Алгебра и анализ
Том. 16 (2004), вып. 2

St. Petersburg Math. J.
Vol. 16 (2005), No. 2, Pages 423–436
S 1061-0022(05)00859-9
Article electronically published on March 10, 2005

SYMMETRIC GEODESICS ON CUBES: ALGORITHMS FOR FINDING THEM

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Abstract. An algorithm is presented for finding simple closed symmetric geodesics on the boundary of an n-cube.

§1. Introduction

1.1. A geodesic γ on the boundary ∂K^n of an n-cube K^n is said to be *symmetric* if γ is closed and symmetric with respect to the center O_n of K^n. The following assertions were proved in [1].

1) The closed geodesics on ∂K^n intersect no $(n-3)$-faces of K^n. Each closed geodesic is a broken line the links of which lie on $(n-1)$-faces (*facets*), and the vertices lie at inner points of $(n-2)$-faces.

2) If a cube K^k, where $2 \le k < n$, is the section of K^n by a k-plane passing through O_n (a *central* section), then each symmetric geodesic on ∂K^k is also a symmetric geodesic on ∂K^n. Conversely, if a symmetric geodesic γ on ∂K^n has no links on at least one of the facets of K^n, then γ is a symmetric geodesic on the boundary of a certain cube K^{n-1} which is a central section of K^n.

Therefore, if we pass from the dimension $n-1$ to the dimension n, the only "new" geodesics are the symmetric geodesics that pass *along each* of the $2n$ facets of K^n. The problem that we solve consists in finding symmetric geodesics on ∂K^n with precisely this property for various n. (For $n = 3$, such a geodesic is the hexagon obtained as the central section of ∂K^3 by the plane orthogonal to one of the four great diagonals of K^3.)

As in [1], we are interested only in the symmetric geodesics that are *simple*, i.e., have neither self-intersections nor self-overlappings.

1.2. The above problem was considered in [1] only for small dimensions $n = 3, 4, 5$. This allowed us to use intuitive geometric methods.

Already for $n = 4$, it turned out that besides a "1-turn" symmetric geodesic γ passing only once along each of the eight facets of K^4, there also exist "3-turn" symmetric geodesics. They "surround" γ, filling (together with γ) a certain torus with a cross-section in the form of an open triangle, and thus transforming this torus into a *filled torus with a Seifert fibration*. (See [2, the beginning of §3].)

For $n = 5$, in [1] we found a 1-turn symmetric geodesic by an artificial trick.

1.3. In the present paper, we present algorithms for finding 1- and many-turn symmetric geodesics on the surfaces of cubes of higher dimensions in finitely many steps.

In §§6 and 7, we consider the cases of $n = 4, 5, 6, 7$ as examples of application of these algorithms.

2000 *Mathematics Subject Classification.* Primary 51M20, 52B05, 52B15.
Key words and phrases. Closed geodesics, cubes.

1.4. In [1], we presented a method for finding symmetric geodesics on ∂K^4. This method involves rolling the unit cube K^4 along the integer lattice in $\mathbb{R}^3$. As a result, a symmetric geodesic on ∂K^4 is unfolding in $\mathbb{R}^3$ along some ray. The ray was found by geometric arguments.

If for $n > 4$ we similarly roll the unit cube K^n along the integer lattice in $\mathbb{R}^{n-1}$, then the required geodesic is also unfolding along a certain ray l.

The crucial role for the present paper was played by the following fact: a thorough examination of the results obtained in [1] for $n = 3, 4$ suggested not only the direction of such a ray, but also the choice of its initial point. This allowed us in §3 to immediately indicate a chain of $2n$ unit $(n-1)$-cubes of the integer lattice in $\mathbb{R}^{n-1}$ along which the cube K^n rolls in the course of rectification of a 1-turn geodesic.

Rolling along the chosen ray confirms that the 1-turn geodesic obtained is symmetric, suggests which many-turn closed geodesics can be expected for a given n, and helps to construct them and to verify whether they (or at least some of them) are symmetric with respect to O_n.

As soon as we have found a symmetric geodesic γ on ∂K^n of a certain shape, the other geodesics of the same shape are obtained from γ via self-congruences of K^n.

1.5. Describing the process of rolling for $n > 4$ involves some specific combinatorial means. For this reason, our notation here differs much from that used in [1].

§2. The initial position of the cube K^n and notation for its vertices and faces

2.1. We assume that K^n is a unit cube lying in $\mathbb{R}^n(x_1, \ldots, x_n)$ so that one of its vertices is at the origin and the edges of K^n are parallel to the coordinate axes. We call this the *initial position* of K^n.

2.2. The cube K^n has 2^n vertices. In the initial position, the coordinates $(x_1, \ldots, x_n)$ of any vertex A of K^n form one of the 2^n possible n-tuples of zeros and units. We denote this n-tuple by σ and denote A by A_σ. The n-tuple σ is the *index* of A_σ. In what follows, we rotate and shift K^n in $\mathbb{R}^n$ as a rigid body. However, we preserve the index σ of a vertex adopted in the initial position. So, we will say: "The vertex A_σ comes to a certain point".

2.3. The cube K^n has $2n$ facets, which are unit $(n-1)$-cubes. We denote them by G_{i0} and G_{i1}, where $1 \leq i \leq n$. The vertices of G_{i0} are the vertices of K^n with $x_i = 0$ in the initial position. For the vertices of G_{i1} we have $x_i = 1$. The facets G_{i0} and G_{i1} are opposite to each other. In the initial position, they both are orthogonal to the x_i-axis.

§3. Preparation for rolling

3.1. Throughout, we let $\mathbb{R}^{n-1}$ be the hyperplane $x_n = 0$ in $\mathbb{R}^n(x_1, \ldots, x_n)$, which we call the *plane*. We regard this plane as a space $\mathbb{R}^{n-1} = \mathbb{R}^{n-1}(x_1, \ldots, x_{n-1}, 0)$, retaining the zero nth coordinate of all points of this $(n-1)$-dimensional space, which is convenient for the algorithms in the sequel. In the initial position in $\mathbb{R}^n$, the facet G_{n0} of K^n lies on an $(n-1)$-cube $Q_1 \subset \mathbb{R}^{n-1}$. In this position, the index σ of any vertex of K^n in G_{n0} coincides with the index σ of the corresponding vertex of $Q_1 \subset \mathbb{R}^{n-1}$.

3.2. In the space $\mathbb{R}^{n-1}(x_1,\dots,x_{n-1},0)$, we consider the $(n-2)$-simplex Δ_0^{n-2} with vertices

$$\begin{gathered} D_1^0(0,0,0,\dots,0,0,0),\\ D_2^0(1,0,0,\dots,0,0,0),\\ D_3^0(1,1,0,\dots,0,0,0),\\ \vdots\\ D_{n-1}^0(1,1,1,\dots,1,0,0). \end{gathered} \tag{1}$$

We denote by p_0 the barycenter of Δ_0^{n-2}:

$$p_0=\Big(\frac{n-2}{n-1},\frac{n-3}{n-1},\dots,\frac{1}{n-1},0,0\Big).$$

3.3. Let l be the ray in $\mathbb{R}^n$ (actually, in $\mathbb{R}^{n-1}$) emanating from p_0 and having the parametric equation

$$\begin{aligned} x_1&=\frac{n-2}{n-1}+t,\\ x_2&=\frac{n-3}{n-1}+t,\\ &\ \vdots \qquad\qquad\qquad\qquad 0\le t<\infty.\\ x_{n-2}&=\frac{1}{n-1}+t,\\ x_{n-1}&=t,\\ x_n&=0. \end{aligned} \tag{2}$$

The ray l passes successively through a sequence of unit $(n-1)$-cubes of the integer lattice in $\mathbb{R}^{n-1}$. We are interested only in the first $2n$ of them, which we denote successively by $Q_1,\dots,Q_{2n}$.

Under passage from the $(n-1)$-cube Q_i to the $(n-1)$-cube Q_{i+1}, the ray l intersects their common $(n-2)$-face at the point p_i corresponding to $t=\frac{i}{n-1}$.

At the points $p_1,\dots,p_{n-1}$, the coordinates $x_1,\dots,x_{n-1}$ successively turn out to be integers. After that, the same order repeats for the points $p_n,\dots,p_{2n-2}$, and once again the coordinate x_1 for p_{2n-1} becomes an integer. Accordingly, the $(n-1)$-cubes $Q_1,\dots,Q_{2n}$ in the integer lattice in $\mathbb{R}^{n-1}$ differ from each other by successive shifts by 1 in the direction of the axes $x_1,\dots,x_{n-1},x_1,\dots,x_{n-1},x_1$.

3.4. We roll the cube K^n as a rigid body along the plane $\mathbb{R}^{n-1}$ so that the $(n-1)$-faces of K^n lie successively onto the $(n-1)$-cubes $Q_1,\dots,Q_{2n}$.

When we pass Q_i, the segment of l contained in Q_i is "printed" on the $(n-1)$-face of K^n that lies on Q_i.

In the favorable case, if the prints of p_0 and p_{2n} on ∂K^n coincide, we obtain a closed and symmetric 1-turn geodesic on ∂K^n with vertices $p_0,p_1,\dots,p_{2n}=p_0$.

3.5. If a ray l' parallel to l emanates not from the point p_0, but from an interior point $p_0'\in\operatorname{Int}\Delta_0^{n-2}$, then l' intersects the same $(n-1)$-cubes $Q_1,\dots,Q_{2n}$ in the same order, and enters and leaves them through the same $(n-2)$-faces as the ray l.

The "flow" of the rays parallel to l and emanating from all points in Δ_0^{n-2} successively projects the simplex Δ_0^{n-2} to the $(n-2)$-faces of $Q_1,\dots,Q_{2n}$ through which the ray l *leaves* these $(n-1)$-cubes. We denote the projections by Δ_1^{n-2}, …, Δ_{2n}^{n-2}.

For $1\le i\le 2n$, the projection $\varphi_i:\Delta_0^{n-2}\to\Delta_i^{n-2}$ is an affine mapping. We denote

$$D_j^i:=\varphi_i(D_j^0)\in\Delta_i^{n-2},\quad j=1,\dots,n-1.$$

Since the mapping φ_i is affine, the barycenter of Δ_i^{n-2} is precisely the point p_i. Therefore, if for each $j = 1, \dots, n-1$ we know which of the vertices A_σ of K^n comes to D_j^i at the moment when, in the course of rolling, the cube K^n stays on the $(n-1)$-cube Q_i, then we shall determine not only the (already known) position of the point p_i on the plane $\mathbb{R}^{n-1}$, but also the "print" p_i' of p_i on ∂K^n in the initial position of K^n. We need precisely these "prints" of the points p_i: they will be the vertices of the required 1-turn geodesic.

3.6. *Remark.* For each i, rolling (see §5) tells us which of the vertices A_σ come to the vertices of Q_i in the course of rolling at the moments when K^n stays on Q_i. There are $2n \cdot 2^{n-1}$ such vertices, while the simplexes $\Delta_1^{n-2}, \dots, \Delta_{2n}^{n-2}$ have only $2n(n-1)$ vertices. It suffices to know what vertices A_σ come to these $2n(n-1)$ points, which makes the algorithms polynomial in n.

§4. Shifting the simplexes Δ_i^{n-2}

4.1. The objective of "Algorithm 1" presented below is as follows: for each i, to determine the coordinates in $\mathbb{R}^n$ acquired by the vertices of the simplex Δ_i^{n-2} if we shift the $(n-1)$-cube Q_i containing Δ_i^{n-2} to the position in $\mathbb{R}^{n-1}$ "initial" for the cubes K^{n-1}.

The algorithm takes into account the fact that the vertices

$$D_j^0 \mapsto D_j^1 \mapsto D_j^2 \mapsto \cdots$$

are related by projection along the vector $(1, 1, \dots, 1, 0)$. The algorithm also takes into account the shift of Q_i relative to Q_{i-1} for $i = 1, \dots, n-1$.

4.2. Algorithm 1. 1) We start with the $((n-1) \times n)$-matrix M_0 the rows of which coincide with the coordinates of the vertices $D_1^0, \dots, D_{n-1}^0$ of Δ_0^{n-2}; see (1).

2) Replacing a unique zero row in M_0 by $(1, 1, \dots, 1, 0)$, we obtain a matrix M_1; it is formed by the coordinates of the vertices of the simplex Δ_1^{n-2} after shifting it.

3) We replace a unique column in M_1 consisting only of units by a zero column, after which a unique zero row arises. Replacing it by $(1, 1, \dots, 1, 0)$, we obtain a matrix M_2, the rows of which are the coordinates of the vertices of the simplex Δ_2^{n-2} after shifting it.

We proceed further as in step 3). Replacing a unique column of units in the matrix M_{i-1} by a zero column and, after that, replacing a unique zero row by $(1, 1, \dots, 1, 0)$, we obtain a matrix M_i.

As a result, Algorithm 1 provides matrices $M_0, M_1, \dots, M_{2n}$ formed by the coordinates of the vertices of the simplexes $\Delta_0^{n-2}, \Delta_1^{n-2}, \dots, \Delta_{2n}^{n-2}$ after shifting them.

§5. Rolling

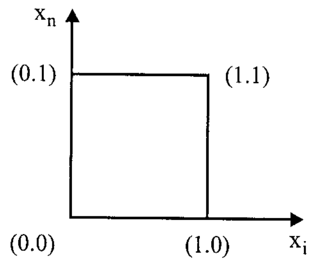
5.1. First, we eliminate the shift component of K^n arising in the course of rolling.

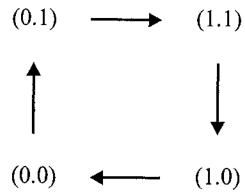
Suppose that in the initial position the cube K^n stays on an $(n-1)$-cube $Q_1 \subset \mathbb{R}^{n-1}$. We roll K^n along the plane $\mathbb{R}^{n-1}$ onto the cell of the integer lattice in $\mathbb{R}^{n-1}$ adjacent to Q_1 in the direction of the x_i-axis, and after that, shift K^n back (by 1 in the direction opposite to that of the x_i-axis) to Q_1. Then K^n undergoes a certain rotation α_i, which gives us a self-congruence of K^n interchanging the vertices of K^n.

Now we describe how, knowing the index σ of a vertex A_σ, we find the index $\alpha_i(\sigma)$ of the vertex $\alpha_i(A_\sigma) = A_{\alpha_i(\sigma)}$. The operator α_i preserves all coordinates of A_σ except x_i and x_n and transforms the pair (x_i, x_n) by the following rule:

$$\alpha_i \; (0,0) \mapsto (0,1) \mapsto (1,1) \mapsto (1,0) \mapsto (0,0).$$

This is illustrated in the figure.





"Algorithm 2" lists the vertices A_σ of the initial cube K^n that come in the course of rolling to the vertices of Δ_i^{n-2} when K^n stays on Q_i. It involves the operators β_i inverse to the operators α_i. The operator β_i transforms a pair of coordinates (x_i, x_n) as follows:

$$\beta_i\,(0,0) \mapsto (1,0) \mapsto (1,1) \mapsto (0,1) \mapsto (0,0). \tag{3}$$

5.2. Algorithm 2. By definition, the operator β_i acts on an $((n-1)\times n)$ matrix M consisting only of zeros and ones as follows. In each row of M, the operator β_i preserves all entries except the ith and the nth, which it transforms by the rule (3).

The objective of Algorithm 2 is as follows: for each $i = 0, 1, \ldots, 2n$, to construct an $((n-1)\times n)$-matrix $A(M_i)$ the rows of which are the indexs σ of the vertices A_σ of K^n that come to the vertices of Δ_i^{n-2} in the course of rolling when K^n stays on Q_i.

After each step of rolling, we shift K^n back to Q_1. Then the entire process of rolling K^n from Q_1 to $Q_2, \ldots, Q_{2n}$ reduces to a sequence of rotations $\alpha_1, \ldots, \alpha_{n-1}, \alpha_1, \ldots, \alpha_{n-1}, \alpha_1$ of K^n about O_n. The simplexes Δ_i^{n-2} together with the $(n-1)$-cubes Q_i have already been shifted to Q_1: Algorithm 1 gave us the matrices M_i of coordinates of their vertices $D_1^i, \ldots, D_{n-1}^i$ thus shifted.

Therefore, Algorithm 2 constructs the matrices $A(M_i)$ by applying the corresponding sequences of operators β_j to the matrices M_i:

$$\begin{aligned}
A(M_0) &= M_0,\\
A(M_1) &= M_1,\\
A(M_2) &= \beta_1(M_2),\\
A(M_3) &= \beta_1\beta_2(M_3),\\
&\ \vdots\\
A(M_n) &= \beta_1\beta_2\cdots\beta_{n-1}(M_n),\\
A(M_{n+1}) &= \beta_1\beta_2\cdots\beta_{n-1}\beta_1(M_{n+1}),\\
A(M_{n+2}) &= \beta_1\cdots\beta_{n-1}\beta_1\beta_2(M_{n+2}),\\
&\ \vdots\\
A(M_{2n-1}) &= \beta_1\cdots\beta_{n-1}\beta_1\cdots\beta_{n-1}(M_{2n-1}),\\
A(M_{2n}) &= \beta_1\cdots\beta_{n-1}\beta_1\cdots\beta_{n-1}\beta_1(M_{2n}).
\end{aligned} \tag{4}$$

5.3. The knowledge of the matrices $A(M_i)$ makes it possible to easily construct a 1-turn symmetric geodesic on ∂K^n with vertices $p_0, p_1, \ldots, p_{2n-1}, p_{2n} = p_0$, as well as obtain many-turn closed geodesics and look for symmetric ones among them. We proceed to examples, where we denote the "prints" of Δ_i^j on ∂K^n by $\bar{\Delta}_i^j$.

§6. 1-turn symmetric geodesics

6.1. For $n = 4$, Algorithm 1 gives us the matrices

$$\overset{M_0}{\begin{pmatrix}0&0&0&0\\1&0&0&0\\1&1&0&0\end{pmatrix}},\quad \overset{M_1}{\begin{pmatrix}1&1&1&0\\1&0&0&0\\1&1&0&0\end{pmatrix}},\quad \overset{M_2}{\begin{pmatrix}0&1&1&0\\1&1&1&0\\0&1&0&0\end{pmatrix}},$$

$$\overset{M_3}{\begin{pmatrix}0&0&1&0\\1&0&1&0\\1&1&1&0\end{pmatrix}},\quad \overset{M_4}{\begin{pmatrix}1&1&1&0\\1&0&0&0\\1&1&0&0\end{pmatrix}},\quad \overset{M_5}{\begin{pmatrix}0&1&1&0\\1&1&1&0\\0&1&0&0\end{pmatrix}},$$

$$\overset{M_6}{\begin{pmatrix}0&0&1&0\\1&0&1&0\\1&1&1&0\end{pmatrix}},\quad \overset{M_7}{\begin{pmatrix}1&1&1&0\\1&0&0&0\\1&1&0&0\end{pmatrix}},\quad \overset{M_8}{\begin{pmatrix}0&1&1&0\\1&1&1&0\\0&1&0&0\end{pmatrix}}.$$

Algorithm 2 transforms them to the matrices

$$\overset{A(M_0)}{\begin{pmatrix}0&0&0&0\\1&0&0&0\\1&1&0&0\end{pmatrix}},\quad \overset{A(M_1)}{\begin{pmatrix}1&1&1&0\\1&0&0&0\\1&1&0&0\end{pmatrix}},\quad \overset{A(M_2)}{\begin{pmatrix}1&1&1&0\\1&1&1&1\\1&1&0&0\end{pmatrix}},$$

$$\overset{A(M_3)}{\begin{pmatrix}1&1&1&0\\1&1&1&1\\0&1&1&1\end{pmatrix}},\quad \overset{A(M_4)}{\begin{pmatrix}0&0&1&1\\1&1&1&1\\0&1&1&1\end{pmatrix}},\quad \overset{A(M_5)}{\begin{pmatrix}0&0&1&1\\0&0&0&1\\0&1&1&1\end{pmatrix}}, \tag{5}$$

$$\overset{A(M_6)}{\begin{pmatrix}0&0&1&1\\0&0&0&1\\0&0&0&0\end{pmatrix}},\quad \overset{A(M_7)}{\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}},\quad \overset{A(M_8)}{\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&0&0\end{pmatrix}}.$$

The rows of these matrices give us the coordinates of the vertices of the 2-simplexes $\bar{\Delta}_0^2, \bar{\Delta}_1^2, \dots, \bar{\Delta}_8^2 \subset \partial K^4$. Their barycenters

$$\begin{aligned} p_8 = p_0\Big(\frac{2}{3},\frac{1}{3},0,0\Big), &\qquad p_4\Big(\frac{1}{3},\frac{2}{3},1,1\Big),\\ p_1\Big(1,\frac{2}{3},\frac{1}{3},0\Big), &\qquad p_5\Big(0,\frac{1}{3},\frac{2}{3},1\Big),\\ p_2\Big(1,1,\frac{2}{3},\frac{1}{3}\Big), &\qquad p_6\Big(0,0,\frac{1}{3},\frac{2}{3}\Big),\\ p_3\Big(\frac{2}{3},1,1,\frac{2}{3}\Big), &\qquad p_7\Big(\frac{1}{3},0,0,\frac{1}{3}\Big) \end{aligned} \tag{6}$$

are the vertices of a closed geodesic on ∂K^4. This geodesic is symmetric, because the midpoints of the segments p_0p_4, p_1p_5, p_2p_6, and p_3p_7 lie at O_4. (All the coordinates of O_4 are equal to 1/2.)

6.2. For $n = 5$, Algorithm 1 gives us the matrices

$$\overset{M_0}{\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\end{pmatrix}},\quad \overset{M_1}{\begin{pmatrix}1&1&1&1&0\\1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\end{pmatrix}},\quad \overset{M_2}{\begin{pmatrix}0&1&1&1&0\\1&1&1&1&0\\0&1&0&0&0\\0&1&1&0&0\end{pmatrix}},$$

$$\overset{M_3}{\begin{pmatrix}0&0&1&1&0\\1&0&1&1&0\\1&1&1&1&0\\0&0&1&0&0\end{pmatrix}},\quad \overset{M_4}{\begin{pmatrix}0&0&0&1&0\\1&0&0&1&0\\1&1&0&1&0\\1&1&1&1&0\end{pmatrix}},\quad \overset{M_5}{\begin{pmatrix}1&1&1&1&0\\1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\end{pmatrix}},$$

$$\overset{M_6}{\begin{pmatrix}0&1&1&1&0\\1&1&1&1&0\\0&1&0&0&0\\0&1&1&0&0\end{pmatrix}},\quad \overset{M_7}{\begin{pmatrix}0&0&1&1&0\\1&0&1&1&0\\1&1&1&1&0\\0&0&1&0&0\end{pmatrix}},\quad \overset{M_8}{\begin{pmatrix}0&0&0&1&0\\1&0&0&1&0\\1&1&0&1&0\\1&1&1&1&0\end{pmatrix}},$$

$$\overset{M_9}{\begin{pmatrix}1&1&1&1&0\\1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\end{pmatrix}},\quad \overset{M_{10}}{\begin{pmatrix}0&1&1&1&0\\1&1&1&1&0\\0&1&0&0&0\\0&1&1&0&0\end{pmatrix}}.$$

Algorithm 2 transforms them to the matrices

$$\overset{A(M_0)}{\begin{pmatrix}0&0&0&0&0\\1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\end{pmatrix}},\quad \overset{A(M_1)}{\begin{pmatrix}1&1&1&1&0\\1&0&0&0&0\\1&1&0&0&0\\1&1&1&0&0\end{pmatrix}},\quad \overset{A(M_2)}{\begin{pmatrix}1&1&1&1&0\\1&1&1&1&1\\1&1&0&0&0\\1&1&1&0&0\end{pmatrix}},$$

$$\overset{A(M_3)}{\begin{pmatrix}1&1&1&1&0\\1&1&1&1&1\\0&1&1&1&1\\1&1&1&0&0\end{pmatrix}},\quad \overset{A(M_4)}{\begin{pmatrix}1&1&1&1&0\\1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\end{pmatrix}},\quad \overset{A(M_5)}{\begin{pmatrix}0&0&0&1&1\\1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\end{pmatrix}},$$

$$\overset{A(M_6)}{\begin{pmatrix}0&0&0&1&1\\0&0&0&0&1\\0&1&1&1&1\\0&0&1&1&1\end{pmatrix}},\quad \overset{A(M_7)}{\begin{pmatrix}0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0\\0&0&1&1&1\end{pmatrix}},\quad \overset{A(M_8)}{\begin{pmatrix}0&0&0&1&1\\0&0&0&0&1\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix}}, \tag{7}$$

$$\overset{A(M_9)}{\begin{pmatrix}1&1&0&0&0\\0&0&0&0&1\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix}},\quad \overset{A(M_{10})}{\begin{pmatrix}1&1&0&0&0\\1&1&1&0&0\\0&0&0&0&0\\1&0&0&0&0\end{pmatrix}}.$$

Their rows give us the coordinates of the vertices of the 3-simplexes $\bar\Delta_0^3, \bar\Delta_1^3, \dots, \bar\Delta_{10}^3 \subset \partial K^5$. Their barycenters

$$(8)\qquad \begin{aligned} p_{10} = p_0\Big(\frac34,\frac12,\frac14,0,0\Big), &\qquad p_5\Big(\frac14,\frac12,\frac34,1,1\Big),\\ p_1\Big(1,\frac34,\frac12,\frac14,0\Big), &\qquad p_6\Big(0,\frac14,\frac12,\frac34,1\Big),\\ p_2\Big(1,1,\frac34,\frac12,\frac14\Big), &\qquad p_7\Big(0,0,\frac14,\frac12,\frac34\Big),\\ p_3\Big(\frac34,1,1,\frac34,\frac12\Big), &\qquad p_8\Big(\frac14,0,0,\frac14,\frac12\Big),\\ p_4\Big(\frac12,\frac34,1,1,\frac34\Big), &\qquad p_9\Big(\frac12,\frac14,0,0,\frac14\Big) \end{aligned}$$

are the vertices of a 1-turn closed symmetric geodesic on ∂K^5.

6.3. For $n = 6$, applying Algorithms 1 and 2, we obtain the matrices

$$(9)\qquad \begin{gathered}
\overset{A(M_0)}{\begin{pmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&0&0\end{pmatrix}},\quad
\overset{A(M_1)}{\begin{pmatrix}1&1&1&1&1&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&0&0\end{pmatrix}},\quad
\overset{A(M_2)}{\begin{pmatrix}1&1&1&1&1&0\\1&1&1&1&1&1\\1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&0&0\end{pmatrix}},\\
\overset{A(M_3)}{\begin{pmatrix}1&1&1&1&1&0\\1&1&1&1&1&1\\0&1&1&1&1&1\\1&1&1&0&0&0\\1&1&1&1&0&0\end{pmatrix}},\quad
\overset{A(M_4)}{\begin{pmatrix}1&1&1&1&1&0\\1&1&1&1&1&1\\0&1&1&1&1&1\\0&0&1&1&1&1\\1&1&1&1&0&0\end{pmatrix}},\quad
\overset{A(M_5)}{\begin{pmatrix}1&1&1&1&1&0\\1&1&1&1&1&1\\0&1&1&1&1&1\\0&0&1&1&1&1\\0&0&0&1&1&1\end{pmatrix}},\\
\overset{A(M_6)}{\begin{pmatrix}0&0&0&0&1&1\\1&1&1&1&1&1\\0&1&1&1&1&1\\0&0&1&1&1&1\\0&0&0&1&1&1\end{pmatrix}},\quad
\overset{A(M_7)}{\begin{pmatrix}0&0&0&0&1&1\\0&0&0&0&0&1\\0&1&1&1&1&1\\0&0&1&1&1&1\\0&0&0&1&1&1\end{pmatrix}},\quad
\overset{A(M_8)}{\begin{pmatrix}0&0&0&0&1&1\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&1&1&1&1\\0&0&0&1&1&1\end{pmatrix}},\\
\overset{A(M_9)}{\begin{pmatrix}0&0&0&0&1&1\\0&0&0&0&0&1\\0&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&1&1&1\end{pmatrix}},\quad
\overset{A(M_{10})}{\begin{pmatrix}0&0&0&0&1&1\\0&0&0&0&0&1\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\end{pmatrix}},\\
\overset{A(M_{11})}{\begin{pmatrix}1&1&1&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\end{pmatrix}},\quad
\overset{A(M_{12})}{\begin{pmatrix}1&1&1&0&0&0\\1&1&1&1&0&0\\0&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\end{pmatrix}}.
\end{gathered}$$

Their rows give us the coordinates of the vertices of the 4-simplexes $\bar{\Delta}_0^4, \bar{\Delta}_1^4, \dots, \bar{\Delta}_{12}^4 \subset \partial K^6$. Their barycenters

$$\begin{aligned}
p_{12} = p_0\Big(\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5},0,0\Big), &\qquad p_6\Big(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1,1\Big),\\
p_1\Big(1,\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5},0\Big), &\qquad p_7\Big(0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},1\Big),\\
p_2\Big(1,1,\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5}\Big), &\qquad p_8\Big(0,0,\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}\Big),\\
p_3\Big(\frac{4}{5},1,1,\frac{4}{5},\frac{3}{5},\frac{2}{5}\Big), &\qquad p_9\Big(\frac{1}{5},0,0,\frac{1}{5},\frac{2}{5},\frac{3}{5}\Big),\\
p_4\Big(\frac{3}{5},\frac{4}{5},1,1,\frac{4}{5},\frac{3}{5}\Big), &\qquad p_{10}\Big(\frac{2}{5},\frac{1}{5},0,0,\frac{1}{5},\frac{2}{5}\Big),\\
p_5\Big(\frac{2}{5},\frac{3}{5},\frac{4}{5},1,1,\frac{4}{5}\Big), &\qquad p_{11}\Big(\frac{3}{5},\frac{2}{5},\frac{1}{5},0,0,\frac{1}{5}\Big)
\end{aligned} \tag{10}$$

are the vertices of a 1-turn closed symmetric geodesic on ∂K^6.

6.4. Let $n = 7$. Applying Algorithm 1 and then Algorithm 2, we obtain the following matrices:

$$\begin{array}{cc}
A(M_0) & A(M_1)\\
\begin{pmatrix}0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&0&0\\1&1&1&1&0&0&0\\1&1&1&1&1&0&0\end{pmatrix}, &
\begin{pmatrix}1&1&1&1&1&1&0\\1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&0&0\\1&1&1&1&0&0&0\\1&1&1&1&1&0&0\end{pmatrix},\\
A(M_2) & A(M_3)\\
\begin{pmatrix}1&1&1&1&1&1&0\\1&1&1&1&1&1&1\\1&1&0&0&0&0&0\\1&1&1&0&0&0&0\\1&1&1&1&0&0&0\\1&1&1&1&1&0&0\end{pmatrix}, &
\begin{pmatrix}1&1&1&1&1&1&0\\1&1&1&1&1&1&1\\0&1&1&1&1&1&1\\1&1&1&0&0&0&0\\1&1&1&1&0&0&0\\1&1&1&1&1&0&0\end{pmatrix},\\
A(M_4) & A(M_5)\\
\begin{pmatrix}1&1&1&1&1&1&0\\1&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&0&1&1&1&1&1\\1&1&1&1&0&0&0\\1&1&1&1&1&0&0\end{pmatrix}, &
\begin{pmatrix}1&1&1&1&1&1&0\\1&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&0&1&1&1&1&1\\0&0&0&1&1&1&1\\1&1&1&1&1&0&0\end{pmatrix},\\
A(M_6) & A(M_7)\\
\begin{pmatrix}1&1&1&1&1&1&0\\1&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&0&1&1&1&1&1\\0&0&0&1&1&1&1\\0&0&0&0&1&1&1\end{pmatrix}, &
\begin{pmatrix}0&0&0&0&0&1&1\\1&1&1&1&1&1&1\\0&1&1&1&1&1&1\\0&0&1&1&1&1&1\\0&0&0&1&1&1&1\\0&0&0&0&1&1&1\end{pmatrix},
\end{array} \tag{11}$$

$$\overset{A(M_8)}{\begin{pmatrix}0&0&0&0&0&1&1\\0&0&0&0&0&0&1\\0&1&1&1&1&1&1\\0&0&1&1&1&1&1\\0&0&0&1&1&1&1\\0&0&0&0&1&1&1\end{pmatrix}},\quad \overset{A(M_9)}{\begin{pmatrix}0&0&0&0&0&1&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\0&0&1&1&1&1&1\\0&0&0&1&1&1&1\\0&0&0&0&1&1&1\end{pmatrix}},$$

$$\overset{A(M_{10})}{\begin{pmatrix}0&0&0&0&0&1&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\0&0&0&1&1&1&1\\0&0&0&0&1&1&1\end{pmatrix}},\quad \overset{A(M_{11})}{\begin{pmatrix}0&0&0&0&0&1&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\0&0&0&0&1&1&1\end{pmatrix}},$$

$$\overset{A(M_{12})}{\begin{pmatrix}0&0&0&0&0&1&1\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&0&0\end{pmatrix}},\quad \overset{A(M_{13})}{\begin{pmatrix}1&1&1&1&0&0&0\\0&0&0&0&0&0&1\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&0&0\end{pmatrix}},$$

$$\overset{A(M_{14})}{\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&1&1&0&0\\0&0&0&0&0&0&0\\1&0&0&0&0&0&0\\1&1&0&0&0&0&0\\1&1&1&0&0&0&0\end{pmatrix}}.$$

Their rows give us the coordinates of the vertices of the 5-simplexes $\bar{\Delta}_0^5, \bar{\Delta}_1^5, \dots, \bar{\Delta}_{14}^5 \subset \partial K^7$. Their barycenters

$$\begin{aligned}
p_{14} = p_0\Big(\tfrac{5}{6}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}, 0, 0\Big), &\qquad p_7\Big(\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1, 1\Big),\\
p_1\Big(1, \tfrac{5}{6}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}, 0\Big), &\qquad p_8\Big(0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1\Big),\\
p_2\Big(1, 1, \tfrac{5}{6}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}\Big), &\qquad p_9\Big(0, 0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}\Big),\\
p_3\Big(\tfrac{5}{6}, 1, 1, \tfrac{5}{6}, \tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{3}\Big), &\qquad p_{10}\Big(\tfrac{1}{6}, 0, 0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}\Big),\\
p_4\Big(\tfrac{2}{3}, \tfrac{5}{6}, 1, 1, \tfrac{5}{6}, \tfrac{2}{3}, \tfrac{1}{2}\Big), &\qquad p_{11}\Big(\tfrac{1}{3}, \tfrac{1}{6}, 0, 0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}\Big),\\
p_5\Big(\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1, 1, \tfrac{5}{6}, \tfrac{2}{3}\Big), &\qquad p_{12}\Big(\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}, 0, 0, \tfrac{1}{6}, \tfrac{1}{3}\Big),\\
p_6\Big(\tfrac{1}{3}, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{5}{6}, 1, 1, \tfrac{5}{6}, \Big), &\qquad p_{13}\Big(\tfrac{2}{3}, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{6}, 0, 0, \tfrac{1}{6}\Big)
\end{aligned} \tag{12}$$

are the vertices of a 1-turn closed symmetric geodesic on ∂K^7.

§7. Many-turn symmetric geodesics

7.1. Suppose the cube K^n, rolled along the ray l on the plane $\mathbb{R}^{n-1}$, runs over $2nk$ (rather than $2n$) $(n-1)$-cubes $Q_1, \ldots, Q_{2nk}$. Let k be the minimum number such that K^n stays on Q_{2nk} in a position obtained from the initial position by a parallel shift. From (2) it follows that this happens when $t = \frac{2nk}{n-1}$ is an integer for the first time. Therefore,

$$k = \begin{cases} n-1 & \text{for } n \text{ even}, \\ \frac{n-1}{2} & \text{for } n \text{ odd}. \end{cases} \tag{13}$$

7.2. If l' is the ray parallel to l and emanating from any point $p'_0 \in \operatorname{Int} \Delta_0^{n-2}$, then the length of the segment of l' from p'_0 to the exit from Q_{2nk} is one and the same, independently of the choice of p'_0. Transferring these segments by Algorithms 1 and 2 to ∂K^n, we obtain k-turn *closed* geodesics. (And 1-turn geodesics of type (6), (8), (10), (12) are passed k times.)

The closed geodesics obtained may be nonsymmetric. For example, for $n = 3$ we have $k = 1$. All these 1-turn geodesics are closed. However, only one of them is symmetric, namely, that emanating from the barycenter p_0 of $\bar{\Delta}_0^2$.

7.3. Necessary conditions for symmetry of a closed k-turn geodesic. Lemma 3.6 in [1] leads to the following two conclusions.

1) The number of turns of a symmetric simple closed geodesic in question is necessarily odd.

2) In order that a k-turn simple closed geodesic in question be symmetric, the point $p'_{nk} \in \partial K^n$ must be symmetric to p'_0 with respect to O_n.

These conditions reduce the class of closed geodesics that must be checked for symmetry.

We proceed to examples where we characterize the initial point p'_0 by its barycentric coordinates $\lambda_1, \ldots, \lambda_{n-1}$ in the simplex $\bar{\Delta}_0^{n-2}$ and denote p'_0 by $p_0(\lambda_1, \ldots, \lambda_{n-1})$.

7.4. Let $n = 4$. We consider the geodesic γ on ∂K^4 emanating in the direction of the ray l from the point

$$p_0(\lambda_1, \lambda_2, \lambda_3) = \lambda_1(0,0,0,0) + \lambda_2(1,0,0,0) + \lambda_3(1,1,0,0).$$

The matrix $A(M_8)$ in (5) shows that after passing along all eight facets of K^4, γ comes to the point

$$p'_8 = \lambda_1(1,0,0,0) + \lambda_2(1,1,0,0) + \lambda_3(0,0,0,0) = p_8(\lambda_3, \lambda_1, \lambda_2).$$

The points p'_0 and p'_8 coincide only for $\lambda_1 = \lambda_2 = \lambda_3$, which brings us back to the 1-turn geodesic (6).

In the other cases, γ goes to the second turn, which starts at the point $p_8(\lambda_3, \lambda_1, \lambda_2)$ and ends at $p_{16}(\lambda_2, \lambda_3, \lambda_1)$, after which the third turn starts, ending at the point $p_{24}(\lambda_1, \lambda_2, \lambda_3) = p'_0$, and γ closes.

The knowledge of the matrices (5) makes it possible to list all the 24 vertices of such a 3-turn geodesic γ. The list confirms the fact already known from [1]: γ is symmetric for any positive barycentric coordinates $\lambda_1, \lambda_2, \lambda_3$ of the initial point p'_0.

7.5. Let $n = 5$. Then $k = 2$, and each geodesic γ emanating from a point $p'_0 = p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \operatorname{Int} \bar{\Delta}_0^3$ closes after 2 turns. Only the geodesics that close already after the first turn can be symmetric.

The matrix $A(M_{10})$ in (6) shows that γ comes to the point $p_{10}(\lambda_3, \lambda_4, \lambda_1, \lambda_2)$ after passing along all ten facets of K^5. It closes if and only if $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_4$. One of such geodesics is the symmetric geodesic (8), for which we have $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{1}{4}$.

Consider a 1-turn geodesic γ such that $\lambda_1 = \lambda_3 = \frac{1}{4} + \varepsilon$ and $\lambda_2 = \lambda_4 = \frac{1}{4} - \varepsilon$, where $0 < |\varepsilon| < \frac{1}{4}$. The matrices $A(M_0)$ and $A(M_5)$ in (6) give us the coordinates of p_0' and p_5', so that we can verify the condition of Subsection 7.3:

$$\begin{aligned} p_0' &= \left(\frac{1}{4}+\varepsilon\right)(0,0,0,0,0) + \left(\frac{1}{4}-\varepsilon\right)(1,0,0,0,0) \\ &\quad + \left(\frac{1}{4}+\varepsilon\right)(1,1,0,0,0) + \left(\frac{1}{4}-\varepsilon\right)(1,1,1,0,0) \\ &= \left(\frac{3}{4}-\varepsilon, \frac{1}{2}, \frac{1}{4}-\varepsilon, 0, 0\right), \\ p_5' &= \left(\frac{1}{4}+\varepsilon\right)(0,0,0,1,1) + \left(\frac{1}{4}-\varepsilon\right)(1,1,1,1,1) \\ &\quad + \left(\frac{1}{4}+\varepsilon\right)(0,1,1,1,1) + \left(\frac{1}{4}-\varepsilon\right)(0,0,1,1,1) \\ &= \left(\frac{1}{4}-\varepsilon, \frac{1}{2}, \frac{3}{4}-\varepsilon, 1, 1\right), \end{aligned}$$

and the midpoint of the segment $p_0'p_5'$ is the point $\left(\frac{1}{2}-\varepsilon, \frac{1}{2}, \frac{1}{2}-\varepsilon, \frac{1}{2}, \frac{1}{2}\right)$, and not $O_5 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Therefore, (8) is a unique symmetric geodesic in the class considered.

7.6. Let $n = 6$. Then $k = 5$. In accordance with the matrix $A(M_{12})$ in (9), the geodesic γ emanating from a point $p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in \operatorname{Int} \bar{\Delta}_0^4$ and parallel to the ray l comes to the point $p_{12}(\lambda_3, \lambda_4, \lambda_5, \lambda_1, \lambda_2)$ after the first turn. It closes after one turn only for

$$\lambda_1 = \lambda_3 = \lambda_5 = \lambda_2 = \lambda_4 = \frac{1}{5},$$

which gives us the geodesic (10).

For all the other initial points p_0, γ closes only after 5 turns, which consist of the vertices $p_0, \dots, p_{11}$, $p_{12}, \dots, p_{23}$, $p_{24}, \dots, p_{35}$, $p_{36}, \dots, p_{47}$, and $p_{48}, \dots, p_{59}$, respectively. Below we present the list of the vertices $p_0, \dots, p_{11}$ of the first turn, which starts at the point $p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$. The vertices $p_{12}, \dots, p_{23}$, $p_{24}, \dots, p_{35}$, $p_{36}, \dots, p_{47}$, and $p_{48}, \dots, p_{59}$ differ from the points $p_0, \dots, p_{11}$ only by the replacement of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ with $(\lambda_3, \lambda_4, \lambda_5, \lambda_1, \lambda_2)$, $(\lambda_5, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$, $(\lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_1)$, and $(\lambda_4, \lambda_5, \lambda_1, \lambda_2, \lambda_3)$, respectively; $p_{60} = p_0$. The list confirms that the points $p_{30}, \dots, p_{59}$ are symmetric to the points $p_0, \dots, p_{29}$ with respect to O_6.

$$\begin{aligned} p_0 &= \lambda_1(000000) + \lambda_2(100000) + \lambda_3(110000) + \lambda_4(111000) + \lambda_5(111100), \\ p_1 &= \lambda_1(111110) + \lambda_2(100000) + \lambda_3(110000) + \lambda_4(111000) + \lambda_5(111100), \\ p_2 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(110000) + \lambda_4(111000) + \lambda_5(111100), \\ p_3 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(111000) + \lambda_5(111100), \\ p_4 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(111100), \\ p_5 &= \lambda_1(111110) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(000111), \\ p_6 &= \lambda_1(000011) + \lambda_2(111111) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(000111), \\ p_7 &= \lambda_1(000011) + \lambda_2(000001) + \lambda_3(011111) + \lambda_4(001111) + \lambda_5(000111), \\ p_8 &= \lambda_1(000011) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(001111) + \lambda_5(000111), \\ p_9 &= \lambda_1(000011) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(100000) + \lambda_5(000111), \end{aligned}$$

$$p_{10} = \lambda_1(000011) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(100000) + \lambda_5(110000),$$
$$p_{11} = \lambda_1(111000) + \lambda_2(000001) + \lambda_3(000000) + \lambda_4(100000) + \lambda_5(110000).$$

The 1-turn geodesic (10) and the family of 5-turn geodesics fill a 5-dimensional topological solid torus $T \subset \partial K^6$ and form a Seifert fibration in T [2]. The simplexes $\bar{\Delta}_0^2, \dots, \bar{\Delta}_{12}^4 = \bar{\Delta}_0^4$ are sections of T by 4-planes *not orthogonal* to rectilinear parts of this family of geodesics.

7.7. Let $n = 7$. Then $k = 3$. The matrix $A(M_{14})$ in (11) shows that the geodesic γ emanating in the direction of l from a point $p_0(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in \bar{\Delta}_0^5$ comes to the point $p_{14}(\lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_1, \lambda_2)$ after the first turn. Such a geodesic γ closes after the first turn only if $\lambda_1 = \lambda_3 = \lambda_5$ and $\lambda_2 = \lambda_4 = \lambda_6$. One such geodesic is the symmetric geodesic (12), for which we have

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \frac{1}{6}.$$

Suppose that

$$\lambda_1 = \lambda_3 = \lambda_5 = \frac{1}{6} + \varepsilon \quad \text{and} \quad \lambda_2 = \lambda_4 = \lambda_6 = \frac{1}{6} - \varepsilon,$$

where $0 < |\varepsilon| < \frac{1}{6}$. The matrices $A(M_0)$ and $A(M_7)$ in (11) give us the coordinates of p_0' and p_7':

$$p_0' = \left(\frac{5}{6} - \varepsilon, \frac{2}{3}, \frac{1}{2} - \varepsilon, \frac{1}{3}, \frac{1}{6}, 0, 0\right),$$
$$p_7' = \left(\frac{1}{6} - \varepsilon, \frac{1}{3}, \frac{1}{2} - \varepsilon, \frac{2}{3}, \frac{5}{6} - \varepsilon, 0, 0\right),$$

so that we can verify the conditions of Subsection 7.3, and the midpoint of the segment p_0p_7 does not lie at $O_7 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Hence, (12) is a unique 1-turn geodesic in the class considered.

All other geodesics, for which the relations $\lambda_1 = \lambda_3 = \lambda_5$ and $\lambda_2 = \lambda_4 = \lambda_6$ are violated, close only after the third turn. Are some of them symmetric? The answer turns out to be rather unexpected.

The necessary condition (see Subsection 7.3) requiring that p_0 and p_{21} be symmetric with respect to O_7 is fulfilled only if

$$\lambda_1 = \lambda_4, \quad \lambda_2 = \lambda_5, \quad \lambda_3 = \lambda_6, \tag{14}$$

i.e., for the initial points of the form

$$p_0\left(\frac{\mu_1}{2}, \frac{\mu_2}{2}, \frac{\mu_3}{2}, \frac{\mu_1}{2}, \frac{\mu_2}{2}, \frac{\mu_3}{2}\right), \tag{15}$$

where $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$, and $\mu_1 + \mu_2 + \mu_3 = 1$. Such points form the interior of the triangle with vertices at the midpoints of the edges $D_1^0D_4^0$, $D_2^0D_5^0$, and $D_3^0D_6^0$ of $\bar{\Delta}_0^4$.

The matrices (11) give us all the 42 vertices of the geodesic γ emanating from the point (15) in the direction of the ray l. The list of these vertices confirms the symmetry of γ independently of the choice of μ_1, μ_2, and μ_3. Here is only the list of vertices $p_0, \dots, p_6$ of the first half of the first turn and of vertices $p_{21}, \dots, p_{27}$ in the middle of the second turn, which are symmetric to $p_0, \dots, p_6$:

$$p_0 = \frac{\mu_1}{2}(1110000) + \frac{\mu_2}{2}(2111000) + \frac{\mu_3}{2}(2211100),$$
$$p_1 = \frac{\mu_1}{2}(2221110) + \frac{\mu_2}{2}(2111000) + \frac{\mu_3}{2}(2211100),$$
$$p_2 = \frac{\mu_1}{2}(2221110) + \frac{\mu_2}{2}(2222111) + \frac{\mu_3}{2}(2211100),$$

$$
\begin{aligned}
p_3 &= \frac{\mu_1}{2}(2221110) + \frac{\mu_2}{2}(2222111) + \frac{\mu_3}{2}(1222211),\\
p_4 &= \frac{\mu_1}{2}(1122221) + \frac{\mu_2}{2}(2222111) + \frac{\mu_3}{2}(1222211),\\
p_5 &= \frac{\mu_1}{2}(1122221) + \frac{\mu_2}{2}(1112222) + \frac{\mu_3}{2}(1222211),\\
p_6 &= \frac{\mu_1}{2}(1122221) + \frac{\mu_2}{2}(1112222) + \frac{\mu_3}{2}(0111222),\\
p_{21} &= \frac{\mu_1}{2}(1112222) + \frac{\mu_2}{2}(0111222) + \frac{\mu_3}{2}(0011122),\\
p_{22} &= \frac{\mu_1}{2}(0001112) + \frac{\mu_2}{2}(0111222) + \frac{\mu_3}{2}(0011122),\\
p_{23} &= \frac{\mu_1}{2}(0001112) + \frac{\mu_2}{2}(0000111) + \frac{\mu_3}{2}(0011122),\\
p_{24} &= \frac{\mu_1}{2}(0001112) + \frac{\mu_2}{2}(0000111) + \frac{\mu_3}{2}(1000011),\\
p_{25} &= \frac{\mu_1}{2}(1100001) + \frac{\mu_2}{2}(0000111) + \frac{\mu_3}{2}(1000011),\\
p_{26} &= \frac{\mu_1}{2}(1100001) + \frac{\mu_2}{2}(1110000) + \frac{\mu_3}{2}(1000011),\\
p_{27} &= \frac{\mu_1}{2}(1100001) + \frac{\mu_2}{2}(1110000) + \frac{\mu_3}{2}(2111000).
\end{aligned}
$$

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Received 23/APR/2003

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