AN EXAMPLE OF A PERIODIC MAGNETIC SCHRÖDINGER OPERATOR WITH DEGENERATE LOWER EDGE OF THE SPECTRUM

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Abstract. The structure of the lower edge of the spectrum of a periodic magnetic Schrödinger operator is investigated. It is known that in the nonmagnetic case the energy depends quadratically on the quasimomentum in a neighborhood of the lower edge of the spectrum of the operator. An example of a magnetic Schrödinger operator is constructed for which energy is partially degenerate with respect to one component of the quasimomentum.

§0. Introduction

0.1. Let $\Gamma$ be a lattice in $\mathbb{R}^d$, $d \geq 1$, and let $\Omega$ be an elementary cell of the lattice $\Gamma$. $\Gamma$-periodic differential operators (DO’s) can be partially diagonalized with the help of the Gelfand transformation (see, e.g., [Sk]). Then the initial DO is represented as a direct integral of a family of DO’s that act on the torus associated with $\Omega$. The operators in this family depend on a parameter $k \in \mathbb{R}^d$ (called the quasimomentum).

We consider lower semibounded selfadjoint DO’s. For most of DO’s of mathematical physics, the spectrum of the corresponding operators acting in $L^2(\Omega)$ is discrete. The eigenvalues $E_j(k)$, $j = 1, 2, \ldots$, arranged in nondecreasing order, depend on $k$ continuously. Then the spectrum of the initial DO has a band structure, with bands corresponding to the band functions $E_j(\cdot)$. For convenience, we assume that the lower edge of the spectrum is $\lambda = 0$. For some problems, it turns out that their solution requires only the knowledge of the approximate behavior of the first band function $E_1(k)$ near its minimum point. In such cases we talk about the threshold effects at the point $\lambda = 0$. It is a nontrivial task to judge whether or not a given effect is a threshold one. As an example, we point out the problem of the discrete spectrum that appears to the left of the point $\lambda = 0$ when a periodic DO is perturbed by a decaying negative potential. If this potential decays not too fast, then the crucial contribution is of threshold nature. Another important case of a threshold effect at the point $\lambda = 0$ is the behavior of a periodic DO in the small period limit.

0.2. The investigation of threshold effects becomes much easier if the periodic DO in question admits an appropriate (“regular”) factorization. In [BSu], a wide class of elliptic periodic second order DO’s was distinguished, and a fairly deep analysis of the lower edge...
of the spectrum was developed for that class. In the scalar case, this class of DO's is
described by the following condition: the operator $M$ admits the representation

$$
M = \sum_{j=1}^{d} D_j b_j,
$$

where $D = \{D_1, \ldots, D_d\} = -i\nabla = -i\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\}$, $f(x)$ is a $\Gamma$-periodic function, $G(x)$ is a $\Gamma$-periodic $(m \times m)$-matrix-valued function $(m \geq d)$, and the $b_j$, $j = 1, 2, \ldots, d$, form a set of linearly independent vectors in $\mathbb{C}^m$. Moreover,

$$
f, f^{-1} \in L_\infty(\mathbb{R}^d); \quad c_0 \leq G(x) \leq c_1 1, \quad x \in \mathbb{R}^d, \quad 0 < c_0 \leq c_1 < \infty.
$$

Under conditions (0.2), the factorization (0.1) will be called a proper factorization. The following statements are direct consequences of the existence of a proper factorization.

1) The first band function $E_1(k)$ has a nondegenerate minimum at the point $k = 0$
(and then $E_1(0) = 0$). The corresponding quadratic form can be estimated explicitly;
the constants in estimates are well controlled.

2) We have $E_1(k) > 0$ for $k \neq 0 \pmod{\Gamma}$.

3) $\min_k E_2(k) > 0$.

Sometimes, the DO in question is given not in the form (0.1), but can be rewritten
properly. In [BSm] it was noted that, in particular, the periodic Schrödinger operator
$-\text{div} \, g(x) \nabla + V(x)$ with a positive matrix-valued function (metric) $g(x)$ admits
a proper factorization. The same is true for the two-dimensional periodic Pauli operator

$$(\mathbf{D} - \mathbf{A})^2 \pm (\partial_1 A_2 - \partial_2 A_1),$$

which is a particular case of the magnetic Schrödinger operator

$$
(D - A)^2 g(x) (D - A) + V(x).
$$

Here $A(x)$ is a $\Gamma$-periodic $\mathbb{R}^2$-valued function.

0.3. We consider the “pure magnetic” Schrödinger operator

$$
H(t) = (\mathbf{D} - t\mathbf{A})^2, \quad t \in \mathbb{R}.
$$

Here, for convenience, we have introduced the parameter $t$. The operator (0.4) is already
factorized, but this factorization is not of the form (0.1). The lower edge of the spectrum
of the operator $H(t)$ is positive for $t \neq 0$ unless the vector-valued function $A$ is potential.
For an appropriate $\theta(t) > 0$, the lower edge of the spectrum of the operator $H(t) - \theta(t)I$
is equal to $\lambda = 0$, while the initial factorization disappears.

For sufficiently small $t$, the operator $H(t) - \theta(t)I$ can be shown to admit a proper
factorization under conditions on the coefficients that are not very restrictive. However,
in general, if $t$ is not assumed to be small, a proper factorization fails.

Observe that the situation may be much different if the electric potential in (0.3) is
coordinated properly with the magnetic potential. For example, the lower edge of the spectrum
of the two-dimensional Pauli operator is always equal to $\lambda = 0$, and a proper
factorization exists.

In the present paper we construct an example of a pure magnetic Schrödinger operator
such that not only does it fail to admit a proper factorization, but also the lower edge of its
spectrum is partially degenerate. Namely, the first band function $E_1(k)$ of this operator
near its minimum point is of fourth (but not second) order in one of the components of
the quasimomentum. Such degeneration of the lower edge of the spectrum leads to some
particular features of the threshold effects that could not be observed in the nonmagnetic
situation. A more detailed discussion of the effects arising in this case will be presented
in another paper, where we shall also examine the existence of a proper factorization for
the operator (0.3) with sufficiently small $A$. 

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§1. THE MAIN RESULT

1.1. Notation. We use the notation \( x = \{x_1, x_2\} \in \mathbb{R}^2 \). Let \( \Omega := [-1/2, 1/2]^2 \), and let \( \tilde{\Omega} := [-\pi, \pi]^2 \) be the cell dual to \( \Omega \). The Sobolev space of functions that are square integrable with all derivatives of the first and second order is denoted by \( H^2(\mathbb{R}^2) \). By \( H^2(\Omega) \) we denote the class of functions whose \( \mathbb{Z}^2 \)-periodic extensions belong to \( H^2_{\text{loc}}(\mathbb{R}^2) \). The symbol \( 1 \) stands for the unit \( (2 \times 2) \)-matrix.

1.2. Construction of the operator. Preliminary remarks. In \( L_2(\mathbb{R}^2) \), we consider the operator

\[
M_t := (D - tA(x))^2 - t^2, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^2.
\]

Here the magnetic potential \( A(x) \) is given by \( A(x) = \text{col}\{0, A_2(x_1)\} \), where

\[
A_2(x_1) := \text{sgn} x_1, \quad x_1 \in [-1/2, 1/2),
\]

\[
A_2(x_1 + 1) = A_2(x_1), \quad x_1 \in \mathbb{R}.
\]

(As we shall see, the term \(-t^2\) ensures that the lower edge of the spectrum of \( M_t \) is equal to zero for sufficiently small \( t \).) Obviously, on the domain \( H^2(\mathbb{R}^2) \) the operator \( M_t, t \in \mathbb{R} \), is selfadjoint and lower semibounded. Using (1.1), we can rewrite the operator \( M_t \) in the form

\[
M_t = D_1^2 + D_2^2 - 2tA_2(x_1)D_2.
\]

We put

\[
\varphi(x_1, x_2) := 2|x_1|, \quad x_1 \in [-1/2, 1/2), \ x_2 \in \mathbb{R},
\]

\[
\varphi(x_1 + 1, x_2) = \varphi(x_1, x_2), \quad x_1 \in \mathbb{R}, \ x_2 \in \mathbb{R}.
\]

Also, we introduce the following family of Hermitian matrices:

\[
G_\gamma(x; t) := \begin{pmatrix}
1 & -i(t\varphi(x) + \gamma) \\
i(t\varphi(x) + \gamma) & 1
\end{pmatrix}, \quad \gamma \in \mathbb{R}.
\]

The identity

\[
M_t = D^*G_\gamma(x; t)D, \quad t \in \mathbb{R}, \ \gamma \in \mathbb{R},
\]

can be verified directly. Observe that the matrix (1.4) generates the representation (1.5) for any \( \gamma \in \mathbb{R} \), though the operator \( M_t \) does not depend on \( \gamma \).

We say that the matrix \( G_\gamma(x; t) \) is uniformly positive (for fixed \( t \) and \( \gamma \)) if for some constant \( c > 0 \) the following estimate is fulfilled:

\[
G_\gamma(x; t) \geq c\mathbf{1}, \quad x \in \mathbb{R}^2, \quad c = c(t, \gamma) > 0.
\]

Lemma 1.1. The matrix \( G_\gamma(x; t) \) is uniformly positive if and only if

\[
|t| < 2 \quad \text{and} \quad \left\{ \begin{array}{ll}
\gamma \in (-1, 1 - t) & \text{for } 0 \leq t < 2, \\
\gamma \in (|t| - 1, 1) & \text{for } -2 < t < 0.
\end{array} \right.
\]

Proof. Clearly, estimate (1.6) is equivalent to the condition

\[
\det G_\gamma(x; t) > 0, \quad x \in \mathbb{R}^2.
\]
By (1.3) and (1.4), inequality (1.7) implies that
\[ 1 - (2t|x_1| + \gamma)^2 > 0, \quad x_1 \in [-1/2, 1/2]. \]

Now, the lemma easily follows. \( \square \)

Remark 1.2. Lemma 1.1 and identity (1.5) imply that for \(|t| < 2\) the operator \(M_t\) admits a proper factorization of the form (0.1). (However, we cannot take one and the same \(\gamma\) ensuring a proper factorization for all \(|t| < 2\) (cf. Lemma 1.1.) As a consequence, the point \(\lambda = 0\) is the lower edge of the spectrum of the operator \(M_t\). In what follows we shall see that the lower edge of the spectrum still coincides with \(\lambda = 0\) up to \(|t| \leq 2\sqrt{3}\).

1.3. Investigation of the spectrum of the operator \(M_t\). In \(L_2(\Omega)\), we consider the family of operators depending on the parameter \(k \in \mathbb{R}^2\) (\(k\) is the quasimomentum):

\[ M_t(k) := (D_1 + k_1)^2 + (D_2 + k_2)^2 - 2t \sgn x_1(D_2 + k_2), \quad k = \{k_1, k_2\} \in \mathbb{R}^2. \]

Defined on the domain \(\tilde{H}^2(\Omega)\), the operators \(M_t(k)\) are selfadjoint and lower semi-bounded. Each of them has discrete spectrum. Let \(\{E_j(k; t)\}\) denote the eigenvalues of the operator \(M_t(k)\); we enumerate them in nondecreasing order:

\[ E_1(k; t) \leq E_2(k; t) \leq \cdots \leq E_n(k; t) \leq \cdots. \]

Let \(\{\psi_j(x; k; t)\}\) be the corresponding (periodic) orthonormal eigenfunctions. The functions \(E_j(k; t)\) are continuous in \(k\) and \(t\) and \((2\pi \mathbb{Z})^2\)-periodic in \(k\). In what follows, we consider only the quasimomenta \(k\) belonging to the dual cell \(\tilde{\Omega}\).

The general Floquet–Bloch theory implies that the spectrum of \(M_t\) has a band structure, and

\[ \sigma(M_t) = \bigcup_{j=1}^{\infty} \mathcal{R}(E_j(\cdot; t)). \]

Here the symbol \(\mathcal{R}(\cdot)\) stands for the range of a function.

In the present subsection we discuss the structure of the lower edge of the spectrum of the operator \(M_t\) in detail. Combined with not too difficult variational arguments, the existence of a proper factorization (1.5) with uniformly positive matrix \(G_x\) leads to some relations for the eigenvalues \(E_j(k; t)\) of the operator \(M_t(k)\) (cf. [BSi]). In particular, for \(|t| < 2\) we have

\[ E_1(k; t) \geq 0, \quad E_2(k; t) > 0, \quad k \in \tilde{\Omega}; \]

\[ E_1(k; t) = 0 \iff k = 0; \quad \psi_1(x; 0; t) = 1. \]

Furthermore, the minimum of the function \(E_1(\cdot, t)\) at the point \(k = 0\) is nondegenerate.

We are interested in the values of \(t\) for which relations (1.10) are true. Using the discrete Fourier transformation in \(x_2\), we see that the operator (1.8) can be decomposed into the orthogonal sum of operators

\[ (D_1 + k_1)^2 + (2\pi n_2 + k_2)^2 - 2t(2\pi n_2 + k_2) \sgn x_1, \quad n_2 \in \mathbb{Z}, \]

with periodic boundary conditions, acting on \(L_2([-1/2, 1/2])\). Thus, the problem concerning the spectrum of the operator \(M_t(k)\) reduces to investigation of the spectrum of the operator family

\[ \tilde{M}_t(k_1, \xi) := (D_1 + k_1)^2 + \xi^2 - 2t \xi \sgn x_1, \quad k_1 \in [-\pi, \pi), \quad \xi \in \mathbb{R}, \]

on the interval \(x_1 \in [-1/2, 1/2]\), with periodic boundary conditions. Let \(\tilde{E}_1(k_1, \xi; t)\), \(k_1 \in [-\pi, \pi)\), be the smallest eigenvalue of the operator \(\tilde{M}_t(k_1, \xi)\). The operator \(\tilde{M}_t(k_1, \xi)\) has the form \((D_1 + k_1)^2 + V_t(x_1)\). Then, as is well known (see, e.g., [RSi]),

\[ \tilde{E}_1(k_1, \xi; t) > \tilde{E}_1(0, \xi; t), \quad k_1 \in [-\pi, \pi) \setminus \{0\}, \quad \xi \in \mathbb{R}. \]
Now we proceed to the study of the spectrum of the operator
\[
\widetilde{M}_t(0, \xi) = -\frac{d^2}{dx_1^2} + \xi^2 - 2t \text{sgn} x_1, \quad \xi \in \mathbb{R},
\]
on the interval \(x_1 \in [-1/2, 1/2]\), with periodic boundary conditions. We distinguish the case where \(\xi = 0\). Obviously, this case corresponds to the relations \(E_1(0; t) = 0, \psi_1(x, 0; t) = 1\). We consider the problem
\begin{equation}
(1.12) \quad \widetilde{M}_t(0, \xi) u(x_1) = 0, \quad u(-1/2) = u(1/2), \quad u'(-1/2) = u'(1/2), \quad \xi \in \mathbb{R}\setminus\{0\}.
\end{equation}
Obviously, the following condition is necessary for the existence of a nontrivial solution of problem (1.12):
\begin{equation}
(1.13) \quad \xi^2 < 2|t\xi|, \quad \xi \in \mathbb{R}\setminus\{0\}.
\end{equation}
We put
\[
\alpha_{\pm} := \sqrt{2|t\xi|} \pm \xi^2, \quad \xi^2 < 2|t\xi|, \quad \xi \in \mathbb{R}\setminus\{0\}.
\]
Simple but cumbersome calculations show that problem (1.12) has a nontrivial solution if and only if relation (1.13) and one of the relations
\begin{equation}
(1.14) \quad \alpha_+ \tanh^{-1} \frac{\alpha_+}{4} = -\alpha_- \tanh^{-1} \frac{\alpha_-}{4}, \quad \alpha_+ \tanh \frac{\alpha_+}{4} = -\alpha_- \tanh \frac{\alpha_-}{4}
\end{equation}
are fulfilled. From the further analysis it is clear that for \(|t| \leq 2\sqrt{3}\) relations (1.14) fail. Combined with (1.9) and (1.11), this implies that for \(|t| \leq 2\sqrt{3}\) the point \(\lambda = 0\) is the lower edge of the spectrum of the operator \(M_t\) and that relations (1.10) are fulfilled.

To complete the study of the lower edge of the spectrum of \(M_t\), we consider again the problem for the first eigenvalue of the operator \(M_t(k)\):
\[
M_t(k)\psi_1(x, k; t) = E_1(k; t)\psi_1(x, k; t).
\]
From the above considerations it follows that \(E_1(0; t) = 0\) and \(\psi_1(x, 0; t) = 1\) for \(|t| \leq 2\sqrt{3}\). Moreover, the point \(\lambda = 0\) is a simple eigenvalue of \(M_t(0)\). It is easy to write down several initial terms of the analytic expansion of the function \(E_1(\cdot; t)\) near the point \(k = 0\):
\[
E_1(k; t) = k_1^2 + (1 - \frac{t^2}{12})k_2^2 + \frac{1}{42} \cdot \frac{t^4}{144} k_2^4 - \frac{1}{10} \cdot \frac{t^2}{12} k_1^2 k_2^2 + O(|k|^6).
\]
In particular, for \(|t| = 2\sqrt{3}\) we have
\[
E_1(k; \pm 2\sqrt{3}) = k_1^2 + \frac{1}{42} k_2^4 - \frac{1}{10} k_1^2 k_2^2 + O(|k|^6).
\]
We summarize the results obtained.

**Theorem 1.3.** 1) For \(|t| < 2\sqrt{3}\) the point \(\lambda = 0\) is the lower edge of the spectrum of the operator \(M_t\); relations (1.10) are fulfilled; and the function \(E_1(\cdot; t)\) is analytic in a neighborhood of \(k = 0\) and has a nondegenerate minimum at the point \(k = 0\). Moreover, for \(|t| < 2\) the operator \(M_t\) admits a proper factorization.

2) For \(|t| = 2\sqrt{3}\) the point \(\lambda = 0\) is the lower edge of the spectrum of the operator \(M_t\); relations (1.10) are fulfilled; and the function \(E_1(\cdot; \pm 2\sqrt{3})\) is analytic in a neighborhood of \(k = 0\) and has a minimum of second order in \(k_1\) and of fourth order in \(k_2\) at the point \(k = 0\).

**Remark 1.4.** 1) It can be proved that, for \(2 \leq |t| \leq 2\sqrt{3}\), there is no proper factorization for the operator \(M_t\). 2) A more detailed analysis shows that, for \(|t| > 2\sqrt{3}\), the operator \(M_t\) is not positive anymore. Herewith, at least if \(|t| \in (2\sqrt{3}, 2\sqrt{3} + \delta)\), where \(\delta > 0\) is sufficiently small, the (negative) minimum of the spectrum of the operator \(M_t\) is reached at two different \((\mod(2\pi Z)^2)\) points \(k^{(1)} = \{0, k_2\}, k^{(2)} = \{0, -k_2\}, k_2 > 0\).
Remark 1.5. Of course, an example similar to that described in the present subsection can be constructed also for $d \geq 3$. We can simply consider the operator

$$M^{(d)}_t := M_t \otimes I' + I \otimes (-\Delta'),$$

where $\Delta' := \left( \frac{\partial^2}{\partial x_3^2} + \cdots + \frac{\partial^2}{\partial x_d^2} \right)$, and $I, I'$ are the identity operators relative to $x_1, x_2$ and $x_3, \ldots, x_d$. Then an analog of Theorem 1.3 is valid.

We have constructed a periodic magnetic Schrödinger operator for which the lower edge of the spectrum is partially degenerate (for one of the components of the quasi-momentum). The author believes that there exist examples of complete degeneration of the lower edge of the spectrum. Such situation seems to come true for a family of two-dimensional operators $M_t := (D - tA)^2 - t^2$, where the $\mathbb{Z}^2$-periodic magnetic potential $A$ is given on $\Omega$ by the formula $A(x) = \text{col} \{ \text{sgn} x_2, \text{sgn} x_1 \}$, $\{x_1, x_2\} \in \Omega$, while the parameter $t \in \mathbb{R}$ is varying.

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