

## ON SPACES OF POLYNOMIAL GROWTH WITH NO CONJUGATE POINTS

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ABSTRACT. The following generalization of the Hopf conjecture is proved: if the fundamental group of an  $n$ -dimensional compact polyhedral space  $M$  without boundary and with no conjugate points has polynomial growth, then there exists a finite covering of  $M$  by a flat torus.

### §1. INTRODUCTION

By an  $n$ -dimensional polyhedral space we mean a metric space  $M$  (with an inner metric) covered by  $n$ -simplexes; each simplex is endowed with a smooth Riemannian metric, and these metrics coincide on the common  $(n - 1)$ -faces of the  $n$ -simplexes. The precise definition is given at the end of this section. In the definitions below, it is assumed that we deal with a fixed triangulation.

A polyhedral pseudomanifold is an  $n$ -dimensional polyhedral space in which the  $(n - 1)$ -simplexes of the triangulation are adjacent to at most two  $n$ -simplexes. The boundary of a polyhedral space is the union of the  $(n - 1)$ -simplexes of the triangulation that are adjacent to only one  $n$ -simplex. We say that  $M$  has no conjugate points if any two points in the universal covering space of  $M$  are connected by a unique geodesic. All polyhedral spaces considered in this paper are assumed to be connected.

Let  $M$  be a compact polyhedral space without boundary and with no conjugate points. It is well known that  $M$  is isometric to the quotient space  $\widetilde{M}/\Gamma$ , where  $\widetilde{M}$  is the universal covering space of  $M$ , and  $\Gamma$  is a subgroup of the group of isometries of  $\widetilde{M}$ ; recall that  $\Gamma$  is isomorphic to  $\pi_1(M)$ .

Our aim in this paper is to prove the following two theorems.

**Theorem 1.** *Let  $M$  be an  $n$ -dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group  $\pi_1(M)$  of  $M$  is nilpotent, then  $M$  is a flat torus.*

**Theorem 2.** *Let  $M$  be an  $n$ -dimensional compact polyhedral space without boundary and with no conjugate points. If the fundamental group  $\pi_1(M)$  of  $M$  is of polynomial growth, then there exists a finite covering of  $M$  by a flat torus.*

Theorem 2 can be derived from Theorem 1. Indeed, let  $M$  satisfy the assumptions of Theorem 1. Then  $\pi_1(M)$  is of polynomial growth. The well-known result by Gromov (see [G2]) says that  $\pi_1(M)$  is virtually nilpotent, i.e.,  $\pi_1(M)$  contains a nilpotent subgroup  $G$  of finite index. Consequently, there exists a finite covering  $\overline{M} \rightarrow M$  such that  $\pi_1(\overline{M}) = G$ .

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2000 *Mathematics Subject Classification.* Primary 57N16.

*Key words and phrases.*  $n$ -dimensional polyhedral space, polyhedral pseudomanifold, fundamental group.

Partially supported by RFBR (grant no. 02-01-00090), by CRDF (grant no. RM1-2381-ST-02), and by SS (grant no. 1914.2003.1).

Since  $\overline{M}$  is a compact polyhedral space without boundary and with no conjugate points,  $\overline{M}$  is flat by Theorem 1. In the remaining part of the paper we prove Theorem 1. The proof is organized as follows.

In §2 we prove that  $M^n$  is a pseudomanifold and that it is homotopy equivalent to an  $n$ -dimensional torus.

In §3 we construct a map  $f : M \rightarrow T^n$ , where  $T^n$  is a flat torus. We show that  $f$  is a local isometry on the complement of the  $(n - 2)$ -skeleton of  $M$ . This step of the proof is similar to a version of the proof of the Hopf conjecture (see [I]). For the first time, the Hopf conjecture was proved by D. Burago and S. Ivanov in [BI].

In §4 we prove that the map  $f : M \rightarrow T^n$  is an isometry. In contrast to the case of Riemannian manifolds considered in [I], this step is not trivial for Riemannian polyhedra.

Now we explain more precisely what we mean by polyhedral spaces.

An  $n$ -dimensional Riemannian simplex is an  $n$ -simplex in  $\mathbb{R}^n$  equipped with a smooth Riemannian metric (as usual, we assume that the metric is defined in a neighborhood of this simplex), as well as any metric space isometric to such a simplex.

An  $n$ -dimensional polyhedral space is a connected metric space that can be obtained by gluing together  $n$ -dimensional Riemannian simplexes along some isometries between their faces.

## §2. HOMOTOPY TYPE OF $M$

In the proof of Theorem 1 we use the following results obtained earlier (see [L1, L2]).

**Claim 1** ([L1]). *Let  $M$  be a compact locally simply connected space without conjugate points. Then every nilpotent subgroup of the fundamental group of  $M$  is Abelian and torsion free.*

**Claim 2** ([L2]). *Let  $M$  be an  $n$ -dimensional compact polyhedral space without boundary and with no conjugate points. If the triangulation of  $M$  contains three  $n$ -simplexes with a common  $(n - 1)$ -face, then the fundamental group  $\pi_1(M)$  is of exponential growth.*

Our aim in this section is to prove the following auxiliary statement.

**Lemma 1.** *Let  $M$  be as in Theorem 1. Then  $M$  is a pseudomanifold that is homotopy equivalent to an  $n$ -dimensional torus.*

*Proof.* Since the fundamental group of a compact metric space with intrinsic metric is finitely generated, from Claim 1 it follows that  $\pi_1(M) = \mathbb{Z}^m$  for some  $m$ . Applying Claim 2, we see that at most two  $n$ -simplexes of  $M$  may have a common  $(n - 1)$ -face, i.e.,  $M$  is a pseudomanifold. Since the universal covering space of  $M$  is contractible, the fundamental group of  $M$  determines the homotopy type of  $M$ . Hence,  $M$  is homotopy equivalent to an  $m$ -torus  $T^m$ . It follows that  $H_k(M, \cdot) = H_k(T^m, \cdot)$  for every  $k$ .

We prove that  $m = n$ , where  $n$  is the dimension of  $M$ .

Suppose that  $n > m$ . Since  $M$  is a pseudomanifold, we have  $H_n(M, \mathbb{Z}_2) = \mathbb{Z}_2$ . This contradicts the relation  $H_n(T^m, \mathbb{Z}_2) = 0$ .

Suppose  $n < m$ ; then  $H_m(M, \mathbb{Z}) = 0$ . This contradicts the relation  $H_m(T^m, \mathbb{Z}) = \mathbb{Z}$ . Thus,  $\pi_1(M) = \mathbb{Z}^n$ .  $\square$

## §3. CONSTRUCTING A LOCAL ISOMETRY

We denote by  $M'$  the complement of the  $(n - 2)$ -skeleton of  $M$ ; then  $M'$  is an open dense subset of  $M$ . In this section we shall prove the following statement.

**Proposition 1.** *Under the assumptions of Theorem 1, there exists a map  $f : M \rightarrow T^n$ , where  $T^n$  is a flat  $n$ -torus, with the following properties:*

- (1)  $f|_{M'}$  is a local isometry on  $M'$ , i.e.,  $f|_{M'}$  is an open map preserving distances;
- (2)  $f$  is Lipschitz;
- (3)  $f$  induces an isomorphism between the corresponding fundamental groups.

We start with several lemmas.

Let  $SM$  denote the space of all unit tangent vectors of  $M$ . A canonical measure  $\mu_L$  on the space  $SM$  is defined in a standard way as the product of two measures: the normalized Riemannian volume on  $M$  and the normalized Riemannian volume on the unit  $(n - 1)$ -sphere. This measure is called the *Liouville measure*.

Since for almost every unit vector  $e \in SM$  there exists a unique generic geodesic  $\gamma$  with  $\gamma'(0) = e$  (see [L1]), the geodesic flow transformation is well defined almost everywhere on  $SM$ , and it is known that the Liouville measure is invariant with respect to this transformation (see [L1]).

We recall that  $M$  is isometric to the quotient space  $\widetilde{M}/\Gamma$ , where  $\widetilde{M}$  is the universal covering space of  $M$  and  $\Gamma$  is a deck transformation group isomorphic to  $\pi_1(M) = \mathbb{Z}^n$  and acting by isometries on  $\widetilde{M}$ .

Consider the vector space  $V = \Gamma \otimes \mathbb{R}$ ; it is isomorphic to  $\mathbb{R}^n$ . There exists a canonical immersion of  $\Gamma = \mathbb{Z}^n \hookrightarrow V$ , and its image is an integral lattice in  $V = \mathbb{R}^n$ . Below we shall denote elements of  $\Gamma$  and the corresponding points of the lattice by the same symbol. Fix a point  $x_0 \in \widetilde{M}$ . The orbit of  $\Gamma$  is a lattice in  $\widetilde{M}$ ; there is a one-to-one correspondence between the points of the lattice and the elements of  $\Gamma$ . For  $k \in \Gamma$  and  $x \in \widetilde{M}$ , we denote by  $x + k$  the image of  $x$  under the isometry  $k$ . When studying distances between remote points, it is convenient to approximate points of  $\widetilde{M}$  by elements of the lattice. We define a map  $\bar{k} : \widetilde{M} \rightarrow \Gamma$  commuting with  $\Gamma$ . For this, we fix a bounded fundamental domain  $F$  containing the point  $x_0$ . For an arbitrary  $x \in \widetilde{M}$ , we put  $\bar{k}(x) = k$ , where  $k$  is a unique element of  $\Gamma$  such that  $x \in F + k$ .

Consider the function  $\|\cdot\| : \Gamma \rightarrow [0, \infty)$  given by the formula

$$\|k\| = \lim_{n \rightarrow \infty} \frac{\widetilde{\rho}(x_0, x_0 + nk)}{n},$$

where  $\widetilde{\rho}$  is the lift of the metric  $\rho$ . The function  $\|\cdot\|$  is well known to be a norm on  $\Gamma$ ; therefore, it extends to a norm on  $V$ , called the stable norm. For a linear function  $L : V \rightarrow \mathbb{R}$  we set  $\|L\| = \max\{L(x) \mid \|x\| = 1\}$ .

**Lemma 2.** *Let  $L : V \rightarrow \mathbb{R}$  be a linear function with  $\|L\| = 1$ . There exists a function  $\widetilde{B}_L : \widetilde{M} \rightarrow \mathbb{R}$  such that*

- 1)  $\widetilde{B}_L$  is Lipschitz with Lipschitz constant 1;
- 2)  $\widetilde{B}_L(x + k) = \widetilde{B}_L(x) + L(k)$  for every  $x \in \widetilde{M}$ ,  $k \in \Gamma$ .

*Proof.* Indeed, let

$$\widetilde{B}_L(x) = \inf_{k \in \Gamma} (L(k) + \rho(x, x_0 + k)).$$

We prove that the function  $\widetilde{B}_L$  is well defined. Since  $\|L\| = 1$ , from the definition of the stable norm it follows that

$$-\rho(x_0 + k, x_0) \leq -\|k\| \leq L(k),$$

whence

$$L(k) + \rho(x, x_0 + k) \geq -\rho(x_0 + k, x_0) + \rho(x, x_0 + k) \geq -\rho(x, x_0).$$

The required properties of  $\widetilde{B}_L$  immediately follow from the definition. □

For a linear function  $L : V \rightarrow \mathbb{R}$ , let  $\tilde{B}_L$  denote the function constructed in Lemma 2. Since  $\tilde{B}_L$  is Lipschitz, it has a gradient almost everywhere; this gradient will be denoted by  $\tilde{v}_L$ .

For  $\tilde{v} \in \widetilde{SM}$ , let  $\tilde{\gamma} : \mathbb{R} \rightarrow \widetilde{M}$  be a geodesic with  $\gamma'(0) = \tilde{v}$ . We define the direction at infinity  $\tilde{R}(\tilde{v}) = \tilde{R}(\tilde{\gamma}) \in V$  by

$$\tilde{R}(\tilde{v}) = \lim_{T \rightarrow \infty} \frac{\bar{k}(\tilde{\gamma}(T)) - \bar{k}(\tilde{\gamma}(0))}{T}.$$

By definition, for  $v \in SM$  we put  $R(v) = \tilde{R}(\tilde{v})$ , where  $\tilde{v}$  is a lifting of  $v$ .

Since  $M$  has no conjugate points, it is clear that  $\|R(v)\| = 1$ .

**Lemma 3.** *The functions  $R$  and  $\tilde{R}$  are defined almost everywhere on  $SM$  and  $\widetilde{SM}$ , respectively.*

*Proof.* Let  $\phi : M \rightarrow V/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n$  be a homotopy equivalence; we may assume that  $\phi$  is simplicial. Since  $\phi$  induces an isomorphism between fundamental groups, the lifting function  $\tilde{\phi} : \widetilde{M} \rightarrow V$  commutes with  $\Gamma$ .

Since the functions  $\tilde{\phi}$  and  $\bar{k}$  commute with  $\Gamma$ , we have  $\|\tilde{\phi} - \bar{k}\| \leq \text{const}$ . Thus, in the definition of  $\tilde{R}$  we can replace  $\bar{k}$  by  $\tilde{\phi}$ . Since the differential  $d\tilde{\phi}$  is defined almost everywhere on  $T\widetilde{M}$  and is  $\Gamma$ -invariant, it is the lift of some measurable function  $\omega : TM \rightarrow V$ . For a geodesic  $\gamma$  in  $M$  and its lifting  $\tilde{\gamma}$ , we have

$$\tilde{\phi}(\tilde{\gamma}(T)) - \tilde{\phi}(\tilde{\gamma}(0)) = \int_0^T d\tilde{\phi}(\tilde{\gamma}') = \int_0^T \omega(\gamma').$$

Thus,  $R(v)$  is equal to the average of  $\omega$  along  $\gamma$ . The Birkhoff ergodic theorem shows that  $R(v)$  is defined for almost all  $v \in SM$ . □

**Lemma 4.** *Let  $L : V \rightarrow \mathbb{R}$  be a linear function with  $\|L\| = 1$ . Recall that  $\tilde{v}_L$  denotes the gradient field of  $B_L$ . Then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle \gamma', \tilde{v}_L \rangle = L(R(\gamma))$$

*if both sides are well defined.*

*Proof.* Since  $B_L \circ \gamma$  is Lipschitz, the Newton–Leibniz formula yields

$$\int_0^T \langle \gamma', \tilde{v}_L \rangle = \int_0^T (B_L \circ \gamma)' = B_L(\gamma(T)) - B_L(\gamma(0)).$$

Since the function  $B_L(x) - L(\bar{k}(x))$  is bounded on the fundamental domain and periodic, it is bounded. This implies that  $B_L(\gamma(T)) - B_L(\gamma(0))$  differs from  $L(\bar{k}(\gamma(T))) - L(\bar{k}(\gamma(0)))$  by a constant. So, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle \gamma', \tilde{v}_L \rangle = \lim_{T \rightarrow \infty} L\left(\frac{\bar{k}(\gamma(T)) - \bar{k}(\gamma(0))}{T}\right) = L(R(\gamma)). \quad \square$$

Let  $F$  denote the unit sphere of the norm  $\|\cdot\|$ , and let  $m$  be the measure on  $F$  that is the image of  $\mu_L$  under  $R : SM \rightarrow F$ .

**Lemma 5.** *If  $L : V \rightarrow \mathbb{R}$  is a linear function with  $\|L\| = 1$ , then*

$$\int_F L^2 dm \leq \frac{1}{n}.$$

*Equality occurs if and only if  $\langle \tilde{v}_L, w \rangle = L(\tilde{R}(w))$  for almost every  $w \in \widetilde{SM}$ .*

*Proof.* Consider the average of  $\langle v_L, \cdot \rangle$  along geodesics. By Lemma 3, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle \gamma', v_L \rangle = L \circ R.$$

By the Schwartz inequality,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle \gamma', v_L \rangle^2 \geq (L \circ R)^2.$$

Since  $R$  is constant on every trajectory of the geodesic flow, we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle v_L, w \rangle^2 = (L \circ R)^2 + \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\langle v_L, \cdot \rangle - L \circ R)^2.$$

Integrating and using the Birkhoff ergodic theorem, we obtain

$$\int_{SM} \langle v_L, \cdot \rangle^2 d\mu_L = \int_{SM} (L \circ R)^2 d\mu_L + \int_{SM} (\langle v_L, \cdot \rangle - L \circ R)^2 d\mu_L.$$

From the inequality  $|v_L| < 1$  it follows that  $\int_{SM} \langle v_L, \cdot \rangle^2 d\mu_L \leq 1/n$ . Consequently,

$$\int_F L^2 dm = \int_{SM} (L \circ R)^2 d\mu_L \leq \frac{1}{n} - \int_{SM} (\langle v_L, \cdot \rangle - L \circ R)^2 d\mu_L.$$

The integral on the right is nonnegative, and it vanishes if and only if  $\langle v_L, w \rangle = L(R(w))$  for almost every  $w \in SM$ . The lemma is proved.  $\square$

We use the following known result (for its proof, see, e.g., [BI]).

**Lemma 6.** *Let  $(V, \|\cdot\|)$  be an  $n$ -dimensional Banach space, let  $F$  be the unit sphere of the norm  $\|\cdot\|$ , and let  $F^*$  be the set of linear functions  $L$  such that  $\|L\| = 1$ . Then there exists an (“inscribed”) quadratic form  $Q : V \rightarrow \mathbb{R}$  representable as a finite sum*

$$Q = \sum a_i L_i^2, \quad L_i \in F^*, \quad a_i > 0, \quad \sum a_i = n,$$

*and such that  $Q(x) \geq \|x\|^2$  for every  $x \in V$ . In particular,  $Q$  is positive.*

*Remark 1.* The unit ball of  $Q$  is the ellipsoid of maximal volume inscribed in  $F$ .

Let  $Q = \sum a_i L_i^2$  be the corresponding (inscribed) quadratic form for the stable norm  $\|\cdot\|$  associated with  $\tilde{\rho}$ . We denote by  $B_i$  the functions constructed as in Lemma 2 for the linear functions  $L_i$ , and by  $\tilde{v}_i$  their gradients.

**Lemma 7.** *For all  $i$ , we have*

$$(1) \quad \langle \tilde{v}_i, w \rangle = L_i(R(w))$$

*for almost every  $w \in S\tilde{M}$ .*

*Proof.* Applying Lemma 5 to  $L_i$ , we obtain

$$\int_F Q dm = \sum a_i \int_F L_i^2 dm \leq \frac{1}{n} \sum a_i = 1.$$

But  $Q|_F \geq 1$  on  $F$ . Therefore,  $\int_F Q dm = 1$ , so that  $\int_F L_i^2 dm = \frac{1}{n}$  for every  $i$ . By Lemma 5, it follows that  $\langle \tilde{v}_i, w \rangle = L_i(\tilde{R}(w))$  for almost every  $w \in S\tilde{M}$ .  $\square$

The lemma just proved implies that (1) is true almost everywhere for almost every trajectory of the geodesic flow; this means that for almost every  $w \in S\tilde{M}$ , if  $\gamma$  is a geodesic with  $\gamma'(0) = w$ , then the function  $\langle \tilde{v}_i, \gamma' \rangle = (B_i \circ \gamma)'$  is defined almost everywhere. Moreover it is equal to the constant  $L_i(R(\gamma))$ . Since this function is Lipschitz, it is linear. Thus,

$$(2) \quad (B_i \circ \gamma)' \equiv L_i(R(\gamma)), \quad t \in \mathbb{R}.$$

Since  $Q|_F \geq 1$ , the relation  $\int_F Q \, dm = 1$  implies that  $m$ -almost everywhere on  $F$  we have  $Q = 1$ . By the definition of  $m$ , this means that

$$(3) \quad Q(R(w)) = 1$$

for almost all  $w \in \widetilde{SM}$ . Since  $Q$  is nondegenerate, there is no loss of generality in assuming that  $L_1, \dots, L_n$  are linearly independent.

Consider the map

$$\tilde{f} = (\tilde{B}_1, \dots, \tilde{B}_n) : \widetilde{M} \rightarrow \mathbb{R}^n.$$

We endow  $\mathbb{R}^n$  with the Euclidean structure corresponding to the quadratic form  $Q$  under the isomorphism

$$I = (L_1, \dots, L_n) : V \rightarrow \mathbb{R}^n.$$

For almost every geodesic  $\gamma : \mathbb{R} \rightarrow \widetilde{M}$  we obtain

$$(\tilde{f} \circ \gamma)' = (L_1(\tilde{R}(\gamma)), \dots, L_n(\tilde{R}(\gamma))) = I(\tilde{R}(\gamma)).$$

Since for almost every geodesic  $\gamma$  the vector  $I(\tilde{R}(\gamma))$  is a unit vector with respect to the new Euclidean structure, the image  $\tilde{f}(\gamma)$  is a straight line with constant unit velocity.

Now we prove Proposition 1.

*Proof.* Since  $\tilde{f}$  commutes with the group  $\Gamma$  of integral translations on  $\widetilde{M}$  and  $\mathbb{R}^n$ ,  $\tilde{f}$  induces a map  $f : M \rightarrow T^n$ , where  $T^n$  is a flat torus. The homomorphism of fundamental groups induced by  $f$  is an isomorphism, which implies statement (3) of Proposition 1. The map  $f$  is Lipschitz because so is  $\tilde{f}$ .

Recall that  $M'$  denotes the complement of the  $(n-2)$ -skeleton of  $M$ .

We show that  $f|_{M'} \rightarrow T^n$  is a local isometry. Consider a convex neighborhood  $U \in M'$  and fix two points  $x, y \in U$ . For any neighborhoods  $U_x, U_y \subset U$  of  $x$  and  $y$ , let  $V(U_x, U_y)$  be the set of initial velocity vectors of all shortest paths starting in  $U_x$  and ending in  $U_y$ . Since for almost every geodesic  $\gamma : [a, b] \rightarrow M$  the image  $f \circ \gamma$  is a straight line with a constant unit speed and  $\mu_L V(U_x, U_y) > 0$ , there exist two points  $x' \in U_x$  and  $y' \in U_y$  such that  $f$  preserves the distance between them. Since  $U_x$  and  $U_y$  are arbitrary and  $f$  is continuous,  $f$  preserves the distance between  $x$  and  $y$ . Thus,  $f|_U$  preserves distances.

Since  $M'$  and  $T^n$  are  $n$ -dimensional manifolds, and  $f|_{M'}$  preserves the distances, for any  $x \in M'$  the image of some neighborhood of  $x$  is a neighborhood of  $f(x)$ , and we see that  $f$  is an open map.  $\square$

#### §4. $f$ IS AN ISOMETRY

The following Lemma 8 is an obvious consequence of Proposition 1(1).

**Lemma 8.**  $f|_{M'}$  preserves the lengths of curves.

**Lemma 9.** The map  $f|_{M'} : M' \rightarrow f(M')$  is bijective, and  $f : M \rightarrow T^n$  is surjective. As a consequence (because  $f|_{M'}$  is a local isometry), the map  $(f|_{M'})^{-1}$  is well defined, is continuous, and preserves the lengths of curves.

*Proof.* Recall that  $M$  is homotopy equivalent to an  $n$ -dimensional torus. Consequently, the  $n$ -homology group of  $M$  is isomorphic to  $\mathbb{Z}$ . We fix an isomorphism between  $H_n(T^n)$  and  $\mathbb{Z}$  and choose a generator of  $H_n(M)$ . The induced homomorphism  $f_* : H_n(M) \rightarrow H_n(T^n) = \mathbb{Z}$  takes the generator of  $H_n(M)$  to some integer; this integer is called the degree of  $f$ . We show that the degree of  $f$  is  $\pm 1$ . Since the universal covering space of  $M$  is contractible, the induced homomorphism  $f_*$  determines the homotopy type of  $f$ . Proposition 1(3) shows that  $f_*$  is an isomorphism; then  $f$  is a homotopy equivalence. Thus, the degree of  $f$  is  $\pm 1$ .

The choice of generators of the homology group fixes orientations of the manifolds  $M' \subset M$  and  $T^n$ . We define the degree of  $f$  at  $x \in M'$  to be equal to 1 if  $d_x f$  preserves the orientations of the tangent spaces at  $x$ , and to  $-1$  if  $d_x f$  reverses the orientations. Suppose  $y \in T^n$  is a regular point, i.e., the preimage  $f^{-1}(y) = x_1, \dots, x_l$  is contained in  $M'$ . As in the case of Riemannian manifolds, it can be proved that the degree of  $f$  is the sum of the degrees of  $f$  at the points  $x_1, \dots, x_l$ . Hence,  $f$  is surjective.

Since  $M$  is a pseudomanifold that is homotopy equivalent to an  $n$ -dimensional torus, the space  $M'$  is connected. Indeed, assume the contrary; then the group  $H_n(M, \mathbb{Z}_2)$  contains two nonzero elements. Since  $f|_{M'}$  is a local isometry, it preserves the orientation of tangent spaces everywhere, or it reverses these orientations. Consequently, the degree of  $f$  is constant at the points  $x_1, \dots, x_l$ . Since the degree of  $f$  is 1, this means that each regular point has a unique preimage. By the definition of a regular point, it follows that all points having two or more preimages are contained in  $f^{-1}(f(M \setminus M'))$ . We put  $J = f^{-1}(f(M \setminus M'))$ . Observe that the dimension of  $J$  does not exceed  $n - 2$ .

Suppose that  $f|_{M'}$  is not injective. Let  $y \in f(M')$  be a point with more than one preimage in  $M'$ , and let  $x_1, x_2$  be two such preimages. Let  $D_{r_0}(x_1), D_{r_0}(x_2)$  be balls centered at  $x_1$  and  $x_2$  and such that the restriction of  $f$  to these balls is an isometry. Since the dimension of  $J$  is at most  $n - 2$ , there exists a point  $x_3 \in D_{r_0}(x_1) \cap M \setminus J$ . The image of this point coincides with an image of some point contained in  $D_{r_0}(x_2)$ , which contradicts the fact that  $f$  is injective on  $M \setminus J$  ( $x_3 \in M \setminus J$ ).  $\square$

We complete the proof of Theorem 1 by the following statement.

**Lemma 10.** *The map  $f : M \rightarrow T^n$  is an isometry.*

*Proof.* We show that  $f$  is noncontracting and nonexpanding. Every path in  $M$  can be approximated by a piecewise differentiable path of almost the same length. We can move each of the corresponding pieces to the interior of an appropriate  $n$ -simplex, leaving the endpoints fixed and almost length preserving.

The map  $f$  preserves the lengths of these pieces (see Lemma 8). Therefore, the map is nonexpanding.

Now we show that  $f$  is noncontracting. Let  $x, y \in M$  be arbitrary points. Given  $\varepsilon > 0$ , we let  $x', y' \in M'$  be points such that  $\rho(x, x') < \varepsilon$  and  $\rho(y, y') < \varepsilon$ . Since  $f$  is nonexpanding, we have  $|(f(x), f(x'))| < \varepsilon$  and  $|(f(y), f(y'))| < \varepsilon$ , where  $|\langle \cdot, \cdot \rangle|$  denotes the metric on the flat torus.

Since  $f$  is Lipschitz and surjective, the Hausdorff dimension of the set  $T^n \setminus f(M')$  does not exceed  $n - 2$ . Therefore, the shortest path  $[f(x'), f(y')] \in T^n$  can be approximated by a path in  $f(M')$  with almost the same length and the same endpoints. Let  $s : [a, b] \rightarrow f(M')$  be a path that joins  $f(x')$  and  $f(y')$  and such that the length of  $s$  differs from  $|f(x'), f(y')|$  by less than  $\varepsilon$ . Since  $(f|_{M'})^{-1}$  preserves distances, the length of the path  $s \circ (f|_{M'})^{-1} : [a, b] \rightarrow M'$ , which joins  $x'$  and  $y'$ , differs from  $|f(x'), f(y')|$  by less than  $\varepsilon$ . Thus,

$$\rho(x, y) < \rho(x', y') + 2\varepsilon < |f(x'), f(y')| + 3\varepsilon < |f(x), f(y)| + 5\varepsilon.$$

Therefore,  $f$  is noncontracting.  $\square$

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Received 18/FEB/2003

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