ON SPACES OF POLYNOMIAL GROWTH
WITH NO CONJUGATE POINTS

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Abstract. The following generalization of the Hopf conjecture is proved: if the fundamental group of an $n$-dimensional compact polyhedral space $M$ without boundary and with no conjugate points has polynomial growth, then there exists a finite covering of $M$ by a flat torus.

§1. Introduction

By an $n$-dimensional polyhedral space we mean a metric space $M$ (with an inner metric) covered by $n$-simplexes; each simplex is endowed with a smooth Riemannian metric, and these metrics coincide on the common $(n-1)$-faces of the $n$-simplexes. The precise definition is given at the end of this section. In the definitions below, it is assumed that we deal with a fixed triangulation.

A polyhedral pseudomanifold is an $n$-dimensional polyhedral space in which the $(n-1)$-simplexes of the triangulation are adjacent to at most two $n$-simplexes. The boundary of a polyhedral space is the union of the $(n-1)$-simplexes of the triangulation that are adjacent to only one $n$-simplex. We say that $M$ has no conjugate points if any two points in the universal covering space of $M$ are connected by a unique geodesic. All polyhedral spaces considered in this paper are assumed to be connected.

Let $M$ be a compact polyhedral space without boundary and with no conjugate points. It is well known that $M$ is isometric to the quotient space $\tilde{M}/\Gamma$, where $\tilde{M}$ is the universal covering space of $M$, and $\Gamma$ is a subgroup of the group of isometries of $\tilde{M}$; recall that $\Gamma$ is isomorphic to $\pi_1(M)$.

Our aim in this paper is to prove the following two theorems.

Theorem 1. Let $M$ be an $n$-dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group $\pi_1(M)$ of $M$ is nilpotent, then $M$ is a flat torus.

Theorem 2. Let $M$ be an $n$-dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group $\pi_1(M)$ of $M$ is of polynomial growth, then there exists a finite covering of $M$ by a flat torus.

Theorem 2 can be derived from Theorem 1. Indeed, let $M$ satisfy the assumptions of Theorem 1. Then $\pi_1(M)$ is of polynomial growth. The well-known result by Gromov (see [G2]) says that $\pi_1(M)$ is virtually nilpotent, i.e., $\pi_1(M)$ contains a nilpotent subgroup $G$ of finite index. Consequently, there exists a finite covering $\tilde{M} \rightarrow M$ such that $\pi_1(\tilde{M}) = G$.

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341
Since $M$ is a compact polyhedral space without boundary and with no conjugate points, $M$ is flat by Theorem 1. In the remaining part of the paper we prove Theorem 1. The proof is organized as follows.

In §2 we prove that $M^n$ is a pseudomanifold and that it is homotopy equivalent to an $n$-dimensional torus.

In §3 we construct a map $f : M \to T^n$, where $T^n$ is a flat torus. We show that $f$ is a local isometry on the complement of the $(n - 2)$-skeleton of $M$. This step of the proof is similar to a version of the proof of the Hopf conjecture (see [I]). For the first time, the Hopf conjecture was proved by D. Burago and S. Ivanov in [BI].

In §4 we prove that the map $f : M \to T^n$ is an isometry. In contrast to the case of Riemannian manifolds considered in [I], this step is not trivial for Riemannian polyhedra.

Now we explain more precisely what we mean by polyhedral spaces.

An $n$-dimensional Riemannian simplex is an $n$-simplex in $\mathbb{R}^n$ equipped with a smooth Riemannian metric (as usual, we assume that the metric is defined in a neighborhood of this simplex), as well as any metric space isometric to such a simplex.

An $n$-dimensional polyhedral space is a connected metric space that can be obtained by gluing together $n$-dimensional Riemannian simplexes along some isometries between their faces.

§2. Homotopy type of $M$

In the proof of Theorem 1 we use the following results obtained earlier (see [L1] [L2]).

Claim 1 ([L1]). Let $M$ be a compact locally simply connected space without conjugate points. Then every nilpotent subgroup of the fundamental group of $M$ is Abelian and torsion free.

Claim 2 ([L2]). Let $M$ be an $n$-dimensional compact polyhedral space without boundary and with no conjugate points. If the triangulation of $M$ contains three $n$-simplexes with a common $(n - 1)$-face, then the fundamental group $\pi_1(M)$ is of exponential growth. Our aim in this section is to prove the following auxiliary statement.

Lemma 1. Let $M$ be as in Theorem 1. Then $M$ is a pseudomanifold that is homotopy equivalent to an $n$-dimensional torus.

Proof. Since the fundamental group of a compact metric space with intrinsic metric is finitely generated, from Claim 1 it follows that $\pi_1(M) = \mathbb{Z}^m$ for some $m$. Applying Claim 2, we see that at most two $n$-simplexes of $M$ may have a common $(n - 1)$-face, i.e., $M$ is a pseudomanifold. Since the universal covering space of $M$ is contractible, the fundamental group of $M$ determines the homotopy type of $M$. Hence, $M$ is homotopy equivalent to an $m$-torus $T^m$. It follows that $H_k(M, \cdot) = H_k(T^m, \cdot)$ for every $k$.

We prove that $m = n$, where $n$ is the dimension of $M$.

Suppose that $n > m$. Since $M$ is a pseudomanifold, we have $H_n(M, \mathbb{Z}_2) = \mathbb{Z}_2$. This contradicts the relation $H_n(T^m, \mathbb{Z}_2) = 0$.

Suppose $n < m$; then $H_m(M, \mathbb{Z}) = 0$. This contradicts the relation $H_m(T^m, \mathbb{Z}) = \mathbb{Z}$. Thus, $\pi_1(M) = \mathbb{Z}^n$. \hfill \Box

§3. Constructing a local isometry

We denote by $M'$ the complement of the $(n - 2)$-skeleton of $M$; then $M'$ is an open dense subset of $M$. In this section we shall prove the following statement.

Proposition 1. Under the assumptions of Theorem 1, there exists a map $f : M \to T^n$, where $T^n$ is a flat $n$-torus, with the following properties:
(1) $f|_{M'}$ is a local isometry on $M'$, i.e., $f|_{M'}$ is an open map preserving distances;
(2) $f$ is Lipschitz;
(3) $f$ induces an isomorphism between the corresponding fundamental groups.

We start with several lemmas.
Let $SM$ denote the space of all unit tangent vectors of $M$. A canonical measure $\mu_L$ on the space $SM$ is defined in a standard way as the product of two measures: the normalized Riemannian volume on $M$ and the normalized Riemannian volume on the unit $(n - 1)$-sphere. This measure is called the Liouville measure.

Since for almost every unit vector $\nu \in SM$ there exists a unique generic geodesic $\gamma$ with $\gamma'(0) = \nu$ (see [L1]), the geodesic flow transformation is well defined almost everywhere on $SM$, and it is known that the Liouville measure is invariant with respect to this transformation (see [L1]).

We recall that $M$ is isometric to the quotient space $\tilde{M}/\Gamma$, where $\tilde{M}$ is the universal covering space of $M$ and $\Gamma$ is a deck transformation group isomorphic to $\pi_1(M) = \mathbb{Z}^n$ and acting by isometries on $\tilde{M}$.

Consider the vector space $V = \Gamma \otimes \mathbb{R}$; it is isomorphic to $\mathbb{R}^n$. There exists a canonical immersion of $\Gamma = \mathbb{Z}^n \hookrightarrow V$, and its image is an integral lattice in $V = \mathbb{R}^n$. Below we shall denote elements of $\Gamma$ and the corresponding points of the lattice by the same symbol. Fix a point $x_0 \in \tilde{M}$. The orbit of $\Gamma$ is a lattice in $\tilde{M}$; there is a one-to-one correspondence between the points of the lattice and the elements of $\Gamma$. For $k \in \Gamma$ and $x \in \tilde{M}$, we denote by $x + k$ the image of $x$ under the isometry $k$. When studying distances between remote points, it is convenient to approximate points of $\tilde{M}$ by elements of the lattice. We define a map $\overline{\Gamma} : \tilde{M} \to \Gamma$ commuting with $\Gamma$. For this, we fix a bounded fundamental domain $F$ containing the point $x_0$. For an arbitrary $x \in \tilde{M}$, we put $\overline{\Gamma}(x) = k$, where $k$ is a unique element of $\Gamma$ such that $x \in F + k$.

Consider the function $\| \cdot \| : \Gamma \to [0, \infty)$ given by the formula

$$\|k\| = \lim_{n \to \infty} \frac{\overline{\rho}(x_0, x_0 + nk)}{n},$$

where $\overline{\rho}$ is the lift of the metric $\rho$. The function $\| \cdot \|$ is well known to be a norm on $\Gamma$; therefore, it extends to a norm on $V$, called the stable norm. For a linear function $L : V \to \mathbb{R}$ we set $\|L\| = \max \{L(x) \|x\| = 1 \}$.

**Lemma 2.** Let $L : V \to \mathbb{R}$ be a linear function with $\|L\| = 1$. There exists a function $\overline{B}_L : \tilde{M} \to \mathbb{R}$ such that

1) $\overline{B}_L$ is Lipschitz with Lipschitz constant $1$;
2) $\overline{B}_L(x + k) = \overline{B}_L(x) + L(k)$ for every $x \in \tilde{M}, k \in \Gamma$.

**Proof.** Indeed, let

$$\overline{B}_L(x) = \inf_{k \in \Gamma}(L(k) + \rho(x, x_0 + k)).$$

We prove that the function $\overline{B}_L$ is well defined. Since $\|L\| = 1$, from the definition of the stable norm it follows that

$$-\rho(x_0 + k, x_0) \leq -\|k\| \leq L(k),$$

whence

$$L(k) + \rho(x, x_0 + k) \geq -\rho(x_0 + k, x_0) + \rho(x, x_0 + k) \geq -\rho(x, x_0).$$

The required properties of $\overline{B}_L$ immediately follow from the definition. \qed
For a linear function $L : V \to \mathbb{R}$, let $\tilde{B}_L$ denote the function constructed in Lemma 2. Since $\tilde{B}_L$ is Lipschitz, it has a gradient almost everywhere; this gradient will be denoted by $\tilde{v}_L$.

For $\bar{v} \in \tilde{S}M$, let $\tilde{\gamma} : \mathbb{R} \to \tilde{M}$ be a geodesic with $\tilde{\gamma}'(0) = \bar{v}$. We define the direction at infinity $R(\bar{v}) = \tilde{R}(\tilde{\gamma}) \in V$ by

$$\tilde{R}(\bar{v}) = \lim_{T \to \infty} \frac{\tilde{F}(\tilde{\gamma}(T)) - \tilde{F}(\tilde{\gamma}(0))}{T}.$$ 

By definition, for $v \in SM$ we put $R(v) = \tilde{R}(\bar{v})$, where $\bar{v}$ is a lifting of $v$.

Since $M$ has no conjugate points, it is clear that $\|R(v)\| = 1$.

**Lemma 3.** The functions $R$ and $\tilde{R}$ are defined almost everywhere on $SM$ and $\tilde{S}M$, respectively.

**Proof.** Let $\phi : M \to V/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n$ be a homotopy equivalence; we may assume that $\phi$ is simplicial. Since $\phi$ induces an isomorphism between fundamental groups, the lifting function $\tilde{\phi} : \tilde{M} \to V$ commutes with $\Gamma$.

Since the functions $\tilde{\phi}$ and $\tilde{F}$ commute with $\Gamma$, we have $\|\tilde{\phi} - \tilde{F}\| \leq \text{const}$. Thus, in the definition of $\tilde{R}$ we can replace $\tilde{F}$ by $\tilde{\phi}$. Since the differential $d\tilde{\phi}$ is defined almost everywhere on $TM$ and is $\Gamma$-invariant, it is the lift of some measurable function $\omega : TM \to V$. For a geodesic $\gamma$ in $M$ and its lifting $\tilde{\gamma}$, we have

$$\tilde{\phi}(\tilde{\gamma}(T)) - \tilde{\phi}(\tilde{\gamma}(0)) = \int_0^T d\tilde{\phi}(\tilde{\gamma}') = \int_0^T \omega(\gamma').$$

Thus, $R(v)$ is equal to the average of $\omega$ along $\gamma$. The Birkhoff ergodic theorem shows that $R(v)$ is defined for almost all $v \in SM$. \hfill $\square$

**Lemma 4.** Let $L : V \to R$ be a linear function with $\|L\| = 1$. Recall that $\tilde{v}_L$ denotes the gradient field of $B_L$. Then

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \gamma', \tilde{v}_L \rangle = L(R(\gamma))$$

if both sides are well defined.

**Proof.** Since $B_L \circ \gamma$ is Lipschitz, the Newton–Leibniz formula yields

$$\int_0^T \langle \gamma', \tilde{v}_L \rangle = \int_0^T (B_L \circ \gamma)' = B_L(\gamma(T)) - B_L(\gamma(0)).$$

Since the function $B_L(x) - L(\bar{k}(x))$ is bounded on the fundamental domain and periodic, it is bounded. This implies that $B_L(\gamma(T)) - B_L(\gamma(0))$ differs from $L(\bar{k}(\gamma(T)) - L(\bar{k}(\gamma(0))))$ by a constant. So, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \gamma', \tilde{v}_L \rangle = \lim_{T \to \infty} L \left( \frac{\bar{k}(\gamma(T)) - \bar{k}(\gamma(0))}{T} \right) = L(R(\gamma)). \hfill \square$$

Let $F$ denote the unit sphere of the norm $\| \cdot \|$, and let $m$ be the measure on $F$ that is the image of $\mu_L$ under $R : SM \to F$.

**Lemma 5.** If $L : V \to R$ is a linear function with $\|L\| = 1$, then

$$\int_F L^2 dm \leq \frac{1}{n}.$$

Equality occurs if and only if $\langle \tilde{v}_L, w \rangle = L(\tilde{R}(w))$ for almost every $w \in \tilde{S}M$. 

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Proof. Consider the average of $\langle v_L, \cdot \rangle$ along geodesics. By Lemma 3, we have
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \gamma', v_L \rangle = L \circ R.
\]
By the Schwartz inequality,
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \langle \gamma', v_L \rangle^2 \geq (L \circ R)^2.
\]
Since $R$ is constant on every trajectory of the geodesic flow, we have
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T (v_L, w)^2 = (L \circ R)^2 + \lim_{T \to \infty} \frac{1}{T} \int_0^T (\langle v_L, \cdot \rangle - L \circ R)^2.
\]
Integrating and using the Birkhoff ergodic theorem, we obtain
\[
\int_{SM} (v_L, \cdot)^2 d\mu_L = \int_{SM} (L \circ R)^2 d\mu_L + \int_{SM} (\langle v_L, \cdot \rangle - L \circ R)^2 d\mu_L.
\]
From the inequality $|v_L| < 1$ it follows that $\int_{SM} (v_L, \cdot)^2 d\mu_L \leq 1/n$. Consequently,
\[
\int_F L^2 dm = \int_{SM} (L \circ R)^2 d\mu_L \leq \frac{1}{n} - \int_{SM} (\langle v_L, \cdot \rangle - L \circ R)^2 d\mu_L.
\]
The integral on the right is nonnegative, and it vanishes if and only if $\langle v_L, w \rangle = L(R(w))$ for almost every $w \in SM$. The lemma is proved. \qed

We use the following known result (for its proof, see, e.g., [BI]).

**Lemma 6.** Let $(V, \| \cdot \|)$ be an $n$-dimensional Banach space, let $F$ be the unit sphere of the norm $\| \cdot \|$, and let $F^*$ be the set of linear functions $L$ such that $\|L\| = 1$. Then there exists an ("inscribed") quadratic form $Q : V \to \mathbb{R}$ representable as a finite sum
\[
Q = \sum a_i L_i^2, \quad L_i \in F^*, \quad a_i > 0, \quad \sum a_i = n,
\]
and such that $Q(x) \geq \|x\|^2$ for every $x \in V$. In particular, $Q$ is positive.

**Remark 1.** The unit ball of $Q$ is the ellipsoid of maximal volume inscribed in $F$.

Let $Q = \sum a_i L_i^2$ be the corresponding (inscribed) quadratic form for the stable norm $\| \cdot \|$ associated with $\bar{p}$. We denote by $B_i$ the functions constructed as in Lemma 2 for the linear functions $L_i$, and by $\bar{v}_i$ their gradients.

**Lemma 7.** For all $i$, we have
\[
(\bar{v}_i, w) = L_i(R(w))
\]
for almost every $w \in SM$.

**Proof.** Applying Lemma 5 to $L_i$, we obtain
\[
\int_F Q dm = \sum a_i \int_F L_i^2 dm \leq \frac{1}{n} \sum a_i = 1.
\]
But $Q|_F \geq 1$ on $F$. Therefore, $\int_F Q dm = 1$, so that $\int_F L_i^2 dm = \frac{1}{n}$ for every $i$. By Lemma 5 it follows that $\langle \bar{v}_i, w \rangle = L_i(R(w))$ for almost every $w \in SM$. \qed

The lemma just proved implies that (1) is true almost everywhere for almost every trajectory of the geodesic flow; this means that for almost every $w \in SM$, if $\gamma$ is a geodesic with $\gamma'(0) = w$, then the function $(\bar{v}_i, \gamma') = (B_i \circ \gamma)'$ is defined almost everywhere. Moreover it is equal to the constant $L_i(R(\gamma))$. Since this function is Lipschitz, it is linear. Thus,
\[
(B_i \circ \gamma)' \equiv L_i(R(\gamma)), \quad t \in \mathbb{R}.
\]
Thus, the degree of \( f \) assuming that \( L \) is linearly independent.

Consider the map
\[
\tilde{f} = (\tilde{B}_1, \ldots, \tilde{B}_n) : \tilde{M} \to \mathbb{R}^n.
\]
We endow \( \mathbb{R}^n \) with the Euclidean structure corresponding to the quadratic form \( Q \) under the isomorphism
\[
I = (L_1, \ldots, L_n) : V \to \mathbb{R}^n.
\]
For almost every geodesic \( \gamma : \mathbb{R} \to \tilde{M} \) we obtain
\[
(\tilde{f} \circ \gamma)' = (L_1(\tilde{R}(\gamma)), \ldots, L_n(\tilde{R}(\gamma))) = I(\tilde{R}(\gamma)).
\]
Since for almost every geodesic \( \gamma \) the vector \( I(\tilde{R}(\gamma)) \) is a unit vector with respect to the new Euclidean structure, the image \( \tilde{f}(\gamma) \) is a straight line with constant unit velocity.

Now we prove Proposition 1.

**Proof.** Since \( \tilde{f} \) commutes with the group \( \Gamma \) of integral translations on \( \tilde{M} \) and \( \mathbb{R}^n \), \( \tilde{f} \) induces a map \( f : M \to T^n \), where \( T^n \) is a flat torus. The homomorphism of fundamental groups induced by \( f \) is an isomorphism, which implies statement (3) of Proposition 1.

The map \( f \) is Lipschitz because so is \( \tilde{f} \).

Recall that \( M' \) denotes the complement of the \((n - 2)\)-skeleton of \( M \).

We show that \( f|_{M'} : M' \to T^n \) is a local isometry. Consider a convex neighborhood \( U \subset M \) and fix two points \( x, y \in U \). For any neighborhoods \( U_x, U_y \subset U \) of \( x \) and \( y \), let \( V(U_x, U_y) \) be the set of initial velocity vectors of all shortest paths starting in \( U_x \) and ending in \( U_y \).

Since for almost every geodesic \( \gamma : [a, b] \to M \) the image \( f \circ \gamma \) is a straight line with a constant unit speed and \( \mu_L V(U_x, U_y) > 0 \), there exist two points \( x' \in U_x \) and \( y' \in U_y \) such that \( f \) preserves the distance between them. Since \( U_x \) and \( U_y \) are arbitrary and \( f \) is continuous, \( f \) preserves the distance between \( x \) and \( y \). Thus, \( f|_U \) preserves distances.

Since \( M' \) and \( T^n \) are \( n \)-dimensional manifolds, and \( f|_{M'} \) preserves the distances, for any \( x \in M' \) the image of some neighborhood of \( x \) is a neighborhood of \( f(x) \), and we see that \( f \) is an open map. \( \Box \)

**§4. \( f \) is an isometry**

The following Lemma 8 is an obvious consequence of Proposition 1.

**Lemma 8.** \( f|_{M'} \) preserves the lengths of curves.

**Lemma 9.** The map \( f|_{M'} : M' \to f(M') \) is bijective, and \( f : M \to T^n \) is surjective. As a consequence (because \( f|_{M'} \) is a local isometry), the map \( (f|_{M'})^{-1} \) is well-defined, is continuous, and preserves the lengths of curves.

**Proof.** Recall that \( M \) is homotopy equivalent to an \( n \)-dimensional torus. Consequently, the \( n \)-homology group of \( M \) is isomorphic to \( \mathbb{Z} \). We fix an isomorphism between \( H_n(T^n) \) and \( \mathbb{Z} \) and choose a generator of \( H_n(M) \). The induced homomorphism \( f_* : H_n(M) \to H_n(T^n) = \mathbb{Z} \) takes the generator of \( H_n(M) \) to some integer; this integer is called the degree of \( f \). We show that the degree of \( f \) is \( \pm 1 \). Since the universal covering space of \( M \) is contractible, the induced homomorphism \( f_* \) determines the homotopy type of \( f \). Proposition 3 shows that \( f_* \) is an isomorphism; then \( f \) is a homotopy equivalence. Thus, the degree of \( f \) is \( \pm 1 \).
The choice of generators of the homology group fixes orientations of the manifolds $M' \subset M$ and $T^n$. We define the degree of $f$ at $x \in M'$ to be equal to 1 if $d_n f$ preserves the orientations of the tangent spaces at $x$, and to $-1$ if $d_n f$ reverses the orientations. Suppose $y \in T^n$ is a regular point, i.e., the preimage $f^{-1}(y) = x_1, \ldots, x_l$ is contained in $M'$. As in the case of Riemannian manifolds, it can be proved that the degree of $f$ is the sum of the degrees of $f$ at the points $x_1, \ldots, x_l$. Hence, $f$ is surjective.

Since $M$ is a pseudomanifold that is homotopy equivalent to an $n$-dimensional torus, the space $M'$ is connected. Indeed, assume the contrary; then the group $H_n(M, \mathbb{Z}_2)$ contains two nonzero elements. Since $f|_{M'}$ is a local isometry, it preserves the orientation of tangent spaces everywhere, or it reverses these orientations. Consequently, the degree of $f$ is constant at the points $x_1, \ldots, x_l$. Since the degree of $f$ is 1, this means that each regular point has a unique preimage. By the definition of a regular point, it follows that all points having two or more preimages are contained in $f^{-1}(f(M \setminus M'))$. We put $J = f^{-1}(f(M \setminus M'))$. Observe that the dimension of $J$ does not exceed $n - 2$.

Suppose that $f|_{M'}$ is not injective. Let $y \in f(M')$ be a point with more than one preimage in $M'$, and let $x_1, x_2$ be two such preimages. Let $D_{\rho_0}(x_1), D_{\rho_0}(x_2)$ be balls centered at $x_1$ and $x_2$ and such that the restriction of $f$ to these balls is an isometry. Since the dimension of $J$ is at most $n - 2$, there exists a point $x_3 \in D_{\rho_0}(x_1) \setminus M' \setminus J$. The image of this point coincides with an image of some point contained in $D_{\rho_0}(x_2)$, which contradicts the fact that $f$ is injective on $M \setminus J$ ($x_3 \in M \setminus J$).

We complete the proof of Theorem 1 by the following statement.

**Lemma 10.** The map $f : M \to T^n$ is an isometry.

**Proof.** We show that $f$ is noncontracting and nonexpanding. Every path in $M$ can be approximated by a piecewise differentiable path of almost the same length. We can move each of the corresponding pieces to the interior of an appropriate $n$-simplex, leaving the endpoints fixed and almost length preserving.

The map $f$ preserves the lengths of these pieces (see Lemma 8). Therefore, the map is nonexpanding.

Now we show that $f$ is noncontracting. Let $x, y \in M$ be arbitrary points. Given $\varepsilon > 0$, we let $x', y' \in M'$ be points such that $\rho(x, x') < \varepsilon$ and $\rho(y, y') < \varepsilon$. Since $f$ is nonexpanding, we have $|(f(x), f(x'))| < \varepsilon$ and $|(f(y), f(y'))| < \varepsilon$, where $|(\cdot, \cdot)|$ denotes the metric on the flat torus.

Since $f$ is Lipschitz and surjective, the Hausdorff dimension of the set $T^n \setminus f(M')$ does not exceed $n - 2$. Therefore, the shortest path $[f(x'), f(y')] \in T^n$ can be approximated by a path in $f(M')$ with almost the same length and the same endpoints. Let $s : [a, b] \to f(M')$ be a path that joins $f(x')$ and $f(y')$ and such that the length of $s$ differs from $|f(x'), f(y')|$ by less than $\varepsilon$. Since $(f|_{M'})^{-1}$ preserves distances, the length of the path $s \circ (f|_{M'})^{-1} : [a, b] \to M'$, which joins $x'$ and $y'$, differs from $|f(x'), f(y')|$ by less than $\varepsilon$. Thus,

$$\rho(x, y) < \rho(x', y') + 2\varepsilon < |f(x'), f(y')| + 3\varepsilon < |f(x), f(y)| + 5\varepsilon.$$

Therefore, $f$ is noncontracting. \qed

**References**


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