

ON THE STABILITY OF AXIALLY SYMMETRIC EQUILIBRIUM FIGURES OF A ROTATING VISCOUS INCOMPRESSIBLE FLUID

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ABSTRACT. It is proved that if the second variation of the energy functional R (see (2.9)) is positive, then the axially symmetric equilibrium figure of a viscous incompressible capillary fluid is stable. The proof is based on the study of a nonstationary free boundary problem for the Navier–Stokes system with initial data close to the rotation regime of the fluid as a rigid body.

§1. INTRODUCTION

Unsteady motion of an isolated liquid mass under the surface tension force on the boundary and the self-gravity forces is governed by the following problem: to find a bounded domain $\Omega_t \subset \mathbb{R}^3$, a vector field $\vec{v}(x, t) = (v_1, v_2, v_3)$, and a function $p(x, t)$, $x \in \Omega_t$, satisfying the relations

$$(1.1) \quad \begin{aligned} \vec{v}_t + (\vec{v} \cdot \nabla)\vec{v} - \nu \nabla^2 \vec{v} + \nabla p &= 0, \quad \nabla \cdot \vec{v} = 0, \quad x \in \Omega_t, \quad t > 0, \\ \vec{v}|_{t=0} &= \vec{v}_0(x), \quad x \in \Omega_0, \end{aligned}$$

$$T(\vec{v}, p)\vec{n} = (\sigma H + \varkappa U(x, t))\vec{n}, \quad V_n = \vec{v} \cdot \vec{n}, \quad x \in \Gamma_t \equiv \partial\Omega_t, \quad t > 0,$$

where Ω_0 is a given domain, \vec{n} is the outward unit normal to the boundary Γ_t of the domain Ω_t , H is twice the mean curvature of Γ_t , $U(x, t) = \int_{\Omega_t} \frac{dy}{|x-y|}$ is the Newtonian potential, and $T(v, p) = -pI + \nu S(\vec{v})$ and $S(v) = \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)_{i,j=1,2,3}$ are the stress tensor and twice the rate-of-strain tensor, respectively; I is the identity tensor. The coefficients ν, σ, \varkappa are given positive constants, but the constant \varkappa can also be set equal to zero, which corresponds to the absence of self-gravitation. By V_n we denote the velocity of the motion of Γ_t in the direction of \vec{n} .

Suppose a drop of a liquid rotates as a rigid body about the axis x_3 with constant angular speed ω_0 . The velocity and pressure are determined by the formulas

$$\begin{aligned} \vec{\mathcal{V}}(x) &= \omega_0(-x_2, x_1, 0) = \omega_0 \vec{e}_3 \times \vec{x}, \\ \mathcal{P}(x) &= \frac{1}{2}|\vec{\mathcal{V}}(x)|^2 + p_0 = \frac{\omega_0^2}{2}(x_1^2 + x_2^2) + p_0, \end{aligned}$$

where $\vec{e}_3 = (0, 0, 1)$ and $p_0 = \text{const}$, while the boundary conditions reduce to a single equation

$$(1.2) \quad \sigma \mathcal{H} + \mathcal{P} + \varkappa \mathcal{U} = 0$$

on the surface of the rotating liquid. The domain filled by the rotating liquid is called an equilibrium figure and is denoted by \mathcal{F} ; its boundary is denoted by \mathcal{G} . In (1.2), by \mathcal{H}

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we mean twice the mean curvature of \mathcal{G} , and

$$\mathcal{U}(x) = \int_{\mathcal{F}} \frac{dy}{|x-y|}.$$

Equation (1.2) serves for determining \mathcal{G} . If \mathcal{F} is axially symmetric with respect to the axis x_3 , then $(\mathcal{F}, \vec{\mathcal{V}}(x), \mathcal{P}(x))$ is a stationary solution of problem (1.1). Clearly, the full angular momentum

$$\vec{\beta} = \int_{\mathcal{F}} (\vec{x} \times \vec{\mathcal{V}}(x)) dx$$

is parallel to the axis x_3 , $\vec{\beta} = \beta \vec{e}_3$; β and ω_0 are related as follows:

$$\beta = \omega_0 \mathcal{I}, \quad \mathcal{I} = \int_{\mathcal{F}} (x_1^2 + x_2^2) dx.$$

In addition, we assume that the barycenter of \mathcal{F} coincides with the origin, i.e.,

$$(1.3) \quad \int_{\mathcal{F}} x_i dx = 0, \quad i = 1, 2, 3.$$

Equation (1.2) is the Euler equation for the functionals

$$(1.4) \quad G = \sigma |\Gamma| - \frac{\omega_0^2}{2} \int_{\Omega} (x_1^2 + x_2^2) dx - \frac{\varkappa}{2} \int_{\Omega} U(x) dx - p_0 |\Omega|,$$

where $U(x) = \int_{\Omega} |x-y|^{-1} dy$, $|\Gamma| = \text{meas } \Gamma$, $|\Omega| = \text{meas } \Omega$, and

$$(1.5) \quad R = \sigma |\Gamma| + \frac{\beta^2}{2 \int_{\Omega} (x_1^2 + x_2^2) dx} - \frac{\varkappa}{2} \int_{\Omega} U(x) dx - p_0 |\Omega|;$$

these functionals are defined on a set of domains Ω that are close to \mathcal{F} and have the same volume and barycenter as \mathcal{F} ; $\Gamma = \partial\Omega$. An arbitrary surface close to \mathcal{G} can be given by an equation of the form

$$(1.6) \quad x = y + N(y)\rho(y), \quad y \in \mathcal{G},$$

where $N(y)$ is the outward normal to \mathcal{G} and $\rho(y)$ is some small function,

$$(1.7) \quad |\rho|_{C^1(\mathcal{G})} \leq \delta \ll 1,$$

so that G and R can be viewed as ρ -dependent functionals. It is easy to verify that

$$\delta_0 G[r] = \delta_0 R[r] = - \int_{\mathcal{G}} (\sigma \mathcal{H} + \mathcal{P} + \varkappa \mathcal{U}(x)) r(x) dS = 0,$$

where

$$\delta_0 G[r] = \frac{\partial}{\partial \lambda} G[\rho + \lambda r] \Big|_{\rho=0, \lambda=0}.$$

As to the stability of \mathcal{F} , the universally recognized stability condition is that the second variations of these functionals should be positive; if we deal with free motion of the fluid with preservation of the angular momentum, then the condition

$$(1.8) \quad \delta_0^2 R[r] = \frac{\partial^2}{\partial \lambda^2} R[\rho + \lambda r] \Big|_{\lambda=0, \rho=0} > 0$$

is more natural (see [1]–[3]).

Comparison of (1.4) and (1.5) shows that

$$\delta_0^2 R - \delta_0^2 G = \frac{\beta^2}{2} \delta_0^2 I^{-1} + \frac{\beta^2}{2\mathcal{I}^2} \delta_0^2 I = \frac{\beta^2}{\mathcal{I}^3} (\delta_0 I)^2 > 0,$$

where $I = \int_{\Omega} (x_1^2 + x_2^2) dx$. Therefore, if $\delta_0^2 G$ is positive, then so is $\delta_0^2 R$. We assume that an equilibrium figure is *stable* if, under initial data (Ω_0, \vec{v}_0) close to $(\mathcal{F}, \vec{\mathcal{V}}(x))$ and such that

$$(1.9) \quad \int_{\Omega_0} \vec{v} dx = 0, \quad \int_{\Omega_0} (\vec{x} \times \vec{v}_0(x)) dx = \beta \vec{e}_3,$$

$$(1.10) \quad |\Omega_0| = |\mathcal{F}|, \quad \int_{\Omega_0} x_i dx = 0, \quad i = 1, 2, 3,$$

problem (1.1) has a solution defined for all $t > 0$ and satisfying $\vec{v}(x, t) - \vec{\mathcal{V}}(x) \rightarrow 0$, $p(x, t) - \mathcal{P}(x) \rightarrow 0$, and $\Gamma_t \rightarrow \mathcal{G}$ as $t \rightarrow \infty$.

In [4]–[6], the stability of the axially symmetric equilibrium figures was proved under the condition $\delta_0^2 G > 0$ (for $\varkappa = 0$ in [4, 5] and for $\varkappa > 0$ in [6]). Here we do the same under the more natural condition (1.8), which has required several modifications in the proofs. Moreover, we remedy some inaccuracies occurred in [5, 6]. The main result is formulated in Theorem 2.1.

§2. AUXILIARY RELATIONS AND STATEMENT OF THE MAIN RESULT

Assume that equation (1.2), which can also be written in the form

$$(2.1) \quad \sigma \mathcal{H}(y) + \frac{\beta^2}{2T^2} (y_1^2 + y_2^2) + \varkappa \mathcal{M}(y) + p_0 = 0,$$

has a solution that determines a smooth axially symmetric equilibrium figure \mathcal{F} satisfying (1.3). We consider the set of surfaces Γ given by equations (1.6) in which the functions ρ satisfy inequality (1.7) and are such that the domains Ω bounded by these surfaces satisfy the conditions

$$(2.2) \quad |\Omega| = |\mathcal{F}|, \quad \int_{\Omega} x_i dx = 0, \quad i = 1, 2, 3.$$

These conditions can be expressed in terms of $\rho(y)$ and have the following form (see [1] and also [5]):

$$(2.3) \quad \int_{\mathcal{G}} \varphi(y; \rho) dS = 0,$$

$$(2.4) \quad \int_{\mathcal{G}} \psi_i(y; \rho) dS = 0, \quad i = 1, 2, 3,$$

where

$$(2.5) \quad \begin{aligned} \varphi(y; \rho) &= \rho(y) - \frac{\rho^2(y)}{2} \mathcal{H}(y) + \frac{\rho^3(y)}{3} \mathcal{K}(y), \\ \psi_k(y; \rho) &= y_k \varphi(y; \rho) + N_k(y) \left(\frac{\rho^2}{2} - \frac{\rho^3}{3} \mathcal{H}(y) + \frac{\rho^4}{4} \mathcal{K}(y) \right), \end{aligned}$$

and $\mathcal{K}(y)$ is the Gaussian curvature of \mathcal{G} . Our basic assumption concerning the functional R is as follows:

there are two positive constants c_1 and c_2 such that

$$(2.6) \quad c_1 \|r^2\|_{W_2^1(\mathcal{G})} \leq \delta_0^2 R[r] \leq c_2 \|r\|_{W_2^1(\mathcal{G})}^2$$

for all $r(y)$ satisfying (1.7), (2.3), and (2.4). If δ is sufficiently small, we can replace conditions (2.3) and (2.4) by

$$(2.7) \quad \int_{\mathcal{G}} r(y) dS = 0, \quad \int_{\mathcal{G}} y_k r(y) dS = 0, \quad k = 1, 2, 3.$$

The first three terms on the right in (1.4), (1.5) correspond to the potential energy related to the capillary, centrifugal, and gravitational forces that act on the fluid in Ω , and the constant p_0 is the Lagrange multiplier corresponding to the restriction on the volume of Ω in (2.2). The other multipliers, related to constraints on the position of the barycenter, vanish. If we multiply (1.2) by $\sum_{i=1}^3 x_i N_i$ and integrate the result over \mathcal{G} , then, after easy calculations, we obtain an expression for p_0 :

$$p_0 = \frac{2\sigma|\mathcal{G}|}{3|\mathcal{F}|} - \frac{5}{6|\mathcal{F}|} \left(\omega_0^2 \mathcal{I} + \varkappa \int_{\mathcal{F}} \mathcal{U}(x) dx \right).$$

In the case of $\sigma = 0$, this expression was found in [1].

The second order variation $\delta_0^2 G[r]$ is defined by the formula

$$\begin{aligned} \delta_0^2 G[r] = & \int_{\mathcal{G}} \left[\sigma(|\nabla_{\mathcal{G}} r|^2 + 2\mathcal{K}r^2) - \frac{\omega_0^2}{2} \left(\frac{\partial}{\partial N} (y_1^2 + y_2^2) - (y_1^2 + y_2^2)\mathcal{H} \right) \eta^2 + p_0 \mathcal{H} r^2 \right] dS \\ (2.8) \quad & + \varkappa \int_{\mathcal{G}} \left(\mathcal{U}\mathcal{H} - \frac{\partial \mathcal{U}}{\partial N} \right) r^2 dS - \varkappa \int_{\mathcal{G}} \int_{\mathcal{G}} r(y)r(z) \frac{dS_y dS_z}{|y-z|}, \end{aligned}$$

which coincides with

$$\begin{aligned} \delta_0^2 G[r] = & \int_{\mathcal{G}} \left[\sigma(|\nabla_{\mathcal{G}} r|^2 + (2\mathcal{K} - \mathcal{H}^2)r^2) - \frac{\omega_0^2}{2} \frac{\partial}{\partial N} (y_1^2 + y_2^2) r^2 \right] dS \\ (2.9) \quad & - \varkappa \int_{\mathcal{G}} \frac{\partial \mathcal{U}}{\partial N} r^2 dS - \varkappa \int_{\mathcal{G}} \int_{\mathcal{G}} r(y)r(z) \frac{dS_y dS_z}{|y-z|} \end{aligned}$$

in view of (1.2); consequently,

$$\begin{aligned} \delta_0^2 R[r] & = \int_{\mathcal{G}} \left[\sigma(|\nabla_{\mathcal{G}} r|^2 + 2\mathcal{K}r^2) - \frac{\omega_0^2}{2} \left(\frac{\partial}{\partial N} (y_1^2 + y_2^2) - (y_1^2 + y_2^2)\mathcal{H} \right) r^2 + p_0 \mathcal{H} r^2 \right] dS \\ & + \varkappa \int_{\mathcal{G}} \left(\mathcal{U}\mathcal{H} - \frac{\partial \mathcal{U}}{\partial N} \right) r^2 dS \\ (2.10) \quad & - \varkappa \int_{\mathcal{G}} \int_{\mathcal{G}} r(y)r(z) \frac{dS_y dS_z}{|y-z|} + \frac{\omega_0^2}{\mathcal{I}} \left(\int_{\mathcal{G}} (y_1^2 + y_2^2) r dS \right)^2 \\ & = \int_{\mathcal{G}} \left[\sigma(|\nabla_{\mathcal{G}} r|^2 + (2\mathcal{K} - \mathcal{H}^2)r^2) - \frac{\omega_0^2}{2} r^2 \frac{\partial}{\partial N} (y_1^2 + y_2^2) \right] dS - \varkappa \int_{\mathcal{G}} \frac{\partial \mathcal{U}}{\partial N} r^2 dS \\ & - \varkappa \int_{\mathcal{G}} \int_{\mathcal{G}} r(y)r(z) \frac{dS_y dS_z}{|y-z|} + \frac{\omega_0^2}{\mathcal{I}} \left(\int_{\mathcal{G}} (y_1^2 + y_2^2) r dS \right)^2, \end{aligned}$$

$\omega_0 = \beta \mathcal{I}^{-1}$. These formulas were deduced in [1] in the case where $\sigma = 0$ and in [7] for $\varkappa = 0$. The proof of (2.8) was also given in [5, 6], and in the case where the surfaces $\Gamma = \partial\Omega$ can be represented by an equation of the form $r = R(\theta, \varphi)$ in spherical coordinates, the formula for the second variation of G (upon perturbing \mathcal{G} along the vector $\vec{e}_r = \frac{\vec{x}}{|\vec{x}|}$) was obtained in [4].

We turn to problem (1.1) and introduce some auxiliary constructions. Assume that (1.1) has a classical solution defined for $t \in [0, T]$, $T \leq \infty$. We recall the following

“conservation laws”:

$$\begin{aligned}
 |\Omega_t| &= |\Omega_0| = |\mathcal{F}|, \\
 \int_{\Omega_t} \vec{v}(x, t) \, dx &= \int_{\Omega_0} \vec{v}_0(x) \, dx = 0, \\
 \int_{\Omega_t} (\vec{x} \times \vec{v}(x, t)) \, dx &= \int_{\Omega_0} (\vec{x} \times \vec{v}_0(x)) \, dx = \beta \vec{e}_3, \\
 \int_{\Omega_t} \vec{x} \, dx &= \int_{\Omega_0} \vec{x} \, dx = 0.
 \end{aligned}
 \tag{2.11}$$

We put

$$\begin{aligned}
 I(t) &= \int_{\Omega_t} (x_1^2 + x_2^2) \, dx, \\
 \omega(t) &= \frac{\beta}{i(t)}, \\
 \vec{v}'(x, t) &= \omega(t)(\vec{e}_3 \times \vec{x}).
 \end{aligned}
 \tag{2.12}$$

In accordance with A. Poincaré [2], for every fixed t , $\vec{v}'(x, t)$ is the velocity of the points of the “equivalent rigid body” occupying the same domain Ω_t as the fluid and rotating about the axis x_3 with the angular speed $\omega(t)$. The relative velocity

$$\vec{v}_r(x, t) = \vec{v}(x, t) - \vec{v}'(x, t)$$

satisfies

$$\int_{\Omega_t} \vec{v}_r \cdot \vec{v}' \, dx = \omega(t) \int_{\Omega_t} \vec{v}_r \cdot [\vec{e}_3 \times \vec{x}] \, dx = 0$$

(see [2]). Indeed, by (2.11)–(2.12) we have

$$\begin{aligned}
 &\int_{\Omega_t} \vec{v}' \cdot (\vec{v} - \vec{v}') \, dx \\
 &= \omega(t) \int_{\Omega_t} \vec{v} \cdot (\vec{e}_3 \times \vec{x}) \, dx - \omega^2(t)I(t) \\
 &= \omega(t) \int_{\Omega_t} \vec{e}_3 \cdot (\vec{x} \times \vec{v}(x, t)) \, dx - \omega^2(t)I(t) = \omega(t)(\beta - \beta) = 0.
 \end{aligned}
 \tag{2.13}$$

Next, we introduce the vector fields

$$\begin{aligned}
 \vec{v}''(x, t) &= \sum_{i=1}^3 \alpha_i(t) \vec{\eta}_i(x) = \vec{\alpha}(t) \times \vec{x}, \\
 \vec{v}^\perp(x, t) &= \vec{v}(x, t) - \vec{v}''(x, t),
 \end{aligned}
 \tag{2.14}$$

where $\vec{\eta}_i(x) = \vec{e}_i \times \vec{x}$, $i = 1, 2, 3$, $\vec{e}_i = (\delta_{ik})_{k=1,2,3}$, and the $\alpha_i(t)$ are determined by the orthogonality conditions

$$\int_{\Omega_t} \vec{v}^\perp \cdot \vec{\eta}_i \, dx = 0, \quad i = 1, 2, 3.
 \tag{2.15}$$

It is easily seen that (2.15) is equivalent to the linear algebraic system

$$\sum_{i=1}^2 S_{ki}(t) \alpha_i(t) = \int_{\Omega_t} \vec{v} \cdot \vec{\eta}_k \, dx = \beta \delta_{k3}, \quad k = 1, 2, 3,$$

where

$$S_{ki}(t) = \int_{\Omega_t} \vec{\eta}_k \cdot \vec{\eta}_i \cdot dx = \int_{\Omega_t} (\delta_{ki}|x|^2 - x_i x_k) \, dx.$$

Thus,

$$(2.16) \quad \alpha_i(t) = \beta S^{i3}(t),$$

where S^{ik} are the elements of the matrix inverse to $(S_{ij})_{i,j=1,2,3} \equiv \mathcal{S}$.

Using the identities

$$\begin{aligned} -(\vec{v}' \cdot \nabla)\vec{v}' &= \frac{1}{2}\nabla|\vec{v}'|^2 = \frac{\beta^2}{2I^2(t)}\nabla(x_1^2 + x_2^2), \\ -(\vec{v}'' \cdot \nabla)\vec{v}'' &= \frac{1}{2}\nabla|\vec{v}''|^2, \end{aligned}$$

it is easy to show that \vec{v}_r and \vec{v}^\perp satisfy the relations

$$\begin{aligned} (2.17) \quad &\vec{v}_{rt} + (\vec{v} \cdot \nabla)\vec{v}_r + (\vec{v}_r \cdot \nabla)\vec{v}' - \nu\nabla^2\vec{v}_r + \nabla p_r \\ &= -\vec{v}'_t(x, t) = -\omega'_t(t)\vec{\eta}_3(x), \\ &\nabla \cdot \vec{v}_r = 0, \quad x \in \Omega_t, \quad t > 0, \\ &\vec{v}_r(x, 0) = \vec{v}_0(x) - \vec{v}'(x, 0), \\ &T(\vec{v}_r, p_r)\vec{n} = (\sigma H + \frac{1}{2}|\vec{v}'|^2 + p_0 + \varkappa U(x, t))\vec{n}, \\ &V_n = \vec{v} \cdot \vec{n}, \quad x \in \Gamma_t, \quad t > 0, \end{aligned}$$

where $\vec{v} = \vec{v}_r + \vec{v}'$, $p_r = p - \frac{1}{2}|\vec{v}'|^2 - p_0$, and

$$\begin{aligned} (2.18) \quad &\vec{v}_t^\perp + (\vec{v} \cdot \nabla)\vec{v}^\perp + (\vec{v}^\perp \cdot \nabla)\vec{v}'' - \nu\nabla^2\vec{v}^\perp + \nabla p^\perp \\ &= -\vec{v}''_t(x, t), \\ &\nabla \cdot \vec{v}^\perp = 0, \quad x \in \Omega_t, \quad t > 0, \\ &\vec{v}^\perp(x, 0) = \vec{v}_0(x) - \vec{v}''(x, 0), \\ &T(\vec{v}^\perp, p^\perp)\vec{n} = (\sigma H + \frac{1}{2}|\vec{v}''|^2 + p_0 + \varkappa U(x, t))\vec{n}, \\ &V_n = \vec{v} \cdot \vec{n}, \quad x \in \Gamma_t, \quad t > 0, \end{aligned}$$

where $\vec{v} = \vec{v}^\perp + \vec{v}''$, $p^\perp = p - \frac{1}{2}|\vec{v}''|^2 - p_0$.

Largely, we shall study problem (2.17), and it is often convenient to use the coordinate system that rotates about the axis x_3 with variable angular speed $\omega(t)$. We rewrite the relations (2.17) in this coordinate system. For this, we make the transformation

$$(2.19) \quad x = \mathcal{Z}(\phi(t))y,$$

$$(2.20) \quad \vec{w}(y, t) = \mathcal{Z}^{-1}(\phi(t))\vec{v}_r(\mathcal{Z}(\phi(t))y, t),$$

where

$$\mathcal{Z}(\lambda) = \begin{pmatrix} \cos \lambda & -\sin \lambda & 0 \\ \sin \lambda & \cos \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\phi(t) = \int_0^t \omega(s) ds$, so that $\phi'_t(t) = \omega(t)$. We use the relations

$$\begin{aligned} (2.21) \quad &\vec{\eta}_3(x) = (\mathcal{Z}\vec{e}_3 \times \mathcal{Z}\vec{y}) = \mathcal{Z}\vec{\eta}_3(y), \\ &\frac{\partial}{\partial t}\vec{v}_r(\mathcal{Z}y, t) = \frac{\partial \vec{v}_r}{\partial t} + \left(\frac{d\mathcal{Z}}{dt}y \cdot \nabla_x\right)\vec{v}_r = \vec{v}_{rt} + (\vec{v}' \cdot \nabla_x)\vec{v}_r, \\ &\mathcal{Z}^{-1}(t)\frac{\partial}{\partial t}\vec{v}_r(\mathcal{Z}y, t) = \frac{\partial}{\partial t}\vec{w}(y, t) - \frac{d\mathcal{Z}^{-1}}{dt}\mathcal{Z}\vec{w} = \vec{w}_t + \omega(t)(\vec{e}_3 \times \vec{w}) \end{aligned}$$

and pass to the coordinates $\{y\}$ in the equation

$$\begin{aligned} &\mathcal{Z}^{-1}(\vec{v}_{rt} + (\vec{v}' \cdot \nabla)\vec{v}_r + (\vec{v}_r \cdot \nabla)\vec{v}' + (\vec{v}_r \cdot \nabla)\vec{v}' - \nu \nabla^2 \vec{v}_r + \nabla p_r) \\ &= -\omega'(t)\mathcal{Z}^{-1}\vec{\eta}_3(x). \end{aligned}$$

After simple calculations, we arrive at the following nonstationary problem with free boundary for the functions $\vec{w}(y, t)$, $s(y, t) = p_r(\mathcal{Z}y, t)$:

$$\begin{aligned} &\vec{w}_t + (\vec{w} \cdot \nabla)\vec{w} + 2\omega(t)(\vec{e}_3 \times \vec{w}) - \nu \nabla^2 \vec{w} + \nabla s \\ &= -\omega'(t)\vec{\eta}_3(y), \\ &\nabla \cdot \vec{w} = 0, \quad y \in \Omega'_t = \mathcal{Z}^{-1}\Omega_t, \quad t > 0, \\ (2.22) \quad &\vec{w}(y, 0) = \vec{v}_0(y) - \vec{v}'_0(y, 0) \equiv \vec{w}_0(y), \\ &T(\vec{w}, s)\vec{n} = \left(\sigma H' + \frac{\beta^2}{2I'^2(t)}(y_1^2 + y_2^2) + p_0 + \kappa U'(y, t) \right) \vec{n}', \\ &V'_n = \vec{w} \cdot \vec{n}', \quad y \in \Gamma'_t = \partial\Omega'_t, \quad t > 0. \end{aligned}$$

Here \vec{n}' is the outward unit normal to Γ'_t , H' is twice the mean curvature of Γ'_t , $U'(y, t) = \int_{\Omega'_t} \frac{dz}{|y-z|}$, V'_n is the velocity of the movement of Γ'_t in the direction of \vec{n}' , and $I'(t) = \int_{\Omega'_t} (y_1^2 + y_2^2) dy$. We deduce the relation $V'_n = \vec{w} \cdot \vec{n}'$ formally. Assume that Γ_t can be given by an equation of the form $x = X(\xi, t)$, where the point ξ runs over some manifold independent of t (as is the case, e.g., in (1.6)). Then $y = Y(\xi, t) = \mathcal{Z}^{-1}X(\xi, t) \in \Gamma'_t$, and since $\vec{n} = \mathcal{Z}\vec{n}'$, we have

$$\begin{aligned} V'_n &= Y_t \cdot \vec{n}' = (\mathcal{Z}^{-1}X)_t \cdot \mathcal{Z}^{-1}\vec{n} = \vec{X}_t \cdot \vec{n} + \frac{d\mathcal{Z}^{-1}}{dt} \mathcal{Z}Y \cdot \vec{n}' \\ &= V_n - \omega(t)\vec{\eta}_3(y) \cdot \vec{n}' = (\vec{v}_r + \vec{v}') \cdot \vec{n} - \omega(t)\vec{\eta}_3(y) \cdot \vec{n} = \vec{v}_r \cdot \vec{n} \\ &= \vec{w} \cdot \vec{n}'. \end{aligned}$$

This implies that $I'_t(t) = \int_{\Omega'_t} \vec{w} \cdot \nabla(y_1^2 + y_2^2) dy$, and

$$(2.23) \quad -\omega'_t(t) = \frac{\beta}{I^2(t)} \int_{\Omega'_t} \vec{w} \cdot \nabla(y_1^2 + y_2^2) dy = \frac{2\beta}{I^2(t)} \int_{\Omega'_t} (w_1 y_1 + w_2 y_2) dy.$$

Now we formulate the main result of the paper.

Theorem 2.1. *Suppose the following:*

- 1) *the surface Γ_0 is given by equation (1.6) with $\rho = \rho_0 \in C^{3+\alpha}(\mathcal{G})$, $\alpha \in (0, 1)$, and conditions (1.7), (1.9), and (1.10) are fulfilled (i.e., ρ_0 satisfies (2.3) and (2.4)).*
- 2) *$\vec{v}_0 \in C^{2+\alpha}(\Omega_0)$ satisfies the compatibility conditions*

$$\begin{aligned} &\nabla \cdot \vec{v}_0(y) = 0, \\ &(S(\vec{v}_0)\vec{n}_0)_{\tan} \equiv S(\vec{v}_0)\vec{n}_0 - \vec{n}_0(\vec{n}_0 \cdot S(\vec{v}_0)\vec{n}_0)|_{y \in \Gamma_0} = 0, \end{aligned}$$

where \vec{n}_0 is the unit outward normal to Γ_0 ;

- 3) *the smallness condition*

$$(2.24) \quad \|\vec{w}_0\|_{L_2(\Omega_0)} + \|\rho_0\|_{W_2^1(\mathcal{G})} \leq \varepsilon \ll 1$$

is fulfilled.

If R satisfies (2.6) for all ρ subject to (1.7) and (2.7), then problem (2.17) has a unique classical solution defined for all $t \geq 0$, the surface Γ_t can be given by equation (1.6) with

$\rho(\cdot, t) \in C^{3+\alpha}(\mathcal{G})$, and

$$(2.25) \quad \begin{aligned} & \sup_{t>0} |\vec{v}_r(\cdot, t)|_{C^\alpha(\Omega_t)} + \sup_{t>0} |\vec{v}_r(\cdot, t)|_{C^{2+\alpha}(\Omega_t)} + \sup_{t>0} |p_r(\cdot, t)|_{C^{1+\alpha}(\Omega_t)} \\ & + \sup_{t>0} |\rho(\cdot, t)|_{C^{3+\alpha}(\mathcal{G})} + \sup_{t>0} |\rho_t(\cdot, t)|_{C^{2+\alpha}(\mathcal{G})} + \sup_{t>0} |\rho_{tt}(\cdot, t)|_{C^\alpha(\mathcal{G})} \\ & \leq ce^{-bt} (|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}), \quad b > 0. \end{aligned}$$

The symbols $C^l(\Omega_t)$ and $C^l(\mathcal{G})$ with nonintegral $l > 0$ denote the standard Hölder spaces. Although the smallness condition is imposed on the L_2 -norm of \vec{w}_0 and the W_2^1 -norm of ρ_0 , interpolation inequalities allow us to show that the norms $|\vec{w}_0|_{C^l(\Omega_0)}$ and $|\rho_0|_{C^{l+1}(\Omega_0)}$ are also small with arbitrary $l < 2 + \alpha$; the latter norms are estimated by ce^γ with some $\gamma \in (0, 1)$ depending on l (see [6]).

Estimate (2.25) indicates that $\vec{v}(x, t) - \vec{V}(x)$, $p(x, t) - \mathcal{P}(x)$ tend exponentially to zero as $t \rightarrow \infty$. This follows from (2.25) and the relations

$$\begin{aligned} \vec{v}(x, t) - \vec{V}(x) &= \vec{v}_r(x, t) + \frac{\beta}{I(t)\mathcal{I}}(\mathcal{I} - I(t))\vec{\eta}_3(x), \\ p(x, t) - \mathcal{P}(x) &= p_r(x, t) + \frac{\beta^2}{2I^2(t)\mathcal{I}^2}(\mathcal{I}^2 - I^2(t))(x_1^2 + x_2^2), \end{aligned}$$

and

$$(2.26) \quad \begin{aligned} I(t) - \mathcal{I} &= \int_{\Omega_t} (x_1^2 + x_2^2) dx - \int_{\mathcal{F}} (y_1^2 + y_2^2) dy \\ &= \int_0^1 ds \int_{\mathcal{G}} \left[(y_1 + sN_1(y)\rho(y, t))^2 + (y_2 + sN_2\rho)^2 \right] \\ & \quad \times \rho(1 - s\rho\mathcal{H} + s^2\rho^2\mathcal{K}) dS_y \end{aligned}$$

(see [5, §2]).

§3. SOLVABILITY OF PROBLEM (2.17) ON A FINITE TIME INTERVAL

In order to prove Theorem 2.1, it suffices to show that problem (2.17) is solvable locally (in time), and to obtain uniform estimates of the solution. For this, we apply the following theorem on the solvability of the linear problem

$$(3.1) \quad \begin{aligned} \vec{v}_t - \nu\nabla^2\vec{v} + \nabla p &= \vec{f}(\xi, t), \\ \nabla \cdot \vec{v} &= g(\xi, t), \quad \xi \in \Omega_0, \quad t \in [0, T], \quad T < \infty, \\ \vec{v}(\xi, 0) &= \vec{v}_0(\xi), \\ \Pi_0 S(\vec{v})\vec{n} &= \vec{b}(\xi, t), \\ \vec{n}_0 \cdot T(\vec{v}, p)\vec{n}_0 - \sigma\vec{n}_0 \cdot \Delta(0) \int_0^t \vec{v}(\xi, \tau) d\tau &= b(\xi, t) + \int_0^t B(\xi, \tau) d\tau, \end{aligned}$$

where $\Pi_0\vec{a} = \vec{a} - \vec{n}_0(\vec{a} \cdot \vec{n}_0)$ is the projection of the vector \vec{a} to the plane tangent to Γ_0 , and $\Delta(0)$ is the Laplace–Beltrami operator on Γ_0 .

Theorem 3.1 ([8, 9]). *For any $\vec{f}(\cdot, t) \in C^\alpha(\Omega_0)$ and any $t \in [0, T]$, $\alpha \in (0, 1)$, $g \in C^{1+\alpha}(\Gamma_0)$, $\vec{v}_0 \in C^{2+\alpha}(\Omega_0)$, $\vec{b} \in C^{1+\alpha, \frac{1+\alpha}{2}}(G_T)$, $G_T = \Gamma \times (0, T)$, $b \in C^{1+\alpha}(\Gamma_0)$, and $B \in C^\alpha(\Gamma_0)$ satisfying the compatibility conditions*

$$\nabla \cdot \vec{v}_0(\xi) = g(\xi, 0), \quad \vec{b}(\xi, t) \cdot \vec{n}_0 = 0, \quad \Pi_0 S(\vec{v}_0)\vec{n}_0 = \vec{b}(\xi, 0),$$

and also the condition

$$g(\xi, t) = \nabla \cdot \vec{h}(\xi, t), \quad \vec{h}_t \in C^\alpha(\Omega_0),$$

problem (3.1) has a unique solution

$$\vec{v}(\cdot, t) \in C^{2+\alpha}(\Omega_0), \quad p(\cdot, t) \in C^{1+\alpha}(\Omega_0), \quad \vec{v}_t \in C^\alpha(\Omega_0),$$

$t \in [0, T]$, and this solution satisfies the inequality

$$\begin{aligned} & \sup_{t < T} |\vec{v}_t(\cdot, t)|_{C^\alpha(\Omega_0)} + \sup_{t < T} |\vec{v}(\cdot, t)|_{C^{2+\alpha}(\Omega_0)} + \sup_{t < T} |p(\cdot, t)|_{C^{1+\alpha}(\Omega_0)} \\ & \leq c(T) \left(\sup_{t < T} |\vec{f}(\cdot, t)|_{C^\alpha(\Omega_0)} + \sup_{t < T} |g(\cdot, t)|_{C^{1+\alpha}(\Omega_0)} \right. \\ (3.2) \quad & \left. + \sup_{t < T} |\vec{h}_t(\cdot, t)|_{C^\alpha(\Omega_0)} \right. \\ & \left. + |\vec{v}_0|_{C^{2+\alpha}(\Omega_0)} + |\vec{b}|_{C^{1+\alpha, \frac{1+\alpha}{2}}(G_T)} \right. \\ & \left. + \sup_{t < T} |b(\cdot, t)|_{C^{1+\alpha}(\Gamma_0)} + \sup_{t < T} |B(\cdot, t)|_{C^\alpha(\Gamma_0)} \right). \end{aligned}$$

Here $G_T = \Gamma_0 \times (0, T)$, and $C^{1+\alpha, \frac{1+\alpha}{2}}(G_T)$ is the “parabolic” anisotropic Hölder space.

In order to prove the solvability of problem (2.17), we rewrite it in the rotating coordinate system. Next, in (2.22) we pass to the Lagrangian coordinates $\xi \in \Omega_0$ related to the Eulerian coordinates $y \in \Omega_t$ by the formula

$$(3.3) \quad \vec{y} = \vec{\xi} + \int_0^t \vec{u}(\xi, \tau) d\tau \equiv \vec{X}(\xi, t),$$

where $\vec{u}(\xi, t) = \vec{w}(X(\xi, t), t)$. This results in the problem

$$\begin{aligned} & \vec{u}_t - \nu \nabla^2 \vec{u} + \nabla q(\xi, t) = F_1(\vec{u}, q) + F_1'(\vec{u}), \\ & \nabla \cdot \vec{u} = F_2(\vec{u}), \quad \xi \in \Omega_0, \\ & \Pi_0 S(\vec{u}) \vec{n}_0 = F_3(\vec{u}), \quad \xi \in \Gamma_0, \\ (3.4) \quad & \vec{n}_0 \cdot T(\vec{u}, q) \vec{n}_0 - \sigma \vec{n}_0 \cdot \Delta(0) \int_0^t \vec{u}(\xi, \tau) d\tau \\ & = M(\xi) + F_4(\vec{u}, q) + F_5(\vec{u}) + \int_0^t F_6(\vec{u}) d\tau, \\ & \vec{u}(\xi, 0) = \vec{w}_0(\xi), \end{aligned}$$

where $q(\xi, t) = s(X(\xi, t), t)$,

$$\begin{aligned} F_1(\vec{u}, q) &= \nu(\nabla_u^2 - \nabla^2)\vec{u} + (\nabla - \nabla_u)q, \\ F_1'(\vec{u}) &= -2\omega(t)[\vec{e}_3 \times \vec{u}] - \frac{d\omega(t)}{dt} \vec{\eta}_3(X), \\ F_2(\vec{u}) &= (\nabla - \nabla_u) \cdot \vec{u} = \nabla \cdot \vec{h}(\vec{u}), \quad \vec{h} = (I - A^T)\vec{u}, \\ F_3(\vec{u}) &= \Pi_0(\Pi_0 S(\vec{u}) \vec{n}_0 - \Pi S_u(\vec{u}) \vec{n}), \\ F_4(\vec{u}, q) &= (\vec{n}_0 \cdot \vec{n} - 1)q - \nu \vec{n}_0 \cdot (S_u(\vec{u}) \vec{n} - S(\vec{u}) \vec{n}_0), \\ F_5(\vec{u}) &= \sigma \int_0^t \vec{n}_0 \cdot \frac{\partial \Delta(\tau)}{\partial \tau} \vec{\xi} d\tau + \left(\frac{\omega^2(t)}{2}(X_1^2 + X_2^2) - \frac{\omega^2(0)}{2}(\xi_1^2 + \xi_2^2) \right) \\ & \quad + \varkappa(U(X) - U_0(\xi)) \vec{n} \cdot \vec{n}_0 - (\sigma H_0(\xi) - M(\xi))(\vec{n} \cdot \vec{n}_0 - 1), \\ F_6(\vec{u}) &= \sigma \vec{n}_0 \cdot \frac{\partial \Delta(t)}{\partial t} \int_0^t \vec{u}(\xi, \tau) d\tau + \sigma \vec{n}_0 \cdot (\Delta(t) - \Delta(0)) \vec{u}(\xi, t), \\ M(\xi) &= \sigma(H_0(\xi) - \mathcal{H}(\eta)) + \frac{\omega^2(0)}{2}(\xi_1^2 + \xi_2^2) - \frac{\omega_0^2}{2}(\eta_1^2 + \eta_2^2) + \varkappa(U_0(\xi) - \mathcal{U}(\eta)), \end{aligned}$$

and η is the point of \mathcal{G} related to $\xi \in \Gamma_0$ by $\xi = \eta + N(\eta)\rho_0(\eta)$. Here we have used the following notation:

$\nabla_u = \mathcal{A}\nabla_\xi$ is the transformed gradient ∇_y ;
 \mathcal{A} is the matrix of cofactors for the elements

$$(3.5) \quad a_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i}{\partial \xi_j} d\tau$$

of the Jacobi matrix of the transformation (3.3) (the determinant of the latter matrix is equal to 1);

\vec{n} is the outward normal to Γ'_t at the point $X(\xi, t)$;

Π is the projection onto the tangent plane at this point, i.e., $\Pi\vec{a} = \vec{a} - \vec{n}(\vec{a} \cdot \vec{n})$;

$$S_u(\vec{w}) = (\nabla_u \vec{w}) + (\nabla_u \vec{w})^T = \left(\sum_{k=1}^3 \left(A_{kj} \frac{\partial w_i}{\partial \xi_j} + A_{ki} \frac{\partial w_j}{\partial \xi_i} \right) \right)_{i,j=1,2,3}$$

is twice the transformed rate-of-strain tensor $S(\vec{w})$;

$\Delta(t)$ is the Laplace–Beltrami operator on Γ'_t ;

$\frac{\partial \Delta(t)}{\partial t}$ is the operator obtained from $\Delta(t)$ by differentiating its coefficients with respect to t ;

$H_0(\xi)$ is twice the mean curvature of Γ_0 ;

$$U_0(\xi) = \int_{\Omega_0} \frac{d\xi'}{|\xi - \xi'|}.$$

Problems (2.22) and (3.4) are equivalent if $\vec{n} \cdot \vec{n}_0 > 0$.

We prove the following theorem on the solvability of (3.4).

Theorem 3.2. *Under the assumptions of Theorem 2.1, problem (3.4) has a unique solution $\vec{u}(\cdot, t) \in C^{2+\alpha}(\Omega_0)$, $q(\cdot, t) \in C^{1+\alpha}(\Omega_0)$, $\vec{u}_t(\cdot, t) \in C^\alpha(\Omega_0)$ defined on some finite time interval $(0, T_0)$ and satisfying the inequality*

$$(3.6) \quad \begin{aligned} & \sup_{\tau < t} |\vec{u}_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} + \sup_{\tau < t} |\vec{u}(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau < t} |q(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)} \\ & \leq c(t)(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(\mathcal{G})}), \quad t \leq T_0. \end{aligned}$$

The value T_0 increases unboundedly as the norms on the right in (3.6) tend to zero; $c(t)$ is a monotone nondecreasing function of t .

Problem (3.4) differs insignificantly from a similar problem with $\omega = \omega_0$ considered in [6]. Along with Theorem 3.1, we shall apply some estimates of the expressions F_i and M ; most of them were obtained in [9].

Lemma 3.1. 1. *If ρ_0 belongs to $C^{3+\alpha}(\mathcal{G})$ and satisfies condition (1.7), then*

$$(3.7) \quad |M|_{C^{1+\alpha}(\Gamma_0)} \leq c|\rho_0|_{C^{3+\alpha}(\mathcal{G})}.$$

2. *Suppose vector fields $\vec{u}(\xi, t)$, $\vec{u}'(\xi, t)$ and functions $q(\xi, t)$, $q'(\xi, t)$ are given. Assume that*

$$(3.8) \quad \begin{aligned} & t \left(\sup_{\tau < t} |\vec{u}_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} + \sup_{\tau < t} |\vec{u}(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} \right) \leq \delta_1, \\ & t \left(\sup_{\tau < t} |\vec{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} + \sup_{\tau < t} |\vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} \right) \leq \delta_1, \\ & t \sup_{\tau < t} |q(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)} \leq \delta_1, \\ & t \sup_{\tau < t} |q'(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)} \leq \delta_1, \end{aligned}$$

where δ_1 is a sufficiently small number. Also, let

$$\vec{u} - \vec{u}'|_{t=0} = 0.$$

Then

$$\begin{aligned}
 (3.9) \quad & \sup_{\tau < t} |F_1(\vec{u}, q) - F_1(\vec{u}', q')|_{C^\alpha(\Omega_0)} + \sup_{\tau < t} |F_2(\vec{u}) - F_2(\vec{u}')|_{C^{1+\alpha}(\Omega_0)} \\
 & + |F_3(\vec{u}) - F_3(\vec{u}')|_{C^{1+\alpha, \frac{1+\alpha}{2}}(G_t)} + \sup_{\tau < t} |F_4(\vec{u}, q) - F_4(\vec{u}', q')|_{C^{1+\alpha}(\Gamma_0)} \\
 & + \sup_{\tau < t} |F_6(\vec{u}) - F_6(\vec{u}')|_{C^\alpha(\Gamma_0)} \\
 & \leq c\delta_1 N_t(\vec{u} - \vec{u}', q - q'),
 \end{aligned}$$

where $G_t = \Gamma_0 \times (0, t)$ and

$$\begin{aligned}
 (3.10) \quad N_t(\vec{u}, q) = & \sup_{\tau < t} |\vec{u}_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} \\
 & + \sup_{\tau < t} |\vec{u}(\cdot, t)|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau < t} |q(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)}.
 \end{aligned}$$

Moreover, under conditions (3.8) we have

$$(3.11) \quad \sup_{\tau < t} |F'_1(\vec{u}) - F'_1(\vec{u}')|_{C^\alpha(\Omega_0)} \leq c \int_0^t |\vec{u}_\tau(\cdot, \tau) - \vec{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} d\tau,$$

$$(3.12) \quad \sup_{\tau < t} |F_5(\vec{u}) - F_5(\vec{u}')|_{C^\alpha(\Omega_0)} \leq c \int_0^t |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} d\tau,$$

and, finally,

$$\begin{aligned}
 (3.13) \quad & \sup_{\tau < t} \left| \frac{\partial}{\partial t} (h(\vec{u}) - h(\vec{u}')) \right|_{C^\alpha(\Omega_0)} \\
 & \leq c\delta_1 \left(\sup_{\tau < t} |\vec{u}_\tau(\cdot, \tau) - \vec{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} \right. \\
 & \quad \left. + \sup_{\tau < t} |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} \right) \\
 & \quad + c|\vec{u}(\cdot, 0)|_{C^\alpha(\Omega_0)} \sup_{\tau < t} |\nabla \vec{u}(\cdot, \tau) - \nabla \vec{u}'(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)}.
 \end{aligned}$$

Proof. We start with the proof of (3.7). Consider the difference

$$\begin{aligned}
 & \frac{\omega^2(0)}{2}(\xi_1^2 + \xi_2^2) - \frac{\omega_0^2}{2}(\eta_1^2 + \eta_2^2) \\
 & = \frac{\omega^2(0) - \omega_0^2}{2}(\xi_1^2 + \xi_2^2) + \frac{\omega_0^2}{2}(2\eta_1 N_1 + 2\eta_2 N_2)\rho_0(\eta) + \frac{\omega_0^2}{2}(N_1^2 + N_2^2)\rho_0^2.
 \end{aligned}$$

It is estimated with the help of the relations

$$\omega^2(0) - \omega_0^2 = \frac{\beta^2}{I^2(0)} - \frac{\beta^2}{\mathcal{I}^2}$$

and (2.26), i.e.,

$$\begin{aligned}
 I(0) - \mathcal{I} & = \int_{\Omega_0} (\xi_1^2 + \xi_2^2) d\xi - \int_{\mathcal{F}} (\eta_1^2 + \eta_2^2) d\eta \\
 & = \int_0^1 ds \int_{\mathcal{G}} \left[(\eta_1 + sN_1\rho_0)^2 + (\eta_2 + sN_2\rho_0)^2 \right] \rho_0(\eta) (1 - s\rho_0\mathcal{H} + s^2\rho_0^2\mathcal{K}) dS_\eta.
 \end{aligned}$$

Obviously, the norm of that difference is dominated by $c|\rho_0|_{C^{3+\alpha}(\mathcal{G})}$. By [6, Proposition 3.1], the expressions $H_0(\xi) - \mathcal{H}(\eta)$ and $U_0(\xi) - \mathcal{U}(\eta)$ are also estimated by $c|\rho_0|_{C^{3+\alpha}(\mathcal{G})}$, and we arrive at inequality (3.7).

In order to prove (3.9), we use the fact that the elements $\delta_{ij} - A_{ij}(\xi, t)$ of the matrix $I - \mathcal{A}$ are sums of linear and quadratic functions of the expressions $\int_0^t \frac{\partial u_k(\xi, \tau)}{\partial \xi_m} d\tau$, and, consequently, $A_{ij}(\xi, t) - A'_{ij}(\xi, t)$ (A'_{ij} is the cofactor for $\delta_{ij} + \int_0^t \frac{\partial u'_i}{\partial \xi_j} d\tau$) are sums of terms that contain the differences $\int_0^t \left(\frac{\partial u_k}{\partial \xi_m} - \frac{\partial u'_k}{\partial \xi_m} \right) d\tau$ as factors. Thus,

$$(3.14) \quad \begin{aligned} |A_{ij}(\cdot, t) - A'_{ij}(\cdot, t)|_{C^{1+\alpha}(\Omega_0)} &\leq c \int_0^t |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} d\tau \\ &\leq ct \sup_{\tau < t} |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)}. \end{aligned}$$

The same can be said about the difference

$$(3.15) \quad \vec{n} - \vec{n}' = \frac{\mathcal{A}\vec{n}_0}{|\mathcal{A}\vec{n}_0|} - \frac{\mathcal{A}'\vec{n}_0}{|\mathcal{A}'\vec{n}_0|}$$

and about the coefficients of the operator $\Delta(t) - \Delta'(t)$, where $\Delta'(t)$ is the Laplace–Beltrami operator on $\Gamma'_t = X'(\Gamma_0, t)$, $\vec{X}'(\xi, t) = \vec{\xi} + \int_0^t \vec{u}'(\xi, \tau) d\tau$. The definition of $\Delta(t)$ implies that

$$\Delta(t) = \frac{1}{\sqrt{g}} \sum_{\alpha, \beta=1}^2 \frac{\partial}{\partial s_\alpha} g^{\alpha\beta} \sqrt{g} \frac{\partial}{\partial s_\beta},$$

where (s_1, s_2) are local coordinates on Γ_0 , $g = \det(g_{\alpha\beta})_{\alpha, \beta=1,2}$, $g_{\alpha\beta} = \frac{\partial X(\xi(s), t)}{\partial s_\alpha} \cdot \frac{\partial X(\xi(s), t)}{\partial s_\beta}$, and the $g^{\alpha\beta}$ are the elements of the matrix inverse to $(g_{\alpha\beta})_{\alpha, \beta=1,2}$. Detailed calculations can be found in [9, §7].

We explain the idea of the proof of inequalities (3.9). Consider, for instance, the expression

$$\begin{aligned} F_2(\vec{u}) - F_2(\vec{u}') &= (\nabla_u - \nabla_{u'}) \cdot \vec{u} + (\nabla - \nabla_{u'}) \cdot (\vec{u} - \vec{u}') \\ &= (\mathcal{A}' - \mathcal{A})\nabla \cdot \vec{u} + (I - \mathcal{A}')\nabla \cdot (\vec{u} - \vec{u}'). \end{aligned}$$

Using (3.8) and (3.14), we obtain

$$\begin{aligned} &\sup_{\tau < t} |F_2(\vec{u}) - F_2(\vec{u}')|_{C^{1+\alpha}(\Omega_0)} \\ &\leq ct \left(\sup_{\tau < t} |\nabla \vec{u}(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)} + \sup_{\tau < t} |\nabla \vec{u}'(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)} \right) \\ &\quad \times \sup_{\tau < t} |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} \\ &\leq cd_1 \sup_{\tau < t} |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)}. \end{aligned}$$

The other Hölder norms with respect to the spatial variables in (3.9) are estimated in a similar way. Next, let

$$[f]_{(0,t)}^{(\lambda)} = \sup_{0 < h < \tau < t} h^{-\lambda} |f(\tau - h) - f(\tau)|$$

be the Hölder constant of a function $f(\tau)$, $\tau \in (0, t)$. The norm

$$\sup_{\Gamma_0} [F_3(\vec{u}) - F_3(\vec{u}')]_{(0,t)}^{(\frac{1+\alpha}{2})}$$

is estimated with the help of the inequalities

$$\begin{aligned} \sup_{F_0} |A_{ij}(\xi, \cdot) - \delta_{ij}|_{(0,t)}^{(\frac{1+\alpha}{2})} &\leq ct^{\frac{1-\alpha}{2}} \sup_{\tau < t} \sup_{\Omega_0} |\nabla \vec{u}(\xi, \tau)|, \\ \sup_{\Gamma_0} [A_{ij}(\xi, \cdot) - A'_{ij}(\xi, \cdot)]_{(0,t)}^{(\frac{1+\alpha}{2})} &\leq ct^{\frac{1-\alpha}{2}} \left(\sup_{\tau < t} \sup_{\Omega_0} |\nabla \vec{u}(\xi, \tau) - \nabla \vec{u}'(\xi, \tau)| \right. \\ (3.16) \quad &\quad \left. + \int_0^t \sup_{\Omega_0} |\nabla \vec{u}(\xi, \tau) - \nabla \vec{u}'(\xi, \tau)| d\tau \right. \\ &\quad \left. \times (\sup_{\tau < t} \sup_{\Omega_0} |\nabla \vec{u}(\xi, \tau)| + \sup_{\tau < t} \sup_{\Omega_0} |\nabla \vec{u}'(\xi, \tau)|) \right). \end{aligned}$$

Recalling conditions (3.8) and the interpolation inequality

$$\begin{aligned} \sup_{\Omega_0} |\nabla \vec{u}(\xi, \tau) - \nabla \vec{u}'(\xi, \tau)| &\leq c \left(t^{\frac{1+\alpha}{2}} |\vec{u} - \vec{u}'|_{C^{2+\alpha}(\Omega_0)} + t^{-\frac{1-\alpha}{2}} |\vec{u} - \vec{u}'|_{C^\alpha(\Omega_0)} \right) \\ (3.17) \quad &\leq ct^{\frac{1+\alpha}{2}} \left(|\vec{u} - \vec{u}'|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau' < \tau} |\vec{u}_{\tau'}(\cdot, \tau') - \vec{u}'_{\tau'}(\cdot, \tau')|_{C^\alpha(\Omega_0)} \right), \end{aligned}$$

we obtain

$$\begin{aligned} \sup_{\Gamma_0} [A_{ij}(\xi_j \cdot) - A'_{ij}(\xi, \cdot)]_{(0,t)}^{(\frac{1+\alpha}{2})} &\leq ct \left(\sup_{\tau < t} |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau < t} |\vec{u}_\tau(\cdot, \tau) - \vec{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} \right). \\ (3.18) \quad &\end{aligned}$$

Inequalities similar to (3.17) and (3.18) are also true for the difference (3.15).

Now we represent the difference $F_3(\vec{u}) - F_3(\vec{u}')$ in the form

$$\begin{aligned} F_3(\vec{u}) - F_3(\vec{u}') &= \Pi_0(\Pi_0 S(\vec{u} - \vec{u}') \vec{n}_0 \\ &\quad - \Pi S_u(\vec{u} - \vec{u}') \vec{n}) - \Pi_0(\Pi S_u(\vec{u}') \vec{n} - \Pi' S_{u'}(\vec{u}') \vec{n}') \end{aligned}$$

and estimate $\sup_{\Gamma_0} [F_3(\bar{u}) - F_3(\bar{u}')]^{(\frac{1+\alpha}{2})}_{(0,t)}$ by using (3.17)–(3.18). We have

$$\begin{aligned} & \sup_{\Gamma_0} [F_3(\bar{u}) - F_3(\bar{u}')]^{(\frac{1+\alpha}{2})}_{(0,t)} \\ & \leq c \sup_{\Gamma_0} [\nabla \bar{u} - \nabla \bar{u}']^{(\frac{1+\alpha}{2})}_{(0,t)} \sup_{\tau < t} \sup_{\Gamma_0} (|I - \mathcal{A}(\xi, \tau)| + |\bar{n}_0 - \bar{n}|) \\ & \quad + c \sup_{\tau < t} \sup_{\Gamma_0} |\nabla \bar{u} - \nabla \bar{u}'| \sup_{\Gamma_0} \left([I - \mathcal{A}]^{(\frac{1+\alpha}{2})}_{(0,t)} + [\bar{n}_0 - \bar{n}]^{(\frac{1+\alpha}{2})}_{(0,t)} \right) \\ & \quad + c \sup_{\Gamma_0} [\nabla \bar{u}']^{(\frac{1+\alpha}{2})}_{(0,t)} \sup_{\tau < t} \sup_{\Gamma_0} (|\mathcal{A}(\xi, \tau) - \mathcal{A}'(\xi, \tau)| + |\bar{n} - \bar{n}'|) \\ & \quad + c \sup_{\tau < t} \sup_{\Gamma_0} |\nabla \bar{u}'(\xi, \tau)| \sup_{\Gamma_0} \left([\mathcal{A} - \mathcal{A}']^{(\frac{1+\alpha}{2})}_{(0,t)} + [\bar{n} - \bar{n}']^{(\frac{1+\alpha}{2})}_{(0,t)} \right) \\ & \leq c \left(\delta_1 \sup_{\Gamma_0} [\nabla \bar{u} - \nabla \bar{u}']_{C^{\frac{1+\alpha}{2}}(0,t)} \right. \\ & \quad \left. + t^{\frac{1-\alpha}{2}} \sup_{\tau < t} \sup_{\Omega_0} |\nabla \bar{u}(\xi, \tau)| \sup_{\tau < t} \sup_{\Omega_0} |\nabla \bar{u}(\xi, \tau) - \nabla \bar{u}'(\xi, \tau)| \right) \\ & \quad + ct \left(\sup_{\Gamma_0} [\nabla \bar{u}']^{(\frac{1+\alpha}{2})}_{(0,t)} + \sup_{\tau < t} \sup_{\Gamma_0} |\nabla \bar{u}'(\xi, \tau)| \right) \\ & \quad \times \left(\sup_{\tau < t} |\bar{u}(\cdot, \tau) - \bar{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau < t} |\bar{u}_\tau(\cdot, \tau) - \bar{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} \right) \\ & \leq c\delta_1 \left(\sup_{\tau < t} |\bar{u}(\cdot, \tau) - \bar{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau < t} |\bar{u}_\tau(\cdot, \tau) - \bar{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} \right), \end{aligned}$$

as required. □

Inequality (3.11) follows from the relations

$$\begin{aligned} F'_1(\bar{u}) - F'_1(\bar{u}') &= -2(\omega(t) - \omega'(t))[e_3 \times \bar{u}] - 2\omega'(t)[\bar{e}_e \times \bar{u} - \bar{u}'] \\ & \quad - \left(\frac{d\omega(t)}{dt} - \frac{d\omega'(t)}{dt} \right) \bar{\eta}_3(X) - \frac{d\omega'(t)}{dt} (\bar{\eta}_3(X) - \bar{\eta}_3(X')), \\ \omega(t) &= \beta/I(t), \quad I(t) = \int_{\Omega_0} (X_1^2(\xi, t) + X_2^2(\xi, t)) d\xi, \\ \omega'(t) &= \beta/I'(t), \quad I'(t) = \int_{\Omega_0} (X_1'^2(\xi, t) + X_2'^2(\xi, t)) d\xi, \\ \frac{dI(\tau)}{dt} &= 2 \int_{\Omega_0} (X_1 u_1 + X_2 u_2) d\xi, \\ \frac{dI'(t)}{dt} &= 2 \int_{\Omega_0} (X_1' u'_1 + X_2' u'_2) d\xi. \end{aligned}$$

We have

$$\begin{aligned} & \sup_{\tau < t} |F'_1(\bar{u}) - F'_1(\bar{u}')|_{C^\alpha(\Omega_0)} \\ & \leq c \left(\sup_{\tau < t} |\bar{u}(\cdot, \tau)|_{C^\alpha(\Omega_0)} \int_0^t |\bar{u} - \bar{u}'|_{C^\alpha(\Omega_0)} d\tau + \sup_{\tau < t} |\bar{u}(\cdot, \tau) - \bar{u}'(\cdot, \tau)|_{C^\alpha(\Omega_0)} \right) \\ & \leq c \int_0^t |\bar{u}_\tau(\cdot, \tau) - \bar{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} d\tau. \end{aligned}$$

The difference $F_5(\vec{u}) - F_5(\vec{u}')$ is estimated in a similar way, with the help of [6, Proposition 3.2] and the inequality

$$\left| \vec{n}_0 \cdot \left(\frac{\partial \Delta(\tau)}{\partial \tau} - \frac{\partial \Delta'(\tau)}{\partial \tau} \right) \vec{\xi} \right|_{C^{1+\alpha}(\Gamma_0)} \leq c |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Gamma_0)}$$

(the lower terms of the operator $\frac{\partial \Delta(\tau)}{\partial \tau} - \frac{\partial \Delta'(\tau)}{\partial \tau}$, which contain the second order derivatives of $\vec{u} - \vec{u}'$, disappear because the vector \vec{n}_0 is orthogonal to the first order derivatives of $\vec{\xi}$ with respect to the tangential variables).

Finally, we consider the difference

$$\frac{\partial \vec{h}(\vec{u})}{\partial t} - \frac{\partial \vec{h}(\vec{u}')}{\partial t} = \frac{\partial(\mathcal{A}^{T'} - \mathcal{A}^T)}{\partial t} \vec{u} + (\mathcal{A}^{T'} - \mathcal{A}^T) \vec{u}_t - \frac{\partial \mathcal{A}^{T'}}{\partial t} (\vec{u} - \vec{u}') + (I - \mathcal{A}^{T'}) (\vec{u}_t - \vec{u}'_t)$$

and write the first term on the right-hand side in the form

$$\frac{\partial(\mathcal{A}^{T'} - \mathcal{A}^T)}{\partial t} \vec{u} = \frac{\partial(\mathcal{A}^{T'} - \mathcal{A}^T)}{\partial t} \left(\vec{u}(\xi, 0) + \int_0^t \vec{u}_\tau(\xi, \tau) d\tau \right).$$

The derivatives $\frac{\partial A_{ij}}{\partial t}$ are linear combinations of terms that contain $\frac{\partial u_k}{\partial \xi_i}$ as factors (the same refers to the leading coefficients of the operator $\frac{\partial \Delta(t)}{\partial t}$, and the coefficients of the first order derivatives depend on $\frac{\partial^2 u_k}{\partial \xi_i \partial \xi_m}$ in a similar way). As a consequence, we have

$$\begin{aligned} & \sup_{\tau < t} \left| \frac{\partial h(\vec{u})}{\partial \tau} - \frac{\partial h(\vec{u}')}{\partial \tau} \right|_{C^\alpha(\Omega_0)} \\ & \leq c \delta_1 \left(\sup_{\tau < t} |\vec{u}_\tau(\cdot, \tau) - \vec{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} + \sup_{\tau < t} |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} \right) \\ & \quad + c |\vec{u}(\cdot, 0)|_{C^\alpha(\Omega_0)} \sup_{\tau < t} |\nabla \vec{u}(\cdot, \tau) - \nabla \vec{u}'(\cdot, \tau)|_{C^{1+\alpha}(\Omega_0)}, \end{aligned}$$

and the last term here does not exceed the quantity

$$ct^{1/2} |\vec{u}(\cdot, 0)|_{C^\alpha(\Omega_0)} \left(\sup_{\tau < t} |\vec{u}(\cdot, \tau) - \vec{u}'(\cdot, \tau)|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau < t} |\vec{u}_\tau(\cdot, \tau) - \vec{u}'_\tau(\cdot, \tau)|_{C^\alpha(\Omega_0)} \right).$$

The lemma is proved. □

Proof of Theorem 3.2. As in [6], we rewrite problem (3.4) in the form of the equation

$$(3.19) \quad (\vec{u}, q) = \phi(\vec{u}, q) + (\vec{u}_1, q_1),$$

where (\vec{u}_1, q_1) is a solution of problem (3.1) with $\vec{f} = 0$, $g = 0$, $\vec{v}_0 = \vec{w}_0$, $\vec{b} = 0$, $B = 0$, and $b = M(\xi)$, and $\phi(\vec{u}, q) \equiv (\vec{u}', q')$ is a solution of the same problem with $\vec{f} = F_1(\vec{u}, q) + F'_1(\vec{u})$, $g = F_2(\vec{u})$, $\vec{v}_0 = 0$, $\vec{b} = F_3(\vec{u})$, $b = F_4(\vec{u}, q) + F_5(\vec{u})$, and $B = F_6(\vec{u})$. To solve equation (3.19), we apply the successive approximation method:

$$(\vec{u}_{m+1}, q_{m+1}) = \phi(\vec{u}_m, q_m) + (\vec{u}_1, q_1),$$

where $m = 1, \dots$ (\vec{u}_1 and q_1 are defined as before). Let

$$N_t(\vec{u}, q) = \sup_{\tau < t} |\vec{u}_t(\cdot, t)|_{C^\alpha(\Omega_0)} + \sup_{\tau < t} |\vec{u}(\cdot, t)|_{C^{2+\alpha}(\Omega_0)} + \sup_{\tau < t} |q(\cdot, t)|_{C^{1+\alpha}(\Omega_0)}.$$

From (3.2) and (3.7) it follows that

$$(3.20) \quad N_t(\vec{u}_1, q_1) \leq c(t)(|\rho_0|_{C^{3+\alpha}(\mathcal{G})} + |\vec{w}_0|_{C^{2+\alpha}(\Omega_0)}).$$

The differences $(\vec{u}_{m+1} - \vec{u}_m, q_{m+1} - q_m)$ satisfy the equations

$$(\vec{u}_{m+1} - \vec{u}_m, q_{m+1} - q_m) = \phi(\vec{u}_m, q_m) - \phi(\vec{u}_{m-1}, q_{m-1}),$$

$m = 2, 3, \dots$. By (3.2), we have

$$\begin{aligned} & N_t(\vec{u}_{m+1} - \vec{u}_m, q_{m+1} - q_m) \\ & \leq c(t) \left(\sup_{\tau < t} |F_1(\vec{u}_m, q_m) - F_1(\vec{u}_{m-1}, q_{m-1})|_{C^\alpha(\Omega_0)} \right. \\ & \quad + \sup_{\tau < t} |F_1'(\vec{u}_m) - F_1'(\vec{u}_{m-1})|_{C^\alpha(\Omega_0)} + \sup_{\tau < t} |F_2(\vec{u}_m) - F_2(\vec{u}_{m-1})|_{C^{1+\alpha}(\Omega_0)} \\ & \quad + \sup_{\tau < t} \left| \frac{\partial h(\vec{u}_m)}{\partial t} - \frac{\partial h(\vec{u}_{m-1})}{\partial t} \right|_{C^\alpha(\Omega_0)} + |F_3(\vec{u}_m) - F_3(\vec{u}_{m-1})|_{C^{1+\alpha, \frac{1+\alpha}{2}}(G_t)} \\ & \quad + \sup_{\tau < t} |F_4(\vec{u}_m, q_m) - F_4(\vec{u}_{m-1}, q_{m-1})|_{C^{1+\alpha}(\Gamma_0)} \\ & \quad \left. + \sup_{\tau < t} |F_5(\vec{u}_m) - F_5(\vec{u}_{m-1})|_{C^{1+\alpha}(\Gamma_0)} + \sup_{\tau < t} |F_6(\vec{u}_m) - F_6(\vec{u}_{m-1})|_{C^\alpha(\Gamma_0)} \right). \end{aligned}$$

Now we use Lemma 3.1. If

$$(3.21) \quad tN_t(\vec{u}_k, q_k) \leq \delta_1, \quad k = m, m-1,$$

then, by (3.9)–(3.13),

$$\begin{aligned} & N_t(\vec{u}_{m+1} - \vec{u}_m, q_{m+1} - q_m) \\ & \leq c_1 \delta_2 N_t(\vec{u}_m - \vec{u}_{m-1}, q_m - q_{m-1}) \\ & \quad + c_2 \int_0^t N_\tau(\vec{u}_m - \vec{u}_{m-1}, q_m - q_{m-1}) d\tau, \quad \delta_2 = \delta_1 + |\vec{w}_0|_{C^\alpha(\Omega_0)} t^{1/2}, \end{aligned}$$

and if (3.21) is true for all $k = 1, \dots, m$, then

$$\begin{aligned} & \sum_{k=1}^m N_t(\vec{u}_{k+1} - \vec{u}_k, q_{k+1} - q_k) \\ (3.22) \quad & \leq c_1 \delta_2 \sum_{k=2}^m N_t(\vec{u}_k - \vec{u}_{k-1}, q_k - q_{k-1}) \\ & \quad + c_2 \int_0^t \sum_{k=2}^m N_\tau(\vec{u}_k - \vec{u}_{k-1}, q_k - q_{k-1}) d\tau + N_t(\vec{u}_2 - \vec{u}_1, q_2 - q_1). \end{aligned}$$

Now suppose that δ_2 is so small that $c_1 \delta_2 < 1/2$ (this is the case for small δ_1 and ε). Then, applying Gronwall's lemma, from (3.22) we can easily deduce the estimate

$$(3.23) \quad \sum_{k=1}^m N_t(\vec{u}_{k+1} - \vec{u}_k, q_{k+1} - q_k) \leq c(t) N_t(\vec{u}_2 - \vec{u}_1, q_2 - q_1) \leq c(t) N_t(\phi(\vec{u}_1, q_1)).$$

Consequently,

$$(3.24) \quad N_t(\vec{u}_{m+1}, q_{m+1}) \leq c(t) \phi(\vec{u}_1, q_1) + N_t(\vec{u}_1, q_1),$$

and if we require the inequality

$$(3.25) \quad T_0(c(T_0)N_{T_0}(\phi(\vec{u}_1, q_1) + N_{T_0}(\vec{u}_1, q_1))) \leq \delta_1,$$

then condition (3.21) will remain true for $k = m+1$. Hence, it will hold for all $k \geq 1$, and (3.23) will be fulfilled for $m = \infty$. The sequence (\vec{u}_m, q_m) will converge to a solution of equation (3.19), i.e., of problem (3.4). By using (3.2) and (3.20), it is not difficult to verify that

$$(3.26) \quad \phi(\vec{u}_1, q_1) \leq c(|\vec{v}_0|_{C^{2\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(G)}).$$

Thus, (3.25) is valid for

$$T_0(|\vec{v}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{3+\alpha}(G)}) \leq \delta_3 \ll 1.$$

This implies that $T_0 \rightarrow \infty$ as the norms of \vec{v}_0 and ρ_0 tend to zero. Estimate (3.4) follows from (3.24), (3.20), and (3.26). The uniqueness of the solution just constructed can be easily proved with the help of (3.2) and (3.9)–(3.13). The theorem is proved. \square

The solvability of problem (3.4) implies that of (2.22), together with the estimate

$$(3.27) \quad \sup_{\tau < t} |\vec{w}_\tau(\cdot, \tau)|_{C^\alpha(\Omega'_\tau)} + \sup_{\tau < t} |\vec{w}_\tau(\cdot, \tau)|_{C^{2+\alpha}(\Omega'_\tau)} + \sup_{\tau < t} |s(\cdot, \tau)|_{C^{1+\alpha}(\Omega'_\tau)} \leq c(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{1+\alpha}(\mathcal{G})}), \quad t \leq T_0,$$

for its solution. Moreover, if ε is sufficiently small, then Γ'_t is given by equation (1.6) with $\rho = \rho'(y, t)$, and moreover, $\rho'(y, 0) = \rho_0(y)$, ρ' satisfies (1.7), and

$$(3.28) \quad \sup_{\tau < t} |\rho'(\cdot, \tau)|_{C^{3+\alpha}(\mathcal{G})} + \sup_{\tau < t} |\rho'_t(\cdot, \tau)|_{C^{1+\alpha}(\mathcal{G})} + \sup_{\tau < t} |\rho'_{\tau\tau}(\cdot, \tau)|_{C^\alpha(\mathcal{G})} \leq c(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{1+\alpha}(\mathcal{G})})$$

(see [6, Theorem 3.3]). But then problem (2.17) is also solvable on the interval $(0, T_0)$, and $\Gamma_t = \mathcal{Z}\Gamma'_t$ is determined by equation (1.6) with $\rho = \rho(y, t) = \rho'(\mathcal{Z}^{-1}y, t)$. By (3.27), (3.28), we obtain the estimate

$$\begin{aligned} & \sup_{\tau < t} |\vec{v}_\tau(\cdot, \tau)|_{C^\alpha(\Omega_\tau)} + \sup_{\tau < t} |\vec{v}_\tau(\cdot, \tau)|_{C^{2+\alpha}(\Omega_\tau)} + \sup_{\tau < t} |p_\tau(\cdot, \tau)|_{C^{1+\alpha}(\Omega_\tau)} \\ & \quad + \sup_{\tau < t} |q(\cdot, \tau)|_{C^{3+\alpha}(\mathcal{G})} + \sup_{\tau < t} |\rho_\tau(\cdot, \tau)|_{C^{2+\alpha}(\mathcal{G})} + \sup_{\tau < t} |\rho_{\tau\tau}(\cdot, \tau)|_{C^\alpha(\mathcal{G})} \\ & \leq c(|\vec{w}_0|_{C^{2+\alpha}(\Omega_0)} + |\rho_0|_{C^{1+\alpha}(\mathcal{G})}), \quad t \leq T_0. \end{aligned}$$

§4. ESTIMATION OF THE SOLUTION OF PROBLEM (2.17)

Along with Theorem 3.2, we need uniform (in time) estimates for the solution of problem (2.17). One of them is an estimate for the generalized energy. The following auxiliary assertion will help us to deduce it.

Lemma 4.1. *Suppose Γ_t is given by equation (1.6) with a function $\rho(y, t)$ satisfying conditions (1.7) and having first order derivatives with respect to t and second order derivatives with respect to the spatial variables. There exists a solenoidal vector field $\vec{W}(x, t)$ on Ω_t with the following properties:*

1)

$$(4.1) \quad \vec{W}(x, t) \cdot \vec{n}(x) = m(y; \rho(y, t))\varphi(y; \rho(y, t)), \quad y \in \mathcal{G},$$

where $x = y + N(y)\rho(y, t) \in \Gamma_t$, $m(y; \rho)$ is a positive function to be defined later (see (4.6)), and $\varphi(y; \rho)$ is as in (2.5);

2)

$$(4.2) \quad \int_{\Omega_t} \vec{W}(x, t) \cdot \vec{\eta}_i(x) dx = 0, \quad \vec{\eta}_i = \vec{e}_i \times \vec{x}, \quad i = 1, 2, 3;$$

3) the inequalities

$$(4.3) \quad \begin{aligned} \|\vec{W}(\cdot, t)\|_{W_2^1(\Omega_t)} & \leq c\|\rho(\cdot, t)\|_{W_2^{1/2}(\mathcal{G})}, \\ \|\vec{W}(\cdot, t)\|_{L_2(\Omega_t)} & \leq c\|\rho(\cdot, t)\|_{L_2(\mathcal{G})}, \\ \|\vec{W}_t(\cdot, t)\|_{L_2(\Omega_t)} & \leq c(\|\rho_t(\cdot, t)\|_{L_2(\mathcal{G})} + \|\rho(\cdot, t)\|_{W_2^{1/2}(\mathcal{G})}) \end{aligned}$$

are true with constants independent of t (but depending on the maxima of the absolute values of the derivatives of ρ mentioned above).

Proof. Obviously, there exists a solenoidal vector field $\vec{W}_0(y, t)$, $y \in \mathcal{F}$, such that

$$\begin{aligned}
 \vec{W}_0 \cdot \vec{N} &= \varphi(y; \rho), \quad y \in \mathcal{G}, \\
 \|\vec{W}_0\|_{W_2^1(\mathcal{F})} &\leq c\|\varphi\|_{W_2^{1/2}(\mathcal{G})} \leq c\|\rho\|_{W^{1/2}(\mathcal{G})}, \\
 \|\vec{W}_0\|_{L_2(\mathcal{F})} &\leq c\|\varphi\|_{L_2(\mathcal{G})} \leq c\|\rho\|_{L_2(\mathcal{G})}, \\
 \|\vec{W}_{0t}\|_{L_2(\mathcal{F})} &\leq c\|\varphi_t\|_{L_2(\mathcal{G})} \leq c\|\rho_t\|_{L_2(\mathcal{G})}.
 \end{aligned}
 \tag{4.4}$$

We map \mathcal{F} onto Ω_t by the transformation

$$x = y + N^* \rho^* \equiv e_\rho(y), \tag{4.5}$$

where N^* and ρ^* are the following extensions of the functions N and ρ from \mathcal{G} to \mathcal{F} : at the points $z = y + N(y)\lambda \in \mathcal{F}$ ($y \in \mathcal{G}$, $\lambda < \lambda_0$) close to \mathcal{G} , we put

$$N^*(z) = N(y), \quad \rho^*(z) = \rho(y)\psi(\lambda),$$

where $\psi(\lambda)$ is a smooth cut-off function equal to zero for $|\lambda| \geq \frac{3\lambda_0}{4}$ and to one for $|\lambda| < \frac{\lambda_0}{2}$; outside the λ_0 -neighborhood of \mathcal{G} , we set $\rho = 0$ and extend N^* in an arbitrary way. Clearly, for such an extension we have

$$\frac{\partial N^*}{\partial N} \Big|_{\mathcal{G}} = 0, \quad \frac{\partial \rho^*}{\partial N} \Big|_{\mathcal{G}} = 0.$$

Let $\mathcal{L}_\rho(y, \rho) = \left(\frac{\partial e_\rho(y)}{\partial y}\right) = \left(\delta_{ij} + \frac{\partial N_i^* \rho^*}{\partial y_j}\right)_{i,j=1,2,3}$ be the Jacobi matrix of the transformation (4.5), let $L_\rho = \det \mathcal{L}_\rho$, and let $\hat{\mathcal{L}}_\rho = L_\rho \mathcal{L}_\rho^{-1} = (\hat{L}_{ij}(y; \rho))_{i,j=1,2,3}$ be the matrix reciprocal to \mathcal{L}_ρ (for δ sufficiently small the transformation (4.4) is obviously invertible and the matrix \mathcal{L}_ρ^{-1} is well defined). Let

$$\vec{W}'(x, t) = \frac{1}{L_\rho(y)} \mathcal{L}_\rho \vec{W}_0(y, t) \Big|_{y=e_\rho^{-1}(x)}$$

(i.e., $\vec{W}_0 = \hat{\mathcal{L}}_\rho \vec{W}'$). The Piola identity

$$\sum_{j=1}^3 \frac{\partial}{\partial y_j} \hat{L}_{jk} = 0, \quad k = 1, 2, 3,$$

implies that $\nabla_x \cdot \vec{W}'(x, t) = 0$; moreover, since

$$\vec{n}(x) = \frac{1}{|\hat{\mathcal{L}}_\rho^T \vec{N}|} \hat{\mathcal{L}}_\rho^T \vec{N}(y),$$

we see that condition (4.1) is fulfilled,

$$\vec{W}' \cdot \vec{n} = \frac{\varphi(y, \rho)}{|\hat{\mathcal{L}}_\rho^T \vec{N}(y)|} \Big|_{y=e_\rho^{-1}(x)}, \quad x \in \Gamma_t, \tag{4.6}$$

with $m(y; \rho) = |\hat{\mathcal{L}}_\rho^T \vec{N}|^{-1}$. Obviously,

$$\|\vec{W}'(\cdot, t)\|_{W_2^1(\Omega_t)} \leq c\|\vec{W}_0(\cdot, t)\|_{W_2^1(\mathcal{F})} \leq c\|\rho\|_{W_2^{1/2}(\mathcal{G})}, \tag{4.7}$$

$$\|\vec{W}'(\cdot, t)\|_{L_2(\Omega_t)} \leq c\|\vec{W}_0(\cdot, t)\|_{L_2(\mathcal{F})} \leq c\|\rho\|_{L_2(\mathcal{G})}; \tag{4.8}$$

moreover, the relation

$$\frac{\partial \vec{W}'(x, t)}{\partial t} = \frac{\partial \vec{W}'(e_\rho(y), t)}{\partial t} - \sum_{k=1}^3 \frac{\partial \vec{W}'}{\partial x_k} N_k^* \frac{\partial \rho^*}{\partial t}$$

and (4.4) imply

$$(4.9) \quad \left\| \frac{\partial \vec{W}'(\cdot, t)}{\partial t} \right\|_{L_2(\Omega_t)} \leq c(\|\vec{W}_0(\cdot, t)\|_{W_2^1(\Omega)} + \|\vec{W}_{0t}\|_{L_2(\Omega_t)}) \\ \leq c(\|\rho_t(\cdot, t)\|_{L_2(\mathcal{G})} + \|\rho\|_{W_2^{1/2}(\mathcal{G})}).$$

Finally, we set

$$(4.10) \quad \vec{W}(x, t) = \vec{W}'(x, t) - \sum_{i=1}^3 c_i(t) \operatorname{curl} \vec{e}_i \mathcal{A}(x),$$

where $\mathcal{A}(x)$ is a $C_0^\infty(\Omega_t)$ -function with $\int_{\Omega_t} \mathcal{A}(x) dx = \frac{1}{2}$, and

$$c_i(t) = \int_{\Omega_t} \vec{W}'(x, t) \cdot \vec{\eta}_i(x) dx.$$

Since $\operatorname{curl} \vec{\eta}_j(x) = 2\vec{e}_j$, we have

$$\int_{\Omega_t} \vec{W}(x, t) \cdot \vec{\eta}_j(x) dx = c_j - \sum_{i=1}^3 c_i \vec{e}_i \cdot \int_{\Omega_t} \mathcal{A}(x) \operatorname{curl} \vec{\eta}_j(x) dx = 0;$$

estimates (4.3) follow from (4.7)–(4.9). The lemma is proved. \square

Theorem 4.1. *Assume that problem (2.17) has a classical solution defined for $t \in [0, T]$, $T \leq \infty$, and that Ω_t satisfies the assumptions of Lemma 4.1. If condition (2.6) is fulfilled, then there exists a function $E(t)$ satisfying the inequalities*

$$(4.11) \quad c_1(\|\vec{v}_r(\cdot, t)\|_{L_2(\Omega_t)}^2 + \|\rho(\cdot, t)\|_{W_2^1(\mathcal{G})}^2) \leq E(t) \leq c_2(\|\vec{v}_r(\cdot, t)\|_{L_2(\Omega_t)}^2 + \|\rho(\cdot, t)\|_{W_2^1(\mathcal{G})}^2),$$

$$(4.12) \quad E(t) \leq e^{-bt} E(0), \quad t \in [0, T],$$

with constants $c_1, c_2, b > 0$ independent of t .

Proof. First, we prove the energy inequality for the relative velocity \vec{v}_r . For this, we multiply the first equation in (1.1) by \vec{v} and integrate over Ω_t , taking into account (2.13) and also the relations

$$\begin{aligned} \frac{\partial}{\partial t} \int_{\Omega_t} U dx &= \int_{\Omega_t} (U_t + (\vec{v} \cdot \nabla)U) dx \\ &= \int_{\Omega_t} dx \int_{\Omega_t} (\vec{v}(y, t) \cdot \nabla_y) \frac{1}{|x-y|} dy + \int_{\Omega_t} (\vec{v} \cdot \nabla_x)U(x, t) dx \\ &= 2 \int_{\Omega_t} (\vec{v} \cdot \nabla_x)U(x, t) dx = 2 \int_{\Gamma_t} U \vec{v} \cdot \vec{n} dS, \\ \frac{d|\Gamma_t|}{dt} &= \frac{d}{dt} \int_{\mathcal{G}} |\widehat{\mathcal{L}}^T \vec{N}(y)| dS_y \\ &= \int_{\mathcal{G}} \sum_{i,k,m=1}^3 N_k \widehat{L}_{ki} N_m \frac{\partial \widehat{L}_{mi}}{\partial t} \frac{dS_y}{|\widehat{\mathcal{L}}^T \vec{N}|} = \sum_{i,m=1}^3 \int_{\mathcal{G}} n_i N_m \frac{\partial \widehat{L}_{mi}}{\partial t} dS_y \\ &= \sum_{i,k,m,s=1}^3 \int_{\mathcal{G}} \left(-n_i N_m \widehat{L}_{mk} \frac{\partial^2 N_k \rho}{\partial y_s \partial t} l^{si} + n_i N_m \widehat{L}_{mi} \frac{\partial^2 N_k \rho}{\partial y_s \partial t} l^{ks} \right) dS_y \\ &= \int_{\Gamma_t} \left(- \sum_{i,k=1}^3 n_i n_k \frac{\partial^2 N_k \rho}{\partial x_i \partial t} + \sum_{k=1}^3 \frac{\partial^2 N_k \rho}{\partial x_k \partial t} \right) dS_x = \sum_{k=1}^3 \int_{\Gamma_t} \partial_k (N_k \rho_t) dS \end{aligned}$$

$$= \sum_{k=1}^3 \int_{\Gamma_t} (\vec{n} \cdot \vec{N}) H \rho_t dS = - \int_{\Gamma_t} V_n H dS = - \int_{\Gamma_t} \vec{v} \cdot \vec{n} H dS,$$

where the $l^{si} = \frac{\partial y^s}{\partial x_i}$ are the elements of the matrix \mathcal{L}_ρ^{-1} , and $\partial_k = \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial n} = \vec{e}_k \cdot \nabla_{\Gamma_t}$ (compare with [5, §2]). As a result, we obtain

$$(4.13) \quad \frac{1}{2} \frac{d}{dt} \|\vec{v}\|_{L_2(\Omega_t)}^2 + \frac{\nu}{2} \|S(\vec{v})\|_{L_2(\Omega_t)}^2 = -\sigma \frac{d}{dt} |\Gamma_t| + \frac{\varkappa}{2} \frac{d}{dt} \int_{\Omega_t} U(x, t) dx.$$

Since $S(\vec{v}) = S(\vec{v}_r)$ and

$$\|\vec{v}\|_{L_2(\Omega_t)}^2 = \|\vec{v}_r\|_{L_2(\Omega_t)}^2 + \|\vec{v}'\|_{L_2(\Omega_t)}^2 = \|\vec{v}_r\|_{L_2(\Omega_t)}^2 + \frac{\beta^2}{I(t)},$$

we can rewrite (4.13) in the form

$$(4.14) \quad \frac{d}{dt} \left(\frac{1}{2} \|\vec{v}_r(\cdot, t)\|_{L_2(\Omega_t)}^2 + R[\rho] - R[0] \right) + \frac{\nu}{2} \|S(\vec{v}_r)\|_{L_2(\Omega_t)}^2 = 0,$$

where $R[0]$ is the value of the functional (1.5) for $\rho = 0$ (i.e., for $\Omega = \mathcal{F}$). This identity can also be obtained by multiplying the first equation in (2.17) by \vec{v}_r and integrating over Ω_t , or by multiplying the first equation in (2.22) by \vec{v} and then integrating over Ω'_t . The difference

$$\begin{aligned} R[\rho] - R[0] &= \int_0^1 \frac{d}{ds} R[s\rho] ds = \int_0^1 \left(\frac{d}{ds} R[s\rho] - \frac{d}{ds'} R[s'\rho]_{s'=0} \right) ds \\ &= \int_0^1 ds \int_0^s \frac{d^2}{d\lambda^2} R[\lambda\rho] d\lambda = \int_0^1 (1-\lambda) \frac{d^2}{d\lambda^2} R[\lambda\rho] d\lambda \\ &= \frac{1}{2} \delta_0^2 R[\rho] + \int_0^1 (1-\lambda) \left(\frac{d^2}{d\lambda^2} R[\lambda\rho] - \frac{d^2}{d\lambda^2} R[\lambda\rho] \Big|_{\lambda=0} \right) d\lambda \end{aligned}$$

can be estimated from above and from below by $c\|\rho\|_{W_2^1(\mathcal{G})}^2$ if δ is sufficiently small. However, (4.14) does not imply (4.12) yet. As the second step, we multiply the first equation in (2.18) by the vector \vec{W} constructed in Lemma 4.1 and integrate over Ω_t . Using the orthogonality condition (2.15), we obtain

$$\begin{aligned} &\frac{d}{dt} \int_{\Omega_t} \vec{v}^\perp \cdot \vec{W} dx - \int_{\Omega_t} \vec{v}^\perp \cdot (\vec{W}_t + (\vec{v} \cdot \nabla) \vec{W}) dx \\ &\quad + \int_{\Omega_t} (\vec{v}^\perp \cdot \nabla) \vec{v}'' \cdot \vec{W} dx + \frac{\nu}{2} \int_{\Omega_t} S(\vec{v}^\perp) : S(\vec{W}) dx \\ (4.15) \quad &= \int_{\Gamma_t} \left(\sigma H + \frac{1}{2} |\vec{v}''|^2 + p_0 + \varkappa U(x, t) \right) \vec{W} \cdot \vec{n} dS_x \\ &= \int_{\Gamma_t} \left(\sigma H(x) + \frac{1}{2} |\vec{v}''(x)|^2 + p_0 + \varkappa \mathcal{U}(x, t) \right) \frac{\varphi(y, \rho)}{|\widehat{\mathcal{L}}^T \vec{N}|} dS_x, \end{aligned}$$

where $x = y + N\rho(y, t)$. Invoking equation (2.1), we rewrite the last integral in the form

$$(4.16) \quad \int_{\mathcal{G}} \left[\sigma(H(x) - \mathcal{H}(y)) + \frac{1}{2} (|\vec{v}''(x)|^2 - \omega_0^2 (y_1^2 + y_2^2)) + \varkappa(U(x, t) - \mathcal{U}(y)) \right] \varphi dS_y \equiv I_{\mathcal{G}},$$

and then we add (4.14) to (4.15) multiplied by small $\gamma > 0$. This yields

$$\frac{d}{dt} E(t) + E_1(t) = 0,$$

where

$$\begin{aligned}
 E(t) &= \frac{1}{2} \|\vec{v}_r(\cdot, t)\|_{L_2(\Omega_t)}^2 + (R[\rho] - R[0]) + \gamma \int_{\Omega_t} \vec{v}^\perp \cdot \vec{W} \, dx, \\
 (4.17) \quad E_1(t) &= \frac{\nu}{2} \|S(\vec{v}^\perp)\|_{L_2(\Omega)}^2 + \frac{\gamma\nu}{2} \int_{\Omega} S(\vec{v}^\perp) : S(\vec{W}) \, dx \\
 &\quad + \gamma \int_{\Omega_t} (\vec{v}^\perp \cdot \nabla) \vec{v}'' \cdot \vec{W} \, dx - \gamma \int_{\Omega_t} \vec{v}^\perp \cdot (\vec{W}_t + (\vec{v} \cdot \nabla) \vec{W}) \, dx - \gamma I_G,
 \end{aligned}$$

because $S(\vec{v}_r) = S(\vec{v}^\perp)$. By using (2.6) and (4.3), it is not difficult to prove estimate (4.11) if the number $\gamma > 0$ is sufficiently small. To prove (4.12), it suffices to obtain the inequality

$$(4.18) \quad E_1(t) \geq bE(t)$$

with some constant $b > 0$. Consider the integral $-I_G$, which contains one of the main positive terms in $E_1(t)$. We rewrite it as the sum of four terms:

$$\begin{aligned}
 I_G &= I^{(1)} + I^{(2)} + I^{(3)} + I^{(4)}, \\
 I^{(1)} &= \int_{\mathcal{G}} \left[\sigma(H(x) - \mathcal{H}(y)) + \frac{\omega_0^2}{2}(x_1^2 + x_2^2) - \frac{\omega_0^2}{2}(y_1^2 + y_2^2) + \varkappa(U(x) - \mathcal{U}(y)) \right] \rho \, dS_y, \\
 I^{(2)} &= \int_{\mathcal{G}} \left(\frac{\omega^2(t)}{2} - \frac{\omega_0^2}{2} \right) (x_1^2 + x_2^2) \rho(y) \, dS_y, \\
 I^{(3)} &= \frac{1}{2} \int_{\mathcal{G}} (|\vec{v}''(x, t)|^2 - |\vec{v}'(x, t)|^2) \rho \, dS_y, \\
 I^{(4)} &= \int_{\mathcal{G}} \left[\sigma(H(x) - \mathcal{H}(y)) + \frac{1}{2} (|\vec{v}''(x)|^2 - \omega_0^2(y_1^2 + y_2^2)) + \varkappa(U(x) - \mathcal{U}(y)) \right] \\
 &\quad \times (\varphi(y, \rho) - \rho) \, dS_y, \quad x = e_\rho(y).
 \end{aligned}$$

In [5, 6] it was shown that

$$-I^{(1)} = \delta_0^2 G[\rho] + K_1,$$

where $\delta_0^2 G$ is as in (2.8), and K_1 satisfies

$$(4.19) \quad |K_1| \leq c\delta \|\rho\|_{W_2^1(\mathcal{G})}^2.$$

Now we turn to the integral $I^{(2)}$. Since

$$\frac{\omega^2(t)}{2} - \frac{\omega_0^2}{2} = \frac{\beta^2}{2} \left(\frac{1}{I^2(t)} - \frac{1}{I^2} \right) = -\frac{\beta^2}{2} (I(t) - \mathcal{I}) \left(\frac{1}{I(t)\mathcal{I}^2} + \frac{1}{\mathcal{I}I^2(t)} \right),$$

and the leading linear (with respect to ρ) part of the difference

$$I(t) - \mathcal{I} = \int_{\Omega_t} (x_1^2 + x_2^2) \, dx - \int_{\mathcal{F}} (y_1^2 + y_2^2) \, dy$$

is equal to

$$\delta_0 I = \int_{\mathcal{G}} (y_1^2 + y_2^2) \rho(y, t) \, dS_y$$

(this follows from (2.26)), we have

$$-I^{(2)} = \frac{\beta^2}{\mathcal{I}^3} (\delta_0 I)^2 + K_2,$$

where K_2 also satisfies (4.19). Consider the integrand of $-I^{(3)}$:

$$\begin{aligned} |\vec{v}'(x)|^2 - |\vec{v}''(x)|^2 &= \omega^2(t)(x_1^2 + x_2^2) - (\alpha_2 x_3 - \alpha_3 x_2)^2 \\ &\quad - (\alpha_3 x_1 - \alpha_1 x_3)^2 - (\alpha_1 x_2 - \alpha_2 x_1)^2 \\ &= (\omega^2(t) - \alpha_3^2)(x_1^2 + x_2^2) - (\alpha_1^2 + \alpha_2^2)x_3^2 \\ &\quad + 2(\alpha_1 \alpha_2 x_1 x_2 + \alpha_2 \alpha_3 x_2 x_3 + \alpha_3 \alpha_1 x_1 x_3). \end{aligned}$$

We use formulas (2.16) for α_j , and the relations

$$\begin{aligned} S^{13} &= \frac{S_{12}S_{23} - S_{22}S_{13}}{\det \mathcal{S}}, \\ S^{23} &= \frac{S_{21}S_{13} - S_{11}S_{23}}{\det \mathcal{S}}, \\ \omega(t) - \alpha_3 &= \beta \left(\frac{1}{S_{33}} - S^{33} \right) = \frac{\beta}{S_{33}} \sum_{j=1}^2 S_{3j} S^{j3}. \end{aligned}$$

Since $\int_{\mathcal{F}} y_j y_3 dy = 0$, $j = 1, 2$, and $\delta_0 S_{j3} = -\int_{\mathcal{G}} y_j y_3 \rho dS_y$ (see [5, §2]), we have

$$(4.20) \quad S_{j3} = -\int_{\mathcal{G}} y_j y_3 \rho(y, t) dS_y + S'_{j3}, \quad |S'_{j3}| < c \|\rho\|_{L_2(\mathcal{G})}^2.$$

Moreover,

$$(4.21) \quad |\omega^2(t) - \alpha_3^2| + |\alpha_1^2 + \alpha_2^2| + |\alpha_1 \alpha_2| \leq c \|\rho\|_{L_2(\mathcal{G})}^2.$$

Next, by (2.16) and (4.20), we have

$$(4.22) \quad \int_{\mathcal{G}} (\alpha_1 \alpha_3 x_1 x_3 + \alpha_2 \alpha_3 x_2 x_3) \rho(y, t) dS_y = \beta \alpha_3 \frac{S_{22}S_{13}^2 + S_{11}S_{23}^2 - 2S_{12}S_{13}S_{23}}{\det \mathcal{S}} + K_3,$$

where K_3 satisfies (4.19). Since $|S_{12}| \leq \sqrt{S_{11}S_{12}}$, the first term on the right in (4.22) is nonnegative, whence

$$-I^{(3)} \geq K_4, \quad |K_4| \leq c\delta \|\rho\|_{W_2^1(\mathcal{G})}^2.$$

Finally, expressing the difference $H - \mathcal{H}$ in terms of ρ by the formula

$$\begin{aligned} H(x) - \mathcal{H}(y) &= \vec{n} \cdot \Delta_{\Gamma_t} \vec{x} - \vec{N} \cdot \Delta_{\mathcal{G}} \vec{y} \\ &= \vec{n} \cdot \Delta_{\Gamma_t} (\vec{N} \rho) + (\vec{n} - \vec{N}) \cdot \Delta_{\Gamma_t} \vec{y} + \vec{N} \cdot (\Delta_{\Gamma_t} - \Delta_{\mathcal{G}}) \vec{y} \end{aligned}$$

and integrating by parts, we easily verify that

$$|I^{(4)}| \leq c\delta \|\rho\|_{W_2^1(\mathcal{G})}.$$

Thus,

$$-I_{\mathcal{G}} \geq \delta_0^2 R[\rho] + K,$$

where K satisfies (4.19).

Now we consider the volume integrals in (4.17). By (4.3),

$$\left| \int_{\Omega_t} \vec{v}^\perp \cdot (\vec{W}_t + (\vec{v} \cdot \nabla) \vec{W}) dx \right| \leq c \|\vec{v}^\perp\|_{L_2(\Omega_t)} (\|\rho_t\|_{L_2(\mathcal{G})} + \|\rho\|_{W_2^{1/2}(\mathcal{G})}).$$

In order to estimate ρ_t , we use the kinematic boundary condition $V_n = \vec{v} \cdot \vec{n}$, which can be written equivalently in the form

$$\begin{aligned}
 (4.23) \quad \rho_t(y, t) &= \frac{\vec{v}(x, t) \cdot \vec{n}(x)}{\vec{n}(x) \cdot \vec{N}(y)} \\
 &= \frac{1}{\vec{n} \cdot \vec{N}} \left(\vec{v}^\perp \cdot \vec{n} + \sum_{j=1}^2 \alpha_j(t) \vec{\eta}_j(x) \cdot \vec{n}(x) + \alpha_3(t) (\vec{\eta}_3(x) \cdot \vec{n}(x) - \vec{\eta}_3(y) \cdot \vec{N}(y)) \right), \\
 & \quad y \in \mathcal{G},
 \end{aligned}$$

where $x = y + N(y)\rho(y, t)$ (we have applied the relation $\vec{\eta}_3(y) \cdot \vec{N}(y) = 0$). The estimate

$$\|\rho_t(\cdot, t)\|_{L_2(\mathcal{G})} \leq c(\|\vec{v}^\perp(\cdot, t)\|_{L_2(\Gamma_t)} + \|\rho(\cdot, t)\|_{W_2^1(\mathcal{G})})$$

follows from (4.23) and (4.21).

Recalling Korn's inequality

$$\|\vec{v}^\perp\|_{W_2^1(\Omega_t)} \leq c\|S(\vec{v}^\perp)\|_{L_2(\Omega_t)},$$

we easily verify that the first four terms in (4.17) are dominated by

$$\left(\frac{\nu}{2} - c\gamma\right)\|S(\vec{v}^\perp)\|_{L_2(\Omega_t)}^2 - c\gamma\|S(\vec{v}^\perp)\|_{L_2(\Omega_t)}\|\rho\|_{W_2^1(\mathcal{G})},$$

whence

$$E_1(t) \geq \left(\frac{\nu}{2} - c\gamma\right)\|S(\vec{v}^\perp)\|_{L_2(\Omega_t)}^2 - c\gamma\|S(\vec{v}^\perp)\|_{L_2(\Omega_t)}\|\rho\|_{W_2^1(\mathcal{G})} + \gamma\delta_0^2 R[\rho] - c\delta\gamma\|\rho\|_{W_2^1(\mathcal{G})}^2.$$

Choosing δ and γ sufficiently small (but independent of t , $\|\rho\|_{W_2^1(\mathcal{G})}$, and $\|S(\vec{v}^\perp)\|_{L_2(\Omega_t)}$) and using (2.6), we obtain inequality (4.18) with some $b > 0$. This completes the proof of the theorem. \square

The following theorem concerns uniform estimates for the Hölder norms of the solution of problem (2.17).

Theorem 4.2. *Under the assumptions of Theorem 1.1, the solution of problem (2.17) satisfies the estimate*

$$\begin{aligned}
 (4.24) \quad & |\vec{v}_{rt}(\cdot, t)|_{C^\alpha(\Omega_t)} + |\vec{v}_r(\cdot, t)|_{C^{2+\alpha}(\Omega_t)} + |p_r(\cdot, t)|_{C^{1+\alpha}(\Omega_t)} \\
 & + |\rho(\cdot, t)|_{C^{3+\alpha}(\mathcal{G})} + |\rho_t(\cdot, t)|_{C^{2+\alpha}(\mathcal{G})} + |\rho_{tt}(\cdot, t)|_{C^\alpha(\mathcal{G})} \\
 & \leq c \left(\sup_{t-2\tau_0 \leq t' \leq t} \|\vec{v}_r(\cdot, t')\|_{L_2(\Omega_{t'})} + \sup_{t-2\tau_0 \leq t' \leq t} \|\rho(\cdot, t')\|_{W_2^1(\mathcal{G})} \right),
 \end{aligned}$$

where τ_0 is a small number, $t > 2\tau_0$, and c is a constant independent of t .

The proof of this theorem does not differ from that of Theorem 4.1 in [6], where the solution of (2.22) with $\omega = \omega_0$ was estimated. To obtain (4.24), we can use both the fixed and the rotating coordinate systems, due to relations (2.23) and

$$\omega'(t) = -\frac{2\beta}{I^2(t)} \int_{\Omega_t} (v_1x_1 + v_2x_2) dx = -\frac{2\beta}{I^2(t)} \int_{\Omega_t} (v_{r1}x_1 + v_{r2}x_2) dx.$$

Therefore, inequality (4.4) in [6] is true both for \vec{v}_r and for \vec{w} .

Having estimates (4.12) and (4.13) at hand, we can extend the solution to the entire infinite time interval $t \geq 0$ in precisely the same way as this was done in [6], thus completing the proof of Theorem 2.1.

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