $\tilde{\gamma}$ is symmetric, and in the (F) case the length function \tilde{a} is positive. Now, Lemma 5.16 provides the required solution of the BKN-equation over M . □

5.5. Implementing the virtual properties. Here we prove that the conditions of Theorem 3.1 are sufficient to implement the virtual properties. We start with the key Ascent lemma.

5.5.1. *Ascent lemma.* Let $g : S \rightarrow M$ be a horizontal immersion of a closed surface S in a manifold M of class \mathfrak{M} , and let \mathcal{T} be the JSJ-surface in M . Its preimage $\mathcal{T}_g = g^{-1}(\mathcal{T})$ is a finite disjoint collection of simple, closed, and noncontractible curves on S . We may assume that the image of every curve $T \subset \mathcal{T}_g$ is a simple closed curve $g(T)$ on the corresponding torus in \mathcal{T} .

Let Γ_g be the graph dual to the decomposition defined by \mathcal{T}_g on the surface S , $S = \bigcup_{v \in V_g} S_v$. In other words, the vertex set V_g of the graph Γ_g is the set of connected components of the splitting $S|\mathcal{T}_g$ (which are called the blocks S_v of S), and the set of (nonoriented) edges can be identified with the collection \mathcal{T}_g . The set of oriented edges of the graph Γ_g is denoted by W_g , and for the edges $w \in W_g$ we shall use the same agreements and notation as for the oriented edges $w \in W$ of the graph $\Gamma = \Gamma_M$. The immersion g induces a map $\tilde{g} : \Gamma_g \rightarrow \Gamma$ of graphs, which takes vertices to vertices and edges to edges.

Fixing an orientation of the manifold M , fiber orientations of its maximal blocks, and orientations of the curves $T \subset \mathcal{T}_g$, to each edge $w \in W_g$ we assign the number $d_w = |[g(T)] \wedge_{\tilde{g}(w)} f_{\tilde{g}(w)}|$, where the curve $T \subset \mathcal{T}_g$ corresponds to the nonoriented edge $(w, -w)$ of the graph Γ_g . In other words, d_w is equal to the degree with which the curve T is mapped onto the corresponding boundary component of the base orbifold $\mathcal{O}M_{v'}$ of the block $M_{v'} \subset M$ to which the block S_w of S corresponding to the initial vertex of w is mapped under the composition $\pi_{v'} \circ g$, where $\pi_{v'} : M_{v'} \rightarrow \mathcal{O}M_{v'}$ is the canonical projection.

All oriented cycles in Γ_g that we consider below consist of oriented edges of the graph Γ_g (with compatible orientations).

Lemma 5.18. *The immersion g is a virtual embedding, i.e., $g = \tilde{g} \circ p$ for some finite covering $p : \tilde{M} \rightarrow M$ and a horizontal embedding $\tilde{g} : S \rightarrow \tilde{M}$ if and only if the following condition of cyclic balance is fulfilled:*

$$\prod_{w \in c} \frac{d_w}{d_{-w}} = 1$$

for every oriented cycle c in the graph Γ_g .

Sketch of the proof. Since the maximal blocks of M are Seifert bundles, it is easy to show that the cover space of the covering associated with the subgroup $g_*(\pi_1(S)) \subset \pi_1(M)$ is homeomorphic to the product $S \times \mathbb{R}$. Next, we may assume that the covering $q : S \times \mathbb{R} \rightarrow M$ itself coincides with the immersion g over the zero section $S \times \{0\} \subset S \times \mathbb{R}$. In what follows, we identify S with $S \times \{0\}$.

For every curve $T \subset \mathcal{T}_g$, the infinite cylinder $T \times \mathbb{R} \subset S \times \mathbb{R}$ covers the corresponding torus $T_{|\tilde{g}(w)|} \subset \mathcal{T}$, where $T = (w, -w)$. Since $g(T) \subset T_{|\tilde{g}(w)|}$ is a simple closed curve, the preimage $q^{-1}(g(T)) \cap T \times \mathbb{R}$ consists of countably many parallel copies of the curve T . These copies can be labeled by integers $n \in A \subset \mathbb{Z}$ so that any finite subcylinder in $T \times \mathbb{R}$ bounded by T and by its copy $T_{w,n}$ is mapped into the torus $T_{|\tilde{g}(w)|}$ with degree $|n| \cdot d_w$. We shall say that $T_{w,n}$ is the copy of level n . Then $T = T_{w,0}$.

The curve T is a boundary component of a block $S_v \subset S$, where $w \in \partial v \subset V_g$. Then its parallel copy $T_{w,n}$ is a boundary component of a copy $S'_v \subset S_v \times \mathbb{R}$ of the block S_v , and $q(S'_v) = g(S_v)$. Using the Seifert bundle structure over the block $M_{\tilde{g}(v)} \subset M$ and the fact that the degree is homotopy invariant, it is easy to check that the level of every component of the boundary $\partial S'_v$ is one and the same. Thus, we have a collection

$S_{v,n}$, $n \in A$, of copies of the block S_v that lie in the cylinder $S_v \times \mathbb{R}$ with images $q(S_{v,n}) = g(S_v)$ for every vertex $v \in V_g$. Furthermore, for adjacent vertices $v = w^-$, $v' = w^+$ and copies $S_{v,n}$, $S_{v',n'}$ of the corresponding blocks S_v , $S_{v'}$ adjacent along a common boundary component, we have a key relation

$$(deg) \quad n \cdot d_w = n' \cdot d_{-w},$$

in which the left- and the right-hand sides are different representations of the degree of the map of the corresponding cylinder in the torus $T_{|\tilde{g}(w)|}$.

If g is a virtual embedding, then, in its turn, the manifold \tilde{M} involved in the corresponding covering $\tilde{M} \rightarrow M$, which fibers over the circle, can be covered by the product $S \times S^1$, because the holonomy of the gluing of the manifold \tilde{M} from the cylinder $S \times [0, 1]$ has a finite order (up to isotopy). Therefore, the covering $q : S \times \mathbb{R} \rightarrow M$ can be pushed through the covering $S \times S^1 \rightarrow \mathbb{R}$, and the preimage $q^{-1}(g(S))$ contains a parallel copy S' of the surface S . Its block decomposition is $S' = \bigcup_{v \in V_g} S_{v,n(v)}$. By the said above, for every edge $w \in W_g$ we have $n_{w^-} \cdot d_w = n_{w^+} \cdot d_{-w}$. Thus, for every oriented cycle $c \subset \Gamma_g$ we obtain

$$\prod_{w \in c} \frac{d_w}{d_{-w}} = \prod_{w \in c} \frac{n_{w^+}}{n_{w^-}} = 1,$$

because $w^+ = w'^-$ for each pair of consecutive edges w, w' of the cycle c .

Conversely, suppose that for a horizontal immersion $g : S \rightarrow M$ the cyclic balance condition is fulfilled. Together with formula (deg), this makes it possible, starting with some copy $S_{v,n}$, $n \in A \setminus \{0\}$, and using the obvious continuation method, to find a copy of the surface S distinct from S and lying in the preimage $q^{-1}(g(S)) \subset S \times \mathbb{R}$. This implies the existence of the required covering $p : \tilde{M} \rightarrow M$ and of an embedding $\tilde{g} : S \rightarrow \tilde{M}$ with $\tilde{g} \circ p = g$. □

5.5.2. Compatible symmetric solutions. Here we formulate a proposition that will be employed to complete the proof of Theorems 2.3 and 3.1.

A compatible symmetric solution (a, γ) of the BKN-equation over a manifold $M \in \mathfrak{M}$ is called an NPC-solution if it satisfies the condition (NPC) of Theorem 3.1, i.e., the length function is positive, $a > 0$, and for the angle function we have $|\gamma| < 1$.

Proposition 5.19. *If the BKN-equation over $M \in \mathfrak{M}$ has a compatible symmetric solution (a, γ) , then the same equation has a compatible symmetric solution (a', γ') that satisfies one of the following two conditions:*

- (1) (a', γ') is an NPC-solution;
- (2) the angle function γ' takes values in $\{0, \pm 1\}$, and $|\gamma'_w| = 1$ for every edge $w \in W$ with $a'_{w^-} \cdot a'_{w^+} \neq 0$.

Furthermore, if the length function is positive, $a > 0$, then the BKN-equation has a compatible symmetric solution (a', γ') satisfying (1) or

- (3) the length function is positive, $a' > 0$, and the angle function γ' takes values in $\{0, \pm 1\}$.

We prove this proposition in Subsection 5.6.

Remark 5.20. To implement the virtual properties (VE) and (VF), we need rational solutions of the BKN-equation. In cases (2) and (3), the existence of a rational solution (a', γ') follows from the first part of Proposition 5.10. Therefore, the problem of rationality reduces to the case of an NPC-solution. There are manifolds $M \in \mathfrak{M}$ for which the BKN-equation has an NPC-solution, but has no rational NPC-solution. However, as will be shown in Subsection 5.8 (see Proposition 5.26), there always exists a characteristic

cover of such a manifold for which there is a rational NPC-solution. This suffices for our purposes because we consider virtual properties (see Remark 5.27).

5.5.3. *Implementing (VF)*. Here we complete the proof of Theorem 3.1 (VF). Assume that the BKN-equation over a manifold $M \in \mathfrak{M}$ has a compatible symmetric solution (a, γ) with positive length function $a > 0$. We show that M is virtually fibered over the circle.

Proposition 5.19 and Remark 5.20 allow us to assume, taking an appropriate characteristic covering if necessary, that (a, γ) is a rational solution. By Proposition 5.2, there are compatible integral cohomology classes $\{l_v : v \in V\}$ on M such that $l_v(f_v) > 0$ for all vertices $v \in V$ and $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$ for every edge $w \in W$ (we assume that the fiber orientations of maximal blocks and an orientation of M are fixed). For edges $w \in W$ with $|\gamma_w| < 1$ we have $|l_{-w}(f_w)| < l_w(f_w)$, and for the edges with $|\gamma_w| = 1$ we have $l_{-w} = \pm l_w$, by the symmetry property of γ and the compatibility of $\{l_v : v \in V\}$.

Using this collection, we find a horizontal immersion $g : S \rightarrow M$ as in Subsection 5.3.4. Namely, for every edge $w \in W$ we put $l_w^+ = l_w + l_{-w}$, $l_w^- = l_w - l_{-w}$. Then the elements $c_w^+, c_w^- \in H_1(T_{|w|}; \mathbb{Z})$ dual to l_w^+, l_w^- satisfy the conditions of Lemma 5.6. Note that the numbers $\alpha_w^\pm = c_w^\pm \wedge_w f_w = l_w(f_w) \pm l_{-w}(f_w)$ are positive for the edges $w \in W$ with $|\gamma_w| < 1$, and for the edges with $|\gamma_w| = 1$ we have $\alpha_w^- \cdot \alpha_w^+ = 0$. Furthermore, $\alpha_w^- \cdot \alpha_w^+ = 0$ if and only if $\alpha_{-w}^- \cdot \alpha_{-w}^+ = 0$ by the symmetry of γ .

Recall that $c_{-w}^+ = c_w^+$, $c_{-w}^- = -c_w^-$, and if $\alpha_w^+ \cdot \alpha_w^- = 0$, then $c_{-w} = \pm c_w$. Therefore, the horizontal immersions $g_v : S_v \rightarrow M_v$ provided by Lemma 5.6 for every maximal block M_v can easily be pasted together on every gluing torus $T_{|w|}$ at the expense of an appropriate multiplicity d universal for all blocks, and they give a horizontal immersion $g : S \rightarrow M$ as required.

From the construction of Lemma 5.6 it follows that the graph Γ_g defined for the immersion g of Subsection 5.5.1 has the same vertex set as the graph $\Gamma = \Gamma_M$, i.e., $V_g = V$, and its edge set is twice that of Γ , $W_g = 2W$, in the sense that every edge $w \in W$ corresponds to two edges $w_\pm \in W_g$ with the same endpoints as w . The number d_{w_\pm} (defined in Subsection 5.5.1) is equal to $d \cdot \alpha_w^\pm = d \cdot (l_w(f_w) \pm l_{-w}(f_w))$ if $\alpha_w^+ \cdot \alpha_w^- \neq 0$, and $d_{w_-} = d_{w_+} = d \cdot l_w(f_w)$ if $\alpha_w^+ \cdot \alpha_w^- = 0$, where the factor $d > 0$ is one and the same for all $w \in W$. The symmetry condition $l_{-w}(f_w) \cdot l_{-w}(f_{-w}) = l_w(f_w) \cdot l_w(f_{-w})$, $w \in W$, implies that

$$\frac{d_w}{d_{-w}} = \frac{l_v(f_v)}{l_{v'}(f_{v'})}$$

for all $w \in W_g$, where v is the beginning and v' is the end of w .

This immediately yields the cyclic balance condition for g . Consequently, by Lemma 5.18, g is a virtual horizontal embedding. Thus, the manifold M virtually fibers over the circle. \square

5.5.4. *Implementing (VE) = (NPC) \cup (E)*. Here we complete the proof of Theorem 3.1 (VE). Assume that the BKN-equation over a manifold $M \in \mathfrak{M}$ has a compatible symmetric solution (a, γ) . We show that M has property (VE), and that (VE) = (NPC) \cup (E). By Proposition 5.19, we may assume that the solution (a, γ) satisfies one of conditions (1), (2). In the first case, taking an appropriate characteristic covering if necessary, we also assume (by Remark 5.20) that the solution is rational. In the second case, we can also assume (by the first part of Proposition 5.10) that the solution is rational (there is no need to take any finite covering in this case, which is important for the proof of the fact that (VE) = (NPC) \cup (E)).

In the first case, (a, γ) is an NPC-solution. By Subsection 5.5.3, the manifold M virtually fibers over the circle; in particular, it has property (VE). In the second case, the solution (a, γ) satisfies condition (E) in Theorem 3.1. By Subsection 5.3.6, the manifold M has property (E), and hence property (VE). This also proves that $(VE) = (NPC) \cup (E)$. \square

5.5.5. *Historical remarks.* Lemma 5.18 (Ascent lemma) is Theorem 2.3 in [RW], where the authors used this result to give an example showing that a horizontal immersion of a closed surface in a graph-manifold may fail to be a virtual embedding. In [Sv1] it was discovered that the symmetry condition on a solution of the BKN-equation implies the cyclic balance condition for the corresponding horizontal immersion (Subsection 5.5.3), so that the Ascent lemma can be applied to show that a graph-manifold virtually fibers over the circle.

One and the same symmetry condition on the solutions of the BKN-equation ensures two things: isometry of the gluings of maximal blocks in the course of constructing an NPC-metric on a graph-manifold and, at the same time, the cyclic balance condition in the course of finding a virtual embedding of a surface in a graph-manifold. This fact looks somewhat mystic.

5.6. Ideals. Here we prove Proposition 5.19 by using global methods. The notion of an ideal, the decomposition principle, and the lemma on polarized discharge are keys for that. These notions and results play an important role also in the proofs of the spectral criteria in § 6.

A labeled graph $X = (\Gamma, |B|, K)$ is called an ideal if the BKN-equation over every connected labeled graph X' containing X as a subgraph has an NPC-solution. We describe the most important ideals in the following two lemmas.

Lemma 5.21. *Assume that all charges of a labeled graph X have one and the same sign different from 0. If*

$$(A_X^+ a, a) := \sum_{v \in V} |k_v| a_v^2 - \sum_{w \in W} \frac{a_w^- a_w^+}{|b_w|} < 0$$

for some nonnegative function $a : V \rightarrow \mathbb{R}$, then X is an ideal.

Lemma 5.22. *Let X be a labeled graph over the circle with an odd number $p \geq 1$ of edges, or over a linear graph Γ with $p \geq 2$ edges. Assume that in the circle case at most one vertex has a nonzero charge, and if $p = 1$, i.e., in the case of a loop, we require that $|k \cdot b| < 2$, where k is the charge and b the intersection number. In the case of a linear graph, we assume that only extreme vertices have nonzero charges, which are of the same sign if p is odd, and of different signs if p is even. Then X is an ideal.*

We prove these lemmas in Subsection 5.7. Here we use them to prove the following sufficient condition for the existence of NPC-solutions. Recall that the operator $H_M : \mathbb{R}^V \rightarrow \mathbb{R}^V$ was defined in Subsection 4.1.3.

Lemma 5.23. *Let X be the labeled graph of an oriented manifold $M \in \mathfrak{M}$. If the operator H_M has a negative eigenvalue, or the function $s : U \rightarrow \{0, \pm 1\}$ occurring in its definition is zero, then the BKN-equation over X has an NPC-solution.*

Proof. If the graph of sign components of X is not bipartite, then the graph X contains one of the ideals described in Lemma 5.22. Thus, we assume that the graph of sign components is bipartite.

If the charges k_v are zero for all vertices $v \in V$, then the functions $a \equiv 1$, $\gamma \equiv 0$ form an NPC-solution (a, γ) of the BKN-equation. Thus we may also assume that the function $s : U \rightarrow \{0, \pm 1\}$ takes values in $\{\pm 1\}$.

Since the operator H_M has a negative eigenvalue, there is a sign component $u \in U$ such that the operator $H_u = D_u - J_u$ has a negative eigenvalue. If $u \in U_0$, i.e., if u is a vertex with zero charge, then all edges of the graph Γ_u are loops with common vertex u . In this case, the graph X contains an ideal by Lemma 5.22. Thus, we assume that $u \notin U_0$. If $s(u) \text{sign}(u) = -1$, where $\text{sign}(u)$ is the sign of the charges of the vertices in the component u , then the graph X contains one of the ideals described in Lemma 5.22, because there is a component u_0 with $s(u_0) \text{sign}(u_0) = 1$. The proof is complete in this case. Thus, we can assume that $s(u) \text{sign}(u) = 1$. In this case, the operator H_u coincides with the operator A_u^+ , and the component u is an ideal by Lemma 5.21. \square

Proof of Proposition 5.19. Consider the quadratic form $F : \mathbb{R}^V \rightarrow \mathbb{R}$ defined by $F(x) = (H_M x, x)$. Lemma 5.23 allows us to assume that F is nonnegative on \mathbb{R}^V , and the function $s : U \rightarrow \{0, \pm 1\}$ occurring in the definition of the operator H_M takes values in $\{\pm 1\}$.

On the other hand, for a compatible symmetric solution (a, γ) we have $F(a) \leq 0$ by Lemma 4.3. Hence, $F(a) = 0$, and $s(u)\gamma_w = 1$ for all sign components $u \in U$ and all edges $w \in W_u$ with $a_{w^-} \cdot a_{w^+} \neq 0$. Moreover, the derivative $\partial_v F(a)$ is zero for all $v \in V$. Since

$$\frac{1}{2} \partial_v F(a) = s(u)k_v a_v - \sum_{w \in \partial_0 v} \frac{a_{w^+}}{|b_w|},$$

where $\partial_0 v = \partial v \cap W_u$, and since the derivative vanishes and the length function a is nonnegative, we see that if $a_v = 0$ for some vertex $v \in u$, then the function a vanishes at all vertices of the component u . Since $a \not\equiv 0$, there is a component $u \in U$ on which a is positive. Using the BKN-equation for (a, γ) and recalling again that the derivative vanishes, we obtain

$$(G) \quad \sum_{w \in \partial v \setminus \partial_0 v} \frac{\gamma_w}{|b_w|} a_{w^+} = 0$$

for all $v \in u$.

We define functions $a' : V \rightarrow \mathbb{R}$ and $\gamma' : W \rightarrow \{0, \pm 1\}$ as follows: $a'_v = a_v$ for all $v \in u$ and $a'_v = 0$ for all $v \notin u$; $\gamma'_w = \gamma_w$ for all $w \in W_u$ and $\gamma'_w = 0$ for all $w \notin W_u$. Identity (G) shows that (a', γ') is a compatible symmetric solution of the BKN-equation over X , and from the above we see that $|\gamma'_w| = 1$ for all edges $w \in W$ with $a'_{w^-} \cdot a'_{w^+} \neq 0$. This is case (2).

Now, assume that $a_v > 0$ for all vertices $v \in V$. Let $a' = a$, and let $\gamma'_w = \gamma_w$ for all components $u \in U$ and all edges $w \in W_u$, and $\gamma'_w = 0$ for all edges w connecting vertices from different sign components. Then, as above, (a', γ') is a compatible symmetric solution of the BKN-equation over X with positive length function $a' > 0$ and with angle function γ' taking values in $\{0, \pm 1\}$. This is case (3). \square

5.7. Decomposition principle. Here we prove Lemmas 5.21 and 5.22. Our main tool is decomposition of labeled graphs $X = (\Gamma, |B|, K)$ into dipoles.

5.7.1. *Dipoles.* The graph with two vertices v, v' and one nonoriented edge $(w, -w)$ between them, $w \in \partial v$, will be called *the dipole*. The BKN-equation over a labeled dipole $D = (k_w, k_{-w}; b_w)$ is

$$k_w a_w = \frac{\gamma_w}{|b_w|} a_{-w}, \quad k_{-w} a_{-w} = \frac{\gamma_{-w}}{|b_w|} a_w,$$

where k_w, k_{-w} are the charges of the vertices v, v' , respectively, and $b_w \in \mathbb{Z} \setminus \{0\}$ is the intersection number. We note that labeled dipoles (as well as other labeled graphs) considered here are not necessarily associated with graph-manifolds.

All NPC-solutions $(a_w, a_{-w}; \gamma_w)$ of this equation can be described as follows. The positive numbers a_w, a_{-w} are defined up to a common positive factor. The condition $a_w \cdot a_{-w} \neq 0$ implies that $\gamma_w^2 = k_w k_{-w} b_w^2$. This means, in particular, that for different signs of the charges, $k_w k_{-w} < 0$, or for $k_w k_{-w} b_w^2 \geq 1$, the BKN-equation of a dipole has no NPC-solution. Thus, we assume that $0 \leq k_w k_{-w} b_w^2 < 1$.

For $k_w k_{-w} > 0$, the sign of γ_w coincides with that of the charges k_w, k_{-w} , and

$$(dip) \quad \left(\frac{a_w}{a_{-w}} \right)^2 = \frac{k_{-w}}{k_w}.$$

If $k_w = k_{-w} = 0$, the equation puts no restriction on (a_w, a_{-w}) . If one of the charges vanishes and the other does not, then obviously there is no NPC-solution.

5.7.2. *Conjunction and decomposition of labeled graphs.* For labeled graphs X', X'' (the case where $X' = X''$ is not excluded), under conjunction operation some of their vertices are glued and their charges are added (notation: $X = X' + X''$). This operation corresponds to toral summation of oriented graph-manifolds along some maximal blocks (for the details, see [BK1]). The converse operation is called a decomposition of a labeled graph.

Roughly speaking, the decomposition principle claims that the compatible solutions of the BKN-equation are parameterized (with some reservations) by decompositions of the labeled graph in question into labeled dipoles satisfying appropriate conditions.

Here is the precise statement. The decomposition principle says the following:

(a) if $(a', \gamma'), (a'', \gamma'')$ are compatible solutions of the BKN-equations over labeled graphs X', X'' , respectively, $X = X' + X''$ is the conjunction along some vertices, and the values of the functions a', a'' at the glued vertices coincide (if $X' = X''$, we require $(a', \gamma') = (a'', \gamma'')$), then (a, γ) is a compatible solution of the BKN-equation over the labeled graph X , where the functions a, γ are determined naturally by the solutions $(a', \gamma'), (a'', \gamma'')$: the values of a, γ coincide with the values of the corresponding functions of the solution that is defined along the vertices or edges under consideration;

(b) conversely, any compatible solution (a, γ) of the BKN-equation over a labeled graph X with positive length function a canonically determines a decomposition of the graph $X = \sum_{e \in E} D_e$ into labeled dipoles $D_e = (k_w, k_{-w}; b_w)$, where $e = (w, -w)$ is a nonoriented edge, in accordance with the following conditions: their intersection indices b_w are the same as in the graph X ; for every edge $w \in W$ the charge of the vertex $v = w^-$ of the decomposition dipole $D_{(w, -w)}$ is defined by $k_w = \frac{\gamma_w}{|b_w|} \cdot \frac{a_{w^+}}{a_{w^-}}$. Furthermore, the solution (a, γ) restricted to any decomposition dipole is a compatible solution of the BKN-equation for that dipole.

Here we have used the notation of an edge w as the subscript in the notation of the vertex charge of a dipole because this does not lead to any ambiguity and distinguishes the resulting charge from the corresponding charge of X . Moreover, this is convenient for writing the relation $k_v = \sum_{w \in \partial v} k_w, v \in V$, between the charge k_v of X and the charges k_w of the dipoles into which X is decomposed by a compatible solution.

5.7.3. *Vertex balance.* The identity $k_v = \sum_{w \in \partial v} k_w, v \in V$, mentioned above will be called the equation of vertex balance. This identity is necessary for a compatible solution of the BKN-equation to determine a decomposition of a labeled graph into dipoles. Moreover, the vertex balance equation together with the BKN-equations over the dipoles of the decomposition $X = \sum_{e \in E} D_e$ for the collections $(a_w, a_{-w}; \gamma_w, \gamma_{-w}), e = (w, -w)$, imply the BKN-equation over X for (a, γ) if we can adjust the collections (a_w, a_{-w}) by positive factors so that their elements coincide at the glued vertices and hence define a required length function a . In particular, if the graph Γ of a labeled

graph $X = (\Gamma, |B|, K)$ is a tree, then any decomposition $X = \sum_{e \in E} D_e$ into labeled dipoles $D_e = (k_w, k_{-w}; b_w)$, $e = (w, -w)$, such that each of them admits a compatible symmetric solution $(a_w, a_{-w}; \gamma_w)$ for its BKN-equation with positive a_w, a_{-w} , gives rise to a compatible symmetric solution (a, γ) of the BKN-equation over X . Namely, $\gamma(w) = \gamma_w$ and $a_v/a_{v'} = a_w/a_{-w}$ for every edge $w \in W$, where $v = w^-$ is the beginning of w , and $v' = w^+$ is the end of w . Since there is no nontrivial circuit, the relations $a_v/a_{v'} = a_w/a_{-w}$ allow us to define inductively a function $a : V \rightarrow \mathbb{R}$ the restriction of which to every dipole D_e differs from (a_w, a_{-w}) , $e = (w, -w)$, possibly by a common positive factor only. Then (a, γ) is the required solution of the BKN-equation, in accordance with item (a) of the decomposition principle.

5.7.4. Polarized discharge of a vertex.

Lemma 5.24. *Let $X = (\Gamma, |B|, K)$ be a labeled graph with a simply connected graph Γ . For any vertex $v \in V$ of it, there are numbers $\delta \in \{0, \pm 1\}$ and $\varepsilon > 0$ such that if $|k'_v| < \varepsilon$, $\text{sign}(k'_v) = \delta$, and the charges of the remaining vertices are the same as in X , $k'_{v'} = k_{v'}$, then the BKN-equation over the labeled graph $X' = (\Gamma, |B|, K')$ has an NPC-solution.*

Proof. We argue by induction on the number of vertices. For a dipole, the claim follows immediately from the description of the NPC-solutions of its BKN-equation. In the general case, if v is an interior vertex of the tree Γ , then X can be represented as the conjunction along v of a finite collection of trees such that the number of vertices of each of them is less than that of X . Then the assertion follows from the induction hypothesis and item (a) of the decomposition principle, because the graph Γ is a tree.

Now, suppose that for the vertex v there is a unique neighboring vertex v' in Γ . Then X can be represented as the conjunction along v' of a dipole D with vertices v, v' and a labeled tree X_1 . By the induction hypothesis, we may assume that the charge $k_{v'}^1$ of the vertex v' of X_1 is so close to zero and the sign of it is such that the BKN-equation over X_1 has an NPC-solution. Then the dipole charge k_v^D is determined from the vertex balance $k_{v'}^D + k_{v'}^1 = k_{v'}$. But whatever the charge k_v^D may be, it is always possible to find a sign $\delta \in \{0, \pm 1\}$ so that if the dipole charge k_v^D at v has sign δ and is sufficiently close to zero, then the BKN-equation over D has an NPC-solution. As before, now the claim follows from the decomposition principle. \square

For the proof of Lemmas 5.21 and 5.22, we shall use the following characteristic of ideals.

Lemma 5.25. *Suppose that a labeled graph $X = (\Gamma, |B|, K)$ has the following property: for every collection $K' \in \mathbb{R}^V$ of charges in some neighborhood in \mathbb{R}^V of K , the BKN-equation over $X' = (\Gamma, |B|, K')$ has an NPC-solution. Then X is an ideal.*

Proof. Assume that X is a subgraph of a connected labeled graph X_0 . We describe how to find an NPC-solution of the BKN-equation over X_0 . The graph X_0 can be represented as the conjunction of X' , a disjoint collection Y of trees, and a collection Z of dipoles with zero charges,

$$X_0 = X' + Y + Z.$$

Furthermore, we have $X' = (\Gamma, |B|, K')$, and for every tree Y_α in Y the charge of the conjunction vertex is sufficiently close to zero and satisfies the hypothesis of Lemma 5.24. By vertex balance, the collection K' can be chosen arbitrarily close to the initial collection $K \in \mathbb{R}^V$. Since the dipoles with zero charges put no restriction on the length function of a solution of the BKN-equation, we complete the proof by applying the decomposition principle and by recalling that the collection Y consists of trees. \square

Proof of Lemma 5.21. For definiteness, we assume that all charges of the labeled graph X are positive. Arguing by induction on the number of vertices, we assume that the assertion is true for all labeled graphs the number of vertices of which is less than that of $X = (\Gamma, |B|, K)$. By assumption, the quadratic form A_X^+ on \mathbb{R}^V has a negative eigenvalue. There is a sufficiently small neighborhood of K in \mathbb{R}^V such that for every K' in this neighborhood the form $A_{X'}^+$, $X' = (\Gamma, |B|, K')$, has a negative eigenvalue.

Consider the following family of forms $A_{X',t}^+$, $0 \leq t \leq 1$:

$$(A_{X',t}^+x, x) := \sum_{v \in V} k'_v x_v^2 - \sum_{w \in W} \frac{t}{|b_w|} x_{w^-} x_{w^+}.$$

In this family, the form corresponding to $t = 0$ is positive definite, and the form corresponding to $t = 1$ has a negative eigenvalue. Thus, there is $t_0 \in (0, 1)$ such that the form A_{X',t_0}^+ has a zero eigenvalue. Let $a_0 \in \mathbb{R}^V$ be a corresponding eigenvector. Changing the sign of $\gamma_w^0 = t_0$ for the edges w that leave the vertices at which a_0 is negative, and simultaneously changing the sign of the negative values of a_0 , we may assume that $a_0 \geq 0$. Then (a_0, γ_0) is a symmetric solution of the BKN-equation over X' , the length function a_0 is nonnegative, and for the angle function we have $|\gamma_0| < 1$. We can also assume that this solution is compatible.

Consider the labeled subgraph $X_0 \subset X'$ spanned by the vertices at which a_0 is positive. The restriction of (a_0, γ_0) to this subgraph is an NPC-solution of the BKN-equation over X_0 (obviously, this subgraph has no isolated vertex, i.e., a vertex without incident edges). It follows that $(A_{X_0}^+ a_0, a_0) < 0$ for the form $A_{X_0}^+$. Thus, this form has a negative eigenvalue.

If X_0 is a proper subgraph in X' , then by the induction hypothesis it is an ideal. Therefore, no matter whether $X_0 = X'$ or $X_0 \neq X'$, the BKN-equation over X' has an NPC-solution. By Lemma 5.25, the labeled graph X is an ideal. \square

Proof of Lemma 5.22. For a linear graph Γ , every labeled graph $X' = (\Gamma, |B|, K')$ can be represented as the conjunction of p labeled dipoles, namely, $D'_e = (k'_w, k'_{-w}; |b_w|)$, where $e = (w, -w)$ is a nonoriented edge satisfying the NPC-condition $0 < \gamma_w^2 := k'_w k'_{-w} b_w^2 < 1$ (see Subsection 5.7), provided the charge collection $K' \in \mathbb{R}^V$ is sufficiently close to K . By the decomposition principle, the BKN-equation over X' has an NPC-solution, because the graph Γ is simply connected. Now, from Lemma 5.25 it follows that X is an ideal.

In the loop case ($p = 1$), the BKN-equation over X has an NPC-solution by the condition $|k \cdot b| < 2$. Since this condition is open in the space of charges, X is an ideal.

Now, we assume that Γ is a circle with odd number $p \geq 3$ of edges. If every charge of X is zero, then the functions $a \equiv 1$, $\gamma \equiv 0$ make up an NPC-solution (a, γ) of the BKN-equation. Let X' be a perturbed graph. We may assume that there is a vertex $v_1 \in V$ with nonzero charge k'_1 . We assume further that the value $|k'_1|$ is maximal over all vertices. We decompose X' into dipoles $D'_e = (k'_w, k'_{-w}; |b_w|)$ with nonzero charges so that the following conditions are fulfilled together with the NPC-condition above: $k'_w = k'_{-w}$ for all edges $e = (w, -w)$ not incident to v_1 . For the edges $e_1 = (w_1, -w_1)$, $e_p = (w_p, -w_p)$ incident to the vertex v_1 , $v_1 = w_1^- = w_p^+$, we require that

$$\frac{k'_{w_1}}{k'_{-w_1}} = \frac{k'_{-w_p}}{k'_{w_p}}.$$

By relation (dip) in Subsection 5.7, these conditions ensure the existence of an NPC-solution of the BKN-equation over X' . It is easily seen that if K' is sufficiently close to K , then a required decomposition of the labeled circle X' into dipoles does exist because $p \geq 3$ is odd. Hence, X' is an ideal. \square

5.7.5. *Historical remarks.* The notion of an ideal, the decomposition principle, and the lemma on polarized discharge of a vertex appeared in [BK1]; Lemma 5.25 is taken from [BK2].

5.8. Symmetric rational solutions. Here, under some restrictions, we construct a rational approximation to the NPC-solutions of the BKN-equation.

Proposition 5.26. *Let $X = (\Gamma, |B|, K)$ be the labeled graph of a manifold $M \in \mathfrak{M}$ such that if the graph Γ is bipartite, then there are at least four vertices lying in one and the same part and with equal nonzero charges. Then every NPC-solution (a, γ) of the BKN-equation over X can be approximated by rational NPC-solutions.*

Remark 5.27. There are graph-manifolds $M \in \mathfrak{M}$ for which the BKN-equation has a real NPC-solution (a, γ) , and has no rational NPC-solution. As such M we can take an “irrational dipole” whose graph consists of two vertices v, v' and one (nonoriented) edge e between them, and the charges and the intersection number satisfy the conditions $0 < k_v k_{v'} b_e^2 < 1$ and $\sqrt{k_v k_{v'}} \notin \mathbb{Q}$ (at least one of the blocks of M with these properties must have singular fibers).

On the other hand, if a manifold M has a nonzero charge, then it is easy to find its characteristic covering $\widetilde{M} \rightarrow M$ such that the (connected) \widetilde{M} satisfies the conditions of Proposition 5.26. In the case where all the charges are equal to zero, the functions $a \equiv 1$ and $\gamma \equiv 0$ form an NPC-solution (a, γ) of the BKN-equation, as was already mentioned in the proof of Lemma 5.23. Therefore, while considering the virtual properties (VE) and (VF), we may assume that the NPC-solutions are rational, by passing to an appropriate characteristic covering if necessary.

The proof of Proposition 5.26 is based on the properties of the *incidence matrix* of the graph Γ , which is defined as the map $I = I(\Gamma) : V \times E \rightarrow \{0, 1, 2\}$ (E is the set of nonoriented edges of Γ) such that $I_{ve} \neq 0$ if and only if the vertex $v \in V$ and the edge $e \in E$ are incident, and $I_{ve} = 2$ for the loops e with vertex v and only for them. Also, the incidence matrix can be identified with a linear map $I : \mathbb{R}^E \rightarrow \mathbb{R}^V$, by viewing every restriction $I_v = I|_{v \times E}$ as the row corresponding to the vertex $v \in V$.

Lemma 5.28. *If a connected graph Γ is not bipartite, then the rank of the incidence matrix I is equal to the number $|V|$ of vertices. If the graph Γ is bipartite, then the rank of I is equal to $|V| - 1$.*

Proof. Assume that there is a zero linear combination of rows, $\sum_{v \in V} c_v I_v = 0$. Then for every edge $e \in E$ we have $c_v + c_{v'} = 0$, where v, v' are the (possibly, coinciding) ends of e , because the column of I corresponding to e has nonzero elements for the vertices v and v' only. It follows that if the graph Γ has a circuit with odd number of edges, i.e., it is not bipartite, then $c_v = 0$ for all vertices of that circuit, and, consequently, for all vertices $v \in V$, because Γ is connected. In this case, the rank of I is equal to $|V|$.

Now, assume that the graph Γ is bipartite, and let $V = V_0 \cup V_1$ be a decomposition of the vertex set into (nonempty) parts. Then

$$\sum_{v \in V_0} I_v - \sum_{v \in V_1} I_v = 0$$

is a unique (up to a nonzero factor) nontrivial combination of rows of I . Consequently, the rank of I is equal to $|V| - 1$. \square

There are several interpretations of the BKN-equation. For symmetric rational approximations, the following is most convenient. For a positive length function a , the

BKN-equation over X can be written in a more symmetric equivalent form:

$$k_v a_v^2 = \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} a_{w^-}, \quad v \in V.$$

If, moreover, the angle function γ is symmetric, then, in its turn, this form is equivalent to the equation

$$(+) \quad A = I(D),$$

where $I : \mathbb{R}^E \rightarrow \mathbb{R}^V$ is the incidence matrix, $A \in \mathbb{R}^V$, $A_v = k_v a_v^2$, and $D \in \mathbb{R}^E$, $D_e = \frac{\gamma_w}{|b_w|} a_{w^+} a_{w^-}$ for every edge $e = (w, -w)$.

Proof of Proposition 5.26: the graph Γ is not bipartite. In this case we have $|V| \leq |E|$, because the graph Γ is not simply connected. Since the rank of the matrix I equals $|V|$, there is a linearly independent collection E_0 of its rows, $E_0 \subset E$, $|E_0| = |V|$, and by a classical theorem of linear algebra, the solutions of the symmetrized BKN-equation (+) can be parameterized by $|V|$ linear functions L_e , $e \in E_0$, with rational coefficients depending on $|E| = |V| + (|E| - |V|)$ arguments A_v , $v \in V$, and $D_{e'}$, $e' \in E \setminus E_0$,

$$\frac{\gamma_w}{|b_w|} a_{w^+} a_{w^-} = L_e(A_v, D_{e'}), \quad e = (w, -w) \in E_0.$$

Approximating the numbers a_v , $v \in V$, and $\gamma_{w'}$, $(w', -w') \in E \setminus E_0$, by rational numbers a'_v and $\gamma'_{w'}$, respectively, we obtain a rational approximation

$$\gamma'_w = \frac{|b_w|}{a'_{w^+} a'_{w^-}} L_e(A'_v, D'_{e'})$$

of γ_w for all $(w, -w) \in E_0$. This gives a required approximation of the solution (a, γ) of the BKN-equation by compatible, symmetric, rational solutions. \square

Now, let the graph Γ be bipartite, let $V = V_0 \cup V_1$ be the decomposition of its vertex set into parts, and let $U \subset V_0$ be a subset consisting of four vertices with equal charges, $k_v = k \neq 0$ for all $v \in U$. Consider the set $Q = \{x \in \mathbb{R}^V : \sum_{v \in V_0} k_v x_v^2 = \sum_{v \in V_1} k_v x_v^2\}$. The relation $\sum_{v \in V_0} I_v = \sum_{v \in V_1} I_v$ implies that, for any solution (a, γ) of the BKN-equation (+), the length function a lies in Q ; in particular, the set Q contains points with all coordinates different from zero.

In the proof of the following lemma, we use the condition $|U| = 4$ to apply the classical result of Lagrange which says that every natural number can be represented as the sum of four squares of integers.

Lemma 5.29. *The points with rational coordinates are dense in Q .*

Proof. There is no loss of generality in assuming that $k_v \neq 0$ for all $v \in V$. For brevity, we put $\kappa_v := -k_v$ for $v \in V_0$ and $\kappa_v := k_v$ for $v \in V_1$. Then $Q = \{x \in \mathbb{R}^V : \sum_{v \in V} \kappa_v x_v^2 = 0\}$.

The Lagrange theorem implies that the set Q contains a point $c \neq 0$ with rational coordinates. Indeed, approximating a point $a \in Q$ with $a_v \neq 0$ for all $v \in V$ by rational points in \mathbb{R}^V , we find a rational number $q > 0$ representable as

$$q = \frac{1}{k} \sum_{v \in V \setminus U} \kappa_v c_v^2,$$

where $c_v \in \mathbb{Q}$ for all $v \in V \setminus U$. By the Lagrange theorem, we can represent q as the sum of four squares of rationals, $q = \sum_{v \in U} c_v^2$, obtaining a required point $c \in Q$.

It suffices to show that the rational points are dense in the subset $Q_0 \subset Q$ of points all coordinates of which are different from zero. We take $b \in Q_0$, assuming that $c \neq b$. Moreover, changing if necessary the signs of coordinates of c , we can assume that $A := \sum_{v \in V} \kappa_v b_v c_v > 0$.

Consider the line σ in \mathbb{R}^V through the points c and b , $\sigma(t) = c + t\xi$, where $\xi = b - c$, $t \in \mathbb{R}$. The set Q is the zero set of the function $F : \mathbb{R}^V \rightarrow \mathbb{R}$ given by $F(x) = \sum_{v \in V} \kappa_v x_v^2$. Since the gradient of F is different from zero at any point $x \in Q \setminus \{0\}$, the set $Q \setminus \{0\}$ is a regular hypersurface in \mathbb{R}^V . The restriction of F to the line σ is a quadratic polynomial, $F(t) = F \circ \sigma(t)$, which vanishes at $t = 0, 1$. Thus, $F(t) = A't(t - 1)$ for some coefficient $A' \in \mathbb{R}$. A direct computation shows that $A' = -2A \neq 0$. Consequently, the line σ transversally intersects Q at b . Now, approximating the vector ξ by vectors $\xi' \in \mathbb{R}^V$ with rational coordinates, we obtain lines σ' , $\sigma'(t) = c + t\xi'$, such that the points where these lines intersect Q give rational approximations to b . \square

Proof of Proposition 5.26 in the case where Γ is bipartite. The argument for the nonbipartite case can still be applied with the following modifications: a linearly independent collection of rows $E_0 \subset E$ consists of $|V| - 1$ elements, and the length function is chosen in the set $Q \subset \mathbb{R}^V$, $a \in Q$. Then Lemma 5.29 provides a required rational approximation. \square

§6. PROOF OF SPECTRAL CRITERIA

6.1. Spectral criterion for (I)=(HI). Here we prove Theorem 4.4.

The “only if” part. Assume that the BKN-equation over M has a compatible solution (a, γ) . We can also assume that the length function is positive, $a > 0$. By Lemma 4.1, the operator A_M^+ has a nonpositive eigenvalue, and if it has no negative eigenvalues, then the length function $a \in \mathbb{R}^V$ lies in the kernel of this operator and $\gamma_w = \text{sign}(k_{w-})$ for every edge $w \in W$. Since $\gamma_w \cdot \gamma_{-w} \neq -1$ by the compatibility of the solution, all charges of M have one and the same sign. \square

The “if” part. First, consider the case where the operator A_M^+ has no negative eigenvalue and all charges of M are of one and the same sign. We may assume that all charges are positive. By assumption, the operator A_M^+ is singular. Let $a \in \mathbb{R}^V$ be a nonzero vector in its kernel. Then the functions $|a|$ and $\gamma_w = \text{sign}(a_{w-}) \cdot \text{sign}(a_{w+})$, $w \in W$, are, respectively, the length function and the angle function of a compatible solution of the BKN-equation over M . By Theorem 3.1, the manifold M has property (I)=(HI).

Now, we assume that the operator A_M^+ has a negative eigenvalue. Then an argument similar to the proof of Lemma 5.21 shows that for some $t \in [0, 1)$ the deformation $A_{M,t}^+$, $0 \leq t \leq 1$, of A_M^+ has a zero eigenvalue, which easily gives a compatible solution of the BKN-equation. Thus, in this case M also has property (I)=(HI). \square

6.2. Spectral criterion for (E). Here we prove Theorem 4.5.

The “only if” part. Suppose a manifold $M \in \mathfrak{M}$ has property (E). By Theorem 3.1(E), there is a compatible symmetric solution (a, γ) of the BKN-equation over M such that if $a_{w-} \cdot a_{w+} \neq 0$, then $\gamma_w = \pm 1$ for every edge $w \in W$. We define a cocycle $\lambda : W \rightarrow \mathbb{Z}_2$ by putting $\lambda_w = \gamma_w$ if $a_{w-} \cdot a_{w+} \neq 0$ and $\lambda_w = 1$ otherwise. We show that the operator A_λ is weakly singular.

Suppose $a_v \neq 0$ for some $v \in V$. It suffices to show that $(A_\lambda a)_v = 0$. We have

$$(A_\lambda a)_v = k_v a_v - \sum_{w \in \partial v} \frac{\lambda_w}{|b_w|} a_{w+} = k_v a_v - \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w+} = 0$$

by the BKN-equation, because $\lambda_w a_{w+} = \gamma_w a_{w+}$ for $w \in \partial v$, no matter whether a_{w+} is equal to zero or not. \square

The “if” part. Suppose the operator A_λ is weakly singular for some class $\lambda \in H^1(\Gamma; \mathbb{Z}_2)$. Taking a nonzero vector $x \in \mathbb{R}^V$ such that $(A_\lambda x)_v = 0$ for all vertices $v \in V$ with $x_v \neq 0$, we define a function $\gamma : W \rightarrow \{0, \pm 1\}$ by $\gamma_w = \lambda_w \text{sign}(x_{w^-} \cdot x_{w^+})$, $w \in W$. Then $(|x|, \gamma)$ is a compatible symmetric solution of the BKN-equation over M satisfying condition (E) in Theorem 3.1. Thus, M has property (E). \square

6.3. Spectral criterion for (VE). Here we prove Theorem 4.6.

The “only if” part. Suppose a manifold $M \in \mathfrak{M}$ has property (VE). By Theorem 3.1, there is a compatible symmetric solution (a, γ) of the BKN-equation over M . Then the operator H_M has a nonpositive eigenvalue by Lemma 4.3. \square

The “if” part. Suppose the operator H_M has a nonpositive eigenvalue. By Theorem 3.1 (VE), it suffices to show that the BKN-equation over M has a compatible symmetric solution. If H_M has a negative eigenvalue, or the function s occurring in its definition is zero, then this assertion follows from Lemma 5.23. Thus, we assume that the operator H_M is singular and that the function s takes its values in $\{\pm 1\}$. Let $a \in \mathbb{R}^V$ be a nonzero vector in the kernel of H_M . Then

$$s(u)k_v a_v - \sum_{w \in \partial v \cap W_u} \frac{a_{w^+}}{|b_w|} = 0$$

for all sign components $u \in U$ and all vertices $v \in V_u$. We define $\gamma_w = s(u) \text{sign}(a_{w^-} \cdot a_{w^+})$ for $w \in W_u$ and $\gamma_w = 0$ for every edge $w \in W$ connecting vertices from different sign components. Then $(|a|, \gamma)$ is a compatible symmetric solution of the BKN-equation over M . \square

6.4. Spectral criterion for (F). Here we prove Theorem 4.7.

The “only if” part. By Theorem 3.1(F), there is a function $\varepsilon : V \rightarrow \{\pm 1\}$ and a compatible solution (a, γ) of the BKN-equation such that $a_v > 0$ for all $v \in V$ and $\gamma_w = \varepsilon_{w^-} \varepsilon_{w^+} \text{sign}(b_w)$ for all $w \in W$. It suffices to show that the vector $x = (x_v) \in \mathbb{R}^V$ defined by $x_v = \varepsilon_v a_v$ lies in the kernel of the operator A_ρ . By Lemma 4.2, we may assume that

$$J_M^\rho x = \sum_{v \in V} \left(\sum_{w \in \partial v} \frac{1}{b_w} x_{w^+} \right) v.$$

Thus,

$$A_\rho x = \sum_{v \in V} \left(k_v x_v - \sum_{w \in \partial v} \frac{1}{b_w} x_{w^+} \right) v = \sum_{v \in V} \varepsilon_v \left(k_v a_v - \sum_{w \in \partial v} \frac{\gamma_w}{|b_w|} a_{w^+} \right) v = 0. \quad \square$$

The “if” part. Suppose the operator $A_\rho : \mathbb{R}^V \rightarrow \mathbb{R}^V$ is supersingular. Let $a \in \mathbb{R}^V$ be an element in its kernel with nonzero coordinates, $a_v \neq 0$, $v \in V$. Then, for the function $\varepsilon : V \rightarrow \{\pm 1\}$, $\varepsilon_v = \text{sign}(a_v)$, the functions $|a| : V \rightarrow \mathbb{R}$ and $\gamma : W \rightarrow \{\pm 1\}$, $\gamma_w = \varepsilon_{w^+} \varepsilon_{w^-} \text{sign}(b_w)$, are, respectively, the length function and the angle function of a solution of the BKN-equation, and this solution satisfies condition (F) in Theorem 3.1. Thus, the manifold M fibers over the circle. \square

6.5. Spectral criterion for (VF). Here we prove Theorem 4.8.

The “only if” part. We assume that a manifold $M \in \mathfrak{M}$ is virtually fibered over the circle. Then, by Theorem 3.1, there is a compatible symmetric solution (a, γ) of the BKN-equation over M with positive length function, $a > 0$. Suppose the operator H_M

has no negative eigenvalue. Then $(H_M a, a) \geq 0$. On the other hand, $(H_M a, a) \leq 0$ by Lemma 4.3. So, $(H_M a, a) = 0$. Since H_M is positive semidefinite, the vector a lies in its kernel, so H_M is supersingular. \square

The “if” part. By Theorem 3.1(VF), it suffices to show that the BKN-equation over M has a compatible symmetric solution with positive length function. If the operator H_M has a negative eigenvalue, or the function s occurring in its definition is equal to zero, then this assertion follows from Lemma 5.23.

We may assume that the operator H_M is positive semidefinite and supersingular, and that the function s takes its values in $\{\pm 1\}$. Choosing $a \in \mathbb{R}^V$ in the kernel of H_M so that $a_v \neq 0$ for all $v \in V$, we define a function $\gamma : W \rightarrow \{0, \pm 1\}$ by putting $\gamma_w = 0$ for all edges $w \in W$ connecting vertices from different sign components, and $\gamma_w = s(u) \text{sign}(a_{w^-} \cdot a_{w^+})$ for all sign components $u \in U$ and all $w \in W_u$. Then $(|a|, \gamma)$ is a compatible symmetric solution of the BKN-equation over M with positive length function. \square

6.6. Spectral criterion for (NPC). Here we prove Theorem 4.9.

The “only if” part. Suppose $M \in \mathfrak{M}$ carries an NPC-metric. Then, by Theorem 3.1 (NPC), there is a compatible symmetric solution (a, γ) of the BKN-equation over M with length function $a > 0$, and angle function γ such that $|\gamma| < 1$.

Suppose the operator H_M has no negative eigenvalue. Then $(H_M a, a) \geq 0$. On the other hand, $(H_M a, a) \leq 0$ by Lemma 4.3. Hence, $(H_M a, a) = 0$, and moreover $s(u)\gamma_w = 1$ for every sign component $u \in U$ and every edge $w \in W_u$. Combined with the condition $|\gamma| < 1$, this means that every sign component consists of one point, and $W_u = \emptyset$. In this case, $(H_M x, x) = \sum_{v \in V} s(v)k_v x_v^2$ for every $x \in \mathbb{R}^V$.

Suppose the function s takes its values in $\{\pm 1\}$. Then $s(v_0) = 1$ for some vertex $v_0 \in V$ with positive charge, $k_{v_0} > 0$. Since $0 = (H_M a, a) = \sum_{v \in V} s(v)k_v a_v^2$, this means that the operator H_M has a negative eigenvalue, a contradiction. Thus, $s \equiv 0$. \square

The “if” part. By Theorem 3.1(NPC), it suffices to check that the BKN-equation over M has an NPC-solution. This was proved in Lemma 5.23. \square

6.6.1. *Historical remarks.* As was already mentioned, the spectral criterion for (NPC) obtained in [BK2] was inaccurate. Namely, in terms of the present survey, it claims that property (NPC) is equivalent to the condition that either the operator H_M has a negative eigenvalue, or $H_M \equiv 0$. As a counterexample, we can take a graph-manifold $M \in \mathfrak{M}$ that is the mapping torus of a Dehn twist for an orientable surface of genus at least two along a simple closed curve not homological to zero. The graph of such a manifold is a loop, the charge of the vertex equals $k = 2/|b|$, where the intersection number b coincides with the order of the twist, and $H_M \equiv 0$. However, M admits no NPC-metric because $s \not\equiv 0$. Nevertheless, all the results of [BK2] that employ the criterion mentioned above remain valid; in particular, this is true for the criterion for the mapping tori of Dehn twists to have an NPC-metric (Theorem 4.6.3).

A similar inaccuracy occurred in the criterion, obtained in [BK3], for the property of an infinite graph-manifold to carry an NPC-metric with bounded sectional curvatures and finite volume. The error is also not of principal nature, and it can be corrected similarly (in terms of this survey) by replacing the condition $H_M \equiv 0$ with the condition $s \equiv 0$.

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