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# INTEGRAL REPRESENTATIONS AND EMBEDDING THEOREMS FOR FUNCTIONS DEFINED ON THE HEISENBERG GROUPS $\mathbb{H}^n$

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Abstract. Integral representations of Sobolev type are obtained for functions defined on the Heisenberg group $\mathbb{H}^n$. These representations are employed to prove embedding theorems, Poincaré inequalities, and coercive estimates for functions defined on regions in $\mathbb{H}^n$.

## §0. Introduction

**0.1.** It is well known that integral representations of functions defined on regions in Euclidean spaces have considerable application to function spaces, partial differential equations, cubature formulas, and other areas. An intensive study of these fields was initiated by S. L. Sobolev in his fundamental papers in 1936–1938. The theory of spaces of functions with distributional derivatives is described in Sobolev's book [1], and also in the monographs by J. Nečas [2], S. M. Nikol′skiĭ [3], I. M. Stein [4], O. V. Besov, V. P. Il′in, and S. M. Nikol′skiĭ [5], V. M. Gol′dshteĭn and Yu. G. Reshetnyak [6], V. G. Maz′ya [7], D. R. Adams and L. I. Hedberg [8], V. I. Burenkov [9], Yu. G. Reshetnyak [10], and in several books by other authors. For different ways of deriving integral representations, see also the papers [11]–[17].

The topicality of the theory of Sobolev spaces on Heisenberg groups is explained by numerous applications of it to the study of solutions of subelliptic differential equations, quasiconformal analysis, and many other related problems (see, e.g., [18]–[24]). The Heisenberg groups $\mathbb{H}^n$ represent the best known and, in many respects, a model case of the Carnot–Carathéodory spaces. The latter are smooth manifolds with distinguished tangent subbundle satisfying certain algebraic conditions. The vector fields of this subbundle are called horizontal. The curves for which the tangent vectors are contained in the distinguished subbundle are also called horizontal. The Carnot–Carathéodory distance between two points is equal to the infimum of the lengths of the horizontal curves connecting these points. The Carnot–Carathéodory metric is not equivalent to the Riemann metric. The geometry of the Carnot–Carathéodory spaces was studied by M. Gromov in [25, 26], A. Nagel, E. M. Stein, and S. Wainger in [27], P. Pansu in [26, 28], and by other authors.

The Sobolev classes of functions on regions of Carnot–Carathéodory spaces are defined by derivatives along vector fields of the distinguished subbundle. The development of the theory of such function spaces was stimulated by the study of regularity properties of subelliptic differential equations. In particular, to generalize the iterative technique of Moser, we need to prove the Poincaré and Sobolev inequalities for functions defined on a ball in the Carnot–Carathéodory space. This line of investigation is related to the

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papers of D. Jerison [29]–[31], B. Franci [32]–[37], R. L. Wheeden [33, 36, 37], L. Capogna [38], D. Danielli [39, 40], N. Garofallo [38], [40]–[42], D. M. Nhieu [40, 42], P. Hajłazs [43, 44], J. Heinonen [45], P. Koskela [43, 45], G. Lu [36, 37], [46]–[49], O. Martio [44], S. K. Vodopyanov [50]–[58], A. V. Greshnov [54]–[56], [59], and other authors.

At present, by integral representations of functions defined in Carnot–Carathéodory spaces, some authors mean inequalities of the form

$$|f(x) - C_1| \le C_2 \int_{B(z,C_3 r)} \frac{|\nabla_{\mathcal{L}} f|(y)}{\rho(x,y)^{\nu-1}}\, dy,$$

where $x \in B(z,r)$, and the constants $C_2$ and $C_3$ are independent of $x$, $r$, and $f$. The Poincaré and Sobolev inequalities are obtained from this relation. However, many subtler results cannot be derived from this inequality. Among those are, for example, coercive estimates for differential operators, which are expressed in terms of linear combinations of derivatives of some order along vector fields in the standard basis of the horizontal subbundle. In the sequel, we call such operators homogeneous linear differential operators with constant coefficients in the sense of the standard basis of the horizontal subbundle.

It is well known that coercive estimates provide an important tool for the study of systems of partial differential equations. By a coercive estimate for a differential operator $Q$, we usually mean either an inequality of the form

$$\|u\|_{W_p^k(\Omega)} \le C(\Omega, Q)\big(\|Qu\|_{L_p(\Omega)} + \|u\|_{L_1(\Omega)}\big)$$

for an arbitrary vector-valued function $u$, or an inequality of the form

$$\|u\|_{W_p^k(\Omega)} \le C(\Omega, Q)\|Qu\|_{L_p(\Omega)}$$

for a compactly supported vector-valued function. In 1907, Korn [60] proved that such inequalities are valid for the operator $Q_1 = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$ (the *stress tensor*) for $p = 2$. Much later, in the papers [61, 62] by N. Aronszajn and K. T. Smith and also in O. V. Besov's paper [63], these inequalities were proved for a sufficiently large class of operators acting on functions defined on regions of Euclidean spaces (see also [64]–[66]).

For the operators

$$Q_1 = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$$

and

$$Q_2 = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big) - \frac{1}{n}\operatorname{Tr}\nabla u,$$

Yu. G. Reshetnyak [10, 14] obtained a stronger version of coercive estimates. These estimates were used in quasiconformal analysis for proving stability theorems [10].

In the present paper, we prove the strong version of coercive estimates for the homogeneous linear differential operators with constant coefficients and finite-dimensional kernel in the sense of the standard basis of the horizontal subbundle of the group $\mathbb{H}^n$.

We use the ideas and classical approaches to the theory of spaces of functions with distributional derivatives, initiated by S. L. Sobolev, O. V. Besov, V. I. Burenkov, V. G. Maz′ya, Yu. G. Reshetnyak, and other authors.

**0.2.** We present the basic definitions used in the paper. The points of the group $\mathbb{H}^n$ will be identified with points of the space $\mathbb{R}^{2n+1}$. The group multiplication is given by the formula

$$(x', x'', x_{2n+1}) \cdot (y', y'', y_{2n+1}) = (x' + y', x'' + y'', x_{2n+1} + y_{2n+1} - 2\langle x', y''\rangle + 2\langle x'', y'\rangle),$$

where $x' = (x_1, \dots, x_n)$, $x'' = (x_{n+1}, \dots, x_{2n})$, $y' = (y_1, \dots, y_n)$, $y'' = (y_{n+1}, \dots, y_{2n})$, and $\langle x', y''\rangle = x_1 y_{n+1} + \dots + x_n y_{2n}$.

The Heisenberg metric is defined as follows: $\rho(p,q) = |p^{-1}\cdot q|$, where

$$|(x', x'', x_{2n+1})| = ((x'^2 + x''^2)^2 + x_{2n+1}^2)^{1/4}.$$

This metric is not equivalent to the Euclidean one. It is easily seen that the topology of the Heisenberg group is equivalent to the Euclidean topology. Accordingly, a region in $\mathbb{H}^n$ is identified with an open connected subset of $\mathbb{R}^{2n+1}$.

The left-invariant vector fields $X_i = \frac{\partial}{\partial x_i} + 2x_{i+n}\frac{\partial}{\partial x_{2n+1}}$, $i = 1,\dots,n$, $X_i = \frac{\partial}{\partial x_i} - 2x_{i-n}\frac{\partial}{\partial x_{2n+1}}$, $i = n+1,\dots,2n$, form the standard basis of the horizontal subbundle. Together with the vector field $X_{2n+1} = \frac{\partial}{\partial x_{2n+1}}$, they form the standard basis of the Lie algebra corresponding to the group $\mathbb{H}^n$. We have the following nontrivial commutation relations:

$$[X_j, X_{j+n}] = -4X_{2n+1}, \quad j = 1,\dots,n. \tag{0.1}$$

It can easily be seen that the left shift $l_x : y \mapsto x\cdot y$ is a diffeomorphism from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+1}$, and $\det(Dl_x) \equiv 1$.

By $\nu$ we denote the Hausdorff dimension of $\mathbb{H}^n$ with respect to the Heisenberg metric, which is equal to $2n+2$.

The family of mappings $\delta_t : (x', x'', x_{2n+1}) \mapsto (tx', tx'', t^2x_{2n+1})$, $t > 0$, is called the one-parametric family of dilations. Obviously, $|\delta_t(x)| = t|x|$.

The biinvariant Haar measure on the group $\mathbb{H}^n$ coincides with Lebesgue measure in the space $\mathbb{R}^{2n+1}$. It is easy to show that $|\delta_t(S)| = t^\nu|S|$ for every measurable subset $S$ in $\mathbb{H}^n$.

To an arbitrary $(2n+1)$-dimensional multiindex $\alpha$, we assign the number $|\alpha|_h = \alpha_1 + \dots + \alpha_{2n} + 2\alpha_{2n+1}$. The symbol $X^\alpha$ will denote the differential operator $X_1^{\alpha_1}\cdots X_{2n+1}^{\alpha_{2n+1}}$.

Consider the following norms:

$$\|f\|_{V_p^k(\Omega)} = \sum_{|\alpha|_h \le k} \|X^\alpha f\|_{L_p(\Omega)},$$
$$\|f\|_{L_p^k(\Omega)} = \sum_{|\alpha|_h = k} \|X^\alpha f\|_{L_p(\Omega)},$$
$$\|f\|_{W_p^k(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|_h = k} \|X^\alpha f\|_{L_p(\Omega)}.$$

For an arbitrary homogeneous linear $k$th order differential operator $Q$ with constant coefficients in the sense of the standard basis of the horizontal subbundle, we consider the norms

$$\|f\|_{W_{Q,p}(\Omega)} = \|f\|_{L_p(\Omega)} + \|Qf\|_{L_p(\Omega)},$$
$$\|f\|_{V_{Q,p}(\Omega)} = \|f\|_{V_p^{k-1}(\Omega)} + \|Qf\|_{L_p(\Omega)}.$$

We define the space $W_p^k(\Omega)$ as the completion of the set $\{f \in C^\infty(\Omega) \mid \|f\|_{W_p^k(\Omega)} < \infty\}$ with respect to the norm $\|f\|_{W_p^k(\Omega)}$. The spaces $L_p^k(\Omega)$, $V_p^k(\Omega)$, $W_{Q,p}(\Omega)$, and $V_{Q,p}(\Omega)$ are defined in a similar way.

The space $V_0^{k,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{V_p^k(\Omega)}$.

The space $C^s(\overline{\Omega})$ consists of all functions for which the horizontal derivatives of order $[s]$ are uniformly Hölder continuous with respect to the Heisenberg metric with exponent $s - [s]$ ($[s]$ is the integral part of $s$).

Now, we define the main classes of regions considered in the paper.

A region $\Omega \subset \mathbb{H}^n$ satisfies the *cone condition* with constant $R > 0$ if, for each point $x \in \Omega$, there is a ball $B_x \subset \Omega$ such that the cone $\{x\cdot\delta_t(x^{-1}\cdot y) \mid y \in \overline{B}_x,\ 0 < t \le 1\}$ lies

in $\Omega$ and the radii of the balls $B_x$ are uniformly bounded below by a positive constant $R$. An equivalent definition can be found in [38].

A region $\Omega \subset \mathbb{H}^n$ satisfies the *strong Lipschitz condition* if every point $x^0 \in \partial\Omega$ has a neighborhood $U_{x^0}$ such that, in some Cartesian coordinates, the set $U_{x^0} \cap \Omega$ can be represented by an inequality $x_N > f_{x^0}(x_1, \dots, x_{N-1})$ $(N = 2n+1)$, where either the function $f_{x^0}$ satisfies the Lipschitz condition with constant $L(f_{x^0})$ and the cone $x_N - x_N^0 > L(f_{x^0})\|(x_1, \dots, x_{N-1}) - (x_1^0, \dots, x_{N-1}^0)\|$ has nonempty intersection with the horizontal plane attached to the point $x^0$, or the function $f_{x^0}$ belongs to the class $C^{1,1}(U_{x^0})$.

A region $\Omega \subset \mathbb{H}^n$ satisfies the $(\varepsilon, \delta)$-*condition* if, for arbitrary $x$, $y \in \Omega$ such that $\rho(x, y) < \delta$, there is a curve $\gamma$ connecting $x$ and $y$, rectifiable in the sense of the Heisenberg metric, and satisfying the inequalities

$$l(\gamma) \le \frac{1}{\varepsilon}\rho(x, y), \quad \operatorname{dist}(z, \partial\Omega) \ge \epsilon \min(\rho(x, z), \rho(y, z)) \text{ for all } z \in \gamma,$$

where $l(\gamma)$ is the length of $\gamma$ in the sense of the Heisenberg metric.

We say that an open set $U \subset \mathbb{H}^n$ is *star-like* in a region $\Omega$ with respect to a ball $B \Subset \Omega$ if for arbitrary $x \in U$ and $y \in B$ the point $x \cdot \delta_t(x^{-1} \cdot y)$ belongs to $\Omega$ for all $t \in (0, 1]$.

By a *horizontal polynomial* of degree at most $l$ we mean a function for which all horizontal derivatives of order $(l+1)$ (i.e., the derivatives along the horizontal vector fields $X_i$, $i = 1, \dots, 2n$) are zero.

## §1. Integral representations of functions of Sobolev classes in regions of Heisenberg groups

### 1.1. Integral representations by first horizontal derivatives for functions on bounded regions in the group $\mathbb{H}^n$.

**Theorem 1.** *Let $\Omega' \subset \mathbb{H}^n$ be a star-like region in $\Omega \subset \mathbb{H}^n$ with respect to a Heisenberg ball $B(a, r)$, and let $\varphi \in C_0^\infty(B(a, r))$ be a function satisfying $\int_\Omega \varphi(x)\,dx = 1$. Then for each function $f$ of class $C^\infty(\Omega)$ we have the following integral representation in $\Omega'$:*

$$\begin{aligned} f(x) = &\int_\Omega f(y)\varphi(y)\,dy \\ &+ \int_\Omega \Gamma(x, y; \varphi) \\ &\quad \times \Bigg(\sum_{i=1}^{2n}(y_i - x_i)X_i f(y) \\ &\qquad + 2(y_{2n+1} - x_{2n+1} + 2\langle x', y''\rangle - 2\langle x'', y'\rangle)X_{2n+1} f(y)\Bigg)\,dy, \end{aligned} \tag{1.1}$$

*where $\Gamma(x, y; \varphi) = -\int_1^\infty \varphi(x \cdot \delta_t(x^{-1} \cdot y))t^{2n+1}\,dt$.*

*Proof.* For every $x \in \mathbb{R}^{2n+1}$, we consider the mapping

$$v_x : (0, \infty) \times [0, 2\pi) \times [0, \pi)^{2n-2} \times (-\infty, \infty) \to \mathbb{R}^{2n+1}$$

given by the formula

$$v_x(r, \overline{\alpha}, \beta) = l_x(v(r, \overline{\alpha}, \beta)) = l_x((1 + \beta^2)^{-1/4}\theta(r, \overline{\alpha}), (1 + \beta^2)^{-1/2}\beta r^2), \tag{1.2}$$

where $\overline{\alpha} \in [0, 2\pi) \times [0, \pi)^{2n-2}$, and $\theta : (0, \infty) \times [0, 2\pi) \times [0, \pi)^{2n-2} \to \mathbb{R}^{2n}$ is the mapping that gives rise to the standard polar coordinate system in the Euclidean space. In the sequel, we need the following properties of $\theta$:

1) $\theta|_{(0,\infty)\times(0,2\pi)\times(0,\pi)^{2n-2}}$ is a diffeomorphism;

2) $\|\theta(r,\overline{\alpha})\| = r$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{2n}$;
3) $\det(D\theta(r,\overline{\alpha})) = A(\overline{\alpha})r^{2n-1}$;
4) $\frac{\partial}{\partial r}\theta(r,\overline{\alpha}) = \frac{1}{r}\theta(r,\overline{\alpha})$.

We note that

$$\begin{aligned} v(r,\overline{\alpha},\beta) &= \delta_r\big(v(1,\overline{\alpha},\beta)\big), \\ \rho(x, v_x(r,\overline{\alpha},\beta)) &= \rho(0, v(r,\overline{\alpha},\beta)) \\ &= (\|(1+\beta^2)^{-1/4}\theta(r,\overline{\alpha})\|^4 + ((1+\beta^2)^{-1/2}\beta r^2)^2)^{1/4} = r. \end{aligned}$$

Moreover, if $v(r,\overline{\alpha},\beta) = (z,t)$ $(z \in \mathbb{R}^{2n}, t \in \mathbb{R})$, then $r = |(z,t)|$, $\beta = \frac{t}{\|z\|^2}$, and $\overline{\alpha}$ is the vector of angles that are involved in the coordinates of $z$ in the polar system. Thus, the mapping $v_x|_\Pi$ is a diffeomorphism, where $\Pi = (0,\infty)\times(0,2\pi)\times(0,\pi)^{2n-2}\times(-\infty,\infty)$.

Now, it is easy to calculate $\det(Dv(r,\overline{\alpha},\beta))$. For this, we put

$$\theta(r,\overline{\alpha}) = (\theta_1(r,\overline{\alpha}),\dots,\theta_{2n}(r,\overline{\alpha})).$$

We obtain

$$Dv(r,\overline{\alpha},\beta) = \begin{pmatrix} \frac{1}{(1+\beta^2)^{\frac14}}\frac{\partial\theta_1}{\partial r} & \frac{1}{(1+\beta^2)^{\frac14}}\frac{\partial\theta_1}{\partial\alpha_1} & \dots & \frac{1}{(1+\beta^2)^{\frac14}}\frac{\partial\theta_1}{\partial\alpha_{2n-1}} & \frac{\partial}{\partial\beta}((1+\beta^2)^{-\frac14})\theta_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(1+\beta^2)^{\frac14}}\frac{\partial\theta_{2n}}{\partial r} & \frac{1}{(1+\beta^2)^{\frac14}}\frac{\partial\theta_{2n}}{\partial\alpha_1} & \dots & \frac{1}{(1+\beta^2)^{\frac14}}\frac{\partial\theta_{2n}}{\partial\alpha_{2n-1}} & \frac{\partial}{\partial\beta}((1+\beta^2)^{-\frac14})\theta_{2n} \\ \frac{2\beta r}{(1+\beta^2)^{\frac12}} & 0 & \dots & 0 & \frac{\partial}{\partial\beta}(\beta(1+\beta^2)^{-\frac12})r^2 \end{pmatrix}, \tag{1.3}$$

$\frac{\partial}{\partial\beta}((1+\beta^2)^{-\frac14}) = -\frac{\beta}{2}(1+\beta^2)^{-\frac54}$, $\frac{\partial}{\partial\beta}(\beta(1+\beta^2)^{-\frac12}) = (1+\beta^2)^{-\frac12} - \beta^2(1+\beta^2)^{-\frac32} = (1+\beta^2)^{-\frac32}$.

We find the determinant of the above matrix expanding it along the last row. We have

$$\begin{aligned} \det(Dv(r,\overline{\alpha},\beta)) &= (1+\beta^2)^{-\frac32}r^2\Big((1+\beta^2)^{-\frac14}\Big)^{2n}\det(D\theta(r,\overline{\alpha})) \\ &\quad + 2\beta r(1+\beta^2)^{-\frac12}\Big((1+\beta^2)^{-\frac14}\Big)^{2n-1}\Big(-\frac{\beta}{2}(1+\beta^2)^{-\frac54}\Big)\det M, \end{aligned}$$

where

$$M = \begin{pmatrix} \frac{\partial\theta_1}{\partial\alpha_1} & \dots & \frac{\partial\theta_1}{\partial\alpha_{2n-1}} & \theta_1 \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial\theta_{2n}}{\partial\alpha_1} & \dots & \frac{\partial\theta_{2n}}{\partial\alpha_{2n-1}} & \theta_{2n} \end{pmatrix}.$$

Using the identity $\theta_i = r\frac{\partial}{\partial r}\theta_i$ and interchanging the first and the last rows, we see that

$$\det M = (-1)^{2n-1}r\det\big(D\theta(r,\overline{\alpha})\big) = -A(\overline{\alpha})r^{2n}.$$

Finally, we obtain the relation

$$\begin{aligned} &\det\big(Dv(r,\overline{\alpha},\beta)\big) \\ &= (1+\beta^2)^{-\frac32-\frac{n}{2}}A(\overline{\alpha})r^{2+(2n-1)} + \beta^2(1+\beta^2)^{-\frac12-\frac{2n-1}{4}-\frac54}A(\overline{\alpha})r^{2n+1} \\ &= A(\overline{\alpha})r^{2n+1}((1+\beta^2)^{-\frac{n+3}{2}} + \beta^2(1+\beta^2)^{-\frac{n+3}{2}}) = A(\overline{\alpha})(1+\beta^2)^{-\frac{n+1}{2}}r^{2n+1}, \end{aligned}$$

whence

$$\det(Dv_x(r,\overline{\alpha},\beta)) = \det(D(l_x\circ v)(r,\overline{\alpha},\beta)) = A(\overline{\alpha})(1+\beta^2)^{-\frac{n+1}{2}}r^{2n+1}. \tag{1.4}$$

We fix a smooth function $\varphi$ with support in the Heisenberg ball $B(a,R)\subset\mathbb{R}^{2n+1}$ and satisfying the condition $\int_\Omega \varphi(x)\,dx = 1$.

Let $f$ be an arbitrary smooth function $f$ defined on $\Omega$, and let $x_0 \in \Omega'$. We introduce the functions

$$\begin{aligned}
\overline{\Phi}(x_0; r, \overline{\alpha}, \beta) &= -\int_r^\infty \varphi(v_{x_0}(s, \overline{\alpha}, \beta)) \det(Dv_{x_0}(s, \overline{\alpha}, \beta))\, ds, \\
\Phi(x_0; y) &= \overline{\Phi}(x_0; v_{x_0}^{-1}(y)), \\
\overline{f} &= f \circ v_{x_0}, \\
\overline{\Psi}(x_0; r, \overline{\alpha}, \beta) &= \overline{\Phi}(x_0; r, \overline{\alpha}, \beta) \cdot \overline{f}(r, \overline{\alpha}, \beta).
\end{aligned}$$

Then

$$\frac{\partial \overline{\Psi}}{\partial r} = \overline{f} \frac{\partial \overline{\Phi}}{\partial r} + \overline{\Phi} \frac{\partial \overline{f}}{\partial r}. \tag{1.5}$$

It is easily seen that the support of $\overline{\Phi}(x_0, \cdot)$ lies in $v_{x_0}^{-1}(\Omega)$, because the domain $\Omega'$ is star-like in $\Omega \subset \mathbb{H}^n$ with respect to $B(a, r)$. Therefore, the functions $\frac{\partial \overline{\Psi}}{\partial r}$, $\overline{f} \frac{\partial \overline{\Phi}}{\partial r}$, and $\overline{\Phi} \frac{\partial \overline{f}}{\partial r}$ can be extended by zero to continuous functions on the entire set $\Pi \setminus v_{x_0}^{-1}(\Omega)$. Integrating (1.5) from $r$ to $\infty$ and letting $r \to 0$, we obtain

$$f(x_0) \int_0^\infty \varphi(v_{x_0}(s, \overline{\alpha}, \beta)) \det(Dv_{x_0}(s, \overline{\alpha}, \beta))\, ds = \int_0^\infty \left( \overline{f} \frac{\partial \overline{\Phi}}{\partial r} + \overline{\Phi} \frac{\partial \overline{f}}{\partial r} \right) dr. \tag{1.6}$$

Now, integrating both sides of (1.6) with respect to $\alpha_1$ from 0 to $2\pi$, with respect to $\alpha_i$ from 0 to $\pi$, $i = 2, \ldots, 2n-1$, and with respect to $\beta$ from $-\infty$ to $\infty$, we get

$$f(x_0) \int_\Pi \varphi(v_{x_0}(r, \overline{\alpha}, \beta)) \det(Dv_{x_0}(r, \overline{\alpha}, \beta))\, dr d\overline{\alpha} d\beta = \int_\Pi \left( \overline{f} \frac{\partial \overline{\Phi}}{\partial r} + \overline{\Phi} \frac{\partial \overline{f}}{\partial r} \right) dr d\overline{\alpha} d\beta.$$

In the integral on the left, we make the change of variables $r, \overline{\alpha}, \beta \mapsto x_1, \ldots, x_{2n+1}$, obtaining

$$f(x_0) = f(x_0) \int_{v_{x_0}(\Pi)} \varphi(x)\, dx = \int_\Pi \left( \overline{f} \frac{\partial \overline{\Phi}}{\partial r} + \overline{\Phi} \frac{\partial \overline{f}}{\partial r} \right) dr d\overline{\alpha} d\beta.$$

Next, we have $\frac{\partial \overline{\Phi}}{\partial r} = \varphi(v_{x_0}(r, \overline{\alpha}, \beta)) \det(Dv_{x_0}(r, \overline{\alpha}, \beta))$, whence

$$\int_\Pi \overline{f} \frac{\partial \overline{\Phi}}{\partial r}\, dr d\overline{\alpha} d\beta = \int_{v_{x_0}(\Pi)} f(x) \varphi(x)\, dx = \int_\Omega f(x) \varphi(x)\, dx.$$

Since $\frac{\partial \overline{f}}{\partial r} = \frac{\partial}{\partial r} f(v_{x_o}(r, \overline{\alpha}, \beta)) = \left\langle \nabla(f \circ l_{x_0})(v(r, \overline{\alpha}, \beta)), \frac{\partial}{\partial r} v(r, \overline{\alpha}, \beta) \right\rangle$, we can write

$$\begin{aligned}
\int_\Pi \overline{\Phi} \frac{\partial \overline{f}}{\partial r}\, dr d\overline{\alpha} d\beta &= \int_{v_{x_0}(\Pi)} \frac{1}{|\det(Dv_{x_0}(v_{x_0}^{-1}(x)))|} \Phi(x_0; x) \\
&\qquad \times \left\langle \nabla(f \circ l_{x_0})(x_0^{-1} \cdot x), \left[ \frac{\partial}{\partial r} v \right](v_{x_0}^{-1}(x)) \right\rangle dx \\
&= \int_\Omega \frac{1}{|\det(Dv_{x_0}(v_{x_0}^{-1}(x)))|} \Phi(x_0; x) \\
&\qquad \times \left\langle \nabla(f \circ l_{x_0})(x_0^{-1} \cdot x), \left[ \frac{\partial}{\partial r} v \right](v^{-1}(x_0^{-1} \cdot x)) \right\rangle dx,
\end{aligned}$$

because the function $\Phi$ is identically zero outside $\Omega$. The last two equalities imply

$$\begin{aligned} f(x_0) = &\int_\Omega f(x)\varphi(x)\,dx \\ &+ \int_\Omega \frac{1}{|\det(Dv_{x_0}(v_{x_0}^{-1}(x)))|}\Phi(x_0;x) \\ &\qquad \times \left\langle \nabla(f\circ l_{x_0})(x_0^{-1}\cdot x), \left[\frac{\partial}{\partial r}v\right](v^{-1}(x_0^{-1}\cdot x))\right\rangle dx. \end{aligned} \tag{1.7}$$

We rewrite (1.7), replacing $x_0$ by $x$ and using (1.4):

$$\begin{aligned} f(x) = &\int_\Omega f(y)\varphi(y)\,dy \\ &+ \int_\Omega \frac{1}{A(\overline{\alpha})g(\beta)|x^{-1}\cdot y|^{2n+1}}\Phi(x,y) \\ &\qquad \times \left\langle \nabla(f\circ l_{x})(x^{-1}\cdot y), \left[\frac{\partial}{\partial r}v\right](v^{-1}(x^{-1}\cdot y))\right\rangle dy, \end{aligned} \tag{1.8}$$

where $\Phi(x,y) = -\int_r^\infty \varphi(v_x(s,\overline{\alpha},\beta))A(\overline{\alpha})g(\beta)s^{2n+1}\,ds$, $(r,\overline{\alpha},\beta) = v_x^{-1}(y)$, $g(\beta) = (1+\beta^2)^{-\frac{n+1}{2}}$, and $\cdot$ denotes multiplication in $\mathbb{H}^n$.

Observe that $v_x(s,\overline{\alpha},\beta) = x\cdot v(s,\overline{\alpha},\beta) = x\cdot\delta_{\frac{s}{r}}(v(r,\overline{\alpha},\beta)) = x\cdot\delta_{\frac{s}{r}}(x^{-1}\cdot y)$. Therefore,

$$\Phi(x,y) = -\int_r^\infty \varphi(x\cdot\delta_{\frac{s}{r}}(x^{-1}\cdot y))A(\overline{\alpha})g(\beta)s^{2n+1}\,ds.$$

Using the change of variables $t = \frac{s}{r}$, $s = tr$, $ds = rdt$, we obtain

$$\Phi(x,y) = -\int_1^\infty \varphi(x\cdot\delta_t(x^{-1}\cdot y))A(\overline{\alpha})g(\beta)t^{2n+1}r^{2n+2}\,dt.$$

Substituting the latter expression in (1.8) yields

$$\begin{aligned} f(x) = &\int_\Omega f(y)\varphi(y)\,dy \\ &+ \int_\Omega |x^{-1}\cdot y|\left(-\int_1^\infty \varphi(x\cdot\delta_t(x^{-1}\cdot y))t^{2n+1}\,dt\right) \\ &\qquad \times \left\langle \nabla(f\circ l_{x})(x^{-1}\cdot y), \left[\frac{\partial}{\partial r}v\right](v^{-1}(x^{-1}\cdot y))\right\rangle dy. \end{aligned} \tag{1.9}$$

It is easy to verify that the vector $[\frac{\partial}{\partial r}v](v^{-1}(x))$ is equal to $(\frac{x_1}{|x|},\dots,\frac{x_{2n}}{|x|},2\frac{x_{2n+1}}{|x|})$. Indeed, from (1.2) and (1.3) it follows that

$$\begin{aligned} \frac{\partial}{\partial r}v(r,\overline{\alpha},\beta) &= \left((1+\beta^2)^{-\frac{1}{4}}\frac{\partial\theta_1(r,\overline{\alpha})}{\partial r},\dots,(1+\beta^2)^{-\frac{1}{4}}\frac{\partial\theta_{2n}(r,\overline{\alpha})}{\partial r},(1+\beta^2)^{-\frac{1}{2}}2\beta r\right) \\ &= \left((1+\beta^2)^{-\frac{1}{4}}\frac{1}{r}\theta_1(r,\overline{\alpha}),\dots,(1+\beta^2)^{-\frac{1}{4}}\frac{1}{r}\theta_{2n}(r,\overline{\alpha}),(1+\beta^2)^{-\frac{1}{2}}2\beta r\right) \\ &= \left(\frac{1}{r}v_1(r,\overline{\alpha},\beta),\dots,\frac{1}{r}v_{2n}(r,\overline{\alpha},\beta),\frac{2}{r}v_{2n+1}(r,\overline{\alpha},\beta)\right). \end{aligned}$$

Now, it can easily be seen that for every smooth function $g$ we have

$$\sum_{i=1}^{2n} x_i\frac{\partial}{\partial x_i}g(x) = \sum_{i=1}^{2n} x_iX_ig(x). \tag{1.10}$$

Since the vector fields $X_i$ and $X_{2n+1}$ are left-invariant, we have

$$\begin{aligned}\Big\langle \nabla(f\circ l_x)(x^{-1}\cdot y), \Big[\frac{\partial}{\partial r}v\Big](v^{-1}(x^{-1}\cdot y))\Big\rangle \\ &= \sum_{i=1}^{2n}\frac{(x^{-1}\cdot y)_i}{|x^{-1}\cdot y|}X_i(f\circ l_x)(x^{-1}\cdot y) + 2\frac{(x^{-1}\cdot y)_{2n+1}}{|x^{-1}\cdot y|}X_{2n+1}(f\circ l_x)(x^{-1}\cdot y)\\ &= \sum_{i=1}^{2n}\frac{(x^{-1}\cdot y)_i}{|x^{-1}\cdot y|}X_i f(y) + 2\frac{(x^{-1}\cdot y)_{2n+1}}{|x^{-1}\cdot y|}X_{2n+1}f(y).\end{aligned}$$

Substituting this in (1.8), we arrive at the required integral representation. □

**Lemma 1.** *Let $\Omega'\subset\mathbb{H}^n$ be a star-like region in a region $\Omega\subset\mathbb{H}^n$ with respect to a Heisenberg ball $B(a,r)$. Suppose $\Omega$ is bounded, $\Omega\subset B(0,R)$. Let $\varphi_0\in C_0^\infty(B(0,1))$ be a fixed function satisfying $\int_{B(0,1)}\varphi_0(x)\,dx=1$, and let $\varphi(x)=\frac{1}{r^{2n+2}}\varphi_0\big(\delta_{r^{-1}}(a^{-1}\cdot x)\big)$. Then the function $\Gamma(x,y;\varphi)=-\int_1^\infty\varphi(x\cdot\delta_t(x^{-1}\cdot y))t^{2n+1}\,dt$ is of class $C^\infty(\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\setminus\{x=y\})$, and is compactly supported with respect to the second argument for a fixed $x\in\Omega'$, with support in $\Omega$. Moreover, $\Gamma$ satisfies the estimates*

$$\begin{aligned}&|\Gamma(x,y;\varphi)|\\ &\quad\le \sup_{x\in\Omega}|\varphi(x)|(\operatorname{diam}(\Omega))^{2n+2}\frac{1}{|x^{-1}\cdot y|^{2n+2}}\\ &\quad= \sup_{x\in\Omega}|\varphi_0(x)|\frac{(\operatorname{diam}(\Omega))^{2n+2}}{r^{2n+2}}\frac{1}{|x^{-1}\cdot y|^{2n+2}},\quad x\in\Omega',\ \ y\in B(a,r),\\ &|X_x^\alpha X_y^\beta\Gamma(x,y;\varphi)|\\ &\quad\le C(\alpha,\beta,\varphi,R)\frac{1}{|x^{-1}\cdot y|^{2n+2+|\alpha|_h+|\beta|_h}}\\ &\quad= C\Big(\alpha,\beta,\varphi_0,R,\frac{\operatorname{diam}(\Omega)}{r}\Big)\frac{1}{|x^{-1}\cdot y|^{2n+2+|\alpha|_h+|\beta|_h}},\quad x\in\Omega',\ \ y\in B(a,r),\end{aligned}\tag{1.11}$$

*where* $\operatorname{diam}(\Omega)$ *is the diameter of* $\Omega$ *in the sense of the Heisenberg metric, and* $C$ *increases with* $R$ *and* $\frac{\operatorname{diam}(\Omega)}{r}$.

*Proof.* Since

$$\rho(x,x\cdot\delta_t(x^{-1}\cdot y))=t|x^{-1}\cdot y|,\quad t=\frac{\rho(x,x\cdot\delta_t(x^{-1}\cdot y))}{|x^{-1}\cdot y|},$$

we see that $\varphi(x\cdot\delta_t(x^{-1}\cdot y))$ is zero for all $t>\frac{\operatorname{diam}(\Omega)}{|x^{-1}\cdot y|}$. Consequently,

$$\begin{aligned}|\Gamma(x,y;\varphi)| &\le \left(\frac{\operatorname{diam}(\Omega)}{|x^{-1}\cdot y|}-1\right)\sup_{x\in\Omega}|\varphi(x)|\left(\frac{\operatorname{diam}(\Omega)}{|x^{-1}\cdot y|}\right)^{2n+1}\\ &\le \sup_{x\in\Omega}|\varphi(x)|\left(\frac{\operatorname{diam}(\Omega)}{|x^{-1}\cdot y|}\right)^{2n+2}.\end{aligned}$$

Now, taking into account that the fields $X_i$ are left-invariant and for an arbitrary differentiable function $g$ we have $X_i(g\circ\delta_t)(x)=t(X_ig)(\delta_t x)$, $i=1,\dots,2n$, we obtain

$$\begin{aligned}[X_{i,y}(\varphi\circ l_x\circ\delta_t\circ l_{x^{-1}})](y) &= [X_{i,y}(\varphi\circ l_x\circ\delta_t)](x^{-1}\cdot y)\\ &= t[X_{i,y}(\varphi\circ l_x)](\delta_t(x^{-1}\cdot y)) = t[X_i\varphi](x\cdot\delta_t(x^{-1}\cdot y)).\end{aligned}$$

Therefore,

$$X_y^\alpha\Gamma(x,y;\varphi)=-\int_1^\infty[X^\alpha\varphi](x\cdot\delta_t(x^{-1}\cdot y))t^{2n+1+|\alpha|_h}\,dt.\tag{1.12}$$

We assume that $y$ is fixed and put $v_y(x) = x \cdot \delta_t(x^{-1} \cdot y)$.

First, we observe that the following relations are valid for all $\tau, \sigma \in \mathbb{H}^n$: $\delta_t(\tau \cdot \sigma) = \delta_t\tau \cdot \delta_t\sigma$, $(\delta_t\tau)^{-1} = \delta_t(\tau^{-1})$, $\tau^{-1} = -\tau$, where $-\tau = (-\tau_1, \dots, -\tau_{2n+1})$. We have

$$v_y(x) = ((\delta_t y)^{-1} \cdot (\delta_t x^{-1})^{-1} \cdot x^{-1})^{-1} = ((\delta_t y)^{-1} \cdot \delta_t x \cdot x^{-1})^{-1} = (p(y) \cdot w(x))^{-1},$$

where $p(y) = \delta_t(y^{-1})$ is a fixed point in $\mathbb{H}^n$ and $w(x) = \delta_t x \cdot (x^{-1})$.

Let $g$ be a smooth function defined on $\mathbb{H}^n$. We denote by inv the mapping $x \mapsto x^{-1}$. Let $i \in \{1, \dots, 2n\}$. Then

$$[X_i(g \circ v_y)](x) = [X_i(g \circ ((p(y) \cdot w)^{-1}))](x) = [X_i(g \circ \mathrm{inv} \circ l_{p(y)} \circ w)](x). \tag{1.13}$$

Next, for any differentiable $h$ we have $[X_i(h \circ w)](x) = \sum_{j=1}^{2n+1} [\frac{\partial h}{\partial x_j}](w(x)) X_i w_j(x)$. On the other hand,

$$\begin{aligned} w_j(x) &= (t-1)x_j, & X_i w_j &\equiv (t-1)\delta_{ij}, \\ w_{2n+1}(x) &= (t^2-1)x_{2n+1}, & X_i w_{2n+1}(x) &= 2\,\mathrm{sgn}(n-i)(t^2-1)x_{i+n\,\mathrm{sgn}(n-i)}, \end{aligned}$$

$j = 1, \dots, 2n$. Consequently, since

$$\begin{aligned} &\left[\frac{\partial h}{\partial x_i}\right](w(x)) + 2\,\mathrm{sgn}(n-i)(t-1)x_{i+n\,\mathrm{sgn}(n-i)}\left[\frac{\partial h}{\partial x_{2n+1}}\right](w(x)) \\ &\quad = \left[\frac{\partial h}{\partial x_i}\right](w(x)) + 2\,\mathrm{sgn}(n-i)w_{i+n\,\mathrm{sgn}(n-i)}(x)\left[\frac{\partial h}{\partial x_{2n+1}}\right](w(x)) \\ &\quad = \Big[X_i h\Big](w(x)), \end{aligned}$$

we obtain

$$\begin{aligned} &[X_i(h \circ w)](x) \\ &\quad = (t-1)\left(\left[\frac{\partial h}{\partial x_i}\right](w(x)) + 2\,\mathrm{sgn}(n-i)(t+1)x_{i+n\,\mathrm{sgn}(n-i)}\left[\frac{\partial h}{\partial x_{2n+1}}\right](w(x))\right) \\ &\quad = (t-1)([X_i h](w(x)) + 4\,\mathrm{sgn}(n-i)x_{i+n\,\mathrm{sgn}(n-i)}[X_{2n+1}h](w(x))). \end{aligned}$$

Therefore,

$$|[X^\alpha(h \circ w)](x)| \le t^{|\alpha|_h} C(|x|) \max_{|\beta|_h \le 2|\alpha|_h} |[X^\beta h](w(x))|, \tag{1.14}$$

where $t \in [0, \infty)$, and $C$ increases with $|x|$.

Let $g$ be a smooth function with compact support contained in the ball $B(0, R)$. Since $X_i$ is left-invariant, we have

$$[X_i(g \circ \mathrm{inv} \circ l_{p(y)})](z) = [X_i(g \circ \mathrm{inv})](p(y)z).$$

Since

$$\begin{aligned} [X_i(g \circ \mathrm{inv})](x) &= \sum_{j=1}^{2n+1} \left[\frac{\partial g}{\partial x_j}\right](-x) X_i(-x_j) \\ &= -\left[\frac{\partial g}{\partial x_i}\right](-x) + 2\,\mathrm{sgn}(n-i)x_{i+n\,\mathrm{sgn}(n-i)}(-1)\left[\frac{\partial g}{\partial x_{2n+1}}\right](-x) \\ &= -[X_i g](-x) - 4\,\mathrm{sgn}(n-i)x_{i+n\,\mathrm{sgn}(n-i)}[X_{2n+1}g](-x), \end{aligned}$$

we have

$$
\begin{aligned}
&[X_i(g\circ \operatorname{inv}\circ l_{p(y)})](z)\\
&\qquad = -[X_i g + 4\operatorname{sgn}(n-i)(p(y)z)_{i+n\operatorname{sgn}(n-i)}X_{2n+1}g]((p(y)z)^{-1}),\\
&|[X^\beta(g\circ \operatorname{inv}\circ l_{p(y)})](z)|\\
&\qquad \le C(R)\max_{|\gamma|_h\le 2|\beta|_h}|[X^\gamma g]((p(y)z)^{-1})|,
\end{aligned}
\tag{1.15}
$$

where $C$ increases with $R$.

Comparing formulas (1.13), (1.14), and (1.15), we see that the following inequality is valid for every smooth function $g$:

$$
|X_x^\alpha[g(x\cdot\delta_t(x^{-1}\cdot y))]| \le C(R)t^{|\alpha|_h}\max_{|\beta|_h\le 4|\alpha|_h}|[X^\beta g](x\cdot\delta_t(x^{-1}\cdot y)|,
$$

where $C$ increases with $R$.

Substituting this in (1.12) and plugging $g = X^\alpha\varphi$, we obtain

$$
|X_x^\alpha X_y^\beta\Gamma(x,y;\varphi)| \le C(R)\max_{|\gamma|_h\le 4|\beta|_h}\biggl(\int_1^\infty |[X^{\alpha+\gamma}\varphi](x\cdot\delta_t(x^{-1}\cdot y))|t^{2n+1+|\alpha|_h+|\beta|_h}\,dt\biggr).
$$

Since $\varphi(x\cdot\delta_t(x^{-1}\cdot y))$ is zero for all $t > \frac{\operatorname{diam}(\Omega)}{|x^{-1}\cdot y|}$, the relationship between $\varphi$ and $\varphi_0$ implies the required inequality (1.11). □

**Theorem 2.** *Suppose that the assumptions of Lemma* 1 *are fulfilled for regions* $\Omega$ *and* $\Omega'$ *and a function* $\varphi_0$. *Then, for every function* $f$ *of class* $C^\infty(\Omega)$ *in* $\Omega'$, *the following integral representation is valid*:

$$
f(x) = \int_\Omega f(y)\varphi(y)\,dy + \int_\Omega\sum_{i=1}^{2n}K_i(x,y)X_if(y)\,dy, \quad x\in\Omega',
\tag{1.16}
$$

*where* $\varphi(x) = \frac{1}{r^{2n+2}}\varphi_0\bigl(\delta_{r^{-1}}(a^{-1}\cdot x)\bigr)$, $K_i\in C^\infty(\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\setminus\{x=y\})$, *and the functions* $K_i$ *have compact support with respect to the second argument and satisfy the inequality*

$$
|X_x^\alpha X_y^\beta K_i(x,y)| \le C\biggl(\alpha,\beta,\varphi_0,R,\frac{\operatorname{diam}(\Omega)}{r}\biggr)|x^{-1}\cdot y|^{-(2n+1+|\alpha|_h+|\beta|_h)},
$$

*where* $C$ *increases with* $R$ *and* $\frac{\operatorname{diam}(\Omega)}{r}$.

*Proof.* We rewrite formula (1.1) with the help of (0.1), obtaining

$$
\begin{aligned}
f(x) &= \int_\Omega f(y)\varphi(y)\,dy + \int_\Omega\Gamma(x,y;\varphi)\biggl(\sum_{i=1}^{2n}(y_i-x_i)X_if(y)\biggr)\,dy\\
&\quad - \int_\Omega\frac{1}{2}\Gamma(x,y;\varphi)\\
&\qquad\times\Bigl(y_{2n+1}-x_{2n+1}+2\langle x',y''\rangle-2\langle x'',y'\rangle)[X_j,X_{j+n}]f(y)\Bigr)\,dy,
\end{aligned}
\tag{1.17}
$$

where $j\le n$ is arbitrary.

Now, from the definition of a homogeneous norm we deduce the estimates

$$
\begin{gathered}
|y_i-x_i| = |(x^{-1}\cdot y)_i| \le |x^{-1}\cdot y|, \quad i=1,\dots,2n,\\
|y_{2n+1}-x_{2n+1}+2\langle x',y''\rangle-2\langle x'',y'\rangle| = |(x^{-1}\cdot y)_{2n+1}| \le |x^{-1}\cdot y|^2.
\end{gathered}
\tag{1.18}
$$

Taking into account the equalities

(1.19)
$$X_{j,y}(y_{2n+1} - x_{2n+1} + 2\langle x', y''\rangle - 2\langle x'', y'\rangle) = 2(y_{j+n} - x_{j+n}), \quad j = 1, \dots, n;$$
$$X_{j,y}(y_{2n+1} - x_{2n+1} + 2\langle x', y''\rangle - 2\langle x'', y'\rangle) = 2(x_{j-n} - y_{j-n}), \quad j = n+1, \dots, 2n,$$

and relations (1.11) and (1.18), we obtain the inequality

$$\left|X_x^\alpha X_y^\beta\big((x^{-1}\cdot y)_{2n+1}\Gamma(x,y;\varphi)\big)\right| \le C\left(\varphi_0, R, \frac{\operatorname{diam}(\Omega)}{r}\right)|x^{-1}\cdot y|^{-(2n+|\alpha|_h+|\beta|_h)}, \tag{1.20}$$

where $C$ increases with $R$ and $\frac{\operatorname{diam}(\Omega)}{r}$.

Since the function $\Gamma(x;\cdot)$ has compact support, we can integrate by parts in the last term on the right in (1.17). This yields

$$\begin{aligned} f(x) = \int_\Omega f(y)\varphi(y)\,dy + \int_\Omega \Gamma(x,y;\varphi)\left(\sum_{i=1}^{2n}(y_i - x_i)X_i f(y)\right)dy \\ + \int_\Omega \left(\sum_{i=1}^{2n}\Theta_i(x,y)X_i f(y)\right)dy, \end{aligned} \tag{1.21}$$

where the $\Theta_i(x,y)$ are of class $C^\infty(\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\setminus\{x=y\})$ and satisfy

$$|X_x^\alpha X_y^\beta \Theta_i(x,y)| \le C\left(\varphi_0, R, \frac{\operatorname{diam}(\Omega)}{r}\right)|x^{-1}\cdot y|^{-(\nu-1+|\alpha|_h+|\beta|_h)},$$

where $C$ increases with $R$ and $\frac{\operatorname{diam}(\Omega)}{r}$.

Combining the second and third terms on the right-hand side of (1.21), we obtain the required integral representation. □

**Lemma 2.** *Let $K$ be a $C^\infty(\mathbb{H}^n\setminus\{0\})$-function positively homogeneous of degree $-\nu+l$ ($l\in\mathbb{N}$), i.e., $K(\delta_t(x)) = t^{-\nu+l}K(x)$ for all $t>0$. Then, for each multiindex $\alpha$ such that $|\alpha|_h = l$, the function $X^\alpha K$ is positively homogeneous of degree $-\nu$ and satisfies the relation $\int_{S(0,1)} X^\alpha K(z)\,dz = 0$, where $S(0,1)$ is the Heisenberg sphere centered at 0 and of radius 1.*

*Proof.* The case where $l=1$ is typical. For $t>0$ and $j\in\{1,\dots,2n\}$ we have

$$[X_j(K\circ\delta_t)](z) = t[X_jK](tz).$$

On the other hand, $[X_j(K\circ\delta_t)](z) = t^{-\nu+1}[X_jK](z)$. Thus,

$$[X_jK](tz) = t^{-\nu}[X_jK](z).$$

We prove that $\int_{S(0,1)} X_jK(z)\,dz = 0$. Let $P_1 = (-1,1)^{2n+1}$ and $P_\varepsilon = (-\varepsilon,\varepsilon)^{2n}\times(-\varepsilon^2,\varepsilon^2)$. We consider the region $U_\varepsilon$ and the vector field $\overline{V}$ with components $\delta_{ij}K(x)$, $i=1,\dots,2n$, $2\operatorname{sgn}(n-j)x_{j+n\operatorname{sgn}(n-j)}K(x)$. Using the divergence theorem (the Ostrogradskiĭ–Liouville formula) for the vector field $\overline{V}$, we obtain

$$\begin{aligned} \int_{U_\varepsilon}\operatorname{div}\overline{V}(y)\,dy &= \int_{U_\varepsilon}[X_jK](y)\,dy, \\ \int_{\partial U_\varepsilon}\overline{V}\cdot\overline{n}\,d\sigma &= \int_{G_j}K(z)\,dz \\ &\quad - \int_{G_{j,\varepsilon}}K(z)\,dz + 2\operatorname{sgn}(n-j) \\ &\qquad \times\left(\int_{G_{2n+1}} z_{j+n\operatorname{sgn}(n-j)}K(z)\,dz - \int_{G_{2n+1,\varepsilon}} z_{j+n\operatorname{sgn}(n-j)}K(z)\,dz\right), \end{aligned}$$

where $G_j$ is the union of the faces $\{x_j = \text{const}\}$ of the parallelepiped $P_1$, and $G_{j,\varepsilon}$ is the union of the faces $\{x_j = \text{const}\}$ of the parallelepiped $P_\varepsilon$.

Since

$$\frac{\left|G_{j,\varepsilon}\right|}{\left|G_j\right|} = \varepsilon^{2n+1}, \ \ j = 1, \dots, 2n, \quad \frac{\left|G_{2n+1,\varepsilon}\right|}{\left|G_{2n+1}\right|} = \varepsilon^{2n},$$

the function $K(x)$ is homogeneous of degree $-\nu+1$, and the functions $x_{j+n\,\mathrm{sgn}(n-j)}K(x)$ are homogeneous of degree $-\nu+2$, we see that $\int_{\partial U_\varepsilon} \overline{V} \cdot \overline{n}\, d\sigma = 0$. Finally,

$$\int_{U_\varepsilon} [X_j K](y)\, dy = 0.$$

It remains to use the fact that the function $X_j K$ is homogeneous of degree $-\nu$, which shows that

$$\sup_\varepsilon \left| \int_{U_\varepsilon} [X_j K](y)\, dy \right| < \infty \iff \int_{S(0,1)} [X_j K](z)\, dz = 0. \qquad \square$$

**Lemma 3.** *Suppose the kernel $K(x, y^{-1} \cdot x) = K(x, z)$ is of class $C^\infty(\mathbb{H}^n \times (\mathbb{H}^n \setminus \{0\}))$ and is homogeneous of degree $-\nu + l$ ($l \in \mathbb{N}$) with respect to $z$. Let $\alpha$ be a multiindex such that $|\alpha|_h = l$. Let $\Omega$ be a bounded region. We assume that $X_x^\alpha K(x, z) \le C|z|^{-\nu+s}$, $j = 1, \dots, 2n$, where $s > 0$, and put $K_j(x, z) = K_j(x, y^{-1} \cdot x) = X_x^\alpha K_j(x, y^{-1} \cdot x)$. Then the operator taking a function $f$ to the function $PV \int_{\mathbb{H}^n} K_j(x, y^{-1} \cdot x) \overline{f}(y)\, dy$, where $\overline{f} = f$ in $\Omega$ and $\overline{f} = 0$ off $\Omega$, is bounded in the space $L_p(\Omega)$, $1 < p < \infty$.*

*Proof.* The case where $l = 1$ is typical. Let $j \in \{1, \dots, 2n\}$. Since the vector field $X_j$ is left-invariant, we have

$$K_j(x, z) = X_{j,x} K(x, z) + X_{j,z} K(x, z) = K_j'(x, z) + K_j''(x, z).$$

Since the order of singularity of $K_j'(x, z)$ does not exceed $\nu - s$, the operator $f \mapsto \int_{\mathbb{H}^n} K_j'(x, y^{-1} \cdot x) \overline{f}(y)\, dy$ is bounded in $L_p(\Omega)$. Now, we note that, for a fixed $x$, the function $K(x, z)$ satisfies the assumptions of Lemma 2. Consequently, the kernel $K_j''(x, z)$ is homogeneous of degree $-\nu$ with respect to $z$ and satisfies $\int_{S(0,1)} K_j''(x, z)\, dz = 0$. Thus, for the kernel $K_j''(x, z)$, the hypotheses of the generalization of the Zygmund–Calderón theorem to the case of the groups $\mathbb{H}^n$ are fulfilled (see [20, 22, 24]). Finally, we conclude that the operator determined by the kernel $K_j''(x, z)$ is bounded in the sense of $L_p(\Omega)$. $\square$

For the boundedness of such operators on arbitrary Carnot groups, see [20, 24].

**Proposition 1.** *The kernels $K_i(x, z)$ in the integral representation* (1.16) *can be written in the form $K_i'(x, z) + K_i''(x, z)$, where $K_i'(x, z) \in C^\infty(\mathbb{H}^n \times \mathbb{H}^n)$ and $K_i''(x, z)$ is of class $C^\infty(\mathbb{H}^n \times (\mathbb{H}^n \setminus \{0\}))$ and is homogeneous of degree $-\nu + 1$ with respect to $z$.*

*Proof.* We have

$$\begin{aligned} \Gamma(x, y; \varphi) &= \int_0^1 \varphi(x \cdot \delta_t(x^{-1} \cdot y)) t^{2n+1+|\alpha|_h}\, dt - \int_0^\infty \varphi(x \cdot \delta_t(x^{-1} \cdot y)) t^{2n+1+|\alpha|_h}\, dt \\ &= \Gamma'(x, y^{-1} \cdot x) + \Gamma''(x, y^{-1} \cdot x). \end{aligned}$$

It is easily seen that $\Gamma'(x, z) \in C^\infty(\mathbb{H}^n \times \mathbb{H}^n)$ and that the kernel $\Gamma''(x, z)$ is of class $C^\infty(\mathbb{H}^n \times (\mathbb{H}^n \setminus \{0\}))$ and is homogeneous of degree $-\nu$ with respect to $z$. It remains to use the fact that the product of homogeneous functions is a homogeneous function and that differentiation along the vector field $X_j$, $j = 1, \dots, 2n$, reduces the degree of homogeneity by one. $\square$

**1.2. Integral representations by horizontal derivatives of arbitrary order for functions on regions in Heisenberg groups.** In this section, we modify the method of obtaining integral representations developed by Yu. G. Reshetnyak and V. I. Burenkov in the Euclidean case. Since the commutation relations for vector fields of the horizontal subbundle are nontrivial, the application of this method to the Heisenberg groups is not straightforward. More specifically, the analog of Taylor's expansion on $\mathbb{H}^n$ fails to have the properties of Taylor's expansion in $\mathbb{R}^n$.

**Theorem 3.** *Let $\Omega' \subset \mathbb{H}^n$ be a star-like region in a region $\Omega \subset \mathbb{H}^n$ with respect to a Heisenberg ball $B(a,r)$. Suppose $\Omega$ is bounded, $\Omega \subset B(0,R)$ and $\varphi_0 \in C_0^\infty(B(0,1))$ satisfies the relation $\int_\Omega \varphi_0(x)\,dx = 1$. Then, for every function $f$ of class $C^\infty(\Omega)$ and every positive integer $k$, we have the following integral representation in $\Omega'$:*

$$f(x) = \int_\Omega P_k(x,y)f(y)\,dy + \int_\Omega \sum_{i_1,\dots,i_k=1}^{2n} K_{i_1\cdots i_k}(x,y)X_{i_1}\cdots X_{i_k}f(y)\,dy, \tag{1.22}$$

*where $x \in \Omega'$, $P_k(\cdot,y)$ is a horizontal polynomial of order $k-1$ such that $\operatorname{supp} P_k(x,\cdot) \subset B$ and $|P_k(x,y)| \le C_k'(r,R,\varphi_0)$, $K_{i_1\cdots i_k} \in C^\infty(\mathbb{R}^{2n+1}\times\mathbb{R}^{2n+1}\setminus\{x=y\})$, and the functions $K_{i_1\cdots i_k}$ have compact support with respect to the second argument and satisfy the inequality*

$$|X_x^\alpha X_y^\beta K_{i_1\cdots i_k}(x,y)| \le C\biggl(\alpha,\beta,\varphi_0,R,\frac{\operatorname{diam}(\Omega)}{r}\biggr)|x^{-1}\cdot y|^{-(\nu-k+|\alpha|_h+|\beta|_h)}, \tag{1.23}$$

*where $C$ increases with $R$ and $\frac{\operatorname{diam}(\Omega)}{r}$.*

*Proof.* For $i$, $j \in \mathbb{N}$, we consider the differential operators

$$A_j : h(x,y) \mapsto \sum_{i_1,\dots,i_j=1}^{2n} (x_{i_1}-y_{i_1})(x_{i_2}-y_{i_2})\cdots(x_{i_j}-y_{i_j})X_{i_1,y}X_{i_2,y}\cdots X_{i_j,y}h(x,y),$$

$$B_i : h(x,y) \mapsto (y^{-1}\cdot x)^i_{2n+1}X^i_{2n+1,y}h(x,y).$$

Since $A_1((x_{i_1}-y_{i_1})(x_{i_2}-y_{i_2})\cdots(x_{i_k}-y_{i_k})) = -k(x_{i_1}-y_{i_1})(x_{i_2}-y_{i_2})\cdots(x_{i_k}-y_{i_k})$, $B_1((y^{-1}\cdot x)^k_{2n+1}) = -k(y^{-1}\cdot x)^k_{2n+1}$, the above operators satisfy the recurrence relations $A_1A_k = A_{k+1}-kA_k$, $B_1B_k = B_{k+1}-kB_k$. Moreover, these operators commute, $A_jB_i = B_iA_j$. Indeed,

$$\begin{aligned}(A_jB_i - B_iA_j)h(x,y) = \sum_{i_1,\dots,i_j=1}^{2n} &(x_{i_1}-y_{i_1})(x_{i_2}-y_{i_2})\cdots(x_{i_j}-y_{i_j})\\ &\times X_{i_1,y}X_{i_2,y}\cdots X_{i_j,y}((y^{-1}\cdot x)^i_{2n+1})X^i_{2n+1,y}h(x,y).\end{aligned}$$

Using (1.19), we see that the terms

$$(x_{i_1}-y_{i_1})(x_{i_2}-y_{i_2})\cdots(x_{i_j}-y_{i_j})X_{i_1,y}X_{i_2,y}\cdots X_{i_j,y}((y^{-1}\cdot x)^i_{2n+1})$$

cancel.

Now, we proceed directly to the deduction of the integral representation. Let $f$ be a function of class $C^\infty(\Omega)$. For the function

$$g_k(x,y) = \sum_{2i+j\le k} \frac{1}{i!j!}B_iA_jf(y)$$

of two variables, we have $g_k(x,x) = f(x)$. Viewing $g_k$ as a function of $y$, we use the integral representation (1.1) at $y = x$, obtaining

$$f(x) = g_k(x,y)|_{y=x} = \int_\Omega g_k(x,y)\varphi(y)\,dy - \int_\Omega \Gamma(x,y)(A_1+2B_1)g_k(x,y)\,dy.$$

We have

$$\begin{aligned}A_1 g_k(x,y) &= \sum_{2i+j\le k}\frac{1}{i!j!}B_iA_1A_jf(y) = \sum_{i=0}^{[k/2]}\frac{1}{i!}B_i\sum_{j=0}^{k-2i}\frac{1}{j!}A_1A_jf(y)\\ &= \sum_{i=0}^{[k/2]}\frac{1}{i!}B_i\sum_{j=0}^{k-2i}\frac{1}{j!}(A_{j+1}-jA_j)f(y)\\ &= \sum_{i=0}^{[k/2]}\frac{1}{i!}B_i\frac{A_{k-2i+1}}{(k-2i)!}f(y).\end{aligned}$$

Similarly,

$$\begin{aligned}B_1 g_k(x,y) &= \sum_{2i+j\le k}\frac{1}{i!j!}A_jB_1B_if(y) = \sum_{j=0}^{k}\frac{1}{j!}A_j\sum_{i=0}^{[(k-j)/2]}\frac{1}{i!}B_1B_if(y)\\ &= \sum_{j=0}^{k}\frac{1}{j!}A_j\sum_{i=0}^{[(k-j)/2]}\frac{1}{i!}(B_{i+1}-iB_i)f(y)\\ &= \sum_{i=0}^{k}\frac{A_j}{j!}\frac{B_{[(k-j)/2]+1}}{([(k-j)/2])!}f(y).\end{aligned}$$

As a result, we obtain

$$\begin{aligned}f(x) &= \int_\Omega g_k(x,y)\varphi(y)\,dy\\ &\quad - \int_\Omega\biggl(\sum_{i=0}^{[k/2]}\frac{B_i}{i!}\frac{A_{k-2i+1}}{(k-2i)!}f(y) + \sum_{i=0}^{k}\frac{A_j}{j!}\frac{B_{[(k-j)/2]+1}}{([(k-j)/2])!}f(y)\biggr)\Gamma(x,y)\,dy.\end{aligned}\tag{1.24}$$

Integrating by parts, we can rewrite the first term in (1.24) as $\int_\Omega P_k(x,y)f(y)\,dy$. It can easily be seen that the function $P_k(x,y)$ satisfies all assumptions of the theorem.

The following inequality generalizes estimate (1.20) and can easily be proved:

$$\begin{aligned}&X_x^\alpha X_y^\beta((x^{-1}\cdot y)^\gamma\Gamma(x,y))\\ &\qquad\le C_{\alpha,\beta,\gamma}\biggl(\varphi_0, R, \frac{\operatorname{diam}(\Omega)}{r}\biggr)|x^{-1}\cdot y|^{-(\nu+|\alpha|_h+|\beta|_h-|\gamma|_h)},\end{aligned}\tag{1.25}$$

where $\alpha,\beta,\gamma\in\mathbb{N}^{2n+1}$ and, for $z\in\mathbb{H}^n$, the symbol $z^\gamma$ denotes $z_1^{\gamma_1}z_2^{\gamma_2}\cdots z_{2n+1}^{\gamma_{2n+1}}$.

It is easily seen that $B_iA_{k-2i+1}f(y)$ is a linear combination of horizontal derivatives of $f(y)$ of order $(k+1)$. Further, for odd $(k-j)$, the expression $A_jB_{[(k-j)/2]}$ is also a linear combination of horizontal derivatives of $f(y)$ of order $(k+1)$. For even $(k-j)$, the expression $A_jB_{[(k-j)/2]+1}$is a linear combination of horizontal derivatives of $f(y)$ of order $(k+2)$.

Finally, taking into account that the kernel $K(x,y)$ has compact support with respect to $y$, and integrating by parts the terms containing $A_jB_{[(k-j)/2]+1}$, where $(k-j)$ is even, from the integral representation (1.24) we obtain an integral representation of $f$ in terms of horizontal derivatives of order $(k+1)$. Estimates (1.23) for the kernel in the latter integral representation easily follow from (1.25). □

**Lemma 4.** *The kernels $K_{i_1\cdots i_k}(x,z)$ in the integral representation* (1.22) *can be written as $K'_{i_1\cdots i_k}(x,z)+K''_{i_1\cdots i_k}(x,z)$, where $K'_{i_1\cdots i_k}(x,z)\in C^\infty(\mathbb{H}^n\times\mathbb{H}^n)$, and $K''_{i_1\cdots i_k}(x,z)$ is*

*of class* $C^\infty(\mathbb{H}^n \times (\mathbb{H}^n \setminus \{0\}))$ *and is homogeneous of degree* $-\nu + l$ *with respect to* $z$. *Furthermore, for each multiindex* $\alpha$ *such that* $|\alpha|_h = l$, *we have* $X_x^\alpha K''_{i_1 \cdots i_k}(x,z) \le C|z|^{-\nu+s}$, *where* $s > 0$.

The proof of Lemma 4 is similar to that of Proposition 1 in Subsection 1.1.

**1.3. On some classes of regions in Heisenberg groups.** Let $B(a,r)$ denote the Heisenberg ball centered at $a$ and of radius $r$.

**Lemma 5.** *The ball* $B(a,R)$ *is star-like in the ball* $B(a,3R)$ *with respect to the ball* $B(a,R)$.

*Proof.* Indeed, let $x, y \in B(a,R)$. We must prove that $x\delta_t(x^{-1}\cdot y) \in B(a,3R)$, $0 \le t \le 1$. We have

$$x^{-1}\cdot y = x^{-1}\cdot a \cdot a^{-1}\cdot y = (a^{-1}\cdot x)^{-1}\cdot a^{-1}\cdot y.$$

Next, $a^{-1}\cdot x, a^{-1}\cdot y \in B(0,R)$. Therefore, $(a^{-1}\cdot x)^{-1} \in B(0,R)$. The triangle inequality implies that $x^{-1}\cdot y \in B(0,2R)$. Consequently, $\delta_t(x^{-1}\cdot y) \in B(0,2R)$. Using again the triangle inequality on the group $\mathbb{H}^n$ (see [16]), we obtain $x\delta_t(x^{-1}\cdot y) \in B(a,3R)$. □

The definition of the *cone condition*, which we use in the next lemma, was given in Subsection 0.2.

**Lemma 6.** *Suppose a bounded region* $\Omega \subset \mathbb{H}^n$ *satisfies the cone condition. Then there is a finite collection of open sets* $U_i$ *that cover* $\Omega$ *and are star-like in* $\Omega$ *with respect to certain balls.*

*Proof.* Let $x$ be an arbitrary point in $\Omega$. There is a ball $B_x$ of radius at least $C(\Omega)$ such that the cone $K_x = \{x\cdot\delta_t(x^{-1}\cdot y) \mid y \in \overline{B}_x,\ 0 \le t \le 1\}$ lies in $\Omega$. Since the sets $K_x$ and $\partial\Omega$ are compact and disjoint, we have $\operatorname{dist}(K_x, \partial\Omega) > 0$. Therefore, there is a neighborhood $U_x$ of $x$ such that the set $\{z\cdot\delta_t(z^{-1}\cdot y) \mid y \in B_x,\ z \in U_x,\ 0 < t \le 1\}$ is also contained in $\Omega$. This means that the set $U_x$ is star-like in the region $\Omega$ with respect to $B_x$. Consequently, there exists a countable collection $\{U_i\}$ of subsets of $\Omega$ such that each $U_i$ is star-like in $\Omega$ with respect to the balls $B_i$, and the radii of the balls are bounded below by a positive constant.

Without loss of generality, we may assume that all balls $B_i$ have the same radius $R$, $B_i = B(a_i, R)$. We fix a positive number $r < R$ and form the open set $V_1 = \bigcup_j U_j$, where the union is taken over all $j$ such that the distance from the center of $B_j$ to the center of $B_1$ does not exceed $r$. Obviously, all $B_j$ contain the ball $B_1' = B(a_1, R-r)$, which implies immediately that the set $V_1$ is star-like in $\Omega$ with respect to $B_1'$. Next, we choose a set $U_k$ that does not occur among the above $U_j$. Repeating the same argument, we construct an open set $V_2$ that is star-like in $\Omega$ with respect to the ball $B_2' = B(a_k, R-r)$. We continue this process. It will stop after several steps, because the centers of the $B_i'$ are contained in a bounded region and the distance between any two of them is greater than $r$. □

An analog of Lemma 6 in the Euclidean case was proved by V. P. Glushko in [67].

The definition of the strong Lipschitz condition occurring in Lemma 7 can be found in Subsection 0.2.

**Lemma 7.** *Let* $\Omega \subset \mathbb{H}^n$ *be a bounded region satisfying the strong Lipschitz condition. Then the regions* $\Omega$ *and* $C\Omega$ *satisfy the cone condition.*

*Proof.* Two classes of points can be distinguished among the points of the boundary $\partial\Omega$. For every point of the first class, there is a neighborhood in which $\Omega$ can be represented, in some coordinate system, as an epigraph of a Lipschitz function (as described in the

definition of the strong Lipschitz condition). For every point of the second class, there exists a neighborhood in which the boundary $\partial\Omega$ is $C^{1,1}$-smooth and the tangent planes to $\partial\Omega$ at such a point coincides with the horizontal plane attached to it.

Let $x \in \overline{\Omega}$. We choose an open ball $B_x$ centered at $x$ as follows. If $x \in \Omega$, then $2B_x \subset \Omega$. If $x$ lies on $\partial\Omega$ and belongs to the first class, then we assume that $2B_x \subset U_x$ and the cone $x_N - y_N > L(f_x)\|(x_1, \dots, x_{N-1}) - (y_1, \dots, y_{N-1})\|$ has nonempty intersection with the horizontal plane attached to $y$ for each $y \in B_x$. If $x \in \partial\Omega$ belongs to the second class, then we require that $2B_x \subset U_x$ and that the region $\Omega$ be representable in the form of either an epigraph or a subgraph of a $C^{1,1}$-smooth function $f_x$ in the initial coordinate system. For sufficiently small balls, all these conditions can be satisfied.

The balls $B_x$ are open and cover the compact set $\overline{\Omega}$. We choose a finite subcover $B_i$, $i = 1, \dots, m$.

Now, we verify the cone condition for the region $\Omega$. We fix an arbitrary point $x \in \Omega$. It is contained in one of the balls $B_j$ of the subcover. We consider three cases. In the first case, $2B_j \subset \Omega$. Then as $K_x$ we can take the Heisenberg ball centered at $x$ and of radius coinciding with that of $B_j$.

In the second case, a Euclidean cone with fixed opening and height and nonempty intersection with the horizontal plane attached to $x$ also lies in $\Omega$. We choose a ball $B$ contained in this cone. Since the orbits $\{x \cdot \delta_t(x^{-1} \cdot y)\}$ are tangent to the horizontal plane at $x$ and are concave in the direction of that plane, we see that the Heisenberg cone with base $B$ and vertex at $x$ lies in $\Omega$.

Finally, in the third case, we use the formula for the group operation in $\mathbb{H}^n$ to conclude that, in the standard system of coordinates, the set $l_{x^{-1}}(2B_j \cap \Omega)$ can be defined either by $x_{2n+1} > f(x_1, \dots, x_{2n})$, or by $x_{2n+1} < f(x_1, \dots, x_{2n})$, where $f$ is a function of class $C^{1,1}$. Consider the first possibility. We construct a Heisenberg cone with vertex at 0 and lying above the graph of $f$. We have $f(0) < 0$. Fixing a point $y$ in the domain of $f$ and using the Taylor formula, we obtain

$$\begin{aligned} f(y) &\le f(0) + \big(\nabla f(0)\big)y + L(\nabla f)\|(y_1, \dots, y_{2n})\|^2 \\ &\le \big(\nabla f(0)\big)y + L(\nabla f)\|(y_1, \dots, y_{2n})\|^2, \end{aligned}$$

where $L(\nabla f)$ is the Lipschitz constant of $\nabla f$. It follows that if $\nabla f(0) = 0$, then the required cone is the paraboloid

$$x_{2n+1} > L(\nabla f)\|(x_1, \dots, x_{2n})\|^2.$$

If $\nabla f \neq 0$, then, as a required cone, we can take the half of this paraboloid cut along the axis $\{x_1 = \dots = x_{2n} = 0\}$ that corresponds to the directions $v$ in the space of $x_1, \dots, x_{2n}$ for which $\big(\nabla f(0)\big)v < 0$.

Obviously, the intersection of the cone obtained in this way with the ball $l_{x^{-1}}(B_j)$ contains a closed Heisenberg ball $\overline{B}$ the radius of which depends only on the constant $L(\nabla f)$ and the radius of $B_j$. Since the left shift takes a Heisenberg ball into a Heisenberg ball of the same radius, the cone $\{x \cdot \delta_t(x^{-1} \cdot y) \mid y \in l_x(\overline{B}),\ 0 < t \le 1\}$ is as required.

The cone condition for the region $C\Omega$ can be verified in a similar way. □

**Corollary.** *Every region $\Omega \subset \mathbb{H}^n$ representable as a finite union of bounded regions with $C^{1,1}$-smooth boundary satisfies the cone condition.*

## §2. Coercive estimates

We recall that by a region we mean a connected open set.

**Theorem 4.** *Let $\Omega$ be a bounded region satisfying the cone condition, and let $Q$ be the linear differential operator that takes a smooth vector-valued function $f : \Omega \to \mathbb{R}^m$ to the*

*vector-valued function with the components*

$$\sum_{i=1}^{m} \sum_{\alpha:|\alpha|_h=k} C_{i,\alpha}^{j} X^{\alpha} f_i(x), \quad x \in \Omega, \ \ j = 1, \dots, l,$$

*where $\alpha$ is a multiindex and the $C_{i,\alpha}^{j}$ are constants. Assume that the operator $Q$ has a finite-dimensional kernel. Then there exists a family of projection operators $P_{Q,\varphi}$ taking the smooth $m$-vector-valued functions to functions lying in* $\ker(Q)$ *and satisfying the inequalities*

1) $\|P_{Q,\varphi} u\| \le C(\Omega, \varphi, Q)\|u\|_{L_1(\Omega)}$;
2) $\|u - P_{Q,\varphi} u\|_{W_p^k(\Omega)} \le C(\Omega, \varphi, Q)\|Qu\|_{L_p(\Omega)}$.

*Remark.* The parameter $\varphi$ in the statement of the theorem is related to the fact that the operator we construct in the proof of the theorem depends on the averaging function in the integral representation of Sobolev type that we use.

*Proof.* It is easily seen that if $|\alpha|_h = l$, then $X^{\alpha} g = \text{const}$ for every homogeneous horizontal polynomial $g$ of degree $l$. We have $X^{\alpha} g = 0$ for all multiindices $\alpha$ satisfying $|\alpha|_h = l$ if and only if $g \equiv 0$.

Since the kernel of $Q$ is finite-dimensional, there exists a positive integer $l$ such that $\ker Q \cap G_l = 0$, where $G_l$ is the linear space of all homogeneous horizontal polynomials of degree $l$.

We denote by $D_{\mathcal{L}}^{l-k} Q$ the operator that takes an $m$-vector-valued function $f$ to the vector-valued function the components of which are all possible horizontal $(l-k)$th order derivatives of the components of $Qf$. Obviously, $D_{\mathcal{L}}^{l-k} Qf = A \nabla_{\mathcal{L}}^{l} f$, where $A$ is a matrix with constant entries and $\nabla_{\mathcal{L}}^{l} f$ is the vector-valued function whose components are all horizontal $l$th order derivatives of the components of $f$.

Now, it is easy to see that the linear mapping $\nabla_{\mathcal{L}}^{l} : g \mapsto \nabla_{\mathcal{L}}^{l} g$ establishes a one-to-one correspondence between the space of homogeneous polynomials and the space of vectors of the corresponding dimension.

Since $\ker Q \cap G_l = 0$, we have $\ker D_{\mathcal{L}}^{l-k} Q \cap G_l = 0$. Viewing $\nabla_{\mathcal{L}}^{l}$ as a mapping that determines a coordinate system in $G_l$, we see that the matrix $A$ is invertible. Thus, $\nabla_{\mathcal{L}}^{l} f = A^{-1} D_{\mathcal{L}}^{l-k} Qf$. It follows that all horizontal $l$th order derivatives of the components of $f$ can be represented in the form of linear combinations of horizontal $(l-k)$th order derivatives of components of $Qf$.

We represent $\Omega$ as the union of a finite family of regions $U_i$ satisfying Lemma 6. In each $U_i$, we use an integral representation of Sobolev type. We have

$$f(x) = (P_{l,\varphi}^{i} f)(x) + \int_{\Omega} K_l^i(x,y) \nabla_{\mathcal{L}}^{l} f(y)\, dy, \quad x \in U_i.$$

Consequently,

$$f(x) = (P_{l,\varphi}^{i} f)(x) + \int_{\Omega} K_l^i(x,y) A^{-1} D_{\mathcal{L}}^{l-k} Qf(y)\, dy, \quad x \in U_i.$$

Integrating by parts $l-k$ times in the second term of the latter equation, we obtain

$$f(x) = (P_{l,\varphi}^{i} f)(x) + \int_{\Omega} H_i(x,y) Qf(y)\, dy, \quad x \in U_i, \tag{2.1}$$

where $H_i(x,y)$ is a matrix-valued function. It is easy to show that the components $H_i(x,y)$ satisfy the conditions of the generalization of the Zygmund–Calderón theorem to the case of the groups $\mathbb{H}^n$ [20, 22]; see Lemmas 2 and 3 in Subsection 1.1 and Lemma 4 in Subsection 1.2.

The projections $P^i_{l,\varphi}$ that take the functions in $L_1(B_i)$ to horizontal polynomials of degree at most $l-1$ correspond to the same smooth finite function $\varphi$ to within a dilation and a left shift.

We claim that there exists a linear projection $P_{l,Q}$ defined on the space of polynomials of degree at most $l$ and satisfying the following conditions:

$$\begin{aligned} &1)\ P_{l,Q}g \in \ker Q;\\ &2)\ \|P_{l,Q}g\| \le C_0(l,\Omega)\|g\|_{L_1(\Omega)};\\ &3)\ \|g - P_{l,Q}g\|_{W^1_p(\Omega)} \le C_1(l,\Omega)\|Qg\|_{L_p(\Omega)}, \end{aligned} \tag{2.2}$$

where $g$ is an arbitrary polynomial of degree at most $l$.

This claim is a simple consequence of the following lemma.

**Lemma 8.** *Let $V$ be a finite-dimensional linear space, let $A : V \mapsto V$ be a linear mapping, and let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms in $V$. Then there exists a linear projection $P_A$ such that $P_A v \in \ker A$ for all $v \in V$ and the inequality $\|v - P_A v\|_1 \le C\|Av\|_2$ is valid.*

*Proof.* This statement can easily be proved. It suffices to use the fact that all norms are equivalent in a finite-dimensional vector space. Indeed, let $\|\cdot\|$ be the Euclidean norm in $V$, let $P_A$ be the orthogonal projection onto $\ker A$, and let $V_1$ be the orthogonal complement of $\ker A$. Then the vector $v - P_A v$ belongs to $V_1$. The operator $A$ is one-to-one on $V_1$. Consequently, there exists a linear operator $A'$ such that the restriction of $A' \circ A$ to $V_1$ is the identity. Since $V_1$ is finite-dimensional, the operator $A'$ is bounded. Since, $A(v - P_A v) = Av$, we have $v - P_A v = A'(Av)$ and $\|v - P_A v\| \le C\|Av\|$. Since any two norms in $V$ are equivalent, we obtain the required inequality. □

We note that the choice of $\Omega$ influences the corresponding norms.

As a projection defined on the set of all smooth $m$-vector-valued functions on $B_i$, we take $P^i_{Q,\varphi} = P_{Q,l} \circ P^i_{l,\varphi}$. Since the operators $P_{Q,l}$ and $P^i_{l,\varphi}$ are linear and bounded, so is the operator $P^i_{Q,\varphi}$.

Putting

$$R_i f = \int_\Omega H_i(x,y) Qf(y)\,dy,$$

for any $x \in U_i$ we can write $f(x) = P^i_{l,\varphi} f(x) + R_i f(x)$.

From (2.1) and (2.2) it follows that

$$\begin{aligned} \|f - P^i_{Q,\varphi} f\|_{W^k_p(U_i)} &= \|f - P_{Q,l} \circ P^i_{l,\varphi} f\|_{W^k_p(U_i)}\\ &\le \|f - P^i_{l,\varphi} f\|_{W^k_p(U_i)} + \|P^i_{l,\varphi} f - P_{Q,l} \circ P^i_{l,\varphi} f\|_{W^k_p(U_i)}\\ &\le \|R_i f\|_{W^k_p(U_i)} + C\|Q(P^i_{l,\varphi} f)\|_{L_p(U_i)}\\ &\le C\|Qf\|_{L_p(\Omega)} + C\|Q(f - R_i f)\|_{L_p(U_i)} \le C\|Qf\|_{L_p(\Omega)}. \end{aligned}$$

Without loss of generality, we may assume that $10B_i \subset \Omega$ for all $i$. We fix two arbitrary balls $B_i$ and $B_j$ in the collection in question and connect them by a chain of balls $S_1, \dots, S_m$ such that $S_1 = B_i$, $S_m = B_j$, and the sets $S_k$ are star-like in $\Omega$ with respect to the balls $S_k$ and $S_{k+1}$. Since $\|G\|_{W^k_p(\Omega)} \le C\|G\|_{W^k_p(S_i)}$, $i = 1, \dots, m$, for each polynomial $G$ of degree at most $l-1$, the inequalities proved before yield

$$\|P^i_{Q,\varphi} f - P^j_{Q,\varphi} f\|_{W^k_p(\Omega)} \le C\|Qf\|_{L_p(\Omega)}.$$

Finally, each of the operators $P^i_{Q,\varphi}$ can be taken as the projection $P_{Q,\varphi}$. The theorem is proved. □

Theorem 4 and the method of its proof are new even in the Euclidean case (see [68]).

**Corollary.** *Let $\Omega$ be a bounded region satisfying the cone condition. Let $Q$ be the linear differential operator that takes a smooth vector-valued function $f:\Omega\to\mathbb{R}^m$ to the vector-valued function with the components*

$$\sum_{i=1}^{m}\sum_{\alpha:|\alpha|_h=k} C^j_{i,\alpha}X^\alpha f_i(x),\quad x\in\Omega,\ \ j=1,\dots,l,$$

*where $\alpha$ is a multiindex and the $C^j_{i,\alpha}$ are constants. Assume that the operator $Q$ has a finite-dimensional kernel. Then the standard norms of the spaces $W_{Q,p}(\Omega)$ and $W^k_p(\Omega)$ are equivalent.*

## §3. Embedding theorems

### 3.1. Generalized Poincaré inequalities.

**Theorem 5.** *Suppose $\Omega$ is a bounded connected region satisfying the cone condition with constant $r$ and contained in $B(0,R)$. Let $1<p\le\infty$. Then for each $k\in\mathbb{N}$ there exists a projection $P_k$ that takes the functions of class $W^k_p(\Omega)$ to horizontal polynomials of degree at most $k-1$ and satisfies the inequalities*

$$\begin{aligned}&\|X^\alpha(f-P_kf)\|_{L_q(\Omega)}\\&\quad\le C\biggl(R,\frac{\operatorname{diam}(\Omega)}{r},\alpha\biggr)r^{k-|\alpha|_h-\frac{\nu}{p}+\frac{\nu}{q}}\biggl\|\sum_{i_1,\dots,i_k=1}^{2n}X_{i_1}\cdots X_{i_k}f\biggr\|_{L_p(\Omega)};\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\|X^\alpha(f-P_kf)\|_{L_\infty(\Omega)}\\&\quad\le C\biggl(R,\frac{\operatorname{diam}(\Omega)}{r},\alpha\biggr)r^{k-|\alpha|_h-\frac{\nu}{p}}\\&\qquad\times\biggl\|\sum_{i_1,\dots,i_k=1}^{2n}X_{i_1}\cdots X_{i_k}f\biggr\|_{L_p(\Omega)}\qquad\text{if } k-|\alpha|_h>\frac{\nu}{p};\end{aligned}$$

$$\begin{aligned}&\|X^\alpha(f-P_kf)\|_{L_p(\Omega)}\\&\quad\le C\biggl(R,\frac{\operatorname{diam}(\Omega)}{r},\alpha\biggr)r^{k-|\alpha|_h}\biggl\|\sum_{i_1,\dots,i_k=1}^{2n}X_{i_1}\cdots X_{i_k}f\biggr\|_{L_p(\Omega)},\end{aligned}\tag{3.2}$$

*where $\alpha$ is a multiindex, $|\alpha|_h\le k$, $1<p\le q$, $q\ne\infty$, and $q\le\frac{\nu p}{\nu-(k-|\alpha|_h)p}$ if $p<\frac{\nu}{k-|\alpha|_h}$.*

*Proof.* First, we note that (3.2) is a special case of (3.1).

By Lemma 6 in Subsection 1.3, we can represent $\Omega$ as the union of a finite family of open sets $U_i$ that are star-like in $\Omega$ with respect to some balls $B_i$. The number of the $U_i$ can be estimated by the constant $\frac{\operatorname{diam}(\Omega)}{r}$. Using the integral representation (1.22), estimates (1.23), the Hölder and Young inequalities, and some well-known results concerning Riesz potentials on the groups $\mathbb{H}^n$ (see, e.g., [51]), we can easily prove the inequality

$$\begin{aligned}&\|X^\alpha f\|_{L_q(U_i)}\\&\quad\le C\biggl(R,\frac{\operatorname{diam}(\Omega)}{r},\alpha\biggr)\\&\qquad\times\biggl(r^{k-|\alpha|_h-\frac{\nu}{p}+\frac{\nu}{q}}\biggl\|\sum_{i_1,\dots,i_k=1}^{2n}X_{i_1}\cdots X_{i_k}f\biggr\|_{L_p(\Omega)}+r^{-|\alpha|_h}\|f\|_{L_q(B_i)}\biggr).\end{aligned}\tag{3.3}$$

Without loss of generality, we assume that $10B_i\subset\Omega$ for all $i$. Since the region $\Omega$ is connected, each ball $B_i$ can be connected with $B_1$ by a finite chain of pairwise disjoint

balls $B_i^j$ such that $5B_i^j \subset \Omega$. Since the balls $B_i^j$ are star-like in $3B_i^j$ with respect to each ball lying in the intersection $B_i^j \cap B_i^{j+1}$, we obtain the inequality

$$\|f\|_{L_q(B_i)} \le C\Big(R, \frac{\operatorname{diam}(\Omega)}{r}\Big)\Big(r^{k-\frac{\nu}{p}+\frac{\nu}{q}}\Big\|\sum_{i_1,\dots,i_k=1}^{2n} X_{i_1}\cdots X_{i_k} f\Big\|_{L_p(\Omega)} + \|f\|_{L_q(B_1)}\Big).$$

By (3.3), this implies that

$$\begin{aligned}\|X^\alpha f\|_{L_q(\Omega)} &\\ &\le C\Big(R, \frac{\operatorname{diam}(\Omega)}{r}, \alpha\Big)\\ &\quad\times\Big(r^{k-|\alpha|_h-\frac{\nu}{p}+\frac{\nu}{q}}\Big\|\sum_{i_1,\dots,i_k=1}^{2n} X_{i_1}\cdots X_{i_k} f\Big\|_{L_p(\Omega)} + r^{-|\alpha|_h}\|f\|_{L_q(B_1)}\Big).\end{aligned} \tag{3.4}$$

Now, we use the integral representation (1.22) for $f$ in the ball $B_1$, obtaining

$$f(x) - P_k^{B_1} f(x) = \int_{3B_1} \sum_{i_1,\dots,i_k=1}^{2n} K_{i_1\cdots i_k}(x,y) X_{i_1}\cdots X_{i_k} f(y)\,dy,$$

where $P_k^{B_1} f(x) = \int_\Omega P_k(x,y) f(y)\,dy$ is a horizontal polynomial of degree at most $k-1$, and the kernels $K_{i_1\cdots i_k}(x,y)$ satisfy (1.23). We have

$$\|f(x) - P_k^{B_1} f(x)\|_{L_q(B_1)} \le C r^{k-\frac{\nu}{p}+\frac{\nu}{q}}\Big\|\sum_{i_1,\dots,i_k=1}^{2n} X_{i_1}\cdots X_{i_k} f\Big\|_{L_q(5B_1)}. \tag{3.5}$$

Putting $P_k = P_k^{B_1}$ and substituting $f - P_k f$ for $f$ in (3.4), and using (3.5), we arrive at the first of the required inequalities. The second inequality is proved similarly. □

*Remark.* In the case where $k = 1$ and $\Omega$ is a ball, Theorem 5 was proved by other methods in a number of papers (see, e.g., [29, 32, 33, 36, 43]).

**Corollary.** *Suppose $\Omega$ is a bounded connected region satisfying the cone condition. Let $p > 1$. Then the following inequality is valid for every projection $\Pi_k$ bounded in $L_p(\Omega)$ and taking the functions of class $W_p^k(\Omega)$ to horizontal polynomials of degree at most $k-1$:*

$$\|f - \Pi_k f\|_{L_q(\Omega)} \le C(\Omega, \Pi_k)\Big\|\sum_{i_1,\dots,i_k=1}^{2n} X_{i_1}\cdots X_{i_k} f\Big\|_{L_p(\Omega)},$$

*where $1 < p \le q$ and $q \le \frac{\nu p}{\nu - kp}$ if $p < \frac{\nu}{k}$.*

*Proof.* First, we note that, since the space of all horizontal polynomials of degree at most $k$ is finite-dimensional, the boundedness of $\Pi_k$ in the sense of $L_p$ implies its boundedness in the sense of $L_q$ for all $q \ge p$. Let $P_k$ be a projection satisfying the assumptions of Theorem 5. The triangle inequality yields

$$\begin{aligned}\|f - \Pi_k f\|_{L_q(\Omega)} &\le \|f - P_k f\|_{L_q(\Omega)} + \|P_k f - \Pi_k f\|_{L_q(\Omega)}\\ &= \|f - P_k f\|_{L_q(\Omega)} + \|\Pi_k(f - P_k f)\|_{L_q(\Omega)} \le (1 + \|\Pi_k\|)\|f - P_k f\|_{L_q(\Omega)}.\end{aligned}$$

Now, the required inequality follows directly from Theorem 5. □

Invoking the well-known results concerning Riesz potentials on metric spaces, we obtain the following statement.

**Proposition 2.** *Suppose $\Omega$ is a bounded connected region satisfying the cone condition. Let $p > 1$, and let $\mu$ be a measure on $\mathbb{H}^n$ such that $\mu(B(R)) \le CR^s$, where $B(R)$ is a ball of radius $R$ in the Heisenberg metric. Then the following inequality is valid for every projection $\Pi_k$ bounded in $L_{p,\mu}(\Omega)$ and taking functions of the class $W_p^k(\Omega)$ to horizontal polynomials of degree at most $k-1$:*

$$\|f - \Pi_k f\|_{L_{q,\mu}(\Omega)} \le C(\Omega, \mu, \Pi_k) \left\| \sum_{i_1,\dots,i_k=1}^{2n} X_{i_1} \cdots X_{i_k} f \right\|_{L_p(\Omega)},$$

*where $1 < p \le q$ and $q \le \frac{sp}{s-kp}$ if $p < \frac{s}{k}$.*

**Theorem 6.** *Suppose $\Omega$ is a bounded connected region satisfying the cone condition with constant $r$ and contained in $B(0,R)$. Let $p > 1$, and let $Q$ be the linear differential operator that takes a smooth vector-valued function $f : \Omega \to \mathbb{R}^m$ to the vector-valued function with the components*

$$\sum_{i=1}^{m} \sum_{\alpha:|\alpha|_h=k} C_{i,\alpha}^j X^\alpha f_i(x), \quad x \in \Omega, \ \ j = 1, \dots, l,$$

*where $\alpha$ is a multiindex and the $C_{i,\alpha}^j$ are constants. If $Q$ has a finite-dimensional kernel, then there exists an operator $P_Q$ that takes smooth $m$-vector-valued functions defined on $\Omega$ to horizontal polynomials of degree at most $s(Q)$ and satisfies the inequalities*

$$\|X^\alpha(f - P_Q f)\|_{L_q(\Omega)} \le C\left(R, \frac{\operatorname{diam}(\Omega)}{r}, \alpha\right) r^{k-|\alpha|_h-\frac{\nu}{p}+\frac{\nu}{q}} \|Qf\|_{L_p(\Omega)},$$
$$\|X^\alpha(f - P_Q f)\|_{L_p(\Omega)} \le C\left(R, \frac{\operatorname{diam}(\Omega)}{r}, \alpha\right) r^{k-|\alpha|_h} \|Qf\|_{L_p(\Omega)},$$

*where $\alpha$ is a multiindex, $|\alpha|_h \le k$, $1 < p \le q < \infty$, and $q \le \frac{\nu p}{\nu-(k-|\alpha|_h)p}$ if $p < \frac{\nu}{k-|\alpha|_h}$.*

The proof of this theorem repeats almost word for word that of Theorem 5. The only difference is that now we use the integral representation (2.1).

**3.2. Embedding theorems.** We recall that $\nu$ denotes the homogeneous dimension of $\mathbb{H}^n$. In the sequel, we denote by $L_M$ the Orlicz space constructed starting with the function $M(x) = e^{|x|^2} - |x|^2 - 1$.

**Lemma 9.** *Suppose $f$ is a smooth function defined on a bounded region $\Omega$ with the cone condition. Let $p > 1$, and let $l$ be a nonnegative integer not exceeding $k$. Then*

$$\|\nabla_{\mathcal{L}}^l f\|_{L_{\frac{\nu p}{\nu-(k-l)p}}(\Omega)} \le C\|f\|_{W_p^k(\Omega)} \quad \text{if } p < \frac{\nu}{k-l};$$
$$\|\nabla_{\mathcal{L}}^l f\|_{L_M(\Omega)} \le C\|f\|_{W_p^k(\Omega)} \quad \text{if } p = \frac{\nu}{k-l};$$
$$\|\nabla_{\mathcal{L}}^l f\|_{C(\Omega)\cap L_\infty(\Omega)} \le C\|f\|_{W_p^k(\Omega)} \quad \text{if } p > \frac{\nu}{k-l}.$$

*Proof.* By Lemma 6, there is a finite family of open sets $U_i$ that cover $\Omega$ and are star-like in $\Omega$ with respect to some balls. From the integral representation (1.22), estimates (1.23), and generalizations of the well-known results on Riesz potentials to the Heisenberg groups (see, e.g., [17]), it follows immediately that for each multiindex $\alpha$ with $|\alpha|_h < k$

we have

$$\|X^\alpha f\|_{L_{\frac{\nu p}{\nu-(k-|\alpha|_h)p}}(U_i)} \le C_i\|f\|_{W_p^k(\Omega)} \quad \text{if } p < \frac{\nu}{k-|\alpha|_h};$$
$$\|X^\alpha f\|_{L_M(U_i)} \le C_i\|f\|_{W_p^k(\Omega)} \quad \text{if } p = \frac{\nu}{k-|\alpha|_h};$$
$$\|X^\alpha f\|_{C(U_i)\cap L_\infty(U_i)} \le C_i\|f\|_{W_p^k(\Omega)} \quad \text{if } p > \frac{\nu}{k-|\alpha|_h}.$$

Since all these norms satisfy the subadditivity condition, we obtain the required estimates for the norm of $\nabla^l_{\mathcal{L}} f$. □

The following lemma is proved similarly.

**Lemma 10.** *Let $\mu$ be a measure on $\mathbb{H}^n$ such that $\mu(B(R)) \le CR^l$, where $B(R)$ is a ball of radius $R$ in the Heisenberg metric. Let $\Omega$ be a bounded region satisfying the cone condition, and let $p > 1$. Then the following inequality is valid for every smooth function $f \in C^\infty(\Omega)$:*

$$\|f\|_{L_{\frac{sp}{s-kp}}(\Omega,\mu)} \le C\|f\|_{W_p^k(\Omega)}, \quad p < \frac{s}{k}.$$

Now, it is easy to prove the main result of this section.

**Theorem 7.** *Suppose $\Omega \subset \mathbb{H}^n$ is a bounded region satisfying the cone condition. Let $\mu$ be a measure on $\mathbb{H}^n$ such that $\mu(B(R)) \le CR^s$, where $B(R)$ is a ball of radius $R$ in the Heisenberg metric. Let $p > 1$. Then we have the following embeddings of function spaces:*

$$\begin{aligned}
&W_p^k(\Omega) \subset L_{\frac{\nu p}{\nu-kp}}(\Omega) && \text{if } p < \frac{\nu}{k};\\
&W_p^k(\Omega) \subset L_M(\Omega) && \text{if } p = \frac{\nu}{k};\\
&W_p^k(\Omega) \subset C(\Omega) && \text{if } p > \frac{\nu}{k};\\
&W_p^k(\Omega) \subset V^l_{\frac{\nu p}{\nu-(k-l)p}}(\Omega) && \text{if } l \le k \text{ and } p < \frac{\nu}{k-l};\\
&W_p^k(\Omega) \subset L_{\frac{sp}{s-kp}}(\Omega,\mu) && \text{if } p < \frac{s}{k}.
\end{aligned} \tag{3.6}$$

*Proof.* We fix a function $f$ in the Sobolev class $W_p^k$. Let $q = \frac{\nu p}{\nu-kp}$. By definition, there exists a sequence of smooth functions on $\Omega$ that converges to $f$ in the norm of the corresponding Sobolev space. This sequence has a subsequence converging to $f$ almost everywhere. All elements of the latter sequence belong to $L_q$ if $p < \frac{\nu}{k}$, and to $L_M$ if $p = \frac{\nu}{k}$. This sequence is fundamental in $L_q$ if $p < \frac{\nu}{k}$, in $L_M$ if $p = \frac{\nu}{k}$, and in $C(\overline{U})$ for every $U \Subset \Omega$ if $p > \frac{\nu}{k}$. Since all these spaces are complete, the sequence converges to a function of class $L_q$ if $p < \frac{\nu}{k}$, of class $L_M$ if $p = \frac{\nu}{k}$, and of class $C(\overline{U})$ for all $U \Subset \Omega$ if $p > \frac{\nu}{k}$. Passing to a subsequence once again, we obtain a sequence that converges to $f$ almost everywhere. As a result, we see that $f$ belongs to $L_q$ if $p < \frac{\nu}{k}$, belongs to $L_M$ if $p = \frac{\nu}{k}$, and coincides almost everywhere with a function of class $C(\Omega)$ if $p > \frac{\nu}{k}$.

The other inclusions in (3.6) are proved similarly. □

**Corollary.** *Suppose $\Omega$ is a bounded region satisfying the cone condition. Let $p > 1$. Then the function spaces $V_p^k(\Omega)$ and $W_p^k(\Omega)$ coincide.*

**Proposition 3.** *If $\Omega$ is a bounded region and $p > 1$, then the function spaces $V_0^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$, and $L_0^{k,p}(\Omega)$ coincide.*

*Proof.* We assume that $\Omega$ lies in a Heisenberg ball $B(0,R)$ and fix a ball $B(a,r) \subset B(0,3R) \setminus B(0,2R)$. Viewing $B(0,9R)$ as the domain of $f$, we observe that $\Omega$ is a star-like region in $B(0,9R)$ with respect to $B(a,r)$. To prove the required statement, it suffices to use the integral representation (1.22) and estimates (1.23). $\square$

The proof of the following assertions is based on application of an extension theorem in [42].

**Theorem 8.** *Suppose $\Omega \subset \mathbb{H}^n$ is a bounded region satisfying the $(\varepsilon,\delta)$-condition. Let $p > \frac{\nu}{k}$. Then we have the following embedding of function spaces*:

$$V_p^k(\Omega) \subset C^{k-\frac{\nu}{p}}(\overline{\Omega}).$$

*Proof.* Assuming that $\Omega$ satisfies the hypothesis of the theorem, consider a function $f$ of class $V_p^k(\Omega)$. Let $E_k$ be a bounded extension operator. We must prove that

$$\frac{|X^\alpha f(x) - X^\alpha f(y)|}{\rho(x,y)^{k-|\alpha|_h-\frac{\nu}{p}}} \le C(f) \tag{3.7}$$

for almost all $x$ and $y$ in $\Omega$, where $\alpha$ is a multiindex such that $|\alpha|_h = \left[k - \frac{\nu}{p}\right]$.

Obviously, in inequality (3.7) we can replace the function $f$ by $E_k f$. We fix a smooth function $g$ with compact support and close to $E_k f$ in the space $V_0^{k,p}(\mathbb{H}^n)$. Using the generalized Poincaré inequality (3.1) in the ball $B = B(x, 3\rho(x,y))$, we obtain

$$\|X^\alpha(g - P_k g)\|_{L_\infty(B)} \le C(\Omega,\alpha) r^{k-|\alpha|_h-\frac{\nu}{p}} \left\| \sum_{i_1,\dots,i_k=1}^{2n} X_{i_1}\cdots X_{i_k} g \right\|_{L_p(\Omega)}.$$

It follows that

$$\begin{aligned} |X^\alpha g(x) - X^\alpha g(y)| &\le \left|X^\alpha P_k g(x) - X^\alpha P_k g(y)\right| + C(\Omega,\alpha) r^{k-|\alpha|_h-\frac{\nu}{p}} \|g\|_{L_0^{k,p}(\mathbb{H}^n)} \\ &\le C'(\Omega,\alpha) r^{k-|\alpha|_h-\frac{\nu}{p}} \|E_k f\|_{L_0^{k,p}(\mathbb{H}^n)} \\ &\le C''(\Omega,\alpha) r^{k-|\alpha|_h-\frac{\nu}{p}} \|f\|_{V_p^k(\mathbb{H}^n)}. \end{aligned}$$

Consequently,

$$\frac{|X^\alpha g(x) - X^\alpha g(y)|}{\rho(x,y)^{k-|\alpha|_h-\frac{\nu}{p}}} \le C(\Omega,\alpha)\|f\|_{V_p^k(\mathbb{H}^n)}.$$

Passing to the limit as $g$ tends to $E_k f$ in the norm of the space $V_p^k(\mathbb{H}^n)$, we see that

$$\frac{|X^\alpha E_k f(x) - X^\alpha E_k f(y)|}{\rho(x,y)^{k-|\alpha|_h-\frac{\nu}{p}}} \le C(\Omega,\alpha)\|f\|_{V_p^k(\mathbb{H}^n)}$$

for almost all $x$ and $y$ in $\mathbb{H}^n$. Since the function $E_k f$ coincides with $f$ almost everywhere in $\Omega$, this inequality implies the assertion. $\square$

**Corollary 1.** *Suppose $\Omega \subset \mathbb{H}^n$ is a bounded region satisfying the cone condition and the $(\varepsilon,\delta)$-condition. Let $p > \frac{\nu}{k}$. Then we have the following embedding of function spaces*:

$$W_p^k(\Omega) \subset C^{k-\frac{\nu}{p}}(\overline{\Omega}).$$

**Corollary 2.** *Suppose $\Omega \subset \mathbb{H}^n$ is a bounded region with $C^2$-smooth boundary. Let $p > \frac{\nu}{k}$. Then we have the following embedding of function spaces*:

$$W_p^k(\Omega) \subset C^{k-\frac{\nu}{p}}(\overline{\Omega}).$$

*Remark.* In my paper [70], there is an error in the statement of Theorem 1. The first term in the corresponding integral representation must be replaced by the expression

$$\int_\Omega P_k(x,y;\varphi)f(y)\,dy,$$

where $P_k(\cdot,y;\varphi)$ is a horizontal polynomial of degree $k-1$, $|P_k(x,y;\varphi)| \le C(\Omega,\varphi)$, $\operatorname{supp} P_k(x,\cdot;\varphi) \subset B$.

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